Poisson distribution: Let λ be the probability per unit time of an event, say the decay of a nucleus. To calculate $p_n(t)$ the probability of exactly n events in the time interval t. One has

$$p_n(t+t') = \sum_{a+b=n} p_a(t)\, p_b(t') \quad \Rightarrow \quad \sum s^n p_n(t+t') = \sum s^n p_n(t) \sum s^n p_n(t')$$

because events in disjoint time intervals are independent. For a small time ε we have

$$\begin{aligned}\sum s^n p_n(\varepsilon) &= p_0(\varepsilon) + s p_1(\varepsilon) + O(\varepsilon^2) \\ &= 1 - \lambda\varepsilon + s\lambda\varepsilon = e^{(\lambda s - \lambda)\varepsilon}\end{aligned}$$

so

$$\sum s^n p_n(t) = e^{(\lambda s-\lambda)t} = e^{-\lambda t} \sum \frac{\lambda^n t^n}{n!} s^n$$

or
$$p_n(t) = e^{-\lambda t} \frac{\lambda^n t^n}{n!}$$

Consider an oscillator $H_0 = \mu a^* a$ with perturbation $V = \text{const}\cdot q = c(a+a^*)$, or better take $V = J(t)a^* + \bar{J}(t)a$. Then the S-matrix for this perturbation is

$$S = T\, e^{-i\int (J(t)e^{i\mu t} a^* + \bar{J}(t) e^{-i\mu t} a)\,dt}$$

$$= \underbrace{e^{-\int_{t>t'} \bar{J}(t) e^{-i\mu(t-t')} J(t')}}_{\langle 0|S|0\rangle} e^{\left(-i\int \bar{J}(t) e^{+i\mu t} dt\right) a^*} e^{\left(-i\int \bar{J}(t) e^{-i\mu t} dt\right) a}$$

Put
$$\gamma = -i\int J(t) e^{i\mu t}\,dt .$$

Then
$$\begin{aligned} S|0\rangle &= \langle 0|S|0\rangle\, e^{\gamma a^*}|0\rangle \\ &= \langle 0|S|0\rangle \sum_n \frac{\gamma^n}{n!} (a^*)^n |0\rangle \end{aligned}$$

Hence the amplitude for producing n-particles is

$$\langle n | S | 0 \rangle = \langle 0 | S | 0 \rangle \frac{\gamma^n}{\sqrt{n!}}$$

and the probability of producing n-particles is

$$p_n = |\langle n | S | 0 \rangle|^2 = |\langle 0 | S | 0 \rangle|^2 \frac{(|\gamma|^2)^n}{n!}$$

This is a probability distribution since S is unitary, so

$$|\langle 0 | S | 0 \rangle|^2 = e^{-|\gamma|^2}$$

so we get the Poisson distribution

$$p_n = e^{-|\gamma|^2} \frac{(|\gamma|^2)^n}{n!}$$

Suppose now that $V = J(t)\, a^* + \bar{J}(t)\, a$ is a constant for $0 \le t \le T$ and zero elsewhere, say $V = c(a^* + a)$ in $[0, T]$. Then

$$\gamma = -i \int_0^T c\, e^{i\mu t}\, dt = -ic \frac{e^{i\mu T} - 1}{i\mu} = c \frac{1 - e^{i\mu T}}{\mu}$$

$$\lambda = |\gamma|^2 = \left(\frac{2c}{\mu}\right)^2 \left(\sin \frac{\mu T}{2}\right)^2 \sim c^2 T^2 \qquad \text{for } T \text{ small}$$

Thus $p_n(T)$ is not a Poisson distribution in time T because λ is proportional to T^2.

February 9, 1980

Start with H_0 on $\mathcal{H}_1$ and extend it to fermion Fock space $\Lambda\mathcal{H}_1$. Assume 0 is not an eigenvalue of H_0, so that there is a unique ground state in $\Lambda\mathcal{H}_1$ obtained as follows. Split $\mathcal{H}_1$ into $\mathcal{H}_1^- \oplus \mathcal{H}_1^+$ where $\mathcal{H}_1^+$ is the sum of the positive eigenspaces of H_0. Then the ground state is the line $\Lambda^p\mathcal{H}_1^-$ where $p = \dim \mathcal{H}_1^-$.

Let us be given an orthonormal basis $|n\rangle$ for $\mathcal{H}_1$ and let a_n, a_n^* be the corresponding annihilation + creation operators on $\Lambda\mathcal{H}_1$. The Green's matrix is

$$G_0(m,t;n,t') = \langle T[a_m(t)\, a_n^*(t')]\rangle$$

Let $\quad H_0 = \sum \mu_{mn} a_m^* a_n \quad , \qquad \mu_{mn} = \langle m|H_0|n\rangle$. Then

$$[H_0, a_m] = \sum' \mu_{m'n} \underbrace{[a_{m'}^* a_n, a_m]}_{-\{a_{m'}^*, a_m\} a_n} = -\sum \mu_{m'n}\delta_{mm'} a_n = -\sum_n \mu_{mn} a_n$$

so that
$$\frac{d}{dt} a_m(t) = \frac{d}{dt} e^{tH_0} a_m e^{-tH_0} = e^{tH_0}[H_0, a_m]e^{-tH_0}$$
$$= -\sum' \mu_{mn} a_n(t)$$

So this shows

$$\frac{d}{dt} G_0(m,t;p,t') = -\sum' \mu_{mn}\, G_0(n,t;p,t') \qquad t \neq t'$$

and as t crosses t' one gets the identity matrix. Thus G_0 is the L^2-inverse for the matrix operator $\frac{d}{dt} + H_0$ on $\mathcal{H}_1$.

It seems to be consistent to use the matrix notation

$$G_0(t,t') = \langle T[a(t)a^*(t')]\rangle$$

and formulas like

$$\frac{\int e^{-\int \bar\psi \left(\frac{d}{dt}+H_0\right)\psi\, dt}\ \psi(t)\bar\psi(t')}{\int e^{-\int \bar\psi\left(\frac{d}{dt}+H_0\right)\psi\, dt}} = G(t,t')$$

[redacted] provided we interpret $\psi(t)$ as a kind of column vector and $\bar\psi(t)$ as a row vector.

[redacted] Next notice that from the decomposition $\mathcal{H}_1 = \mathcal{H}_1^- \oplus \mathcal{H}_1^+$ we get a decomposition

$$\Lambda \mathcal{H}_1 = \Lambda \mathcal{H}_1^- \otimes \Lambda \mathcal{H}_1^+$$

that is, an intrinsic decomposition [redacted] labelled by the number of particles and number of holes. [redacted] Also we can describe the 1-particle space as spanned by the elements $a_n^* \Phi_0$ and the 1-hole space as spanned by the elements $a_n \Phi_0$.

How does this picture persist if H_0 is perturbed slightly. First consider the case where $H = H_0 + H'$ [redacted] on $\Lambda \mathcal{H}_1$ comes from $\mathcal{H}_1$, i.e. is a derivation. Since H' is small we don't change the number of negative + positive eigenvalues in going from H_0 to H. Since the above discussion [redacted] goes the same for H as for H_0, we get the 1-particle states by applying a_n^* to Φ_0 the ground state for H, etc. Φ_0 is a decomposable vector of $\Lambda^p \mathcal{H}_1$, so that its[redacted] annihilator in the space W spanned by the operators a_n, a_n^* is half the dimension. In fact the space W has a [redacted] natural quadratic form $\{a,b\} = ab + ba$, and the annihilator of a decomposable vector is a maximal isotropic subspace of W.

The question arises as to what the story is when the perturbation is no longer a one-particle operator. The Green's function is defined by the same formula

$$G(t,t') = \langle \Psi_0 | T[a(t)a^*(t')] | \Psi_0 \rangle .$$

I don't think $a(t)$ lies in W; here $a(t) = e^{tH} a e^{-tH}$.

Let's recall the Lehmann type structure for the Green's function. If $t > t'$ then

$$G(t,t') = \langle \Psi_0 | a e^{-tH} e^{t'H} a^* | \Psi_0 \rangle e^{+E_0(t-t')}$$

I am going to suppose that H' commutes with $\hat{N} = \sum a_n^* a_n$. Let $\Psi_0 \in \Lambda^p$, so that $a^*\Psi_0 \in \Lambda^{p+1}$ and let Ψ_n^{p+1} run over the H eigenvectors in Λ^{p+1} with eigenvalues E_n^{p+1}. Then for $t > t'$

$$G(t,t') = \sum_n e^{-(E_n^{p+1} - E_0)(t-t')} \underbrace{\langle \Psi_0 | a | \Psi_n^{p+1} \rangle}_{\text{column vector}} \underbrace{\langle \Psi_n^{p+1} | a^* | \Psi_0 \rangle}_{\text{its conjugate transpose}}$$

and similarly for $t < t'$ one has

$$G(t,t') = \sum_n (-) e^{+(E_n^{p-1} - E_0)(t-t')} \langle \Psi_0 | a^* | \Psi_n^{p-1} \rangle \langle \Psi_n^{p-1} | a | \Psi_0 \rangle$$

Actually the last two factors should be written in the different order so as to have a matrix.

It should be possible to take any H on $\Lambda \mathcal{H}_1$ close to H_0 and write it as a polynomial in the a, a^* operator, since $\mathrm{End}(\Lambda \mathcal{H}_1)$ is Clifford algebra on the $\{a, a^*\}$ space.

So I can arrange that $\Phi_0 \in \Lambda^p$ be indecomposable, or better, you want the spaces of $a^*\Phi_0$ and $a\Phi_0$ to project into all the H eigenspaces of Λ^{p+1} and Λ^{p-1} resp. This seems to imply that the Green's [redacted] matrix will involve more that $\dim(\mathcal{H}_1)$ exponents. It seems that the particle hole division doesn't work nicely in the "non-linear" case.

February 10, 1980

Review EM: Maxwell's equations are

$$\nabla \cdot B = 0 \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot E = \rho \qquad \nabla \times B = j + \frac{\partial E}{\partial t}$$

From $\nabla \cdot B = 0$ we can solve $\boxed{B = \nabla \times A}$. Then from

$$\nabla \times E = -\nabla \times \partial_t A$$

we can solve for A_0 such that

$$\boxed{E = -\nabla A_0 - \partial_t \vec{A}}$$

Finally if I add $(\partial_t f, -\nabla f)$ to (A_0, A), then B, E don't change, but $\partial_t A_0 + \nabla \cdot A$ changes by $(\partial_t^2 - \nabla^2) f$, hence

$$\boxed{\partial_t A_0 + \nabla \cdot A = 0}$$

can be arranged. Hence the first two of Maxwell's equations allow one to find an (A_0, A) satisfying the above boxed equations. The 2nd two of Maxwell's equations are

$$\rho = \nabla(-\nabla A_0 - \partial_t \vec{A}) = (\partial_t^2 - \nabla^2) A_0$$

$$\vec{j} = \underbrace{\nabla \times (\nabla \times A)}_{\nabla(\nabla \cdot A) - \nabla^2 A} + \partial_t(-\nabla A_0 - \partial_t A) = (\partial_t^2 - \nabla^2) A.$$

or $$\boxed{(\partial_t^2 - \nabla^2)(A_0, A) = (\rho, j)}$$

The force on a particle of charge e is given by the Lorentz formula

$$F = e(E + v \times B)$$

$$= e\Big(-\nabla A_0 - \partial_t \vec{A} + \underbrace{v \times (\nabla \times A)}_{\nabla(v\cdot A) - (v\cdot\nabla)A}\Big)$$

$$\frac{d\text{[redacted]}_{kin}}{dt} = e\left(-\nabla A_0 + \nabla(v\cdot A) - \frac{d}{dt}\vec{A}\right) \qquad p_{kin} = \frac{m\text{[redacted]}v}{\sqrt{1-v^2}}$$

$$= \frac{\partial}{\partial v}\left(-m\text{[redacted]}\sqrt{1-v^2}\right)$$

or

$$\frac{d}{dt}\left(\text{[redacted]}_{kin} + e\text{[redacted]}\right) = \nabla\left(-eA_0 + e v\cdot A\right)$$

This is the Lagrange DE for $L = -m\text{[redacted]}\sqrt{1-v^2} - eA_0 + e v\cdot A$.

Then

$$p = \frac{\partial L}{\partial v} = p_{kin} + eA$$

$$H = p\cdot v - L = \frac{m\text{[redacted]}v^2}{\sqrt{1-v^2}} + m\text{[redacted]}\sqrt{1-v^2} + eA_0$$

$$H = \frac{m\text{[redacted]}}{\sqrt{1-v^2}} + eA_0 = E_{kin} + eA_0$$

The momentum-energy relation becomes

$$(E - eA_0)^2 = (p\text{[redacted]}eA)^2 + m^2$$

or

$$E - eA_0 = \pm\sqrt{(p-eA)^2 + m^2}$$

Feb. 10, 1980:

At this point I have a good understanding of how to handle a fermion situation: H_0 on $\Lambda \mathcal{H}_1$, where H_0 is given on $\mathcal{H}_1$. So far I have been thinking of $\mathcal{H}_1$ being finite-dimensional, in any case, H_0 with discrete spectrum. The idea now is to graduate to the case of continuous spectrum and scattering and to understand the renormalization problems that arise.

The good case to look at is Schwinger V: the Dirac field in the presence of an external EM field A of compact support in space-time. The Dirac equation is obtained as follows. We want matrices α, β such that

$$H_0 = \alpha \frac{1}{i}\nabla + \beta m$$

H_0 is hermitian and $\qquad H_0^2 = -\nabla^2 + m^2$, which gives

$$\alpha = \alpha^*, \ \beta = \beta^* \qquad \alpha^2 = 1, \ \beta^2 = 1, \quad \alpha\beta + \beta\alpha = 0.$$

The Green's matrix is given by $\left(\frac{d}{dt} + H_0\right)^{-1}$ or

$$G_0(t,t') = \frac{\int e^{-\int \psi^*\left(\frac{d}{dt}+H_0\right)\psi}\ \psi(t)\psi^*(t')}{\int e^{-\int \psi^*\left(\frac{d}{dt}+H_0\right)\psi}}$$

where ψ has components for each degree of freedom. In this case one gets a degree of freedom for each point $\vec{x}$ of space together with the "spinor" components of ψ. So

$$\int \psi^*\left(\frac{d}{dt} + H_0\right)\psi$$

is an integral over $t, \vec{x}$ and sum over spinor components.

It is customary to put things in Lorentz-invariant notation. Thus the Dirac equation

$$i\frac{\partial\psi}{\partial t} = H_0\psi \qquad \text{or} \qquad \left(\frac{1}{i}\partial_t + \alpha\frac{1}{i}\nabla_{\vec{x}} + \beta m\right)\psi = 0$$

is written

$$\left(\gamma\cdot\frac{1}{i}\partial + m\right)\psi = 0$$

where

$$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha^i \qquad \text{so that}$$

$$(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1, \quad \{\gamma^i,\gamma^j\} = 0 \quad i \neq j.$$

Instead of Dirac I am working with imaginary time

$$\left(\frac{\partial}{\partial t} + H_0\right)\psi = 0 \qquad \text{or}$$

$$\left(\gamma\cdot\frac{1}{i}\partial + m\right)\psi = 0$$

where now $\gamma^0 = i\beta$ satisfies $(\gamma^0)^2 = -1$. Thus the γ's generate the Clifford algebras C_4 which I think is isomorphic to 4×4 matrices over $\mathbb{C}$.

Change

$$\int \psi^*\left(\frac{d}{dt} + H_0\right)\psi \quad = \quad \int \tilde{\psi}\left(\gamma\frac{1}{i}\partial + m\right)\psi$$

where

$$\tilde{\psi} = \psi^*\beta.$$

Now our Green's functions is

$$G_0(x,x') = \frac{\int e^{-\int\tilde{\psi}(\gamma\frac{1}{i}\partial + m)\psi\,dx}\,\psi(x)\tilde{\psi}(x')}{\int e^{-\int\tilde{\psi}(\gamma\frac{1}{i}\partial+m)\psi\,dx}}$$

where now $x, x' \in \mathbb{R}^4$. This is the kernel for the operator $\left[\gamma\frac{1}{i}\partial + m\right]^{-1}$ which should be elliptic.

Consider 2-dimensions where the algebra over $\mathbb{R}$ generated by γ^0, γ^1 is isomorphic to $\mathbb{H}$ with $\gamma^0 = i$ and $\gamma^1 = j$. It seems that the simplest choice is

$$\gamma^0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then for $p = (p_0, p_1)$ real we have that

$$\gamma . p + m = \begin{pmatrix} ip_0 + m & p_1 \\ -p_1 & -ip_0 + m \end{pmatrix}$$

has determinant $\qquad m^2 + p_0^2 + p_1^2 \geq m^2.$

Next I want to understand how the gauge field A enters.

February 11, 1980

Consider the Dirac equation[redacted] in 2 space-time dimensions with an external [redacted] stationary EM field. It is $i\partial_t \psi = H\psi$ where

$$H = eA_0 + \alpha\left(\frac{1}{i}\partial_x - eA_1\right) + \beta m$$

A_0, A_1 are functions of x

where $\alpha^2 = \beta^2 = 1$ and $\alpha\beta + \beta\alpha = 0$. Absorb e into A. Take

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

whence

$$H = \begin{pmatrix} A_0 + \frac{1}{i}\frac{d}{dx} - A_1 & m \\ m & A_0 - \frac{1}{i}\frac{d}{dx} + A_1 \end{pmatrix}$$

The spectrum of H is unchanged if we conjugate by a unitary transformation. Since

$$e^{-i\theta}\,\frac{1}{i}\frac{d}{dx}\,e^{i\theta} = \frac{1}{i}\frac{d}{dx} + \theta'$$

we can remove A_1 by conjugating by a scalar matrix depending on x: $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$

namely we choose θ so that $\theta' = A_1$. Next if we choose $\theta_1' = A_0$ and conjugate by

$$\begin{pmatrix} e^{-i\theta_1} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix}$$

we remove A_0 [redacted] but the off-diagonal entries change giving

$$H = \begin{pmatrix} \frac{1}{i}\frac{d}{dx} & p(x) \\ \overline{p(x)} & -\frac{1}{i}\frac{d}{dx} \end{pmatrix}$$

where $|p(x)| = m$, in fact $p(x) = m e^{2i\theta_1(x)}$.
If $A_0(x)$ has compact support, then $\theta_1(x)$ and hence $p(x)$ must be constant for large $|x|$.

The goal is now to understand the 2nd quantization of the Dirac field with A [redacted] as a perturbation of the case $A = 0$.

$$H = \underbrace{\alpha \tfrac{1}{i}\partial_x + \beta m}_{H_0} + \underbrace{A_0 - \alpha A_1}_{H'}$$

$$H_0 = \begin{pmatrix} \frac{1}{i}\frac{d}{dx} & m \\ m & -\frac{1}{i}\frac{d}{dx} \end{pmatrix} \qquad H' = \begin{pmatrix} A_0 - A_1 & 0 \\ 0 & A_0 + A_1 \end{pmatrix}$$

Both H_0, H operate on $\mathcal{H}_1 = L^2(\mathbb{R})^{\oplus 2}$

$$G_0(t,t') = \left(\frac{d}{dt} + H_0\right)^{-1}.$$

I need to decompose $\mathcal{H}_1$ into eigenstates for H_0. First decompose under translation; this gives states $v e^{ikx}$ of definite momentum k, where $v \in \mathbb{C}^2$. On the space of these states

$$H_0^{(k)} = \begin{pmatrix} k & m \\ m & -k \end{pmatrix}$$

which has eigenvalues $\pm\omega(k)$, $\omega(k) = \sqrt{k^2+m^2}$; the eigenvectors are

$$\left(H_0^{(k)} - \omega\right)\begin{pmatrix} m \\ \omega - k \end{pmatrix} = 0 \qquad \left(H_0^{(k)} + \omega\right)\begin{pmatrix} m \\ -\omega - k \end{pmatrix} = 0$$

So for each momentum k we have two eigenvectors for H_0, one with energy $\omega(k)$ and the other with energy $-\omega(k)$.

Problem: We have these two operators H_0 and H on $\mathcal{H}_1 = L^2(\mathbb{R})^{\oplus 2}$, and the goal is to obtain the Fock space for H from the Fock space for H_0. Recall that the Fock spaces are constructed by splitting $\mathcal{H}_1$ into negative and positive subspaces for the operator and then taking $\Lambda \mathcal{H}_1^- \otimes \Lambda \mathcal{H}_1^+$. From this description it is not clear how you relate the Fock spaces for H_0 and H.

The operator H_0 is self-adjoint so for any μ we can split $\mathcal{H}$ into the part where $H_0 \leq \mu$ and where $H_0 > \mu$. Hence one gets a Fock space for each value of μ. Presumably these are canonically isomorphic provided there are only finitely eigenstates with eigenvalues between μ_1, μ_2.

Question: What does one need to define a Fock space with vacuum state belonging to $\mathcal{H}_1$? $\mathcal{H}_1$ gives one a Clifford algebra: If an orthonormal basis is chosen one gets operators a_n^*, a_n and canonical commutation relations.

February 13, 1980

The goal is to understand 2nd quantization of the Dirac field in the presence of a stationary EM field A as a perturbation of the $A=0$ case. Start with the Dirac operators

$$H_0 = \alpha \frac{1}{i}\frac{\partial}{\partial x} + \beta m$$

$$H = \underbrace{\alpha \frac{1}{i}\frac{\partial}{\partial x} + \beta m}_{H_0} + \underbrace{(A_0 - \alpha A_1)}_{H'}$$

which are self-adjoint operators on $\mathcal{H} = L^2(\mathbb{R})^{\oplus 2}$ (2 dim'l space-time).
The second quantization is constructed by decomposing $\mathcal{H}$ into $\mathcal{H}^- \oplus \mathcal{H}^+$ for the operator, and then constructing the Fock space $\Lambda\mathcal{H}^- \otimes \Lambda\mathcal{H}^+$ which is a representation of the Clifford alg. of $\mathcal{H}$ having a vacuum state killed by the annihilation operators from $\mathcal{H}^+$ and the creation ops from $\mathcal{H}^-$. Starting from H_0 and H we therefore get two Fock spaces.

The idea now is that if H is a sufficiently weak perturbation of H_0, then these Fock spaces are canonically isomorphic. The reason is that the Fock space depends only on the subspace $\mathcal{H}^-$ of $\mathcal{H}$, and that if one has subspaces $\mathcal{H}^- \subset \mathcal{H}_1^-$ with $\mathcal{H}_1^-/\mathcal{H}^-$ finite-dimensional it is clear that the Fock spaces are canonically isomorphic. This should generalize to subspaces $\mathcal{H}^-, \mathcal{H}_1^-$ which are commensurable in a suitable trace class sense.

February 15, 1980

Situation: I start with a self-adjoint operator H_0 on a Hilbert space $\mathcal{H}$ [redacted] such that H_0 does not have 0 in its discrete spectrum. Then I can split $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$ where $H_0 < 0$ on $\mathcal{H}^-$, etc., and I can form Fock space

$$\Lambda \mathcal{H}^- \otimes \Lambda \mathcal{H}^+$$

which is a representation of the canonical commutation relations for creation and annihilation ops. belonging to $\mathcal{H}$. Next I want to consider a perturbation $H = H_0 + H'$ of H_0. I feel that if the perturbation is sufficiently weak, then in the above Fock space there should be a unit vector Ψ which will be the ground state for H.

To be more precise the operator H on $\mathcal{H}$ [redacted] leads to a decomposition $\mathcal{H} = \mathcal{H}_1^- \oplus \mathcal{H}_1^+$ and we want Ψ to be annihilated by the creators from $\mathcal{H}_1^-$ and annihilators from $\mathcal{H}_1^+$.

The above problem is kinematical rather than dynamical. All one needs is the splittings of $\mathcal{H}$. It should be a problem in linear algebra.

Generalize à la Segal. One forms [redacted] the Clifford algebra with generators [redacted] a_v, a_v^* $v \in \mathcal{H}$ satisfying

$$\{a_v, a_{v'}\} = \{a_v^*, a_{v'}^*\} = 0$$

$$\{a_v, a_{v'}^*\} = \langle v | v' \rangle$$

and $v \mapsto a_v^*$ is $\mathbb{C}$-linear. [redacted] Next point is to understand the $*$-autos. of this algebra. I think that this Clifford algebra depends only on the underlying real Hilbert space of $\mathcal{H}$. In effect $*$ gives a conjugation on the span of the operators a_v, a_v^* whose real elements are of the form $a_v + a_v^*$, $v \in \mathcal{H}$. Also

$$(a_v + a_v^*)^2 = \{a_v, a_v^*\} = \langle v|v\rangle.$$

So it follows that the orthogonal group of $\mathcal{H}$ as a real Hilbert space acts as $*$-autos of the Clifford algebra. Next point is to ask about maximal isotropic subspaces of the space of operators $a_v + a_w^*$. Since

$$(a_v + a_w^*)^2 = \{a_v, a_w^*\} = \langle v|w\rangle$$

one is after a subspace of $(v, w) \in \bar{\mathcal{H}} \oplus \mathcal{H}$ such that $\langle v, w\rangle = 0$. Examples include $\mathcal{H}_1^\perp \oplus \mathcal{H}_1$ with $\mathcal{H}_1 \subset \mathcal{H}$ and the graph $\{(Tw, w) \mid w \in \mathcal{H}\}$ where $T : \mathcal{H} \to \mathcal{H}$ is conjugate linear and satisfies $\langle Tw|w\rangle = 0$.

The important situation: Recall we started with H_0 and $H = H_0 + H'$ on $\mathcal{H}$. In a scattering situation we get wave operators which are unitary transformations of $\mathcal{H}$ which intertwine H_0 and H. Thus we have a unitary operator on $\mathcal{H}$ which gives rise to an $*$-automorphism of the Clifford algebra, and the problem is to "implement" this by an operator on the Fock space.

Let C be the above Clifford algebra of $\mathcal{H}$. The complex structure of $\mathcal{H}$ is not used to define C, but is used to define elements a_v, a_v^* of C. Precisely we have the canonical map $\rho : \mathcal{H} \to C$ such that $\rho(v)^2 = \langle v|v\rangle$, $\rho(v)^* = \rho(v)$ and then we can define a_v, a_v^* by

$$\rho(v) = a_v + a_v^*$$
$$\rho(iv) = -ia_v + ia_v^* .$$

The next point is to take a [redacted] maximal isotropic subspace W of $\rho(\mathcal{H})\otimes\mathbb{C}$, whatever this means. I want to be able to construct from W a Hilbert space $\mathcal{F}_W$ with a unit vector Φ and an action of C on $\mathcal{F}_W$ such that [redacted] W kills Φ. Moreover

$$\Lambda\left(\rho(\mathcal{H})\otimes\mathbb{C}/W\right) \hookrightarrow \mathcal{F}_W$$

as a dense subspaces.

$\rho(\mathcal{H})\otimes\mathbb{C}$ is a Hilbert space, because any orthonormal basis for the real Hilbert space $\mathcal{H}$ gives one an orthonormal basis for $\rho(\mathcal{H})\otimes\mathbb{C}$. Also it comes with a non-degenerate quadratic form $v \mapsto \rho(v)^2$. It clear that a maximal isotropic subspace is a closed isotropic subspace [redacted] W such that if W' is defined by

$$0 \longrightarrow W \longrightarrow \rho(\mathcal{H})\otimes\mathbb{C} \longrightarrow W' \longrightarrow 0$$

then the [redacted] quadratic form makes $W' \simeq W^* = \mathrm{Hom}(W,\mathbb{C})$. So [redacted] the question arises as to how this compares with a complex structure on $\mathcal{H}$. You should be able to see what happens in the finite diml case. The space of max isotropic subspace for a quad. form is non-compact, so it's clear we want the complex structures on $\mathcal{H}$.

So we want W to arise from a complex structure [redacted] on $\mathcal{H}$. In this case W, $\overline{W}$ are orthogonal subspaces of $\rho(\mathcal{H})\otimes\mathbb{C}$ and it's pretty clear we get the right Fock space structure.

<u>Summary</u>: Let $\mathcal{H}$ be a real Hilbert space, and C the associated Clifford algebra. Then each complex structure on $\mathcal{H}$ we have a Fock space which is the l^2 completion of $\Lambda\mathcal{H}$ with C acting bia $\rho(v) = a_v + a_v^*$.

<u>Question</u>: When are the Fock space modules over C associated to two complex structures on $\mathcal{H}$ unitarily equivalent?

The examples of interest to me are those complex structures on $\mathcal{H}$ given by unitary operators J with $J^2 = -1$. Notice: my $\mathcal{H}$ comes with a complex structure and such a J corresponds to a splitting $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$, namely the eigenspaces of J.

There seems to be a simpler way of seeing things. Recall that we start with the decomposition

$$\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$$

obtained from H_0. Assuming the perturbation $H = H_0 + H'$ is "weak", the negative eigenspace for H should be the graph of a bdd. linear transformation T from $\mathcal{H}^-$ to $\mathcal{H}^+$. Take this as part of the defn. of "weak". Now choose orth. bases of $\mathcal{H}^-$ and $\mathcal{H}^+$ as nicely as possible. One does this by diagonalizing the operator T^*T on $\mathcal{H}^-$. To begin assume this has discrete spectrum, so there is an orth. basis e_n for $\mathcal{H}^-$ with $T^*Te_n = \lambda_n^2 e_n$ with $\lambda_n \geq 0$. Then $(Te_n, Te_m) = (T^*Te_n, e_m) = \lambda_n^2 \delta_{mn}$ so that the $\frac{1}{\lambda_n} Te_n$ for $\lambda_n \neq 0$ can be extended to an orthonormal basis for $\mathcal{H}^-$. To keep things simple suppose $Te_n = \lambda_n e'_n$ where e'_n is an orth. basis of $\mathcal{H}^-$.

Now by analogy with the finite-dimensional situation I should be able to write down the vector in "$\Lambda\mathcal{H}$" (precisely $\Lambda\mathcal{H}^- \otimes \Lambda\mathcal{H}^+$) corresponding to the negative eigen-subspace for H.

Take image of $e_1 \wedge \cdots \wedge e_N$ under (id, T) which is

$$(e_1 + \lambda_1 e_1') \wedge \cdots \wedge (e_N + \lambda_N e_N')$$

$$= (e_1 \wedge \cdots \wedge e_N) + \sum_{i=1}^{N} (e_1 \wedge \cdots \wedge e_i' \cdots \wedge e_N)\lambda_i + \sum_{i<j} (e_1 \cdots e_i' \; e_j' \cdots e_N)\lambda_i\lambda_j + \cdots$$

~~We are looking at the derivation of $\Lambda\mathcal{H}$ extending $(id \times T)$ on $\Lambda^1\mathcal{H}$ which is~~

$$\cancel{\sum_n (a_n^* + \lambda_n a_n^*) a_n .}$$

~~Here $id \times T$~~

It's obvious how this formula extends in the infinite dimensional case:

$$\Psi = \Phi + \sum_i \lambda_i \, a_{i'}^* a_i \Phi + \sum_{i<j} \lambda_i \lambda_j \, a_{i'}^* a_i a_{j'}^* a_j \Phi + \cdots$$

This will be in the Hilbert space iff

$$1 + \sum \lambda_i^2 + \sum_{i<j} \lambda_i^2 \lambda_j^2 + \cdots = \prod_i (1 + \lambda_i^2) < \infty$$

which is equivalent to $T : \mathcal{H}^- \to \mathcal{H}^+$ being a Hilbert-Schmidt operator.

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We consider a Hilbert space $\mathcal{H}$ and the associated Fock space associated to a splitting $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$. This Fock space is a representation of the Clifford algebra generated by $a(v)$, $a^*(v)$ for $v \in \mathcal{H}$, and it comes with a unit vector Φ whose annihilator is generated by the $a(v)$, $v \in \mathcal{H}^+$ and $a^*(v)$ $v \in \mathcal{H}^-$. We consider a transf. $T : \mathcal{H}^- \longrightarrow \mathcal{H}^+$ and the subspace Γ_T of $\mathcal{H}$ which is the graph of [redacted] T, and the problem is to find the vector Φ in Fock space belonging to this subspace Γ_T.

Recall that Fock space is an infinite-dimensional version of $\Lambda\mathcal{H}$ so that linear mappings on $\mathcal{H}$ extend to $\Lambda\mathcal{H}$ modulo problems of convergence. For example relative to an orth. basis of $\mathcal{H}$, the [redacted] transformation with matrix $[\alpha_{mn}]$ becomes the derivation $\sum \alpha_{mn} a_m^* a_n$.

[redacted] If $m \neq n$, then $(a_m^* a_n)^2 = a_m^* a_n a_m^* a_n = -a_m^{*2} a_n^2 = 0$, so

$$\exp(t a_m^* a_n) = 1 + t a_m^* a_n$$

is a 1-parameter group in the Clifford algebra. If I choose [redacted] a basis e_n for $\mathcal{H}^-$ and let a_n be the annihilation operator, then the operators

$$1 + a^*(Te_n)\, a_n$$

commute with each other, and so modulo convergence problems we can form

$$\prod_n \left(1 + a^*(Te_n) a_n\right) = e^{\sum_n a^*(Te_n) a_n}$$

which will be an invertible (non-unitary) transf. of Fock space. I want to evaluate

$$\left\| e^{\sum_n a^*(Te_n)a_n} |\Phi\rangle \right\|^2$$

so as to find conditions on T guaranteeing this is finite. The answer I found yesterday: suppose $Te_n = \lambda_n e_n'$ where $\lambda_n \geq 0$ and the e_n' for $\lambda_n \neq 0$ are part of an orthonormal basis for $\mathcal{H}^+$. Then I found

$$\left\| e^{\sum \lambda_n a^*_{n'} a_n} \Phi \right\|^2 = \prod (1 + \lambda_n^2)$$

$$= \exp\left\{ \sum \lambda_n^2 - \frac{1}{2}\sum \lambda_n^4 + \frac{1}{3}\sum \lambda_n^6 - \cdots \right\}$$

$$= \exp\left\{ \mathrm{tr}(T^*T) - \frac{1}{2}\mathrm{tr}(T^*T)^2 + \cdots\cdots \right\}$$

$$= \exp \mathrm{tr} \log(1 + T^*T)$$

$$= \det(1 + T^*T)$$

This formula <u>should be true in general</u>.

Let us return to the case where we have a perturbation H of H_0 on $\mathcal{H}$. Then we want to relate the above to the operators H and H_0.

Assume we are in the finite-dimensional case and let Φ be the ground state for H_0 on $\Lambda\mathcal{H}$. Then

$$e^{-\beta H}\Phi = \sum e^{-\beta \tilde{E}_n} |n\rangle\langle n|\Phi\rangle.$$

Let Ψ be the ground state for H on $\Lambda\mathcal{H}$ and suppose that H is close enough to H_0 that $\langle\Phi|\Psi\rangle \neq 0$. Then we can normalize Ψ so that $\langle\Psi|\Phi\rangle = 1$, whence

$$|0\rangle = \frac{\Psi}{\|\Psi\|}$$

and $$e^{-\beta H}\Phi \sim e^{-\beta \tilde{E}_0}|0\rangle\langle 0|\Phi\rangle = \frac{e^{-\beta\tilde{E}_0}|\Psi\rangle\overbrace{\langle\Psi|\Phi\rangle}^{1}}{\langle\Psi|\Psi\rangle}$$

Thus $$\langle\Phi|e^{-\beta H}|\Phi\rangle \sim e^{-\beta\tilde{E}_0}\frac{1}{\langle\Psi|\Psi\rangle}$$

and so $$\frac{\langle\Phi|e^{-\beta H}|\Phi\rangle}{\langle\Phi|e^{-\beta H_0}|\Phi\rangle} \sim e^{-\beta\Delta E}\frac{1}{\langle\Psi|\Psi\rangle}$$

But [redacted] Ψ is the element of $\Lambda\mathcal{H}$ belonging to the negative eigenspace for H, normalized so that its inner product with Φ is 1. Hence

$$\langle\Psi|\Psi\rangle = \det(1+T^*T)$$

with the notation of the preceding page.

Thus we have

$$\boxed{\log\frac{\langle\Phi|e^{-\beta H}|\Phi\rangle}{\langle\Phi|e^{-\beta H_0}|\Phi\rangle} \sim -\beta\Delta E - \log\langle\Psi|\Psi\rangle}$$

with an exponentially small error. [redacted] Notice how much better this is than

$$\frac{\mathrm{Tr}(e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H_0})} = \frac{\prod(1+e^{-\beta\tilde{E}_n})}{\prod(1+e^{-\beta E_n})}$$

E_n = eigenvalues of H_0 on $\mathcal{H}$

which only gives the ground energy shift.

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Setup: We have this perturbation H of H_0, Φ is the ground state of H_0, Ψ the ground state of H. We suppose $\langle \Psi | \Phi \rangle \neq 0$. ~~But I then want to~~ Then as $\beta \to \infty$.

$$e^{-\beta H} |\Phi\rangle \sim e^{-\beta \tilde{E}_0} |\Psi\rangle \langle \Psi | \Phi \rangle$$

$$\langle \Phi | e^{\beta H_0} e^{-\beta H} | \Phi \rangle \sim e^{-\beta \Delta E} |\langle \Phi | \Psi \rangle|^2$$

Now suppose H_0, H are 1-particle operators on Fock space. Then we know by diagram methods that

$$\log \langle \Phi | e^{\beta H_0} e^{-\beta H} | \Phi \rangle = tr(G_0 V^\beta) - \frac{1}{2} tr(G_0 V^\beta)^2 + \frac{1}{3} tr(G_0 V^\beta)^3 - \cdots$$

where G_0 is the 1-particle Green's function $\left(\frac{d}{dt} + H_0\right)^{-1}$ and where V^β represents the interaction turned on for $[-\beta/2, \beta/2]$. Thus

$$V^\beta(t) = \begin{cases} H' & -\beta/2 \le t \le \beta/2 \\ 0 & \text{otherwise.} \end{cases}$$

So

$$tr(G_0 V^\beta) = \int_{-\beta/2}^{\beta/2} dt \; G_0(t^-, t) H'$$

$$G_0(t_1, t_2) = \frac{1}{2\pi} \int dk \frac{e^{ik(t_1 - t_2)}}{ik + H_0} = \begin{cases} e^{-H_0(t_1 - t_2)} P_0^+ & t_1 > t_2 \\ e^{-H_0(t_1 - t_2)} (-P_0^-) & t_1 < t_2 \end{cases}$$

$$tr(G_0 V^\beta) = \beta \int \frac{dk}{2\pi} e^{ik0^-} tr\left(\frac{1}{ik + H_0} H'\right) = \beta\left(-tr(P_0^- H')\right)$$

This has no constant part as $\beta \to \infty$, so there is no first order contribution to $\langle \Phi | \Psi \rangle$.

Consider next

$$tr(G_0 V^\beta)^2 = \int_{-\beta/2}^{\beta/2} dt_1 \int_{-\beta/2}^{\beta/2} dt_2 \, tr\left(G_0(t_1, t_2) H' G_0(t_2, t_1) H'\right)$$

$$= \int_{-\beta/2}^{\beta/2} dt_1 \int_{-\beta/2}^{\beta/2} dt_2 \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \, e^{ik_1(t_1-t_2) + ik_2(t_2-t_1)} \, tr\left(\frac{1}{ik_1+H_0} H' \frac{1}{ik_2+H_0} H' \right)$$

$$\int_{-\beta/2}^{\beta/2} dt_2 \, e^{i(k_2-k_1)t_2} = \frac{e^{i(k_2-k_1)\beta/2} - e^{-i(k_2-k_1)\beta/2}}{2i(k_2-k_1)/2}$$

$$= \frac{\sin \beta(k_2-k_1)/2}{(k_2-k_1)/2}$$

so

$$tr(G_0 V\beta)^2 = \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \left(\frac{\sin \beta(k_2-k_1)/2}{(k_2-k_1)/2} \right)^2 tr\left(\frac{1}{ik_1+H_0} H' \frac{1}{ik_2+H_0} H' \right)$$

In order to understand how this behaves as $\beta \to \infty$ consider

$$\int \left[\frac{\sin(\lambda x)}{x} \right]^2 f(x)\, dx$$

with f smooth. Actually f is rational in the case of interest. If $f(0) = f'(0) = 0$, then

$$\int \sin(\lambda x)^2 \frac{f(x)}{x^2} dx = \int \left[\frac{1 - \cos(2\lambda x)}{2} \right] \frac{f(x)}{x^2} dx$$

$$\longrightarrow \frac{1}{2} \int \frac{f(x)}{x^2} dx$$

It clear the integral is 0 if f is odd. Also

$$\int \frac{\sin(\lambda x)^2}{x^2} dx = \lambda \int \left(\frac{\sin x}{x} \right)^2 dx$$

Now

$$\int_{-1}^{1} \frac{e^{-ixt}}{2} dt = \frac{e^{ix} - e^{-ix}}{2ix} = \frac{\sin x}{x}$$

$$\therefore \int \left(\frac{\sin x}{x} \right)^2 \frac{dx}{2\pi} = \int \left(\frac{1}{2} \chi_{[-1,1]}(t) \right)^2 dt = \frac{1}{2}$$

and so $$\int \left(\frac{\sin x}{x}\right)^2 dx = \pi$$

Thus we have

$$\int \left(\frac{\sin \lambda x}{x}\right)^2 f(x)\,dx \sim \lambda\pi f(0) + P\int \frac{(f(x)-f(0))}{x^2}\,dx$$

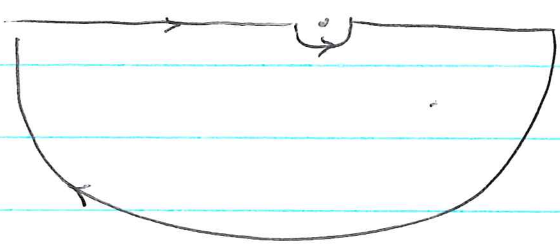
Example: Take $f(x) = \dfrac{1}{ix+a}$ where $\operatorname{Re} a \neq 0$ so that $f(x)$ is defined for $x \in \mathbb{R}$. Then to compute

$$P\int \frac{dx}{2\pi x^2}(f(x)-f(0)) = P\int \frac{dx}{2\pi x^2}\left[\frac{1}{ix+a} - \frac{1}{a}\right]$$

$$= P\,\frac{1}{2\pi}\int \frac{dx}{x^2}\,\frac{-ix}{(ix+a)(a)}$$

$$= P\,\frac{1}{2\pi i}\int \frac{dx}{x}\,\frac{1}{(ix+a)(a)}$$

If $\operatorname{Re}(a) > 0$, then $ix+a$ vanishes at $x = ia \in UHP$ so closing the contour in the LHP

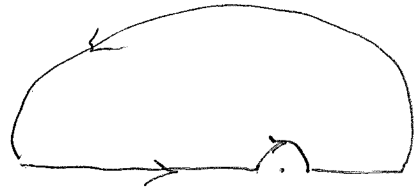


gives $$P\int + \tfrac{1}{2}\operatorname{Res}_0 = 0.$$

Thus

$$P\,\frac{1}{2\pi i}\int \frac{dx}{x}\,\frac{1}{(ix+a)(a)} = -\frac{1}{2}\frac{1}{a^2} \qquad \operatorname{Re}(a) > 0$$

If $\operatorname{Re}(a) < 0$, then closing contour in UHP gives



$$P\int = \frac{1}{2}\,\mathrm{Res}_0 = \frac{1}{2}\frac{1}{a^2}$$

Thus

$$P\,\frac{1}{2\pi}\int \frac{dx}{x^2}\left[\frac{1}{ix+a}-\frac{1}{a}\right] = \begin{cases} -\dfrac{1}{2a^2} & \mathrm{Re}(a)>0 \\[2ex] \dfrac{1}{2a^2} & \mathrm{Re}(a)<0 \end{cases}$$

So return to

$$\mathrm{tr}\left(G_0V^\beta\right)^2 = \int\frac{dk_1\,dk_2}{(2\pi)^2}\left[\frac{\sin\ \beta(k_1-k_2)/2}{(k_1-k_2)/2}\right]^2 \mathrm{tr}\left(\frac{1}{ik_1+H_0}H'\frac{1}{ik_2+H_0}H'\right)$$

$$= \int\frac{dk}{2\pi}\frac{dx}{2\pi}\left[\frac{\sin(\beta x/2)}{x/2}\right]^2 \mathrm{tr}\left(\frac{1}{ix+ik+H_0}H'\frac{1}{ik+H_0}H'\right)$$

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Under the perturbation $H = H_0 + \lambda H'$ the ground state Φ of H_0 goes into $\Psi = \Phi + \lambda\Psi^{(1)} + \lambda^2\Psi^{(2)} + \cdots\cdots$ which is unique provided $\langle\Phi|\Psi\rangle = 1$. (Better to say the terms $\Psi^{(n)}$ are unique.) I want to understand the normalization constant

$$\langle\Psi|\Psi\rangle = 1 + \lambda^2\langle\Psi^{(1)}|\Psi^{(1)}\rangle + \lambda^3\left[\langle\Psi^{(1)}|\Psi^{(2)}\rangle + \langle\Psi^{(2)}|\Psi^{(1)}\rangle\right] + \cdots$$

Let's compute the 2nd order term.

$$(H_0 - E_0)\Psi^{(1)} + (H' - E_0^{(1)})\Phi = 0$$

$$\therefore \quad E_0^{(1)} = \langle\Phi|H'|\Phi\rangle$$

$$\Psi^{(1)} = \frac{1}{E_0 - H_0}(H' - E_0^{(1)})|\Phi\rangle$$

$$= \sum_{n\neq 0} |n\rangle \frac{1}{E_0 - E_n}\langle n|H'|0\rangle$$

hence $$\langle\Psi^{(1)}|\Psi^{(1)}\rangle = \sum_{n\neq 0} \frac{\langle 0|H'|n\rangle\langle n|H'|0\rangle}{(E_0 - E_n)^2}$$

$$(H_0 - E_0)\Psi^{(2)} + (H' - E_0^{(1)})\Psi^{(1)} + (-E_0^{(2)})\Phi = 0$$

$$E_0^{(2)} = \langle\Phi|H'|\Psi^{(1)}\rangle = -\sum_{n\neq 0}\frac{\langle 0|H'|n\rangle\langle n|H'|0\rangle}{E_n - E_0}$$

$$(E_0 - H_0)\Psi^{(2)} = \left(H' - \langle 0|H'|0\rangle\right)\left(\sum_{n\neq 0} \frac{|n\rangle\langle n|H'|0\rangle}{E_0 - E_n}\right) - E_0^{(2)}\Phi$$

$$= \sum_{m,n\neq 0} \frac{|m\rangle\langle m|H'|n\rangle\langle n|H'|0\rangle}{E_0 - E_n} - \sum_{n\neq 0}\frac{|n\rangle\langle n|H'|0\rangle\langle 0|H'|0\rangle}{E_0 - E_n}$$

$$\Psi^{(2)} = \sum_{m,n\neq 0} \frac{|m\rangle\langle m|H'|n\rangle\langle n|H'|0\rangle}{(E_0 - E_m)(E_0 - E_n)} - \sum_{n\neq 0}\frac{|n\rangle\langle n|H'|0\rangle\langle 0|H'|0\rangle}{(E_0 - E_n)^2}$$

$E_0^{(3)} = \langle\Phi|H'|\Psi^{(2)}\rangle$. In general $E_0^{(n)} = \langle\Phi|H'|\Psi^{(n-1)}\rangle$.