November 19, 1980

Let's consider a unitary operator U on the holomorphic representation of the CCR such that conjugation by U preserves the space V of operators spanned by a, a^*. Thus I suppose that

$$U^{-1}\begin{pmatrix} a \\ a^* \end{pmatrix} U = \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{T}\begin{pmatrix} a \\ a^* \end{pmatrix}.$$

Now the space V has a natural symplectic structure given by $[\ ,\ ]$, so it follows that the matrix T is symplectic. Precisely we have

$$I = U^{-1}[a, a^*]U = [U^{-1}aU, U^{-1}a^*U]$$

$$= [Aa + Ba^*, Ca + Da^*]$$

$$I = AD^t - BC^t$$

(This is short hand for

$$\delta_{ij} = U^{-1}[a_i, a_j^*]\, U = \sum_{k,l}[A_{ik}a_k + B_{ik}a_k^*, C_{jk}a_k + D_{jl}a_l^*]$$

$$= \sum_k A_{ik}D_{jk} - B_{ik}C_{jk})$$

Similarly we have

$$0 = AB^t - BA^t$$

$$0 = CD^t - DC^t$$

or

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} = I$$

or

$$T^{-1} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{J^{-1}} T^t \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{J} \implies T^tJT = J$$

which means T is symplectic for J.

The next point is that V has a conjugation $*$ which is preserved by conjugation with U as U is unitary. Thus [redacted]

$$U^* a U = Aa + Ba^* \Longrightarrow U^* a^* U = \bar{A} a^* + \bar{B} a$$

$$\Longrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$$

So we conclude that [redacted] U induces a map on V

$$U^* \begin{pmatrix} a \\ a^* \end{pmatrix} U = \underbrace{\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}}_{T} \begin{pmatrix} a \\ a^* \end{pmatrix}$$

such that

$$T^{-1} = \begin{pmatrix} A^* & -B^t \\ -B^* & A^t \end{pmatrix} \qquad \begin{cases} B^t \bar{A} & \text{symm.} \\ AB^t & \text{symm.} \\ AA^* - BB^* = 1 \end{cases}$$

Given U above we want to compute $\langle e_{\bar{z}} | U | e_\lambda \rangle$, or better Ue_λ. When $\lambda = 0$, $e_0 = |0\rangle$ is ~~[illegible]~~ annihilated by the subspace W in V spanned by the a_i. Hence $U|0\rangle$ is [redacted] annihilated by the $U a_i U^{-1}$! But

$$U a U^{-1} = D^t a - B^t a^*$$

If $f(z)$ is killed by the $U a U^{-1}$, then

$$\left(D^t \frac{d}{dz} - B^t z \right) f = 0$$

$$\Longrightarrow \quad f = c\, e^{\frac{1}{2} z^t (BD^{-1})^t z}$$

Recall that $\quad D^t B - B^t D = 0 \quad$ so $\quad BD^{-1} = (D^t)^{-1} B^t = (BD^{-1})^t$

is symmetric. Thus $U(0) = c\, e^{\frac{1}{2} z^t B D^{-1} z}$.

More generally if $f(z) = \langle e_{\bar z} | U e_\lambda \rangle$, then

$$\left(D^t \frac{d}{dz} - B^t z \right) f = \lambda f$$

so

$$\langle e_{\bar z} \mid U e_\lambda \rangle = c\, e^{\frac{1}{2} z^t (B D^{-1}) z + z^t (D^t)^{-1} \lambda}$$

Similarly if we replace U by U^{-1} we get $U^{-1} e_\lambda$ is an eigenvector for $U^{-1} a U = Aa + Ba^*$, hence $f(z) = \langle e_{\bar z} | U^{-1} e_\lambda \rangle$ satisfies

$$\left(A \frac{d}{dz} + Bz \right) f = \lambda f$$

so

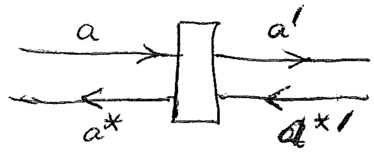
$$\langle e_{\bar z} | U^{-1} e_\lambda \rangle = \text{const} \cdot e^{-\frac{1}{2} z^t (A^{-1} B) z + z^t A^{-1} \lambda}$$

November 21, 1980

Let U be an operator such that conjugation by U preserves the space V of operators spanned by a, a^*. Think of ~~[illegible]~~ the Hilbert space as incoming free states, and U as the scattering operator. Then put

$$\cancel{\begin{matrix} a' = U a U^{-1} \\ a^{*\prime} = U a^* U^{-1} \end{matrix}} \qquad \begin{pmatrix} a' \\ a^{*\prime} \end{pmatrix} = U^{-1} \begin{pmatrix} a \\ a^* \end{pmatrix} U$$

Then a' is a acting after the scattering has taken place, i.e. it is the operator which seems to be denoted a_{out}. Picture:



Let us now assume that the operators $a, a^{*\prime}$ span V, in which case we get a relation

$$(*) \qquad \begin{pmatrix} a' \\ a^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a^{*\prime} \\ a \end{pmatrix}$$

There are two cases where this assumption is valid

1) U is close to the identity.
2) U is unitary.

To see 2) let's recall that on V we have a hermitian form obtained by polarizing the function

$$\theta \longmapsto [\theta, \theta^*]$$

The space span(a) is positive for this form, and span(a^*) is negative. If U is unitary, then U preserves this form, so span(a'^*) is also negative for the hermitian form. Hence span(a) and span(a'^*) can't intersect, so they span V.

So now let's compute the amplitude $\langle e_{\bar z}|U|e_\lambda\rangle$:

$$\frac{\partial}{\partial z}\langle e_{\bar z}|U|e_\lambda\rangle = \langle e_{\bar z}|\underbrace{aU}_{Ua'}|e_\lambda\rangle$$

$$= \langle e_{\bar z}|U(\alpha a^{*\prime}+\beta a)|e_\lambda\rangle$$

$$= \alpha\langle e_{\bar z}|a^* U|e_\lambda\rangle + \beta\lambda\langle e_{\bar z}|U|e_\lambda\rangle$$

$$= (\alpha z+\beta\lambda)\langle e_{\bar z}|U|e_\lambda\rangle$$

$$\frac{\partial}{\partial\lambda}\langle e_{\bar z}|U|e_\lambda\rangle = \langle e_{\bar z}|Ua^*|e_\lambda\rangle$$

$$= \langle e_{\bar z}|U(\gamma a^{*\prime}+\delta a)e_\lambda\rangle$$

$$= (\gamma z+\delta\lambda)\langle e_{\bar z}|U|e_\lambda\rangle$$

One concludes that $\log\langle e_{\bar z}|U|e_\lambda\rangle$ is a quadratic fn. of z,λ and hence that

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \text{ is symmetric} \qquad \begin{matrix}\alpha=\alpha^t\\ \gamma=\beta^t\end{matrix} \quad \delta=\delta^t$$

Thus

$$\boxed{\langle e_{\bar z}|U|e_\lambda\rangle = \text{const}\; e^{\frac{1}{2}z^t\alpha z+z^t\beta\lambda+\frac{1}{2}\lambda^t\delta\lambda}}$$

Now suppose U is unitary, [redacted] whence

$$(a')^* = (a^*)'$$

Applying $*$ to equation $(*)$ on the preceding page, we find

$$\begin{pmatrix}a^{*\prime}\\ a\end{pmatrix} = \begin{pmatrix}\bar\alpha & \bar\beta\\ \bar\gamma & \bar\delta\end{pmatrix}\begin{pmatrix}a'\\ a^*\end{pmatrix}$$

and hence

$$\begin{pmatrix}\bar\alpha & \bar\beta\\ \bar\gamma & \bar\delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}^{-1}$$

Thus we conclude the matrix $S=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is a symmetric matrix, [redacted] which is also unitary, since

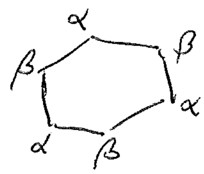
$$S^{-1} = \bar S = S^*.$$

Next
$$U e_0 = U|0\rangle = c\, e^{\frac{1}{2} z^t \alpha z}$$

Now
$$\langle e^{\frac{1}{2} z^t \beta z} | e^{\frac{1}{2} z^t \alpha z} \rangle = \sum_n \frac{1}{(n!)^2\, 2^{2n}} \langle (z^t \beta z)^n | (z^t \alpha z)^n \rangle$$

One evalues the last inner product by Wicks thm, and obtains a sum over ways of connecting n β-vertices to n α-vertices

The connected graphs are



$2n$ vertices
Symmetry factor $2n$

So
$$\langle e^{\frac{1}{2} z^t \beta z} | e^{\frac{1}{2} z^t \alpha z} \rangle = \exp \Big\{ \underbrace{\sum_{n \geq 1} \frac{1}{2n} \operatorname{tr}(\bar{\beta}\alpha)^n}_{-\frac{1}{2} \cancel{\operatorname{tr}} \log(1 - \bar{\beta}\alpha)} \Big\}$$

$$\boxed{\langle e^{\frac{1}{2} z^t \beta z} | e^{\frac{1}{2} z^t \alpha z} \rangle = \det(1 - \bar{\beta}\alpha)^{-\frac{1}{2}}}$$

It follows that
$$1 = \| e_0 \|^2 = \| U e_0 \|^2 = |c|^2 \det(1 - \bar{\alpha}\alpha)^{-1/2}$$

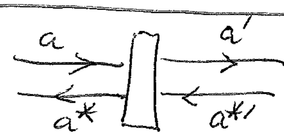
However because $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$ we have

$$\bar{\alpha}\alpha + \bar{\beta}\gamma = 1 \implies 1 - \bar{\alpha}\alpha = \bar{\beta}\beta^t$$

so $\qquad \det(1 - \bar{\alpha}\alpha) = |\det \beta|^2 \quad$ so $\quad |c| = |\det \beta|^{1/2}$

Summary of formulas: (with slight change of notation)

$$\begin{pmatrix} a' \\ a^{*\prime} \end{pmatrix} = U^{-1} \begin{pmatrix} a \\ a^{*} \end{pmatrix} U \text{[redacted]}$$



$$\begin{pmatrix} a' \\ a^{*}\text{[redacted]} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^t & \gamma \end{pmatrix} \begin{pmatrix} a^{*\prime} \\ a \end{pmatrix} \qquad \begin{pmatrix} \alpha & \beta \\ \beta^t & \gamma \end{pmatrix} \text{ is symmetric and unitary}$$

$$\langle e_{\bar{z}} | U | e_\lambda \rangle = \zeta \, |\det \beta|^{1/2} \, e^{\frac{1}{2} z^t \alpha z + z^t \beta \lambda + \frac{1}{2} \lambda^t \gamma \lambda}$$

where $|\zeta| = 1$

In these formulas a stands for the vector (a_i) and a^* for (a_i^*)[redacted].

November 22, 1980

Let's now consider a perturbed harmonic oscillator

$$H = \frac{p^2}{2} + \frac{1}{2}(\omega^2+\varepsilon)q^2$$

where $\varepsilon = \varepsilon(t)$ decays fast as $|t| \to \infty$. Let U be the S-matrix $U_D(\infty, -\infty)$.

First let's understand what happens classically. The equations of motion are

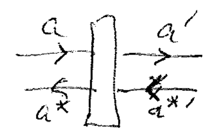
$$\dot{q} = p, \qquad \ddot{q} + (\omega^2+\varepsilon)q = 0$$

Given a solution $q(t)$ we have asymptotic behavior

$$Ae^{-i\omega t} + A^* e^{i\omega t} \xleftrightarrow{\ q(t)\ } A' e^{-i\omega t} + A^{*\prime} e^{i\omega t}$$

for certain numbers $A, A^*, A', A^{*\prime}$. We have

$$\begin{pmatrix} A' \\ A^* \end{pmatrix} = \begin{pmatrix} \tilde{R} & T \\ T & R \end{pmatrix} \begin{pmatrix} A^{*\prime} \\ A \end{pmatrix}$$



where the reflection coeff. R and transmission coefficient T are defined by

$$e^{-i\omega t} + Re^{i\omega t} \longleftrightarrow Te^{-i\omega t}$$

$$Te^{i\omega t} \longleftrightarrow \tilde{R}e^{-i\omega t} + e^{i\omega t}$$

where

$$\tilde{R} = -\frac{\bar{T}}{T}R$$

Now when we quantize, $q(t)$ becomes an operator with the asymptotic behavior

$$\frac{1}{\sqrt{2\omega}}\left(ae^{-i\omega t}+a^{*}e^{i\omega t}\right) \xleftrightarrow{\;g(t)\;} \frac{1}{\sqrt{2\omega}}\left(a'e^{-i\omega t}+a^{*\prime}e^{i\omega t}\right)$$

Here a, a^* are "in" operators and $a', a^{*\prime}$ are "out" operators.

The scattering operator $\mathcal{U}$ transforms "in" into "out":

$$\begin{pmatrix} a' \\ a^{*\prime} \end{pmatrix} = \mathcal{U}^{-1}\begin{pmatrix} a \\ a' \end{pmatrix}\mathcal{U}$$

I can now conclude that the same formula

$$\begin{pmatrix} a' \\ a^{*} \end{pmatrix} = \begin{pmatrix} \tilde{R} & T \\ T^{t} & R \end{pmatrix}\begin{pmatrix} a^{*\prime} \\ a \end{pmatrix}$$

holds for the operators that hold for the numbers.

(The above needs to be made clearer. The idea is [redacted] that [redacted] (Heisenberg) [redacted] operators can always be converted to numbers by setting

$$\langle \mathcal{O} \rangle = \langle \varphi | \mathcal{O} | \psi \rangle .$$

Consequently

$$\begin{pmatrix} a' \\ a^{*} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^{t} & \gamma \end{pmatrix}\begin{pmatrix} a^{*\prime} \\ a \end{pmatrix} \longmapsto \begin{pmatrix} \langle a' \rangle \\ \langle a^{*} \rangle \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^{t} & \gamma \end{pmatrix}\begin{pmatrix} \langle a^{*\prime} \rangle \\ \langle a \rangle \end{pmatrix}.)$$

Formulas for "in" and "out" operators.

$$\begin{aligned} a_H(t) &= \mathcal{U}(0,t)\, a\, \mathcal{U}(t,0) \\ &= \mathcal{U}(0,t)\, e^{-iH_0 t}\, \underbrace{e^{iH_0 t}\, a\, e^{-iH_0 t}}_{a_D(t)}\, e^{iH_0 t}\, \mathcal{U}(t,0) \end{aligned}$$

For $t \ll 0$ or $t \gg 0$: $e^{iH_0 t}\,\mathcal{U}(t,0)$ is independent of t. Put

$$W_{in} = e^{iH_0 t}\,\mathcal{U}(t,0) \qquad t \ll 0$$

and $\quad W_{out} = e^{iH_0 t}\, U(t,0) \qquad t \gg 0.$ Then

$$a_H(t) = W_{in}^{-1}\, e^{iH_0 t} a e^{-iH_0 t}\, W_{in}$$
$$= e^{-i\omega t}\left(W_{in}^{-1}\, a\, W_{in}\right)$$

so that

$$a_{in} = W_{in}^{-1}\, a\, W_{in}$$

and similarly for "out":

$$a_{out} = W_{out}^{-1}\, a\, W_{out}$$

Then $\qquad a_{out} = U^{-1} a_{in}\, U \quad$ provided

$$U = W_{in}^{-1}\, W_{out}$$
$$= U(0,t_i)\, e^{-iH_0 t_i}\, e^{iH_0 t_f}\, U(t_f,0)$$

This U is not $\quad U_D(\infty,-\infty) = e^{iH_0 t_f}\, U(t_f, t_{in})\, e^{-iH_0 t_{in}}$

however U is conjugate to [redacted] $U_D(\infty,-\infty)$.

These formulas are confusing. What's artificial is the use of the Schroedinger description at $t=0$. Really one should work with $a = a_{in}$ i.e. with the perturbation happening in $t \geq 0$.

Question: What the relation between the Green's function and the S-matrix?

Take the case of a perturbed simple harmonic oscillator

$$H = \frac{p^2}{2} + \frac{1}{2}(\omega^2 + \varepsilon)q^2$$

The Green's function is

$$G(t,t') = i \langle 0 | T[q(t)\, q(t')] | 0 \rangle$$

where $|0\rangle$ is the ground state for $H_0 = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$. Then for $t \neq t'$, G satisfies the equation of motion

$$\left(\frac{d^2}{dt^2} + (\omega^2 + \varepsilon) \right) G = 0$$

because the [redacted] operator $q(t)$ satisfies it. On the other hand

$$\frac{d}{dt} G(t,t') = i \langle 0 | T[p(t)\, q(t')] | 0 \rangle$$

$$\left. \frac{d}{dt} G(t,t') \right]_{t'-}^{t'+} = i \langle 0 | p(t') q(t') - q(t') p(t') | 0 \rangle$$
$$= 1.$$

Thus
$$\left(\frac{d^2}{dt^2} + (\omega^2 + \varepsilon) \right) G(t,t') = \delta(t,t')$$

What are the boundary conditions satisfied by G as $t \to \pm\infty$? As $t \to -\infty$ we have

$$q(t) = q_{in}(t) = \frac{1}{\sqrt{2\omega}} \left(e^{-i\omega t} a_{in} + e^{i\omega t} a^{*}_{\text{[redacted]}\,in} \right)$$

and
$$G(t,t') = i \langle 0 | q(t')\, q(t) | 0 \rangle.$$

It seems we want to change the definition of G to

$$G(t,t') = i \langle 0_{out} | T[q(t)\, q(t')] | 0_{in} \rangle \big/ \langle 0_{out} | 0_{in} \rangle$$

Then
$$q(t) |0_{in}\rangle = \frac{1}{\sqrt{2\omega}} e^{i\omega t} a^{*}_{in} |0_{in}\rangle \qquad t \ll 0$$

involves negative frequences as $t \to -\infty$. Therefore we get the standard boundary conditions for the Green's function.

[redacted]

November 23, 1980

Review: [redacted] We consider a perturbed simple harmonic oscillator:
$$H = \frac{p^2}{2} + \frac{1}{2}(\omega^2+\varepsilon(t))q^2$$
When quantized we get an operator $q(t) = U(0,t)\,q\,U(t,0)$ satisfying
$$\dot{q}(t) = p(t)\ ,\quad \ddot{q}(t) + (\omega^2+\varepsilon(t))q(t) = 0$$
and having the asymptotic behavior
$$\frac{1}{\sqrt{2\omega}}\left(e^{-i\omega t}a + e^{+i\omega t}a^*\right) \xleftarrow{\ q(t)\ } \!\!\longrightarrow \frac{1}{\sqrt{2\omega}}\left(e^{-i\omega t}a' + e^{i\omega t}a^{*\prime}\right)$$
The scattering operator U satisfies
$$\begin{pmatrix} a' \\ a^{*\prime} \end{pmatrix} = U^{-1}\begin{pmatrix} a \\ a^* \end{pmatrix} U$$
and is nicely described using the S-matrix for the classical equation of motion: Suppose we have
$$e^{-i\omega t} + Re^{i\omega t} \longleftrightarrow Te^{-i\omega t}$$
$$Te^{i\omega t} \longleftrightarrow \tilde{R}e^{-i\omega t} + e^{i\omega t}.$$
so that
$$\begin{pmatrix} a' \\ a^* \end{pmatrix} = \begin{pmatrix} \tilde{R} & T \\ T & R \end{pmatrix}\begin{pmatrix} a^{*\prime} \\ a \end{pmatrix}.$$
Then I know that U is given by
$$\langle e_{\bar{z}} | U | e_\lambda \rangle = \gamma\,|\text{[redacted]}\,T|^{1/2}\, e^{\frac{1}{2}\tilde{R}z^2 + Tz\lambda + \frac{1}{2}R\lambda^2} \qquad |\gamma| = 1$$
and so U has the form.
$$U = \gamma\,|T|^{1/2}\, e^{\frac{1}{2}\tilde{R}a^{*2}}\, e^{\text{[redacted]}\, a^*(\log T)a}\, e^{\frac{1}{2}Ra^2}$$
It seems that this is the simplest possible description

one can give of the scattering operator U.

Let review Green's fns. techniques as these connect up nicely with path integrals. We add to our Hamiltonian a source term

$$H_J = \frac{p^2}{2} + \frac{1}{2}(\omega^2+\varepsilon)q^2 + Jq$$

and let

$$Z(J) = \frac{\langle 0_{out} | 0_{in} \rangle_J}{\langle 0_{out} | 0_{in} \rangle}$$

$$= \frac{\langle 0 | U_J(t_f, t_i) | 0 \rangle}{\langle 0 | U(t_f, t_i) | 0 \rangle}$$

Then

$$\delta \log Z(J) = (-i)\int dt_1 \frac{\langle 0 | U_J(t_f, t_1)\, \delta J(t_1) q\, U_J(t_1, t_i) | 0 \rangle}{\langle 0 | U_J(t_f, t_i) | 0 \rangle}$$

$$= (-i) \int dt_1\, \delta J(t_1) \langle q(t_1) \rangle_J$$

where

$$\langle q(t) \rangle_J = \frac{\langle 0_{out} | q_H(t) | 0_{in} \rangle}{\langle 0_{out} | 0_{in} \rangle} = \frac{\langle 0 | U_J(t_f, t) q\, U_J(t, t_i) | 0 \rangle}{\langle 0 | U_J(t_f, t_i) | 0 \rangle}$$

Then $\langle q(t) \rangle_J$ has to satisfy the equations of motion w.r.t. H_J:

$$\dot{q} = i[H, q] = p$$

$$\dot{p} = i\left[\tfrac{1}{2}(\omega^2+\varepsilon)q^2 + Jq, p\right] = -(\omega^2+\varepsilon)q - J$$

Thus

$$\left(\frac{d^2}{dt^2} + \omega^2 + \varepsilon\right) \langle q(t) \rangle_J = -J(t)$$

and the boundary conditions are the standard ones. Hence

$$\langle q(t) \rangle_J = -\int G(t,t')\, J(t')\, dt'$$

$$\delta \log Z(J) = i \int dt\, dt'\ \delta J(t)\, G(t,t') J(t')$$

$$\therefore\ Z(J) = \exp\left\{ \frac{i}{2} \int dt\, dt'\ J(t)\, G(t,t') J(t') \right\}$$

Recall that $G(t,t')$ is defined by

$$\left(\frac{d^2}{dt^2} + (\omega^2 + \varepsilon) \right) G(t,t') = \delta(t-t')$$

$$G(t,t') \quad \text{prop. to } e^{-i\omega t} \qquad t \gg 0$$
$$\qquad\quad \text{"} \quad \text{"} \quad e^{i\omega t} \qquad t \ll 0$$

and is given by

$$G(t,t') = \frac{\varphi(t_<)\, \psi(t_>)}{W(\varphi, \psi)}$$

Then

$$e^{-i\omega t} + R e^{i\omega t} \xleftrightarrow{\ \psi\ } T e^{-i\omega t}$$

$$T e^{i\omega t} \xleftrightarrow{\ \varphi\ } \tilde{R} e^{-i\omega t} + e^{i\omega t}$$

and

$$W(\varphi, \psi) = T\, W(e^{+i\omega t}, e^{-i\omega t}) = T(-2i\omega)$$

Thus

$$G(t,t') = \begin{cases} \dfrac{T}{(-2i\omega)}\, e^{-i\omega(t-t')} & t' \ll 0 \ll t \\[2ex] \dfrac{1}{(-2i\omega)} \left(e^{+i\omega t'}\right)\left(e^{-i\omega t} + R e^{i\omega t}\right) & t' < t \ll 0 \\[2ex] \dfrac{1}{(-2i\omega)} \left(\tilde{R} e^{-i\omega t'} + e^{i\omega t'}\right) e^{-i\omega t} & 0 \ll t' < t \end{cases}$$

It's clear that the asymptotic behavior of the Green's function is equivalent to the classical S-matrix.

Now it is time to look at the important case, where one has a continuous harmonic oscillator. For example, consider a string with the equation of motion

$$\ddot{\phi}_x + (-\Delta_x)\phi_x = 0$$

In other words ω^2 is the operator $-\Delta_x$. It might look nicer if I took a discrete string, but the essential point is that the spectrum be continuous.

Next consider a perturbation:

$$(*) \qquad \ddot{\phi} + (-\Delta + V)\phi = 0$$

Here V might depend upon t. I would like it to be time-independent, but technically I want to let it act adiabatically or for a finite time interval.

The important question is how to compute the S-matrix for the above "classical equation of motion" (*).

November 26, 1980

We consider a perturbed wave equation

1) $$\ddot{\psi} + (-\Delta + V)\psi = 0$$

on the half-line $x \geq 0$ where V has compact support and where a boundary condition is given at $x = 0$. Say we have the Neumann bdry condition

$$\frac{\partial \psi}{\partial x} = 0 \quad \text{at} \quad x = 0$$

and that we want to view 1) as a perturbation of the free wave equation

2) $$\ddot{\psi}^0 + (-\Delta)\psi^0 = 0$$

with the same boundary condition.

Solutions of 1) can be described in the form

$$\psi(x,t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \phi(x,\omega) f(\omega)$$

where $\phi(x,\omega)$ is the solution of

$$(-\Delta + V)\phi = \omega^2 \phi$$

$$\frac{\partial \phi}{\partial x} = 0 \quad \text{at} \quad x = 0$$

$$\phi = 2 \quad \text{at} \quad x = 0.$$

I am assuming there are no bound states so that the $\phi(x,\omega)$ for $\omega > 0$ for a complete set of eigenfns. for $-\Delta + V$ on $x \geq 0$ with the Neumann bdry condition.

Notice that $\phi(x,\omega) = \phi(x,-\omega)$ and that the fact that the integration goes for $-\infty < \omega < \infty$ is required so as to get arbitrary initial data $\psi, \dot{\psi}$.

Solutions of the free equation are

3) $$\psi^0(x,t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \left(e^{-i\omega x} + e^{i\omega x} \right) f(\omega)$$
$$= \hat{f}(t+x) + \hat{f}(t-x)$$

Next we want to determine the free asymptotes as $t \to \pm\infty$ of a solution $\psi(x,t)$ of 1). The point is that for x in a compact set, the Riemann-Lebesgue lemma shows (assuming f rapidly decreasing) that $\psi(x,t) \to 0$ as $t \to \pm\infty$. Once x is outside the support of V we have

$$\phi(x,\omega) = A(\omega) e^{-i\omega x} + \overbrace{A(-\omega)}^{\bar{A}(\omega)} e^{i\omega x}$$

hence
$$\psi(x,t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \left(A(\omega) e^{-i\omega x} + \bar{A}(\omega) e^{i\omega x} \right) f(\omega)$$
$$= (Af)^\wedge(t+x) + (\bar{A}f)^\wedge(t-x)$$

Therefore the asymptotes are

$$(Af)^\wedge(t+x) + (Af)^\wedge(t-x) \xleftrightarrow{\quad\psi\quad} (\bar{A}f)^\wedge(t+x) + (\bar{A}f)^\wedge(t-x)$$

Hence if we describe the free solutions by 3) above using $f(\omega)$, the scattering operator is

$$f(\omega) \longmapsto \frac{\bar{A}(\omega)}{A(\omega)} f(\omega).$$

The next step is to connect this up with operators.

Let's take the classical Heisenberg viewpoint. This means our physical quantities are functions on the space $\mathcal{S}$ of solutions to

$$\ddot{\psi} + (-\Delta + V)\psi = 0.$$

$\mathcal{S}$ is the space of classical trajectories. We will identify $\mathcal{S}$ with the space of $f(\omega)$ via the formula.

$$\psi(x,t) = \int \frac{d\omega}{2\pi} \phi(x,\omega) e^{-i\omega t} f(\omega)$$

Then we have the following functions on $\mathcal{S}$:

$$q(x,t) : f \longmapsto \psi(x,t) = \int \frac{d\omega}{2\pi} \phi(x,\omega) e^{-i\omega t} f(\omega)$$

$$p(x,t) : f \longmapsto \dot{\psi}(x,t) = \int \frac{d\omega}{2\pi} \phi(x,\omega) e^{-i\omega t} (-i\omega) f(\omega)$$

These are linear functions on $\mathcal{S}$ satisfying the equations

$$q(x,t)^{\circ} = p(x,t)$$

$$p(x,t)^{\circ} = -(-\Delta_x + V) q(x,t)$$

We also have the functions

$$q_{in}(x,t) : f \longmapsto \int \frac{d\omega}{2\pi} (e^{-i\omega x} + e^{i\omega x}) e^{-i\omega t} A(\omega) f(\omega)$$

satisfying the free equations of motion

$$\ddot{q}_{in} = +\Delta q_{in}$$

and such that for any f

$$q(x,t) f \sim q_{in}(x,t) f \qquad \text{as} \qquad t \to -\infty$$

Similarly

$$q_{out}(x,t) : f \longmapsto \int \frac{d\omega}{2\pi} (e^{-i\omega x} + e^{i\omega x}) e^{-i\omega t} \bar{A}(\omega) f(\omega)$$

In this picture the scattering operator transforms the function q_{in} into q_{out}:

$$q_{out} = q_{in}\, U$$

so

$$\bar{A}(\omega) f(\omega) = A(\omega) (Uf)(\omega)$$

and

$$(Uf)(\omega) = \frac{\bar{A}(\omega)}{A(\omega)} f(\omega)$$

November 29, 1980

Review the program. I am considering the wave equation

$$1) \qquad \partial_t^2\psi + (-\partial_x^2 + V)\psi = 0 \qquad \text{on} \quad x \geq 0$$

with a boundary condition at $x = 0$, say

$$\partial_x \psi = 0 \qquad \text{at} \quad x = 0.$$

This is to be regarded as a perturbation of the problem

$$2) \qquad \begin{cases} \partial_t^2 \psi^\circ + (-\partial_x^2)\psi^\circ = 0 \\ \qquad\qquad \partial_x \psi^\circ = 0 \qquad \text{at } x = 0. \end{cases}$$

Solutions of 2) can be described as follows. The wave equation is an oscillator equation and hence its solutions can be described using normal modes. For each $\omega \geq 0$ there is one normal mode given by solving

$$\begin{cases} (-\partial_x)^2 u = \omega^2 u \\ \partial_x u = 0 \end{cases} \qquad \therefore \; u = e^{-i\omega x} + e^{i\omega x}$$

up to a scalar

Hence the general solution of 2) is

$$\psi^\circ(x,t) = \int \frac{d\omega}{2\pi} \left(e^{-i\omega x} + e^{i\omega x}\right) e^{-i\omega t} f(\omega)$$

for some f satisfying $\overline{f(\omega)} = f(-\omega)$ if ψ° is real. Thus

$$\psi^\circ(x,t) = \int_0^\infty \frac{d\omega}{\pi} (\cos \omega x) \left(e^{-i\omega t} f(\omega) + e^{i\omega t}\overline{f(\omega)}\right)$$

$$3) \qquad \boxed{\psi^\circ(x,t) = \frac{2}{\pi}\int_0^\infty d\omega\, (\cos\omega x)\; \mathrm{Re}\left(e^{-i\omega t} f(\omega)\right).}$$

Let $q(x)$, $p(x)$ be the (coordinate) functions on the solutions of 2) given by

$$q(x):\ \psi^\circ \longmapsto \psi^\circ(x,0)$$
$$p(x):\ \psi^\circ \longmapsto \dot{\psi}^\circ(x,0)$$

and define $q(x,t)$, $p(x,t)$ similarly. Then if we denote by a_ω the function

$$a_\omega:\ \psi^\circ \longmapsto f(\omega)$$

we have from 3)

$$q(x,t) = \frac{2}{\pi}\int_0^\infty d\omega\,(\cos\omega x)\,\mathrm{Re}\left(e^{-i\omega t} a_\omega\right)$$

where $a_{-\omega} = \overline{a_\omega}$. Thus

$$\mathrm{Re}\left(e^{-i\omega t} a_\omega\right) = \int_0^\infty dx\,(\cos\omega x)\,q(x,t)$$

$$\omega\,\mathrm{Im}\left(e^{-i\omega t} a_\omega\right) = \int_0^\infty dx\,(\cos\omega x)\,p(x,t)$$

$$\boxed{a_\omega = \int_0^\infty dx\,(\cos\omega x)\left(q(x) + \frac{1}{(-i\omega)}p(x)\right)}$$

I want to compute the wave operator

(*) $$\lim_{t\to-\infty} U(0,t)\,e^{-iH_0 t}.$$

Solutions of the perturbed wave equation are described by

$$\psi(x,t) = \int \frac{d\omega}{2\pi}\,\phi(x,\omega)\,e^{-i\omega t}\,g(\omega)$$

where g is any function and $\phi(x,\omega)$ is a convenient

Choice of eigenfunction for the perturbed operator

$$\begin{cases} \left(-\partial_x^2 + V\right)\phi(x,\omega) = \omega^2 \phi(x,\omega) \\ \partial_x \phi(x,\omega) = 0 \quad \text{at} \quad x=0 \end{cases}$$

Let $\phi^+(x,\omega)$ be the eigenfn. with the asymptotic behavior

$$\phi^+(x,\omega) \sim e^{-i\omega x} + R(\omega) e^{i\omega x} \qquad x \to \infty$$

Then by Riemann-Lebesgue lemma if $\phi = \phi^+$

$$\psi(x,t) \sim \int \frac{d\omega}{2\pi} e^{-i\omega x} e^{-i\omega t} g(\omega) = \hat{g}(x+t)$$

as $t \to -\infty$. Consequently we conclude that

$$\psi(x,t) = \int \frac{d\omega}{2\pi} \phi^+(x,\omega) e^{-i\omega t} f(\omega)$$

is asymptotic to

$$\psi^0(x,t) = \int \frac{d\omega}{2\pi} \left(e^{-i\omega x} + e^{i\omega x}\right) e^{-i\omega t} f(\omega)$$

as $t \to -\infty$, and so $\psi^0 \longmapsto \psi$ is the wave operator (*)

Let Ω^+ denote the wave operator, and let us now compute the function

$$q(x)' = q(x)\Omega^+ \qquad a_\omega' = a_\omega \Omega^+ \qquad \text{etc.}$$

Clearly

$$q(x)' : \psi^0 \longmapsto \psi(x,0) = \int \frac{d\omega}{2\pi} \phi^+(x,\omega) f(\omega)$$

hence

$$q(x)' = \int \frac{d\omega}{2\pi} \phi^+(x,\omega)\, a_\omega$$

$$p(x)' = \int \frac{d\omega}{2\pi} \phi^+(x,\omega) (-i\omega) a_\omega$$

so

$$a_\omega' = \int_0^\infty dx \,(\cos \omega x) \left(1 \quad \frac{1}{-i\omega}\right) \int \frac{d\tilde\omega}{2\pi} \phi^+(x,\tilde\omega) \begin{pmatrix} 1 \\ -i\tilde\omega \end{pmatrix} a_{\tilde\omega}$$

or

$$\boxed{a'_\omega = \int \frac{d\tilde\omega}{2\pi}\left(1+\frac{\tilde\omega}{\omega}\right)\left(\int_0^\infty dx\,\cos\omega x\cdot\phi^+(x,\tilde\omega)\right) a_{\tilde\omega}}$$

Check: If $H = H_0$, then $\phi^+(x,\tilde\omega) = 2\cos\tilde\omega x$

$$\int_0^\infty dx\,\cos\omega x\,\cos\tilde\omega x = \frac{\pi}{2}\left[\delta(\omega-\tilde\omega)+\delta(\omega+\tilde\omega)\right]$$

so $\quad a'_\omega = \frac{1}{2\pi}\cdot\underline{2\cdot\cancel{2}\frac{\pi}{\cancel{2}}}\,a_\omega = a_\omega.$

Note that the integral has both pos. + neg. $\tilde\omega$ and hence a'_ω involves $a_{\tilde\omega}$ and $\overline{a_{\tilde\omega}}$.

Now we have found the transfer matrix and the next step is to get the "little" S-matrix.

Problem: We were looking at a [redacted] wave equation with potential on $x \geq 0$

$$\partial_t^2 \psi + (-\partial_x^2 + V)\psi = 0$$

and we found that the scattering operator is given by

$$a'_\omega = R(\omega) a_\omega .$$

Here a_ω is the function on the free solutions

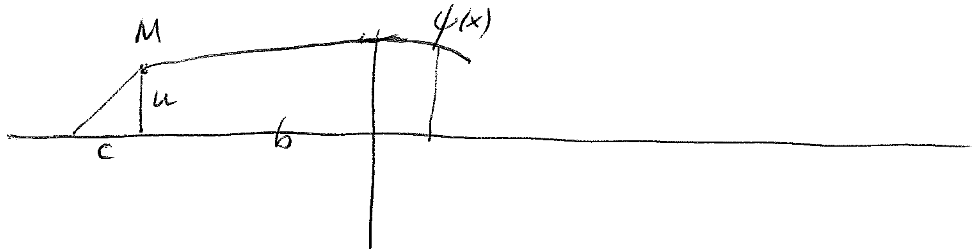
$$\psi^0(x,t) = \int \frac{d\omega}{2\pi} (2\cos\omega x)\, e^{-i\omega t} f(\omega)$$

defined by

$$a_\omega : \psi^0 \longmapsto f(\omega)$$

When the wave equation is quantized as an oscillator, because [redacted] $a' = \{a'_\omega\}_{\omega \geq 0}$ is a $\mathbb{C}$-linear function of $a = \{a_\omega\}_{\omega \geq 0}$, the S-matrix is diagonal for the occupation number basis. Somehow [redacted] the scattering is $\overset{\text{essentially}}{\text{trivial}}$, and won't given me an example of emission and absorption.

So I want to look at the example of the oscillator coupled to a string



$$M\ddot{u} + \frac{1}{c}u = \frac{1}{b}(\psi_0 - u) = (\partial_x \psi)(0) \qquad \begin{array}{l} M = 1 \\ \frac{1}{c} = \omega_0^2 \end{array}$$

$$\partial_t^2 \psi = \partial_x^2 \psi \qquad x > 0.$$

We want to regard the free case as being when $\frac{1}{b} = 0$.

Look classically, i.e. compute the normal modes. When $\frac{1}{b}=0$, they are

$$u = 0 \qquad \psi = \mathrm{Re}\left((2\cos\omega x)\, e^{-i\omega t} f(\omega)\right) \qquad \begin{matrix}\omega > 0 \\ \omega \neq \omega_0\end{matrix}$$

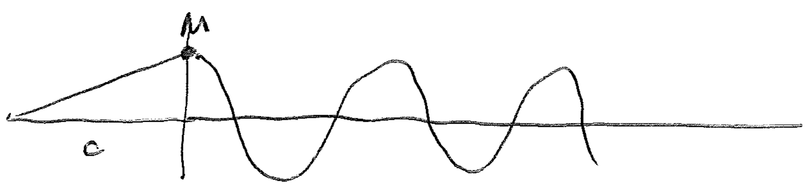
$$u = \mathrm{Re}\left(A e^{-i\omega_0 t}\right) \qquad \psi = 2\cos(\omega_0 x)\, \mathrm{Re}\left(e^{-i\omega_0 t} f(\omega_0)\right)$$

hence there is an extra mode with frequency ω_0. When $\frac{1}{b} \neq 0$, but small, there is exactly one normal mode for each $\omega > 0$.

December 5, 1980

Since Dec. 1 I found a new idea to try on the problem of emission and absorption. This consists in replacing the oscillator (simple) which has discrete spectra with an oscillator connected to a string. If the string is light, then from the viewpoint of the oscillator one has a damped oscillator with sharp resonance, but one has continuous spectrum so that one has a good S-matrix. This situation is analogous to taking a group G and replacing it by the infinite dimensional gadget BG.

So I want to have a simple model of an oscillator damped by being connected to a string. Picture:



The string has density ρ, tension $= T$. The string is described by

$$\rho \partial_t^2 u = \partial_x^2 u$$

and has waves $$e^{i(kx-\omega t)}$$ $$\rho\omega^2 = k^2$$

$$= e^{i(\sqrt{\rho}x - t)\omega} \qquad x = \frac{t}{\sqrt{\rho}}$$

Thus speed of signals on the string is $\frac{1}{\sqrt{\rho}}$.

The equations of motion are

$$M\ddot{u}_0 = -\frac{1}{c}u_0 + (\partial_x u)_0 + g(t) \qquad \rho\partial_t^2 u = \partial_x^2 u$$

$g(t)$: forcing term driving the oscillator

If we rescale by

changing $\begin{cases} x \longmapsto \dfrac{x}{\sqrt{\rho}} \\ \partial_x \longmapsto \sqrt{\rho}\,\partial_x \end{cases}$ we get the equations

$$\begin{cases} \partial_t^2 u = \partial_x^2 u \\ M\ddot{u}_0 + \frac{1}{c} u_0 = \sqrt{\rho}\,(\partial_x u)_0 + g(t) \end{cases}$$

Put $\lambda = \sqrt{\rho}$ and consider $g = Be^{-i\omega t}$. Then

$$u = Ae^{i(x-t)\omega}$$

assuming only outgoing waves, and A satisfies

$$\left(Ms^2 + \frac{1}{c}\right)A = \lambda\underbrace{(i\omega)}_{-s}A + B \qquad s = -i\omega$$

or

$$A = \frac{B}{Ms^2 + \lambda s + \frac{1}{c}}$$

which shows that the oscillator damped by the string is behaving like a damped harm. osc. with damping constant λ. Since $\lambda = \sqrt{\rho}$, this means the string is light and the signal speed $\frac{1}{\lambda}$ is high.

It seems that maybe a simpler example to see emission and absorption is as follows. Let

$$H_0 = \sum' \varepsilon_\alpha a^*_\alpha a_\alpha + \sum' \varepsilon'_\beta b^*_\beta b_\beta$$

describe two generalized oscillators and let the perturbation be

$$H_{int} = \sum' \left(b^*_\beta V_{\beta\alpha} a_\alpha + a^*_\alpha V_{\alpha\beta} b_\beta \right) \qquad V_{\beta\alpha} = V^*_{\alpha\beta}$$

The classical equations are

$$\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \left[iH, \begin{pmatrix} a \\ b \end{pmatrix} \right] = -i \begin{pmatrix} \varepsilon & V \\ V^* & \varepsilon' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

It seems interesting to look at the case where the "b" oscillator is simple, i.e. there is only one b_β. The equation

$$i \frac{\partial}{\partial t} \psi = \begin{pmatrix} \varepsilon & V \\ V^* & \varepsilon' \end{pmatrix} \psi$$

I have encountered in Weinberg's quasi-particle papers. He works with the resolvent

$$\frac{1}{W - \begin{pmatrix} \varepsilon & V \\ V^* & \varepsilon' \end{pmatrix}} = \frac{1}{W - \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix}} + \frac{1}{W - \underbrace{\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix}}_{\mathbb{H}_0}} \underbrace{\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}}_{\mathbb{V}} \frac{1}{W - \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix}} + \cdots$$

and obtains the scattering by letting W approach the real axis from above + below. Actually, better than the resolvent is the T-matrix

$$T(W) = \mathbb{V} + \mathbb{V} \frac{1}{W - \mathbb{H}_0} \mathbb{V} + \cdots\cdots$$

which controls the transitions.

December 10, 1980

Return to the [redacted] S-operator for a simple oscillator with perturbation:

$$H_0 = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$$

$$H = H_0 + \frac{1}{2}\varepsilon q^2 \qquad \varepsilon(t) \text{ has support in } [t_{in}, t_f]$$

The problem is to compute $\langle e_{\bar\mu} | S | e_\lambda \rangle$ for this perturbation, where

$$e_\lambda = e^{\lambda a^*}|0\rangle \qquad q = \frac{a + a^*}{\sqrt{2\omega}}$$

$$a = \frac{1}{\sqrt{2\omega}}(ip + \omega q)$$

Schwinger's method is to add a source term to H to obtain

$$H_J = H + J(t)q$$

and then compute [redacted]

$$\frac{\langle 0 | S_J | 0 \rangle}{\langle 0 | S | 0 \rangle}$$

by infinitesimally varying J. The result is

$$\frac{\langle 0 | S_J | 0 \rangle}{\langle 0 | S | 0 \rangle} = \exp\left\{ \frac{i}{2} \int J(t)\, G(t,t')\, J(t') \right\}$$

where

$$\left[\frac{d^2}{dt^2} + \omega^2 + \varepsilon(t) \right] G(t,t') = \delta(t,t')$$

and

$$G(t,t') = \begin{cases} c\, e^{-i\omega t} & t > t_f \\ c\, e^{-i\omega t} & t < t_{in} \end{cases}$$

(Another way to see the formula is to use functional integrals

$$\frac{\langle 0 | S_J | 0 \rangle}{\langle 0 | S | 0 \rangle} = \frac{\int \mathcal{D}q\, e^{i\int [\frac{1}{2}\dot q^2 - \frac{1}{2}(\omega^2+\varepsilon)q^2 - Jq]\, dt}}{\int \mathcal{D}q\, e^{i\int [\frac{1}{2}\dot q^2 - \frac{1}{2}(\omega^2+\varepsilon)q^2]\, dt}}$$

The numerator is formally the Fourier transf. of Gaussian [redacted]

$e^{-\frac{i}{2} q\cdot Aq}$ where $A = +\partial_t^2 + \omega^2 + \varepsilon$

which is $e^{\frac{iJ}{2}\frac{1}{A}J}$.)

Now the idea will be to choose $J(t)$ to be a δ-function at t_{in} and at t_f. We need S_J for $H_0 + Jq$. The result is

$$S_J = e^{\frac{i}{2}\int J G_0 J}\, e^{-i\int \frac{J(t)e^{+i\omega t}}{\sqrt{2\omega}} a^*}\, e^{-i\int \frac{J(t) e^{-i\omega t}}{\sqrt{2\omega}} a}$$

where $$G_0(t,t') = \frac{e^{-i\omega|t-t'|}}{-2i\omega}$$

Suppose $J(t) = \delta(t-t_0)\, i\sqrt{2\omega}\,\lambda$. Then

$$\frac{i}{2}\int J G_0 J = \frac{i}{2}\frac{1}{-2i\omega}\left(i\sqrt{2\omega}\,\lambda\right)^2 = \frac{\lambda^2}{2}$$

and so

$$S_J = $$

$$= e^{\frac{\lambda^2}{2}}\, e^{\lambda e^{i\omega t_0} a^*}\, e^{\lambda e^{-i\omega t_0} a}$$

Next take H_J where

$$J = \delta(t-t_{in})\, i\sqrt{2\omega}\,\lambda + \delta(t-t_f)\, i\sqrt{2\omega}\,\mu$$

Then $\langle 0|S_J|0\rangle = \langle 0| e^{iH_0 t_{f+}}\, U(t_{f+}, t_{f-})\, U(t_{f-}, t_{in+})\, U(t_{in+}, t_{in-})\, e^{-iH_0 t_{in}} |0\rangle$

or
$$S_J = e^{\frac{\mu^2}{2}}\, e^{\mu e^{i\omega t_f} a^*}\, e^{\mu e^{-i\omega t_f} a}\, S\, e^{\frac{\lambda^2}{2}}\, e^{\lambda e^{i\omega t_{in}} a^*}\, e^{\lambda e^{-i\omega t_{in}} a}$$

so $$\langle 0|S_J|0\rangle = e^{\frac{\mu^2}{2}+\frac{\lambda^2}{2}} \langle e_{e^{+i\omega t_f}\bar{\mu}} |\, S \,| e_{e^{i\omega t_{in}}\lambda} \rangle$$

since

$$\langle 0| e^{\frac{\mu^2}{2}} e^{\mu e^{i\omega t_f} a^*} e^{\mu e^{-i\omega t_f} a}$$

$$= e^{\frac{\mu^2}{2}} \langle 0| e^{\mu e^{-i\omega t_f} a} = e^{\frac{\mu^2}{2}} \langle c_{e^{i\omega t_f}\bar{\mu}}| .$$

Thus

$$\log \langle c_{e^{i\omega t_f}\bar{\mu}} | S | c_{e^{i\omega t_{in}}\lambda} \rangle = -\frac{\mu^2}{2} - \frac{\lambda^2}{2} + \underbrace{\frac{i}{2}\int JGJ} + \log\langle 0|S|0\rangle$$

$$\frac{i}{2}\left(i\sqrt{2\omega}\right)^2 \left[\lambda^2 G(t_{in},t_{in}) + 2\lambda\mu\, G(t_{in},t_f) + \mu^2 G(t_f,t_f)\right]$$

Now

$$G(t,t') = \frac{\phi(t_<)\psi(t_>)}{W(\phi,\psi)}$$

$$Te^{+i\omega t} \xleftrightarrow{\phi} e^{+i\omega t} + Re^{-i\omega t}$$

$$\tilde{R}e^{i\omega t} + e^{-i\omega t} \xleftrightarrow{\psi} Te^{-i\omega t}$$

so

$$G(t_{in},t_{in}) = \frac{Te^{i\omega t_{in}}\left(\tilde{R}e^{i\omega t_{in}} + e^{-i\omega t_{in}}\right)}{T\,(-2i\omega)} = -\frac{1}{2i\omega}\left(1+\tilde{R}e^{2i\omega t_{in}}\right)$$

$$G(t_{in},t_f) = \frac{Te^{i\omega t_{in}}\; Te^{-i\omega t_f}}{T(-2i\omega)}$$

$$G(t_f,t_f) = \frac{\left(e^{i\omega t_f} + Re^{-i\omega t_f}\right)\left(Te^{-i\omega t_f}\right)}{T(-2i\omega)}$$

$$= \frac{1}{-2i\omega}\left(1 + Re^{-2i\omega t_f}\right)$$

So $\log\left(\langle e_{e^{i\omega t_f}\bar\mu} | S | e_{e^{i\omega t_{in}}\lambda}\rangle / \langle 0|S|0\rangle\right)$

$$= \underbrace{\frac{i}{2}(i\sqrt{2\omega})^2 \frac{1}{-2i\omega}}_{\frac{1}{2}} \left[\lambda^2 \tilde{R} e^{2i\omega t_{in}} + 2\lambda\mu T e^{i\omega(t_{in}-t_f)} + \mu^2 R e^{-2i\omega t_f} \right]$$

and so we end up with the nice formula

$$\langle e_{\bar\mu} | S | e_\lambda \rangle = \langle 0|S|0\rangle e^{\frac{1}{2}[\lambda^2\tilde{R} + 2\lambda\mu T + \mu^2 R]}$$

which we knew already.

Next I need to understand the Green's function for the free system:

$$(\partial_t^2 - \partial_x^2)\, G(tx) = \delta(t)\,\delta(x) \qquad \text{on } \mathbb{R}.$$

We think of $-\partial_x^2$ as a positive operator and let $\hat\omega = \sqrt{-\partial_x^2}$ be its positive square root. Then formally

$$G_t = \frac{e^{-i\hat\omega|t|}}{-2i\hat\omega}$$

and so

$$G_t(x-x') = \langle x|G_t|x'\rangle = \int \frac{dk}{2\pi} e^{ik(x-x')} \frac{e^{-i|k||t|}}{-2i|k|}$$

where we have used the orth. basis $\langle x|k\rangle = e^{ikx}$. Thus our first expression is

$$\boxed{G(tx) = \int \frac{dk}{2\pi} e^{ikx} \frac{e^{-i|k||t|}}{-2i|k|}}$$

and it clearly has the property that positive frequencies occur for positive times, etc.

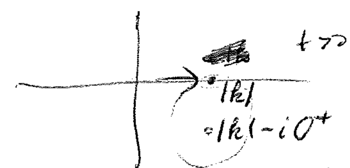
But we can also write

$$G(tx) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, G_\omega(x) \qquad G_\omega(x) = \int dt\, e^{i\omega t}\, G(tx)$$

where $\left(-\omega^2 - \partial_x^2\right) G_\omega(x) = \delta(x)$. To find G_ω we can proceed as follows

$$G(tx) = \int \frac{dk}{2\pi}\, e^{ikx}\, \frac{e^{-i|k||t|}}{-2i|k|} = \int \frac{dk}{2\pi}\, e^{ikx} \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{1}{-\omega^2 + k^2 - i0^+}$$

$$= \int \frac{d\omega}{2\pi}\, e^{-i\omega t} \int \frac{dk}{2\pi}\, e^{ikx}\, \frac{1}{k^2 - \omega^2 - i0^+}$$



$$\frac{-2\pi i\, e^{-i|k|t}}{-2|k|}$$

$$= \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^{-i|\omega||x|}}{-2i|\omega|}$$

Thus

$$\boxed{G_\omega(x) = \frac{e^{-i|\omega||x|}}{-2i|\omega|}}$$

<u>December 11, 1980</u>

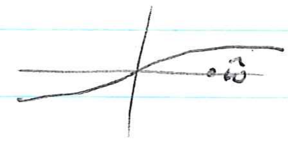
We want to understand the Green's function satisfying:

$$\left[\partial_t^2 + (-\partial_x^2 + V)\right] G_t(x,x') = \delta(t)\,\delta(x-x')$$

and the boundary conditions of pos. frequencies for pos. times, etc.

$$G_t = \frac{e^{-i\hat{\omega}|t|}}{-2i\hat{\omega}} \qquad \text{where} \quad \hat{\omega} = +\sqrt{-\partial_x^2 + V}$$

[redacted] $= \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{-\omega^2 + \hat{\omega}^2 - i0_+}$



$$\therefore \quad G_t(x,x') = \int \frac{d\omega}{2\pi} e^{-i\omega t} \underbrace{\langle x | \frac{-1}{\omega^2 + i0^+ - (-\partial_x^2 + V)} | x' \rangle}_{G_\omega(x,x')}$$

Now $G_\omega(x,x')$ satisfies

$$(-\omega^2 - \partial_x^2 + V)\, G_\omega = \delta$$

and it's the limit of [redacted] L^2 solutions as ω^2 approaches the real axis from above. Thus

$$-G_\omega(x,x') = \frac{\phi_\omega(x_<)\,\psi_\omega(x_>)}{W(\phi_\omega, \psi_\omega)}$$

where

$$\psi_\omega(x) \sim e^{i|\omega|x} \qquad \text{as} \quad x \to +\infty$$

$$\phi_\omega(x) \sim e^{-i|\omega|x} \qquad \text{as} \quad x \to -\infty \qquad (\text{on } \mathbb{R}).$$

Check

$$G_\omega^0(x,x') = -\frac{e^{i|\omega|(x_> - x_<)}}{2i|\omega|} = \frac{e^{i|\omega||x-x'|}}{-2i|\omega|}$$