Consider the resolvent

$$(K^2 - D^2 + V)^{-1} = (K^2-D^2)^{-1} - (K^2-D^2)^{-1}V(K^2-D^2)^{-1} + \dots$$

Recall

$$\langle x | K^2 - D^2 | x' \rangle = \frac{e^{-K|x-x'|}}{2K} = \int \frac{dp}{2\pi} e^{ip(x-x')} \frac{1}{K^2+p^2}$$

Can we describe $\langle x | (K^2-D^2+V)^{-1} | x' \rangle$ nicely for large K. Look at 1rst order term in V which is

$$\frac{1}{(2K)^2} \int e^{-K|x-x_1|} V(x_1) e^{-K|x_1-x'|} dx_1$$

One has $|x-x_1| + |x_1 - x'| \geq |x-x'|$ with equality only if $x' \in [x', x]$. So for large K the part of the integral where $x_1 \notin [x', x]$ is negligible. Thus we have

$$\int e^{-K|x-x_1|} V(x_1) e^{-K|x_1-x'|} dx_1 = e^{-K|x-x'|} \int_{[x',x]} V(x_1) dx_1 + \text{smaller stuff.}$$

We should get an asymptotic expansion in decreasing powers of K involving V and its derivatives at the endpoints x', x. Suppose $x' \leq x$. Then

$$e^{K[x-x']} \int_{x_1 \geq x} e^{+K(x-x_1)} V(x_1) e^{-K(x_1-x')} dx_1$$

$$= \int_{x_1 \geq x} e^{-2K(x_1-x)} V(x_1)\, dx_1 = \int_0^\infty e^{-2Ku} V(x+u)\, du$$

$$\approx \sum_{n \geq 0} \frac{V^{(n)}(x)}{n!} \underbrace{\int_0^\infty e^{-2Ku} u^{n+1} \frac{du}{u}}_{\frac{\Gamma(n+1)}{(2K)^{n+1}}}$$

$$\approx \sum_{n\geq 0} \frac{V^{(n)}(x)}{(2K)^{n+1}}$$

Also

$$e^{K(x-x')} \int_{x_1 \leq x'} e^{-K(x-x_1) - K(x'-x_1)} V(x_1)\,dx_1 \qquad u = x_1 - x'$$

$$= \int_{x_1 \leq x'} e^{-2K(x'-x_1)} V(x_1)\,dx_1 = \int_{-\infty}^{0} e^{2Ku} V(x'+u)\,du$$

$$= \int_0^\infty e^{-2Ku} V(x'-u)\,du \approx \sum_{n\geq 0} \frac{(-1)^n}{(2K)^{n+1}} V^{(n)}(x')$$

Thus for $x' \leq x$

$$\int e^{-K|x-x_1|} V(x_1) e^{-K|x_1-x'|}\,dx_1 \approx e^{-K|x-x'|} \left\{ \int_{x'}^{x} V(x_1)\,dx_1 \right.$$
$$\left. + \sum_{n\geq 0} \frac{1}{(2K)^{n+1}} \left[V^{(n)}(x) + (-1)^n V^{(n)}(x') \right] \right\}$$

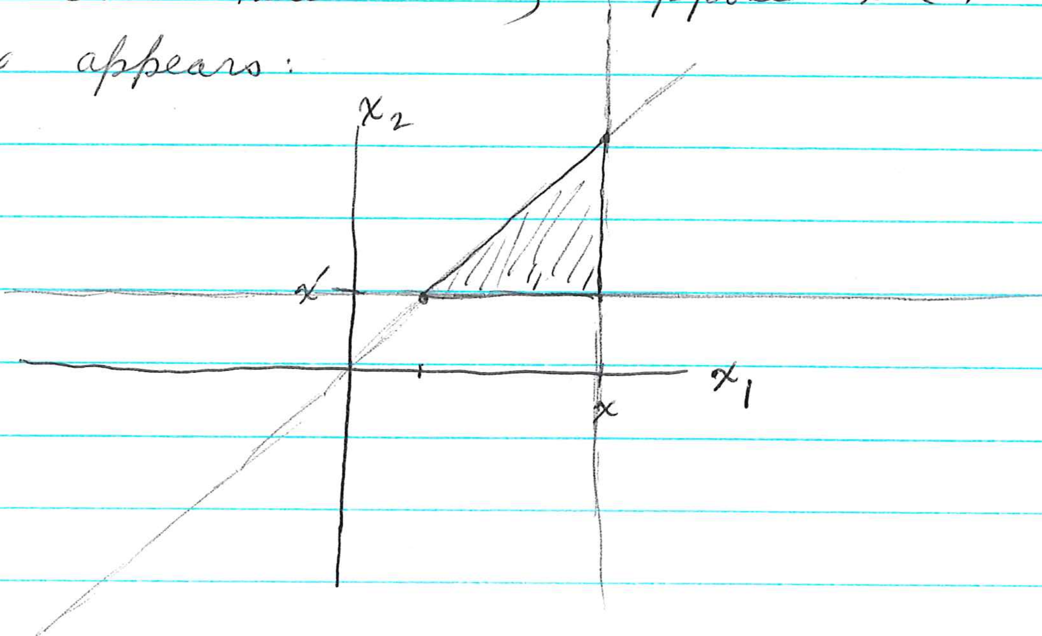
The above expansion probably can be generalized to the higher terms in the expansion for the Green's function, e.g.

$$\frac{1}{(2K)^n} \int e^{-K|x-x_1|} V(x_1) \cdots\cdots V(x_n) e^{-K|x_n-x'|}\,dx_1 \cdots dx_n$$

Since $|x-x_1| + \cdots + |x_n - x'| \geq |x-x'|$ with equality only when $x' \leq x_n \leq \cdots \leq x_1 \leq x$ (supposing $x \leq x'$), its clear that we are going to get an asymptotic expansion of the form

$$\frac{1}{2K^n} e^{-K|x-x'|} \left(\int_{x' \leq x_n \leq \cdots \leq x_1 \leq x} V(x_1) \cdots V(x_n)\,dx_1 \cdots dx_n + O(1) \right)$$

Next point is to see if we can obtain an expression for the higher order terms in the expansion as a sum [redacted] over the faces of the n-simplex we have. Take $n = 2$, suppose $x' \leq x$ so the simplex appears:



January 22, 1980

Suppose we have a system with energy levels E_n, $n=0,1,\ldots$ To "derive" the formula $\langle A\rangle = tr(Ae^{-\beta H})/tr(e^{-\beta H})$ one can use a microcanonical ensemble, i.e. take N copies of the system with total energy $N\varepsilon$, compute the average value of A over all states of this energy and then see what happens as $N\to\infty$. We need a formula for the number of states of the N-fold system of a given energy. The N-fold system has the energy levels $E_{n_1}+\cdots+E_{n_N}$ so we want the measure

$$\sum_{n_1,\ldots,n_N}' \delta(E-(E_{n_1}+\cdots+E_{n_N}))\,dE,$$

whose Laplace transform is $\sum' e^{-\beta(E_{n_1}+\cdots+E_{n_N})} = Z(\beta)^N$ where $Z(\beta)=\sum' e^{-\beta E_n}$ is the partition function for the system. Thus

$$\sum_{n_1,\ldots,n_N}' \delta(N\varepsilon-(E_{n_1}+\cdots+E_{n_N})) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\beta(N\varepsilon)}\,Z(\beta)^N\,d\beta$$

$$= \frac{1}{2\pi i}\int e^{N(\beta\varepsilon+\log Z(\beta))}\,d\beta.$$

The integral is estimated by saddle point method leading to the relation

$$\frac{d}{d\beta}(\beta\varepsilon+\log Z(\beta))=0 \qquad\text{or}\qquad \varepsilon=-\frac{Z'(\beta)}{Z(\beta)}$$

for the average energy ε and temperature β.

<u>Question</u>: What happens ~~if~~, instead of N distinguishable

copies of the system you take indistinguishable copies and count only according to boson (or fermion) statistics?

In order to treat this problem it is convenient to use the grand formalism and allow all different N to occur. Thus one has for boson the measure for N particles:

$$\sum_{n_1 \le \cdots \le n_N}' \delta\left(E - \left(E_{n_1} + \cdots + E_{n_N}\right)\right) dE$$

and the grand partition function is

$$\sum_{N \geq 0} z^N \underbrace{\sum_{n_1 \le \cdots \le n_N}' e^{-\beta\left(E_{n_1} + \cdots + E_{n_N}\right)}}_{\sum_{|\alpha| = N}' \left(e^{-\beta E_0}\right)^{\alpha_0} \cdots \left(e^{-\beta E_n}\right)^{\alpha_n} \cdots}$$

$$= \prod_n \left(\frac{1}{1 - z e^{-\beta E_n}}\right)$$

so

$$\sum_{n_1 \le \cdots \le n_N} \delta\left(N\varepsilon - \left(E_{n_1} + \cdots + E_{n_N}\right)\right)$$

$$= \frac{1}{2\pi i} \oint z^{-N} \frac{dz}{z} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\beta(N\varepsilon)} \prod_n \left(\frac{1}{1 - z e^{-\beta E_n}}\right) d\beta$$

If we use the saddle point method, we find we want the critical point of the function

$$\beta\varepsilon - \log(z) + \log\left(\prod_n \frac{1}{1 - z e^{-\beta E_n}}\right) \qquad \text{boson}$$

$$\beta\varepsilon - \log(z) + \log\left(\prod_n \left(1 + z e^{-\beta E_n}\right)\right) \qquad \text{fermion}$$

$$\beta \varepsilon - \log(z) + \log\left(\frac{1}{1 - z Z(\beta)}\right)$$ classical case.

Let's take the critical point for the classical case in this grand formalism to see that we get the same results as for fixed N.

$$\frac{\partial}{\partial z}\Big(\beta\varepsilon - \log z - \log(1 - zZ(\beta))\Big) = -\frac{1}{z} - \frac{-Z(\beta)}{1 - zZ(\beta)} = 0$$

$$\frac{\partial}{\partial \beta}\Big(\beta\varepsilon - \log z - \log(1 - zZ(\beta))\Big) = \varepsilon - \frac{-zZ'(\beta)}{1 - zZ(\beta)} = 0$$

$$\frac{zZ(\beta)}{1 - zZ(\beta)} = 1 \implies zZ(\beta) = \frac{1}{2}$$

$$\varepsilon + \frac{zZ'(\beta)}{1 - zZ(\beta)} = \varepsilon + \frac{Z'(\beta)}{Z(\beta)} = 0$$ same as before.

In the boson case we get

$$\frac{\partial}{\partial z}\Big(\beta\varepsilon - \log z - \sum_n \log(1 - z e^{-\beta E_n})\Big)$$

$$= -\frac{1}{z} - \sum_n \frac{-e^{-\beta E_n}}{1 - z e^{-\beta E_n}} = 0$$

which gives $$\boxed{\sum_n \frac{z e^{-\beta E_n}}{1 - z e^{-\beta E_n}} = 1}$$

and $$\frac{\partial}{\partial z}(\dots) = \varepsilon - \sum_n \frac{-z e^{-\beta E_n}(-E_n)}{1 - z e^{-\beta E_n}} = 0$$ or

$$\boxed{\sum_n E_n \frac{z e^{-\beta E_n}}{1 - z e^{-\beta E_n}} = \varepsilon}$$

So therefore instead of the Maxwell-Bolzman distribution [redacted] where the probabilities p_n are proportional to $e^{-\beta E_n}$ we get a screwy distribution where

$$p_n = \frac{ze^{-\beta E_n}}{1 - ze^{-\beta E_n}}$$

$$1 + p_n = \frac{1}{1 - ze^{-\beta E_n}} \qquad \frac{1}{1+p_n} = 1 - ze^{-\beta E_n}$$

$$ze^{-\beta E_n} = 1 - \frac{1}{1+p_n} = \frac{p_n}{1+p_n}$$

Thus the distribution has

$$\frac{p_n}{1+p_n} \sim e^{-\beta E_n}$$

Relate resolvent and heat kernel:

$$(s+H)^{-1} = \int_0^\infty e^{-(s+H)t}\,dt = \int_0^\infty e^{-st}\, e^{-tH}\,dt$$

If $Re(s) \to \infty$, then the above integral depends only on e^{-tH} near $t = 0$. In fact because

$$\int_0^\infty e^{-st}\, t^n\, dt = \frac{\Gamma(n+1)}{s^{n+1}} \qquad n > -1$$

an asymptotic expansion for e^{-tH} as $t \downarrow 0$ in powers of t will yield an asymptotic expansion for $(s+H)^{-1}$ as [redacted] $Re(s) \to +\infty$. The converse also seems to be true.

Look [redacted] now at $H = -D^2 - q$ where

$$(s+H)^{-1} = (s-D^2)^{-1} + (s-D^2)^{-1} q \, (s-D^2)^{-1} + \cdots\cdots$$

and put $s = K^2$, and use

$$\langle x \,|\, (K^2 - D^2)^{-1} \,|\, x' \rangle = \frac{e^{-K|x-x'|}}{2K}$$

Then we found for $x' \le x$ that the first order in q term for $\langle x|(K^2-D-q)^{-1}|x'\rangle$

$$\frac{1}{(2K)^2}\int \left(e^{-K|x-x_1|} q(x_1)\, e^{-K|x_1-x'|}\right) dx_1$$

has an asymptotic expansion

$$\frac{e^{-K|x-x'|}}{(2K)^2}\left(\int_{x'}^{x} q(x_1)\,dx_1 + \sum_{n\ge 0}\frac{1}{(2K)^{n+1}}\left[q^{(n)}(x) + (-1)^n q^{(n)}(x')\right]\right)$$

fermion integration: Let V be a finite dimensional vector space over k of dimension $2n$, choose a volume element $\int : \Lambda^{2n} V \xrightarrow{\sim} k$. If $\omega \in \Lambda^2 V$, then we get

$$\int \frac{\omega^n}{n!} \in k .$$

so we get a map $\Lambda^2 V \longrightarrow k$ of of degree n, which is essentially the Pfaffian. For example let V have the basis $v_1, \dots, v_{2n}$ such that $\int v_1 \wedge \dots \wedge v_{2n} = 1$. If $\omega = \frac{1}{2}\sum a_{ij} v_i \wedge v_j$, then

$$\omega^n = \Big(\sum_{i<j} a_{ij} v_i \wedge v_j\Big)^n = n!\, Pf(a_{ij})\, v_1 \cdots v_{2n}$$

where the Pfaffian is a sum over the $\frac{(2n)!}{2^n\, n!} = (2n-1)!!$ different ways of partitioning $\{1, \dots, 2n\}$ in pairs.

Suppose V has the basis $\psi_1, \dots, \psi_n, \tilde{\psi}_1, \dots, \tilde{\psi}_n$ and that the volume element is

$$\psi_1 \tilde{\psi}_1 \cdots \psi_n \tilde{\psi}_n = \psi_1 \cdots \psi_n \tilde{\psi}_n \cdots \tilde{\psi}_1$$

Let $\quad \omega = \sum_{i=1}^{n} \lambda_i \psi_i \wedge \tilde{\psi}_i$. Then

$$\omega^n = n!\, \lambda_1 \cdots \lambda_n\, \psi_1 \tilde{\psi}_1 \cdots \psi_n \tilde{\psi}_n$$

so $\quad \int \omega^n = \lambda_1 \cdots \lambda_n$.

If $\quad \omega = \sum_{i,j=1}^{n} a_{ij} \psi_i \tilde{\psi}_j$, then

$$\omega^n = \sum_{\substack{i_1, i_2, \dots, i_n \\ j_1, \dots, j_n}} a_{i_1 j_1} \cdots a_{i_n j_n}\, \psi_{i_1} \cdots \psi_{i_n} \tilde{\psi}_{j_n} \cdots \tilde{\psi}_{j_1}$$

Only contributions come from $i_1, \dots, i_n$ and $j_1, \dots, j_n$ being permutation of $\{1, \dots, n\}$. So clearly you get

$$\omega^n = n!\, \det(a_{ij})\, \psi_1 \cdots \psi_n \bar{\psi}_n \cdots \bar{\psi}_1$$

Thus the good formula seems to be

$$\int e^{\frac{1}{2}\sum a_{ij} v_i v_j} = Pf(a_{ij}) = \det(a_{ij})^{1/2}$$

which is analogous to the boson formula

$$\int e^{-\frac{1}{2}\sum a_{ij} x_i x_j} \frac{d^n x}{(\sqrt{2\pi})^n} = (\det A)^{-1/2} \qquad (\mathrm{Re}\, A) > 0$$

Let us now suppose that the matrix a_{ij} is skew-symmetric and non-singular so that the denominator in

$$\frac{\int e^{\frac{1}{2}\sum a_{ij} v_i v_j} \prod_{i \in I} v_i}{\int e^{\frac{1}{2}\sum a_{ij} v_i v_j}}$$

makes sense. We need a Wick's thm. to evaluate this quotient. The statement is that one gets a sum over all ways of contracting $\prod v_i$ in pairs of products of what you get for 2 factors.

We should be precise about the signs. So first of all suppose $I = \{i_1, \dots, i_p\}$ where $i_1 < \cdots < i_p$ and that we have a way of contracting I in pairs. Then we rearrange the factors of

$$\prod_I v_i = v_{i_1} \wedge \cdots \wedge v_{i_p}$$

so that the pairs are consecutive, then we take the product of the contracted factors. So what we get is

$$\frac{\int e^{\frac{1}{2}\sum a_{ij} v_i v_j} \prod_I v_i}{\int e^{\frac{1}{2}\sum a_{ij} v_i v_j}} = \sum_{\substack{\sigma \text{ partition} \\ \text{of } I \text{ into pairs } (i_1, j_1)}} \operatorname{sgn}(\sigma) \prod G(i_1, j_1)$$

To prove a formula of this sort, you generalize it so that one has a product $\mathcal{V}_1 \cdots \mathcal{V}_p$ with $\mathcal{V}_i \in \Lambda^1 V$. Both sides are multi-linear, and by invariance considerations you can put $\omega = \frac{1}{2}\sum a_{ij} v_i v_j$ into a standard form: $\omega = \sum_{i=1}^{n} \lambda_i \psi_i \tilde{\psi}_i$. Then you check it on a pair $\psi_j \tilde{\psi}_j$ and you get

$$\frac{\int e^{\sum \lambda_i \psi_i \tilde{\psi}_i} \psi_j \tilde{\psi}_j}{\int e^{\sum \lambda_i \psi_i \tilde{\psi}_i}} = \frac{\int \prod (1 + \lambda_i \psi_i \tilde{\psi}_i) \cdot \psi_j \tilde{\psi}_j}{\int \prod (1 + \lambda_i \psi_i \tilde{\psi}_i)}$$

$$= \frac{\prod_{i \neq j} \lambda_i}{\prod \lambda_i} = \frac{1}{\lambda_j}$$

So the rule is that if the skew-symmetric matrix a_{ij} has the inverse b_{ij}, then

$$G(\alpha, \beta) = \frac{\int e^{\frac{1}{2}\sum' a_{ij} v_i v_j} v_\alpha v_\beta}{\int e^{\frac{1}{2}\sum a_{ij} v_i v_j}} = -b_{\alpha\beta}$$

The reason for the minus sign is that

$$\begin{pmatrix} & & & \lambda_1 \\ & & \lambda_n & \\ & -\lambda_n & & \\ -\lambda_1 & & & \end{pmatrix}^{-1} = \begin{pmatrix} & & & -\frac{1}{\lambda_1} \\ & & -\frac{1}{\lambda_n} & \\ & \frac{1}{\lambda_n} & & \\ \frac{1}{\lambda_1} & & & \end{pmatrix}$$

The good formula: Let (a_{ij}) be a non-singular skew-symmetric matrix with inverse (b_{ij}). Then

$$\frac{\int e^{-\frac{1}{2}\sum a_{ij}v_i v_j}\, v_{k_1}\cdots v_{k_p}}{\int e^{-\frac{1}{2}\sum a_{ij}v_i v_j}} = \mathrm{Pf}\,(b_{k_i k_j})$$

The next step is to apply this to understand fermion path integrals and Feynman diagrams for interacting fermions.

Let us work backwards from the Green's function. Start with $H_0 = \mu a^* a$, where a^*, a are fermion creation and annihilation operators, $\{a,a\} = \{a^*,a^*\} = 0$ $\{a,a^*\} = 1$. Then put

$$G(t,t') = \langle 0 | T\, a(t)\, a^*(t') | 0 \rangle$$

where

$$a^*(t) = e^{tH_0}\, a^*\, e^{-tH_0}$$

$$\frac{d}{dt} a^*(t) = e^{tH_0} \underbrace{[\mu a^* a, a^*]}_{\mu(a^*\underbrace{\{a,a^*\}}_{1} - \underbrace{\{a^*,a^*\}}_{0} a)} e^{-tH_0} = \mu a^*(t)$$

$\therefore$ $a^*(t) = e^{\mu t} a^*$. $a(t) = e^{-\mu t} a$ so

$$G(t,t') = \langle 0 | T(e^{-\mu t} a\, e^{\mu t'} a^*) | 0 \rangle$$

$$= \begin{cases} e^{-\mu(t-t')} & t > t' \\ 0 & t < t' \end{cases}$$

Hence $$\left(\frac{d}{dt} + \mu\right) G(t,t') = \delta(t,t')$$

and so $G(t,t')$ is the inverse of the operator $\frac{d}{dt}+\mu$ on $L^2(\mathbb{R})$.

I need the Wick formula when the skew-symmetric form is $\sum \tilde{\psi}_i a_{ij} \psi_j$ with a_{ij} non-singular. It should be

$$(2) \qquad \frac{\int e^{-\sum \tilde{\psi}_i a_{ij}\psi_j}\ \tilde{\psi}_{m_1}\cdots\tilde{\psi}_{m_p}\psi_{n_p}\cdots\psi_{n_1}}{\int e^{-\sum \tilde{\psi}_i a_{ij}\psi_j}} = \det\left(b_{m_i n_j}\right)$$

where (b_{ij}) is the inverse of (a_{ij}). One way to check this is to introduce extra variables $\tilde{\eta}_j, \eta_j$ and consider the generating function

$$(1) \qquad \frac{\int e^{-\left(\sum \tilde{\psi}_i a_{ij}\psi_j + \sum_i \tilde{\psi}_i \eta_i + \sum_i \tilde{\eta}_i \psi_i\right)}(d\tilde{\psi}\, d\psi)}{\int e^{-\left(\sum \tilde{\psi}_i a_{ij}\psi_j\right)}(d\tilde{\psi}\, d\psi)}$$

If one completes the square in the exponent

$$-\sum_i \left(\tilde{\psi}_i + \sum_j \tilde{\eta}_j b_{ji}\right) a_{ik}\left(\psi_k + \sum_j b_{kj}\eta_j\right) + \sum \tilde{\eta}_j b_{ji}\eta_i$$

then uses invariance of the integral under "translation" one gets for the generating function (1)

$$e^{\sum \tilde{\eta}_j b_{ji}\eta_i}$$

which yields (2).

January 26, 1980

The program is to construct a path integral which involves fermion integration which is associated to a Hamiltonian $H_0 = \sum_k \mu_k a_k^* a_k$, where a_k, a_k^* are annihilation and creation operators on a fermion Fock space. We want to formulate carefully the criteria to be satisfied, and proceed by analogy with the boson case.

To simplify suppose $H_0 = \mu a^* a$. The goal is to explain the Green's functions giving time-ordered expectation values for products of field operators. We follow Schwinger's idea of using a generating function defined as follows. For the time-dependent Hamiltonian

$$H = H_0 + \tilde{J} a + a^* J$$

where $\tilde{J}, J$ are functions of t with compact support. The S matrix is

$$e^{\beta H_0} U^J(\beta, 0) = T e^{-\int_0^\beta \left(\tilde{J}(t) a(t) + a^*(t) J(t)\right) dt}$$

$$= e^{\iint_{t>t'} \tilde{J}(t) e^{-\mu(t-t')} J(t')} \, e^{-\left(\int_0^\beta \tilde{J}(t) e^{\mu t} dt\right) a} \, e^{-\left(\int_0^\beta J(t) e^{-\mu t} dt\right) a^*}$$

the generating function is

$$\langle 0 | S^J | 0 \rangle = \exp\left\{ \iint_{t>t'} \tilde{J}(t) e^{-\mu(t-t')} J(t') \right\}$$

$$= \exp \int \tilde{J} G J$$

where
$$G(t,t') = \begin{cases} e^{-\mu(t-t')} & t > t' \\ 0 & t < t' \end{cases}$$

is the inverse of $\left(\frac{d}{dt}+\mu\right)$ on $L^2(\mathbb{R})$.

Now work backwards from the formula

$$\frac{\int e^{-\frac{1}{2}X\cdot AX \pm J\cdot X}}{\int e^{-\frac{1}{2}X\cdot AX}} = e^{\frac{1}{2}J\cdot A^{-1}J}$$

provided $Re(A) > 0$ and A is a symmetric matrix. The quadratic function

$$\tilde{J}, J \longmapsto \int \tilde{J} G J$$

has positive-definite real part, provided we require $\tilde{J} = \bar{J}$. One sees this via the Fourier transform:

$$J(t) = \int \hat{J}(k)\, e^{ikt} \frac{dk}{2\pi}$$

$$\left(\frac{d}{dt}+\mu\right)^{-1} J = \int \frac{\hat{J}(k)}{ik+\mu} e^{ikt} \frac{dk}{2\pi}$$

$$\int \bar{J}\left(\frac{d}{dt}+\mu\right)^{-1} J = \int \frac{|\hat{J}(k)|^2}{ik+\mu} \frac{dk}{2\pi} \qquad \text{and}$$

$$Re\left(\frac{1}{ik+\mu}\right) = \frac{1}{2}\left(\frac{1}{ik+\mu} + \frac{1}{\mu-ik}\right) = \frac{\mu}{\mu^2+k^2} > 0$$

(Better way: $Re(z) > 0$ is stable under $z \mapsto z^{-1}$)

Because of the formula

$$\frac{\int e^{-\alpha|z|^2 + \bar{J}z + J\bar{z}}}{\int e^{-\alpha|z|^2}} = e^{\frac{|J|^2}{\alpha}}$$

for α real — Wait on 518 you take $\alpha = -in + \omega$??

It's OKAY because both sides are analytic in α with $\mathrm{Re}(\alpha) > 0$. So the conclusion is that on the space of paths $z(t)$ with complex values and $z(t) = 0$ for t large we have a quadratic form

$$\int \bar{z}(t) \left(\frac{d}{dt} + \mu \right) z(t)\, dt$$

with positive definite real part. If I put the Gaussian measure belonging to this form on this path space, then integrating powers of the functions $z \longmapsto z(t)$ or $\bar{z}(t)$ gives me the Green's functions belonging to $H_0 = \mu a^* a$.

So the next step is to find the analogue of the above in the fermion case.

January 27, 1980

Review: The goal is to understand Feynman diagrams, functional integrals, for fermions. We have a Fock space of fermions with $H_0 = \sum \mu_n a_n^* a_n$, and we have a perturbation H', and we are interested in computing the ground energy shift, or the shift in free energy at finite temperature. We use Dyson's expansion

$$(1) \qquad \frac{tr(e^{-\beta H})}{tr(e^{-\beta H_0})} = \langle e^{\beta H_0} e^{-\beta H} \rangle = 1 + \int_0^\beta dt_1 \langle H'(t_1) \rangle + \frac{1}{2!} \int_0^\beta dt_1 \int_0^\beta dt_2 \, \langle T[H'(t_1) H'(t_2)] \rangle + \cdots$$

The perturbation is expressed in terms of the operators a_n^*, a_n and so one needs to evaluate expectation values of time-ordered products of these operators. Do this by Wick's thm:

$$(2) \qquad \langle T[a_{n_1}^*(t_1) \cdots a_{n_p}^*(t_p) \, a_{n_p'}(t_p') \cdots a_{n_1'}(t_1')] \rangle = \det\left(\langle T[a_{n_i}^*(t_i) \, a_{n_j'}(t_j')] \rangle \right)$$

Here T time-orders the product with the appropriate signs.

Important point: The Green's functions (2) are not defined for equal times, because then the T operator is undefined. Example: Take $H_0 = \mu a^* a$ with $\mu > 0$ and vacuum expectation values:

$$\langle T[a^*(t) a(t')] \rangle = e^{\mu(t-t')} \langle 0 | \begin{cases} a^* a & t > t' \\ a a^* & t < t' \end{cases} | 0 \rangle$$

$$= \begin{cases} 0 & t > t' \\ -e^{\mu(t-t')} & t < t' \end{cases}$$

On the other hand when one uses the Dyson expansion, one needs Green's functions [redacted] at equal times. Therefore it looks like we have a renormalization problem already at this level with 0 space dimensions.

We know that the Green's functions (2) can be "explained" by fermion functional integration. Let's work this out. Start from

there's a mistake in signs - see 582

$$\frac{\int e^{-\sum \bar{\psi}_i a_{ij} \psi_j}\, \tilde{\psi}_{m_1} \cdots \tilde{\psi}_{m_p} \psi_{n_p} \cdots \psi_{n_1}}{\int e^{-\sum \bar{\psi}_i a_{ij} \psi_j}} = \det\left(b_{m_i n_j}\right)$$

where $(b_{ij}) = (a_{ij})^{-1}$. Let's apply this to variables $\bar{\psi}(t)$, $\psi(t)$ $\quad t \in [0,\beta)$ and the thermal Green's fn.

$$G(t,t') = \langle T[a^*(t)\, a(t')] \rangle$$

(this will be the matrix b_{ij})

wrong order

$$= e^{\mu(t-t')} \begin{cases} \langle a^* a \rangle & t > t' \\ -\langle a a^* \rangle & t < t' \end{cases}$$

Now

$$\langle a^* a \rangle = \frac{e^{-\beta\mu}}{1+e^{-\beta\mu}}, \qquad + \langle a a^* \rangle = -\langle a^* a \rangle + 1 = \frac{1}{1+e^{-\beta\mu}}$$

Thus

$$G(t,t') = e^{\mu(t-t')} \begin{cases} \dfrac{e^{-\beta\mu}}{1+e^{-\beta\mu}} & t > t' \\ \dfrac{e^{-\beta\mu}}{1+e^{-\beta\mu}} - 1 & t < t' \end{cases}$$

is simply the Green's function for $\frac{d}{dt}-\mu$ on $[0,\beta]$ satisfying the boundary conditions

$$\left.\begin{aligned} G(\beta,t') &= e^{-\mu t'}\left(\frac{1}{1+e^{-\beta\mu}}\right) \\ G(0,t') &= e^{-\mu t'}\left(\frac{e^{-\beta\mu}}{1+e^{-\beta\mu}}-1\right) \end{aligned}\right\} \Rightarrow \quad G(\beta,t') = -G(0,t')$$

Thus it's anti-periodic and hence given by

$$G(t,t') = \frac{1}{\beta}\sum_k \frac{e^{ik(t-t')}}{ik-\mu} \qquad k \in \frac{2\pi}{\beta}\left(\frac{1}{2}+\mathbb{Z}\right)$$

At this point, I am confused by what kind of functional integral to write down. The original idea is to have independent anti-commuting variables $\tilde{\psi}(t), \psi(t)$ for each t on $0 \le t < \beta$, and then the basic 2 form is

$$\int_0^\beta dt \ \tilde{\psi}(t)\wedge\left(\frac{d}{dt}-\mu\right)\psi(t)$$

but this isn't very clear. Note that the above makes sense if $\tilde{\psi}, \psi$ are functions on $[0,\beta]$ with values in $\Lambda^1 V$. Also it is the same as

$$\int_0^\beta dt \ \psi(t)\wedge\left(\frac{d}{dt}+\mu\right)\tilde{\psi}(t)$$

provided

$$\int_0^\beta dt\left(\tilde{\psi}\frac{d\psi}{dt}+\frac{d\tilde{\psi}}{dt}\psi\right) = \left[\tilde{\psi}\psi\right]_0^\beta = 0$$

and this will be the case if $\tilde{\psi}, \psi$ are both periodic or anti-periodic. More generally if $\psi(\beta) = \zeta\psi(0)$, $\tilde{\psi}(\beta) = \zeta^{-1}\tilde{\psi}(0)$.

There's a problem in making sense of $\tilde{\psi}(t), \psi(t)$ being independent anti-commuting variables, which can be solved as follows. You want $\tilde\psi(t)$, $\psi(t)$ to be paths in a vector space $\Lambda' V$ which are anti-periodic. Take the Fourier transform

$$\tilde\psi(t) = \frac{1}{\sqrt{\beta}}\sum_k{}' \tilde\psi_k e^{ikt} \qquad k \in \frac{2\pi}{\beta}\left(\tfrac{1}{2}+\mathbb{Z}\right)$$

$$\psi(t) = \frac{1}{\sqrt{\beta}}\sum_k{}' \psi_k e^{-ikt}$$

and require that the elements $\tilde\psi_k, \psi_k \in \Lambda' V$ be a basis (this hopefully can be made sensible with a suitable topology).

Notice that we have now side-stepped the problem of the functional integral. In some sense the Pfaffian defin. forgets that the integral is an average over classical fields, and it just gives you a number. In any case our Green's functions are

$$\frac{\int e^{-\sum_k \tilde\psi_{-k}(ik-\mu)\psi_k}\ \tilde\psi_{k_1}\cdots\tilde\psi_{k_n}\psi_{k_n'}\cdots\psi_{k_1'}}{\int e^{-\sum_k' \tilde\psi_k(ik-\mu)\psi_k}} = \det\left(\frac{\delta(k_i+k_j')}{ik_j-\mu}\right)$$

The next step is to put in the interaction H'. The simplest case is $H' = g\, a^* a$ which leads to

$$g\int \tilde\psi(t)\psi(t)\,dt = g\sum_k{}' \tilde\psi_{-k}\psi_k$$

So now we want to evaluate

$$(+)\qquad \frac{\int e^{-\sum \tilde\psi_{-k}(ik-\mu)\psi_k - g\sum\tilde\psi_{-k}\psi_k}}{\int e^{-\sum\tilde\psi_{-k}(ik-\mu)\psi_k}}$$

this is wrong because it depends on $\mu-g$ not $\mu+g$. Your G should be $\left(\frac{d}{dt}+\mu\right)^{-1}$

as a power series in g. The answer should be

$$\circledast \qquad \frac{\operatorname{tr} e^{-\beta(\mu a^* a + g a^* a)}}{\operatorname{tr} e^{-\beta(\mu a^* a)}} = \frac{1+e^{-\beta(\mu+g)}}{1+e^{-\beta\mu}}$$

The first order term of (+) in g is

$$-g \underbrace{\sum_k{}' \frac{1}{ik-\mu}}_{\beta G(t,t)}$$

If this series is summed à la Eisenstein

$$\sum_k \frac{1}{ik-\mu} = \frac{1}{2}\sum\left(\frac{1}{ik-\mu}+\frac{1}{-ik-\mu}\right) = -\mu\sum\frac{1}{k^2+\mu^2}$$

This has poles at $\mu = ik = i\frac{2\pi}{\beta}(n+\frac{1}{2})$ so it has poles where $\left(\cosh\frac{\beta\mu}{2}\right)^{-1}$ does. So

$$\sum_k \frac{1}{ik-\mu} = -\frac{\sinh\left(\frac{\beta\mu}{2}\right)}{\cosh\left(\frac{\beta\mu}{2}\right)}\cdot \text{const.}$$

Take residue at $\mu = \frac{\pi}{\beta}i$, $\quad \frac{\beta}{2}\sinh\left(\frac{\pi}{2}i\right) = \frac{\beta}{2}i$. Thus

$$\sum_k \frac{1}{\mu-ik} = \frac{\frac{\beta}{2}\sinh\left(\frac{\beta\mu}{2}\right)}{\cosh\left(\frac{\beta\mu}{2}\right)} = \frac{\beta}{2}\,\frac{e^{\beta\mu}-1}{e^{\beta\mu}+1}$$

So

$$G(t,t) = \frac{1}{\beta}\sum{}'\frac{1}{ik-\mu} = -\frac{1}{2}\,\frac{1-e^{-\beta\mu}}{1+e^{-\beta\mu}}$$

is the average of $G(t^+,t)$ and $G(t^-,t)$. So

$$\begin{array}{c}\text{first order in } g\\ \text{term of } (+)\end{array} = \frac{g\beta}{2}\,\frac{1-e^{-\beta\mu}}{1+e^{-\beta\mu}}$$

$$\begin{array}{c}\text{first order in } g\\ \text{term of } \circledast\end{array} = \frac{e^{-\beta\mu}(-\beta g)}{1+e^{-\beta\mu}} = -\beta g\, G(t^+,t)$$

So the result of this calculation shows that the problems of Green's functions at equal times is not solved by working with energy representation.

Question: Is there some way of modifying things so as to get the correct results from the functional integral?

January 28, 1980

Situation: We have an exterior algebra $\Lambda \mathcal{H}_1$, and a Hamiltonian $H = H_0 + H'$ on $\mathcal{H}_1$ extended to $\Lambda\mathcal{H}_1$ as a derivation. We want to compute

$$\frac{tr\left(e^{-\beta H}\right)}{tr\left(e^{-\beta H_0}\right)} = \langle e^{\beta H_0} e^{-\beta H}\rangle = 1 - \int_0^\beta dt \langle H'(t)\rangle + \frac{1}{2!}\cdots$$

Suppose $|n\rangle$ is an orthonormal basis for $\mathcal{H}_1$ and let a_n^*, a_n be the associated creation and annihilation operators on $\Lambda\mathcal{H}_1$. Then

$$H' = \sum V_{mn}\, a_m^* a_n \qquad \text{where} \quad V_{mn} = \langle m | H' | n\rangle$$

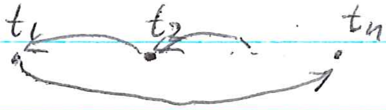
Let's consider the 2nd order term in the Dyson series:

$$\frac{1}{2!}\int_0^\beta\!\!\int_0^\beta dt_1 dt_2 \langle T[H'(t_1)H'(t_2)]\rangle$$

$$= \frac{1}{2!}\sum_{\substack{m_1,n_1,t_1\\ m_2,n_2,t_2}} V_{m_1 n_1} V_{m_2 n_2} \langle T[a^*_{m_1}(t_1) a_{n_1}(t_1)\, a^*_{m_2}(t_2) a_{n_2}(t_2)]\rangle$$

Apply Wick's thm. to evaluate the last expectation value. There is a problem with the equal times; suppose $a_m^*(t)a_n(t)$ is interpreted as $a_m^*(t^+)a_n(t)$, so the creation occurs after the destruction. When you apply Wick's thm. you get a sum of terms each of which you describe by a graph.

Each vertex in the diagram has two edges. We have variables $\psi_n(t)$, $\bar\psi_m(t)$ for each n,t and m,t' to label each edge coming into a vertex with, but the interaction only connects $\bar\psi_m(t)$ with $\psi_n(t')$ when t' comes immediately before t. Consequently one can label each edge at a vertex with an arrow coming in for a ψ_n and going out for a $\tilde{\psi}_m$. Contractions link $\tilde{\psi}_n$ to a ψ so the n-th order contribution is a sum over all directed graphs with vertices $1, \ldots, n$ with one arrow entering and one leaving each vertex. One sums over ways of assigning (m, n, t) at each vertex. The contribution is a $(-V_{mn})$ factor for a vertex of type m, n and a factor $\langle T[a_n(t_1) a_m^*(t_2)] \rangle$ for an edge from (m, t_2) to (n, t_1) and then there is a sign factor. The graph is a bunch of loops so look at a connected graph.

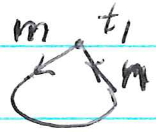


$$\langle T[a^*(t_1)\, a(t_1)\, a^*(t_2)\, a(t_2) \ldots a^*(t_n)\, a(t_n)] \rangle$$

even number

You want to move $a^*(t_1)$ to the far right getting -1. Thus the rule -1 for each loop.

For example take a first order graph:



$$-V_{mn} \underbrace{\langle T[a_m^*(t_1^+)\, a_n(t_1)] \rangle}_{-G_{nm}(t_1, t_1^+)}$$

fermion loop

Anyway it's clear we have cycles for our connected diagrams leading to

$$\log \langle S \rangle = + \operatorname{tr}(GV) - \frac{1}{2} \operatorname{tr}(GV)^2 + \frac{1}{3} \operatorname{tr}(GV)^3 - \cdots$$

$$= \operatorname{tr} \log(1 + GV) = \log(\det(1 + GV))$$

In the above, the term $tr(GV)$ has to be specified because of the discontinuity of G on the diagonal. The other traces should be well-defined, because they are integrals and the equal times are of measure 0.

Correct formula:

$$\frac{\int e^{-\sum \tilde{\psi}_i a_{ij} \psi_j} \psi_k \tilde{\psi}_\ell}{\int e^{-\sum \tilde{\psi}_i a_{ij} \psi_j}} = b_{k\ell} \qquad \text{where } (b_{ij}) = (a_{ij})^{-1}$$

Proof: $LHS = \frac{\partial}{\partial a_{\ell k}} \log \underbrace{\int e^{-\sum \tilde{\psi}_i a_{ij} \psi_j}}_{\pm \det(a_{ij})} = \frac{\ell k\text{-th minor of } |a_{ij}|}{|a_{ij}|} = b_{k\ell}$

Now we want

$$\frac{\int e^{-\int \tilde{\psi} a \psi} \psi(t) \tilde{\psi}(t')}{\int e^{-\int \tilde{\psi} a \psi}} = \langle T[a(t) a^*(t')] \rangle$$

and we have $\langle T[a(t) a^*(t')] \rangle$ = kernel of $\left(\frac{d}{dt} + \mu\right)^{-1}$ because $a(t) = e^{-\mu t} a$ and T jumps by 1 as t crosses t'. So it follows that $a = \left(\frac{d}{dt} + \mu\right)$.

$$\boxed{\frac{\int e^{-\int \tilde{\psi}\left(\frac{d}{dt}+\mu\right)\psi\, dt} \psi(t) \tilde{\psi}(t')}{\int e^{-\int \tilde{\psi}\left(\frac{d}{dt}+\mu\right)\psi\, dt}} = \langle T[a(t) a^*(t')] \rangle}$$

and for higher order products by Wick's thm.

So if we have an interaction $H' = a^* V a$, then

$$\langle S \rangle = \frac{\int e^{-\int \bar{\psi}\left(\frac{d}{dt} + H_0\right)\psi\, dt - \int \bar{\psi} V \psi\, dt}}{\int e^{-\int \bar{\psi}\left(\frac{d}{dt} + H_0\right)\psi\, dt}}$$

So proceeding formally we find

$$\langle S \rangle = \frac{\det\left(\frac{d}{dt} + H_0 + V\right)}{\det\left(\frac{d}{dt} + H_0\right)} = \det\left(1 + G V\right)$$

which is consistent with page 581.