October 26, 1980

The problem I want to look at now is the forced oscillator where $\frac{1}{\omega} J \notin \mathcal{H}$. First note that there are problems connected with the existence of the S-matrix. Ideally it should be a unitary operator commuting with H_0. However if we have a translation operator

$$U = e^{-\frac{1}{2}\gamma^*\gamma} e^{-a^*\gamma} e^{+\gamma^* a}$$

commuting with H_0, then

$$U a U^{-1} = e^{-a^*\gamma} a e^{a^*\gamma} = a + \gamma$$

$$U a^* U^{-1} = a^* + \gamma^*$$

so

$$U H_0 U^{-1} = (a^* + \gamma^*)\omega(a+\gamma)$$

$$= a^*\omega a + \gamma^*\omega a + a^*\omega\gamma + \gamma^*\omega\gamma$$

coincides with H_0 only if

$$\omega\gamma = 0.$$

Now we know that in some sense the S-matrix is given by U when

$$\gamma = i\int dt\, e^{i\omega t} J = 2\pi i\,\delta(\omega) J$$

and the problem is to make sense out of this.

Digression: Suppose $H = H_0 + V$ and we compute S to the first order adiabatically

$$S = 1 - i\int_{-\infty}^{\infty} dt\; e^{iH_0 t - \varepsilon|t|}\, V\, e^{-iH_0 t}$$

$$\langle b|S|a\rangle = \langle b|a\rangle - i\int_{-\infty}^{\infty} dt\; e^{iE_{ba}t - \varepsilon|t|}\, V_{ba}$$

$$- i\left[\frac{1}{iE_{ab} + \varepsilon} - \frac{1}{iE_{ba} - \varepsilon}\right] V_{ba}$$

$$\langle b|S|a\rangle = \langle b|a\rangle - i\underbrace{\left[\frac{2\varepsilon}{E_{ab}^2+\varepsilon^2}\right]}_{\to 2\pi\,\delta(E_{ab})} V_{ba}$$

Thus the [redacted] amplitude to first order for the transition [redacted] $a \mapsto b$ is

$$-i\,\frac{2\varepsilon}{E_{ab}^2+\varepsilon^2}\,V_{ba} \qquad \text{adiabatically}$$

$$-i\,\frac{\sin(E_{ba}T/2)}{(E_{ba}/2)}\,V_{ba} \qquad \text{if } V \text{ acts for } -\tfrac{T}{2}<t<\tfrac{T}{2}$$

$$-i\;2\pi\,\delta(E_{ba})\,V_{ba} \qquad \text{in the limit as either } \varepsilon \downarrow 0 \text{ or } T\to+\infty.$$

In these formulas we understand that b, a range over a continuous family of eigenvectors so that the matrix $\langle b|S|a\rangle$ **is** to be interpreted as a distribution. In some precise sense, we are pulling back via the map $b \longmapsto E_{ba}$ a distribution.

<u>Idea</u>: The pull-back of distributions is not well-defined [redacted] except when suitable transversality conditions hold. Are there higher Tor terms, àla Serre's intersection formula, to make sense for the pull-back of distributions?

<u>Question</u>: Take the case of $H = a^*\omega a + a^*J + J^*a$. Then the eigenstates for H_0 form a space of the type SP[redacted](X) where X is the space of 1-particle states. Does the S-matrix exist in a suitable distributional sense?

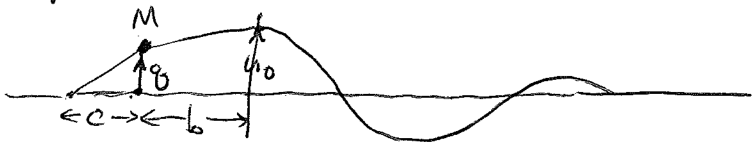
October 27, 1980

The problem: I have a forced oscillator

$$H = a^* \omega a + a^* J + J^* a$$

where ω is a self-adjoint operator and J a vector such that $\frac{1}{\omega} J$ is not normalizable. I want to make sense out of the S-matrix, as far as possible.

Simple model.



The equations of motion are:

$$M\ddot{q} + \frac{1}{c}q = \frac{u_0 - q}{b} = (\partial_x u)_0$$

$$\ddot{u} = \partial_x^2 u \qquad \text{for } x > 0$$

Here $\frac{1}{b}$ is the coupling parameter, so that when $b = \infty$ we have an oscillator with frequency $\omega_0^2 = \frac{1}{Mc}$ uncoupled to a string with free end: $(\partial_x u)_0 = 0$.

Let's determine the reflection coefficient. Start with

$$u(x,t) = \mathrm{Re}\left(A\left(e^{-i\omega x} + Re^{i\omega x}\right)e^{-i\omega t}\right)$$

$$q = \mathrm{Re}\left(\hat{q}\, e^{-i\omega t}\right).$$

$$\widehat{(\partial_x u)_0} = A(i\omega)(-1 + R)$$

$$\hat{u}_0 = A(1 + R)$$

$$bM(-\omega^2 + \omega_0^2)\,\hat{q} = \frac{1}{\cancel{b}}\left[A(1+R) - \hat{q}\right] = bA(i\omega)(-1+R)$$

$$\left\{1 + bM(\omega_0^2 - \omega^2)\right\}\hat{q} = A(1+R)$$

$$\frac{\hat{q}}{1+R} = \frac{A}{1 + bM(\omega_0^2 - \omega^2)} = \frac{\cancel{b}A(i\omega)(R-1)}{\cancel{b}M(\omega_0^2 - \omega^2)(1+R)}$$

Thus

$$i\omega\,\frac{R-1}{R+1} = \frac{M(\omega_0^2 - \omega^2)}{1 + bM(\omega_0^2 - \omega^2)}$$

is the equation for the reflection coefficient. It appears that if bM is large, then we have a sharp resonance at $\omega = \omega_0$. In effect, provided we stay away from ω_0, we have

$$1 + bM(\omega_0^2 - \omega^2) \approx bM(\omega_0^2 - \omega^2)$$

and so

$$i\omega \frac{R-1}{R+1} = \frac{1}{b} \quad \text{or} \quad \frac{R+1}{R-1} = bi\omega$$

which is what one would get by setting $g = 0$, i.e. fixing the weightless segment at $x = -b$.

Maybe the simplest thing to do is to set $b = 1$ and work around $M = \infty$, i.e. with

$$\frac{1}{i\omega}\frac{R+1}{R-1} = b + \frac{1}{M(\omega_0^2 - \omega^2)}$$

This is sufficiently close to the old idea where $b = 0$.

Now, however, I really haven't got to the real point which [redacted] somehow involves quantizing this [redacted] generalized oscillator so that one can see the emission and absorption of quanta.

(Add: There seems to be [redacted] sense in which $b = 0$ is special, in the same way that in Gelfand-Levitan the boundary [redacted] condition $u(0) = 0$ is special in contrast to the condition $u' + hu = 0$ at $x = 0$.)

November 7, 1980

Program: I have decided it is necessary to compute the S-matrix for a quadratic perturbation of a harmonic oscillator:

$$H = \underbrace{a^* \omega a}_{H_0} + \underbrace{H_{int}}_{\text{involves } a^2 \quad a^*a \quad a^{*2}}$$

For example, a [redacted] change in the potential energy of an oscillator:

$$H = \frac{p^2}{2} + (\omega^2 + \varepsilon)\frac{q^2}{2}$$

[redacted] leads to

$$H_{int} = \varepsilon \frac{q^2}{2} = \frac{\varepsilon}{2}\left(\frac{a + a^*}{\sqrt{2\omega}}\right)^2$$

The S-matrix is given by a time-ordered product

$$S = T\left\{ [redacted] \prod_t e^{-i\,dt\,\tilde{H}_{int}(t)} \right\}.$$

I think it should be possible to write the S-matrix in the form

$$S = \underset{\langle 0|S|0\rangle}{\underset{\|}{\text{scalar}}}\; e^{\alpha \frac{(a^*)^2}{2}} e^{\beta a^* a} e^{\gamma \frac{a^2}{2}}$$

for suitable α, β, γ.

Note that the operators $\frac{(a^*)^2}{2}$, a^*a, aa^*, $\frac{a^2}{2}$ span [redacted] a Lie algebra. I know it is the Lie algebra of the symplectic group extended by a [redacted] 1-dimensional center. Brackets are

$$\left[\frac{a^2}{2}, \frac{a^{*2}}{2}\right] = \frac{1}{2}\left\{\left[a, \frac{a^{*2}}{2}\right]a + a\left[a, \frac{a^{*2}}{2}\right]\right\}$$

$$= \frac{1}{2}\,[redacted](a^*a + aa^*) = a^*a + \frac{1}{2}$$

$$\left[\frac{a^*a+aa^*}{2}, \frac{a^2}{2}\right] = \left[a^*a, \frac{a^2}{2}\right] = \left[a^*, \frac{a^2}{2}\right] a$$

$$= -2\frac{a^2}{2}$$

$$\left[\frac{a^*a+aa^*}{2}, \frac{a^{*2}}{2}\right] = \left[a^*a, \frac{a^{*2}}{2}\right] = 2\frac{a^{*2}}{2}$$

So if we put

$$X_+ = i\frac{a^{*2}}{2}, \quad X_- = i\frac{a^2}{2}, \quad H = \frac{a^*a+aa^*}{2}$$

we get

$$[X_+, X_-] = H$$

$$[H, X_+] = 2X_+$$

$$[H, X_-] = 2X_-$$

which are the Sl_2 relations. Note that X_+ is not the adjoint of X_-.

The real point to concentrate on is that a quadratic Hamiltonian gives symplectic equations of motion. So on the space spanned by a, a^* or the p, q's we get a path in the symplectic group. The S matrix will be the lifting of this path into the metaplectic repn. of the symplectic group.

What I want to do now is to learn how to calculate in the metaplectic group using operators in the form

$$(*) \qquad e^{\alpha\frac{a^{*2}}{2}}\, e^{\beta a^*a}\, e^{\gamma\frac{a^2}{2}}$$

First I want to understand how these operators affect the operators a, a^*. Thus I want the matrices of conjugating [redacted] a, a^* by these operators.

$$e^{\alpha \frac{a^{*2}}{2}} \begin{pmatrix} a^* & a \end{pmatrix} e^{-\alpha \frac{a^{*2}}{2}} = \begin{pmatrix} a^* & a \end{pmatrix} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$$

$$e^{\beta a^* a} \begin{pmatrix} a^* & a \end{pmatrix} e^{-\beta a^* a} = \begin{pmatrix} a^* & a \end{pmatrix} \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix}$$

$$e^{\gamma \frac{a^2}{2}} \begin{pmatrix} a^* & a \end{pmatrix} e^{-\gamma \frac{a^2}{2}} = \begin{pmatrix} a^* & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

So now it's clear that the operator $(*)$ belongs to the product of the matrices at the right.

The next thing to get at is the way these operators work on the "coherent" states. In the $L^2(\mathbb{R})$ representation these are the Gaussian functions

$$e^{i\tau \frac{x^2}{2}} \qquad \text{with } \operatorname{Im}(\tau) > 0.$$

In the holomorphic situation it would appear that they are the images of $|0\rangle$ under a symplectic transf. such at $(*)$, and hence are the [redacted] vectors

$$e^{\alpha \frac{(a^*)^2}{2}} |0\rangle = e^{\alpha \frac{z^2}{2}}.$$

which are normalizable. Now

$$\| e^{\alpha z^2/2} \|^2 = \sum_n [redacted] \left\langle \frac{(\alpha z^2/2)^n}{n!} \middle| \frac{(\alpha z^2/2)^n}{n!} \right\rangle$$

$$= \sum_n \frac{|\alpha|^{2n}}{(n!)^2 2^{2n}} (2n)! = \sum_n \frac{|\alpha|^{2n}}{n!} \left(\frac{2n-1}{2}\right) \cdots \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)$$

$$= \sum_n \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!}(\alpha\bar{\alpha})^n = (1-|\alpha|^2)^{-1/2}$$

so we see the "coherent" states are the holomorphic functions $e^{\alpha\frac{(a^*)^2}{2}}|0\rangle = e^{\alpha\frac{z^2}{2}}$ with $|\alpha|<1$.

We know that if we operate on this by one of our operators (*) it stays in this form up to a scalar multiple, because these coherent states are characterized as being killed by a line in the a, a^* space, e.g. $e^{\alpha z^2/2}$ is killed by $a - \alpha a^*$ for which

$$[a-\alpha a^*, a^* - \bar{\alpha} a] = 1 - |\alpha|^2 > 0.$$

Thus we know that

$$(+) \qquad e^{\gamma\frac{a^2}{2}} e^{\alpha\frac{a^{*2}}{2}}|0\rangle = c\, e^{\tau\frac{a^{*2}}{2}}|0\rangle$$

To determine the scalar c take the inner product with $|0\rangle$ and you get

$$c = \langle 0| e^{\gamma\frac{a^2}{2}} e^{\alpha\frac{a^{*2}}{2}}|0\rangle$$

$$= \langle e^{+\bar{\gamma}\frac{z^2}{2}} | e^{\alpha\frac{z^2}{2}}\rangle = (1-\gamma\alpha)^{-1/2}$$

To compute τ notice the left side of $(+)$ is killed by

$$e^{\gamma\frac{a^2}{2}}(a-\alpha a^*)e^{-\gamma\frac{a^2}{2}} = (a^* \ a)\underbrace{\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}\begin{pmatrix}-\alpha\\ 1\end{pmatrix}}_{\begin{pmatrix}-\alpha\\ -\gamma\alpha+1\end{pmatrix}}$$

$$= (1-\gamma\alpha)\left[a - \frac{\alpha}{1-\gamma\alpha}a^*\right]$$

Thus $\qquad \tau = \dfrac{\alpha}{1-\gamma\alpha}$. So we have

$$\boxed{e^{\gamma \frac{a^2}{2}}\, e^{\alpha \frac{a^{*2}}{2}} |0\rangle = (1-\gamma\alpha)^{-1/2}\, e^{\left(\frac{\alpha}{1-\gamma\alpha}\right)\frac{a^{*2}}{2}} |0\rangle}$$

<u>Question:</u> If γ is small, is $e^{\gamma\frac{a^2}{2}}$ a bounded operator on the Hilbert space?

If so it would have to be true that

$$\alpha \longmapsto \frac{\alpha}{1-\gamma\alpha}$$

maps $|\alpha| < 1$ into itself. But this is false because if $|\alpha|$ is close to 1 with argument opposite to that of γ, then $\left|\frac{\alpha}{1-\gamma\alpha}\right| \sim \frac{1}{1-\gamma\alpha} > 1$. Therefore it appears that we might have technical difficulties with writing operators in the form

$$S = e^{\alpha\frac{(a^*)^2}{2}}\, e^{\beta a^* a}\, e^{\gamma\frac{a^2}{2}}$$

However notice that such an operator is in normal form. No. $e^{\beta(a^*a)}$ isn't, but it's easy to see the terms as particle processes. Thus $\gamma\frac{a^2}{2}$ kills pairs of incoming particles, $\beta a^* a$ modifies those that are left, and then $\alpha\frac{(a^*)^2}{2}$ creates new particles. Notice that β can be determined from the 1-particle states:

$$\langle 1|S|1\rangle = \langle 1|e^{\beta a^* a}|1\rangle = e^{\beta}$$

November 8, 1980

Yesterday we found that conjugation on $(a^* \ a)$ space gives

$$e^{\alpha \frac{(a^*)^2}{2}} \longleftrightarrow \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$$

$$e^{\beta a^* a} \longleftrightarrow \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix}$$

$$e^{\gamma \frac{a^2}{2}} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

It would be nice to have formulas for computing products of the operators on the left. First note the identities in $SL_2(\mathbb{C})$

$$\begin{pmatrix} 1 & \frac{b}{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix} = \begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

since $d^{-1} + \frac{bc}{d} = \frac{1+bc}{d} = \frac{ad}{d} = a$, and

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \gamma & 1-\gamma\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{-\alpha}{1-\gamma\alpha} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{1-\gamma\alpha} & 0 \\ 0 & 1-\gamma\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{\gamma}{1-\gamma\alpha} & 1 \end{pmatrix}$$

Thus we get the formal identity

$$e^{\gamma \frac{a^2}{2}} e^{\alpha \frac{(a^*)^2}{2}} = (1-\gamma\alpha)^{-1/2} e^{\left(\frac{\alpha}{1-\gamma\alpha}\right)\frac{(a^*)^2}{2}} e^{\log\left(\frac{1}{1-\gamma\alpha}\right) a^* a} e^{\left(\frac{\gamma}{1-\gamma\alpha}\right)\frac{a^2}{2}}$$

where the scalar $(1-\gamma\alpha)^{-1/2}$ is found by computing the vacuum expectation value.

Somehow the above approach is not going to be effective. It seems that the correct way to deal with symplectic transformations uses an action function $S(q, q')$. So it's necessary to review all this.

November 9, 1980

Let $\quad S(q,q') = \alpha\frac{q^2}{2} + \beta q q' + \gamma\frac{(q')^2}{2} \quad$ with $\alpha, \beta, \gamma \in \mathbb{R}$.
Then
$$e^{iS(q,q')} = e^{i\alpha\frac{q^2}{2} + i\beta q q' + i\gamma\frac{(q')^2}{2}}$$
is essentially the kernel of a unitary operator on $L^2(\mathbb{R})$. In effect $e^{i\alpha\frac{q^2}{2}}$ is a multiplication operator and $e^{iqq'}$ is, up to the scalar $\frac{1}{\sqrt{2\pi}}$, the kernel of the Fourier transform operator. Now
$$\int \beta\, dq\; e^{-iq''\beta q} e^{i\beta q q'} = 2\pi\,\delta(q''-q')$$
hence
$$\sqrt{\frac{\beta}{2\pi}}\; e^{i\beta q q'} \qquad \text{is a unitary kernel.}$$
and we see that one gets a unitary operator with

(*) $\quad \langle q|U|$[redacted]$\rangle = \sqrt{\frac{\beta}{2\pi}}\; e^{i\alpha\frac{q^2}{2}} e^{i\beta q q'} e^{i\gamma\frac{q'^2}{2}}$

We have
$$e^{i\alpha\frac{q^2}{2}} \begin{pmatrix} q & p \end{pmatrix} e^{-i\alpha\frac{q^2}{2}} = \begin{pmatrix} q & p \end{pmatrix}\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$$
$$\mathcal{F} \begin{pmatrix} q & p \end{pmatrix} \mathcal{F}^{-1} = \begin{pmatrix} q & p \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
where $\langle q|\mathcal{F}|q'\rangle = \frac{1}{\sqrt{2\pi}} e^{iqq'}$. Also
$$T_\beta \begin{pmatrix} q & p \end{pmatrix} T_\beta^{-1} = \begin{pmatrix} q & p \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$$
where $\quad (T_\beta f)(x) = \sqrt{\beta}\, f(\beta x)$. $\quad$ Thus the matrix belonging to the operator U in (*) is

$$\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}$$

which is an arbitrary element of the fat cell $B_+ w B_+$ in the Bruhat decomposition.

For future reference, suppose we have a Schroedinger equation with quadratic potential

$$i\frac{\partial\psi}{\partial t} = \left(\frac{p^2}{2} + \frac{1}{2}q^t V q\right)\psi . \qquad V = V(t)$$

Then

$$\psi = e^{\frac{i}{2}x^t A x + c} \qquad A, c \text{ depend on } t$$

is a solution provided $\qquad pp\psi = p[\psi(Ax)] = \psi(Ax)^2 + \psi(\frac{1}{i}\mathrm{tr}A)$

$$\psi\left(-\frac{1}{2}x^t\dot{A}x + i\dot{c}\right) = \psi\left(\frac{1}{2}(Ax)^2 + \frac{1}{2i}\mathrm{tr}A + \frac{1}{2}x^tVx\right)$$

or

$$\begin{cases} \dot{A} + A^2 + V = 0 & \text{(Riccati eqn.)} \\ \dot{c} = -\frac{1}{2}(\mathrm{tr}A) & \end{cases}$$

Recall $SU(1,1)$ consists of matrices $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2 - |\beta|^2 = 1$. It's Lie algebra consists of

$$\begin{pmatrix} ia & b \\ \bar{b} & -ia \end{pmatrix} \qquad \text{with } a \in \mathbb{R}.$$

We have

$$\left[i\left(\alpha\frac{(a^*)^2}{2} + \beta a^* a + \bar{\alpha}\frac{(a)^2}{2}\right), \begin{pmatrix} a^* & a \end{pmatrix} \right] = \begin{pmatrix} a^* & a \end{pmatrix}\begin{pmatrix} i\beta & -i\alpha \\ i\bar{\alpha} & -i\beta \end{pmatrix}$$

Hence we see that a Dirac-style system

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \bar{p} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

is going to involve some sort of path in $SU(1,1)$.

Let's go over the holomorphic repn.

$$\langle g|f\rangle = \int \overline{g(z)}\, f(z)\, e^{-|z|^2}\, \frac{dx\,dy}{\pi}$$

$$a = \frac{d}{dz} \qquad a^* = z.$$

$$f(\lambda) = \sum \frac{\lambda^n}{n!} f^{(n)}(0) \qquad\qquad f(0) = \int f(z) e^{-|z|^2}$$

$$= \sum \frac{\lambda^n}{n!} \langle 0 | a^n f\rangle$$

$$= \sum \frac{\lambda^n}{n!} \langle z^n | f\rangle = \langle e^{\bar{\lambda} z} | f\rangle$$

Thus

$$f(\lambda) = \langle e_{\bar{\lambda}} | f\rangle$$

where $e_{\bar{\lambda}}$ is the exp. fn. $e^{\bar{\lambda} z}$

Also

$$f(\lambda) = \int e^{\lambda \bar{z}} f(z) e^{-|z|^2}$$

or

$$f = \int e_{\bar{z}}\, f(z)\, e^{-|z|^2} = \int e_{\bar{z}} \langle e_{\bar{z}} | f\rangle e^{-|z|^2}$$

which gives the <u>completeness relation</u>

$$id = \int |e_{\bar{z}}\rangle\, e^{-|z|^2} \left(\frac{dx\,dy}{\pi}\right) \langle e_{\bar{z}}|$$

So now let us compute the matrix elements

$$\langle e_\mu | U | e_\lambda \rangle$$

where U is one of our standard operators:

$$\underbrace{\langle e_\mu | \overbrace{e^{\alpha \frac{(a^*)^2}{2}}}^{}}\, e^{\beta a^* a}\, \underbrace{e^{\gamma \frac{a^2}{2}} | e_\lambda \rangle}$$

$$\langle e^{\bar\alpha \frac{a^2}{2}} e_\mu | \, e^{\beta a^* a} \, | e^{\gamma \frac{\lambda^2}{2}} e_\lambda \rangle = e^{\alpha \frac{\bar\mu^2}{2}} \langle e_\mu | e_{e^\beta \lambda} \rangle e^{\gamma \frac{\lambda^2}{2}}$$

$$= e^{\alpha \frac{\bar\mu^2}{2}} e^{\bar\mu e^\beta \lambda} e^{\gamma \frac{\lambda^2}{2}}$$

U corresponds to the matrix $\qquad b^{-1} = e^\beta$

$$\begin{pmatrix} 1 & -\alpha \\ & 1 \end{pmatrix} \begin{pmatrix} b^{-1} & \\ & b^{\text{[redacted]}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ & 1 \end{pmatrix} \begin{pmatrix} b^{\text{[redacted]}} & 0 \\ b^{\text{[redacted]}}\gamma & b^{\text{[redacted]}} \end{pmatrix}$$

$$= \begin{pmatrix} b^{-1} - \alpha b^{\text{[redacted]}} \gamma & -\alpha b^{\text{[redacted]}} \\ b^{\text{[redacted]}} \gamma & b^{\text{[redacted]}} \end{pmatrix}$$

This is in $SU(1,1)$ when

$$b\gamma = -\bar\alpha \bar b \qquad \bar b = b^{-1} - \alpha b \gamma.$$

Note that any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{C})$ is in the form

$$\begin{pmatrix} 1 & -\alpha \\ & 1 \end{pmatrix} \begin{pmatrix} b^{-1} & \\ & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

if and only if $d \neq 0$. In particular a matrix in $SU(1,1)$ has this property, in fact the diagonal entries have $|\;| > 1$.

Conclusion: Any matrix in $SU(1,1)$ gives one an operator $U = e^{\alpha \frac{(a^*)^2}{2}} e^{\beta a^* a} e^{\gamma (\frac{a^2}{2})}$ with α, β, γ essentially determined by matrix $\langle e_\mu | U | e_\lambda \rangle$.

Let's examine the equations $\qquad b = e^{-\beta}$

$$b\gamma = -\bar{\alpha}\bar{b} \qquad \bar{b} = b^{-1} - \alpha b \gamma$$

$$= b^{-1} - \alpha(-\bar{\alpha}\bar{b}) = \text{[redacted]}\ b^{-1} + |\alpha|^2 \bar{b}$$

$$\Longrightarrow \bar{b}\left(1 - |\alpha|^2\right) = b^{-1}$$

$$\Longrightarrow \quad 1 = |\alpha|^2 + \frac{1}{|b|^2} \quad = |\alpha|^2 + e^{2\,\mathrm{Re}(\beta)}$$

This looks like the <u>reflection</u> and transmission coefficients.

Review the[redacted] formulas for transmission + reflection:

$$\begin{cases} e^{-ikx} \longleftrightarrow A e^{-ikx} + B e^{ikx} \\ e^{ikx} \longleftrightarrow \bar{B} e^{-ikx} + \bar{A} e^{ikx} \end{cases} \qquad |A|^2 - |B|^2 = 1.$$

$$\begin{cases} \underbrace{\frac{1}{A}}_{T} e^{-ikx} \longleftrightarrow e^{-ikx} + \underbrace{\frac{B}{A}}_{R} e^{ikx} \\ \left(-\frac{\bar{B}}{A}\right) e^{-ikx} + e^{ikx} \longleftrightarrow \frac{1}{A} e^{ikx} \end{cases}$$

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\bar{B}}{A} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{B}{A} \\ 0 & 1 \end{pmatrix}$$

On the other hand if we associate [redacted] to $U = e^{\alpha \frac{a^{*2}}{2}} e^{\beta a^* a} e^{\gamma \frac{a^2}{2}}$ the matrix of conjugation on $(a \;\; a^*)$ we get the product of matrices

$$\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} e^{-\beta} & 0 \\ 0 & e^{+\beta} \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

It thus appears that [redacted]

$$\gamma = \frac{B}{A} = R, \qquad e^{\beta} = \frac{1}{A} = T, \qquad \alpha = -\frac{\bar{B}}{A} = -\frac{T}{\bar{T}}\bar{R}$$

and so U is unitary when

$$\begin{pmatrix} R & T \\ T & -\frac{T}{\bar{T}}\bar{R} \end{pmatrix} = \begin{pmatrix} \gamma & e^{\beta} \\ e^{\beta} & \alpha \end{pmatrix}$$

is a symmetric unitary matrix.

November 10, 1980 (Jeanie's birthday)

Let's try to see when the operator $\mathcal{U}$ defined by

$$\langle e_\mu | \mathcal{U} | e_\lambda \rangle = e^{\alpha \frac{\bar{\mu}^2}{2} + \bar{\mu} b \lambda + \gamma \frac{\lambda^2}{2}}$$

is unitary up to a suitable scalar which will be determined. I want to do the calculation with several (say n) degrees of freedom. Thus the exponent should be written

$$\frac{1}{2} \bar{\mu}^t \alpha \bar{\mu} + \bar{\mu}^t b \lambda + \frac{1}{2} \lambda^t \gamma \lambda$$

where α, b, γ are matrices with α, γ symmetric.

Let's compute $\mathcal{U}^* \mathcal{U}$ by using the identity

$$id = \int_\mu |e_\mu\rangle e^{-|\mu|^2} \langle e_\mu | \qquad\qquad \int_\mu e^{-|\mu|^2} = 1.$$

Then

$$\langle e_\nu | \mathcal{U}^* | e_\mu \rangle = \overline{\langle e_\mu | \mathcal{U} | e_\nu \rangle} = \exp\left\{ \frac{1}{2} \mu^t \bar{\alpha} \mu + \mu^t \bar{b} \bar{\nu} + \frac{1}{2} \bar{\nu}^t \bar{\gamma} \bar{\nu} \right\}$$

and so

$$\langle e_\nu | \mathcal{U}^* \mathcal{U} | e_\lambda \rangle = \int_\mu e^{\left(\frac{1}{2} \mu^t \bar{\alpha} \mu + \text{[redacted]} \mu^t \bar{b} \bar{\nu} + \frac{1}{2} \bar{\nu}^t \bar{\gamma} \bar{\nu} - |\mu|^2 + \frac{1}{2} \bar{\mu}^t \alpha \bar{\mu} + \bar{\mu}^t b \lambda + \frac{1}{2} \lambda^t \gamma \lambda \right)}$$

This is a Gaussian integral which converges for $|\alpha| < 1$ in some sense. It should be possible to evaluate the [redacted] Gaussian integral by a saddle point method, which means one pushes the contour into complex μ-space. This means the stationary point can be located by treating $\mu, \bar{\mu}$ independently, ~~and we get the equations~~ The quadratic function is

$$\mu^t \underset{\substack{\uparrow \\ \bar{b}\bar{\nu}}}{\tilde{J}} \text{[redacted]} + \frac{1}{2} \mu^t \bar{\alpha} \mu - |\mu|^2 + \frac{1}{2} \bar{\mu}^t \alpha \bar{\mu} + \bar{\mu}^t \underset{\substack{\uparrow \\ b\lambda}}{J}$$

Varying wrt $\mu, \bar{\mu}$ give the equations

$$\begin{cases} \tilde{J} + \bar{\alpha}\mu - \tilde{\mu} = 0 \\ -\mu + \alpha\tilde{\mu} + J = 0 \end{cases}$$

or

$$\begin{pmatrix} J \\ \tilde{J} \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ -\bar{\alpha} & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \tilde{\mu} \end{pmatrix}$$

Multiplying the equations above by $\frac{1}{2}\mu^t$ and $\frac{1}{2}\tilde{\mu}^t$ and adding gives

$$\frac{1}{2}\mu^t \tilde{J} + \frac{1}{2}\mu^t \bar{\alpha}\mu - \underbrace{|\mu|^2}_{\frac{1}{2}\mu^t\tilde{\mu} + \frac{1}{2}\tilde{\mu}^t\mu} + \frac{1}{2}\tilde{\mu}^t \alpha \tilde{\mu} + \frac{1}{2}\tilde{\mu}^t J = 0$$

Hence the value of the Gaussian exponential to be integrated at the critical point $\mu, \tilde{\mu}$ is

$$\frac{1}{2}\mu^t \tilde{J} + \frac{1}{2}\tilde{\mu}^t J = \frac{1}{2}\begin{pmatrix} \tilde{\mu} \\ \mu \end{pmatrix}^t \begin{pmatrix} J \\ \tilde{J} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \tilde{J} \\ J \end{pmatrix}^t \begin{pmatrix} \mu \\ \tilde{\mu} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \tilde{J} \\ J \end{pmatrix}^t \begin{pmatrix} 1 & -\alpha \\ -\bar{\alpha} & 1 \end{pmatrix}^{-1} \begin{pmatrix} J \\ \tilde{J} \end{pmatrix}$$

Now

$$\begin{pmatrix} 1 & -\alpha \\ -\bar{\alpha} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix} + \begin{pmatrix} \alpha\bar{\alpha} & 0 \\ 0 & \bar{\alpha}\alpha \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} \frac{1}{1-\alpha\bar{\alpha}} & \frac{1}{1-\alpha\bar{\alpha}}\alpha \\ \bar{\alpha}\frac{1}{1-\alpha\bar{\alpha}} & \frac{1}{1-\bar{\alpha}\alpha} \end{pmatrix}$$

so after doing the Gaussian integral we should get a scalar times the exponential of

$$\frac{1}{2}\bar{\nu}^t\bar{\gamma}\bar{\nu} + \frac{1}{2}\lambda^t\gamma\lambda + \frac{1}{2}\begin{pmatrix}\bar{b}\bar{\nu}\\ b\lambda\end{pmatrix}^t\begin{pmatrix}\frac{1}{1-\alpha\bar{\alpha}} & \frac{1}{1-\alpha\bar{\alpha}}\alpha\\ \bar{\alpha}\frac{1}{1-\alpha\bar{\alpha}} & \frac{1}{1-\bar{\alpha}\alpha}\end{pmatrix}\begin{pmatrix}b\lambda\\ \bar{b}\bar{\nu}\end{pmatrix}$$

This should be $\bar{\nu}^t\lambda$. If so, we get the equations

$$\frac{1}{2}\bar{\nu}^t\bar{b}^t\frac{1}{1-\alpha\bar{\alpha}}b\lambda + \frac{1}{2}\lambda^t b^t\frac{1}{1-\bar{\alpha}\alpha}\bar{b}\bar{\nu} = \bar{\nu}^t\lambda$$

$$\frac{1}{2}\bar{\nu}^t\bar{\gamma}\bar{\nu} + \frac{1}{2}\bar{\nu}^t\bar{b}^t\frac{1}{1-\alpha\bar{\alpha}}\alpha\bar{b}\bar{\nu} = 0$$

$$\frac{1}{2}\lambda^t\gamma\lambda + \frac{1}{2}\lambda^t b^t\bar{\alpha}\frac{1}{1-\alpha\bar{\alpha}}b\lambda = 0$$

or

$$\bar{b}^t\frac{1}{1-\alpha\bar{\alpha}}b = 1 \implies 1-\alpha\bar{\alpha} = bb^* \quad\text{or}\quad 1=\alpha^*\alpha+b^*b$$

$$\gamma + b^t\bar{\alpha}\underbrace{\frac{1}{1-\alpha\bar{\alpha}}}_{(b^*)^{-1}}b = 0 \implies \gamma b^* + b^t\bar{\alpha} = 0$$

$$\implies \bar{b}\gamma + \bar{\alpha}b = 0$$

$$\bar{\gamma} + \bar{b}^t\frac{1}{1-\alpha\bar{\alpha}}\alpha\bar{b} = 0 \quad \text{same as preceding.}$$

Now $\begin{pmatrix}\alpha & b\\ b^t & \gamma\end{pmatrix}$ unitary means

$$\begin{pmatrix}\alpha & b\\ b^t & \gamma\end{pmatrix}^*\begin{pmatrix}\alpha & b\\ b^t & \gamma\end{pmatrix} = \begin{pmatrix}\bar{\alpha} & \bar{b}\\ b^* & \bar{\gamma}\end{pmatrix}\begin{pmatrix}\alpha & b\\ b^t & \gamma\end{pmatrix} = \begin{pmatrix}\bar{\alpha}\alpha+\bar{b}b^t & \bar{\alpha}b+\bar{b}\gamma\\ b^*\alpha+\bar{\gamma}b^t & b^*b+\bar{\gamma}\gamma\end{pmatrix}$$

is the identity.

$$\alpha\bar{\alpha}+\bar{b}b^t = 1 \implies \bar{\alpha}\alpha + bb^* = 1.$$

so the rest is clear. Thus $\begin{pmatrix}\alpha & b\\ b^t & \gamma\end{pmatrix}$ is unitary.

Next we need the determinant factor in the Gaussian integral.

November 12, 1980

Let's consider a finite time perturbation of a simple harmonic oscillator. The Hamiltonian is

$$H = a^*\omega a + \tfrac{1}{2}\alpha(a^*)^2 + \beta a^* a + \tfrac{1}{2}\gamma a^2 + \delta$$

where $\alpha, \beta, \gamma, \delta$ are compactly supported functions of t with β, δ real and $\alpha = \bar{\gamma}$. With several degrees of freedom, this should be written

$$H = a^*\omega a + a^*\frac{\alpha}{2}(a^*)^t + a^*\beta a + a^t\frac{\gamma}{2}a + \delta$$

where a is a column vector and a^* is a row vector.

The problem is to compute the S-matrix

$$S = T\left\{\prod_t e^{-i\,dt \cdot H_{int}(t)}\right\}.$$

I know, more or less, that S has the form

$$S = \langle 0|S|0\rangle\, e^{A\frac{(a^*)^2}{2}}\, e^{a^*Ba}\, e^{C\frac{a^2}{2}}$$

so the problem is to compute the quantities $\langle 0|S|0\rangle, A, B, C$. in the most efficient manner.

Schwinger's approach is to compute the matrix elements between a-eigenvectors

$$\langle e_\mu|S|e_\lambda\rangle = \langle 0|S|0\rangle\, e^{A\frac{\bar{\mu}^2}{2} + \bar{\mu}B\lambda + C\frac{\lambda^2}{2}}$$

Another of his tricks is to add sources to the Hamiltonian by adding

$$a^*J + \tilde{J}a$$

then he computes $\langle 0|S^J|0\rangle$ and somehow obtains $\langle e_\mu|S|e_\lambda\rangle$ from a suitable choice of $J, \tilde{J}$.

November 14, 1980

The problem is to compute the S matrix for a perturbed oscillator

$$H = \frac{p^2}{2} + \frac{1}{2}(\omega q)^2 + \frac{1}{2} q \cdot \varepsilon(t) q$$

where $\varepsilon(t)$ has compact support. The simplest approach seems to compute the propagator:

$$\langle q_f | U(t_f, t_{in}) | q_{in}\rangle = \int_{\substack{\text{paths from} \\ (qt)_{in} \text{ to } (qt)_{fin}}} e^{iS} \mathcal{D}q$$

where

$$S = \int_{t_{in}}^{t_f} dt \left[\tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2} q^t(\omega^2+\varepsilon) q \right]$$

This is a Gaussian path integral and the answer is

$$\boxed{\langle (qt)_f | (qt)_{in}\rangle = \det\left(\frac{i}{2\pi}\frac{\partial^2 S}{\partial q \partial q'}((tq)_f, (tq)_{in})\right)^{1/2} e^{iS((tq)_f, (tq)_{in})}}$$

Let's review why. Consider the Schroedinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi$$

and put $\psi(tq) = e^{i\tilde{S}(tq)}$. Then

$$i\frac{\partial \psi}{\partial t} = \psi(-\dot{\tilde{S}}) \qquad p^2\psi = p(\psi\, \partial_q \tilde{S}) = \psi\left((\partial_q \tilde{S})^2 + \frac{1}{i}\partial_q^2 \tilde{S}\right)$$

where $\partial_q^2 \tilde{S} = \operatorname{tr}\left(\frac{\partial^2 \tilde{S}}{\partial q_i \partial q_j}\right)$

So we get

$$\mathring{\tilde{S}} + \frac{1}{2}(\partial_q \tilde{S})^2 + \frac{1}{2} q^t V q + \frac{1}{2i} \partial_q^2 \tilde{S} = 0$$

If

$$\tilde{S} = \underbrace{\frac{1}{2} q^t a q + q^t b q' + \frac{1}{2} q'^t c q'}_{S} + d$$

then

$$\frac{1}{2} q^t \mathring{a} q + q^t \mathring{b} q' + \frac{1}{2} q'^t \mathring{c} q' + \mathring{d} + \frac{1}{2}(aq + bq')^2 + \frac{1}{2} q^t V q + \frac{1}{2i} \operatorname{tr}(a) = 0$$

so a, b, c, d must satisfy

$$\mathring{a} + a^2 + V = 0$$

$$\mathring{b} + ab = 0 \qquad \text{(assuming } a = a^t)$$

$$\mathring{c} + b^2 = 0$$

$$(i d)^\circ = -\frac{1}{2} \operatorname{tr}(a)$$

The second equation implies

$$(\det b)^\circ + (\operatorname{tr} a) \det b = 0$$

hence

$$(i d)^\circ = -\frac{1}{2} \operatorname{tr}(a) = \frac{1}{2} (\log \det b)^\circ$$

~~Let's change notation so that~~

$$\cancel{\psi = e^{id} e^{iS(q)}}$$

Consequently we see if

$$\langle t q \text{[redacted]} | t' q' \text{[redacted]} \rangle = e^{id} e^{i \overbrace{}^{}\underbrace{\left(\frac{1}{2} q^t a q + q^t b q' + \frac{1}{2} q'^t c q'\right)}_{S(tq\text{[redacted]},\, t'q')}}$$

then

$$e^{id} = \text{const} \, (\det b)^{1/2}$$

$$= \text{const} \left(\det \frac{\partial^2 S}{\partial q \partial q'} (tq, t'q') \right)^{1/2}$$

In order to determine the constant, let's look at the case of free motion $V=0$ whence

$$\langle tq | t'q' \rangle = \langle q | e^{-i(t-t')\frac{p^2}{2}} | q' \rangle$$

$$= \left(2\pi i (t-t')\right)^{-n/2} e^{\frac{i}{2}\frac{(q-q')^2}{t-t'}}$$

To simplify put $t'=0$. Then

$$S(tq, 0q') = \frac{1}{2}\frac{(q-q')^2}{t} = \frac{1}{2}\frac{1}{t}q^2 - \frac{1}{t}q'q + \frac{1}{2t}q'^2$$

so that
$$b = -\frac{1}{t} \qquad \det(b) = \left(-\frac{1}{t}\right)^n$$

$$\frac{1}{(2\pi i t)^{n/2}} = \det\left(\frac{i}{2\pi} b\right)$$

This should hold in general, whence we get the boxed formula on p. 223

Nov. 15, 1980

Summary: I'm trying to understand the S-matrix for a perturbed harmonic oscillator

$$H = \frac{p^2}{2} + \frac{1}{2} q\cdot(\omega^2+\varepsilon) q$$

We can compute the propagator $U(t,t')$ in terms of the action:

$$\langle tq | t'q' \rangle = \det\left(\frac{i}{2\pi}\frac{\partial^2 S}{\partial q \partial q'}(tq,t'q')\right)^{1/2} e^{iS(tq,t'q')}$$

and now from this we should be able to obtain the S-matrix.

Let's go back to $S = \frac{1}{2} q^t a q + q'^t b q + \frac{1}{2} q'^t c q'$.
It appears that the good matrix is not b but

$$\beta = -(b^t)^{-1}$$

$$\dot{b} + ab = 0$$
$$\dot{b}^t + b^t a = 0$$

In effect

$$\dot\beta = +(b^t)^{-1}\dot{b}^t(b^t)^{-1} = a\beta$$

so

$$\ddot\beta = \dot{a}\beta + a\dot\beta = \dot{a}\beta + a^2\beta = -V\beta$$

or

$$\ddot\beta + V\beta = 0$$

In other words the rows of β are independent solutions of the equation of motion:

$$\ddot{q} = -Vq$$

To get the $\beta = (b^t)^{-1}$ belonging to $S(tq,t'q')$ one lets

$$\beta = (u_1, \dots, u_n)$$

where β is the solution of $\ddot\beta + V\beta = 0$ such that

$$\beta = 0 \quad \text{at} \quad t = t'$$
$$\dot\beta = I \quad \text{at} \quad t = t'.$$

Then $\quad \beta \sim (t-t')$ and from $\dot\beta = a\beta$ we get

$$a \sim \frac{1}{(t-t')} \quad \text{near } t = t'.$$

<u>Check</u>: $V = \omega^2$, $n = 1$. Then $\beta = \frac{\sin\omega t}{\omega}$

and

$$a = \frac{\dot\beta}{\beta} = \omega\frac{\cos\omega t}{\sin\omega t} \quad \text{so that} \quad b = -\frac{\omega}{\sin\omega t}$$

$$S(tq, q') = \frac{1}{2}\frac{\omega}{\sin\omega t}\left((\cos\omega t)q^2 - 2q'q + (?)q'^2\right)$$

The last coeff $\frac{1}{2}cq'^2$ satisfies

$$\dot c = -b^2 = -\frac{\omega^2}{\sin^2\omega t} = -\omega^2\csc^2\omega t$$

$$= +\omega\frac{d}{dt}\cot\omega t \qquad \therefore\ c = \frac{\omega\cos\omega t}{\sin\omega t}$$

Thus

$$S(tq, t'q') = \frac{1}{2}\frac{\omega}{\sin\omega(t-t')}\left(\cos\omega(t-t')(q^2+q'^2) - 2qq'\right)$$

and

$$\langle tq | t'q' \rangle = \frac{1}{\sqrt{2\pi i \sin(\omega(t-t'))}}\, e^{iS(tq,\, t'q')}$$

But we are getting away from the S-matrix.

Rules for calculation: Free motion thru the time t is described by

$$\underset{\substack{\uparrow \\ \text{final} \\ \text{state}}}{\begin{pmatrix} q \\ p \end{pmatrix}} = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \underset{\substack{\uparrow \\ \text{initial} \\ \text{state}}}{\begin{pmatrix} q' \\ p' \end{pmatrix}}$$

on classical states, and by the operator

$$e^{-it\frac{p^2}{2}} \quad \text{on quantum states.}$$

Now

$$e^{-it\frac{p^2}{2}}\,(q \quad p)\,e^{it\frac{p^2}{2}} = (q-tp \quad p) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

so this old way of associating a matrix to an operator is not the one used. Instead we want the rule

$$\underbrace{M(A)}_{\text{the matrix belonging to } A.}\begin{pmatrix} q \\ p \end{pmatrix} = A^{-1}\begin{pmatrix} q \\ p \end{pmatrix} A \qquad (A \text{ unitary})$$

For example if $A = e^{-itp^2/2}$, then

$$e^{it\frac{p^2}{2}}\begin{pmatrix} q \\ p \end{pmatrix} e^{-itp^2/2} = \begin{pmatrix} q+tp \\ p \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

and if $H = \frac{p^2}{2} + \omega^2\frac{q^2}{2}$, then

$$\frac{d}{dt}\, e^{itH}\begin{pmatrix} q \\ p \end{pmatrix} e^{-itH}\Big|_{t=0} = \left[i\left(\frac{p^2}{2}+\omega^2\frac{q^2}{2}\right), \begin{pmatrix} q \\ p \end{pmatrix}\right]$$

$$= \begin{pmatrix} i[p,q]p \\ i\omega^2 q[q,p] \end{pmatrix} = \begin{pmatrix} p \\ -\omega^2 q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

Thus

$$H \longleftrightarrow \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad \text{but better to say}$$

[redacted]

$$e^{-itH} \longleftrightarrow \begin{pmatrix} \cos\omega t & \frac{\sin\omega t}{\omega} \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix}$$

Notice that

$$M(AB)\begin{pmatrix} q \\ p \end{pmatrix} = B^{-1}A^{-1}\begin{pmatrix} q \\ p \end{pmatrix} AB = B^{-1}M(A)\begin{pmatrix} q \\ p \end{pmatrix} B$$

$$= M(A)\,B^{-1}\begin{pmatrix} q \\ p \end{pmatrix} B = M(A)M(B)\begin{pmatrix} q \\ p \end{pmatrix}.$$

Next [redacted] let us consider the operator given by

$$\langle q|U|q'\rangle = \sqrt{\frac{b}{2\pi}}\, e^{i\left(\frac{1}{2}aq^2 + bqq' + \frac{1}{2}cq'^2\right)}$$

One has

$$e^{-ia\frac{q^2}{2}}\begin{pmatrix} q \\ p\end{pmatrix} e^{ia\frac{q^2}{2}} = \begin{pmatrix} q \\ p+aq\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1\end{pmatrix}\begin{pmatrix} q \\ p\end{pmatrix}$$

hence

$$e^{ia\frac{q^2}{2}} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ a & 1\end{pmatrix}$$

Also if

$$(\mathcal{F}f)(x) = \int \frac{dy}{\sqrt{2\pi}}\, e^{-ixy} f(y)$$

then

$$p\mathcal{F}f = -\mathcal{F}(q [redacted] f).$$

hence

$$\mathcal{F}^{-1}\begin{pmatrix} q \\ p\end{pmatrix}\mathcal{F} = \begin{pmatrix} p \\ -q\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} q \\ p\end{pmatrix}$$

Finally if

$$(T_\beta f)(x) = \frac{1}{\sqrt{\beta}} f(\beta^{-1}x)$$

then

$$(pT_\beta f) = \frac{1}{\beta}\frac{1}{\sqrt{\beta}}(pf)(\beta^{-1}x) = \frac{1}{\beta}(T_\beta pf)$$

hence

$$T_\beta^{-1}\begin{pmatrix} q \\ p\end{pmatrix} T_\beta = \begin{pmatrix} \beta q \\ \frac{1}{\beta}p\end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & \frac{1}{\beta}\end{pmatrix}\begin{pmatrix} q \\ p\end{pmatrix}$$

Thus to the operator U with

$$\langle q|U|q'\rangle = \frac{1}{\sqrt{2\pi\beta}}\, e^{ia\frac{q^2}{2} - i\frac{1}{\beta}qq' + ic\frac{q'^2}{2}}$$

belongs the matrix

$$\begin{pmatrix} 1 & 0 \\ a & 1\end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1\end{pmatrix}$$

November 16, 1980

We consider on $L^2(\mathbb{R}^n)$ a operator U of the form

$$\langle q|U|q'\rangle = \det(2\pi i\beta)^{-1/2}\, e^{i\left(\frac{1}{2}q\cdot aq - q'\cdot\frac{1}{\beta}q + \frac{1}{2}q'\cdot cq'\right)}$$

We showed yesterday that the matrix of U, which is defined by $\quad U^{-1}\begin{pmatrix} q \\ p\end{pmatrix}U = M(u)\begin{pmatrix} q \\ p\end{pmatrix}$, is given by

$$M(u) = \begin{pmatrix} 1 & 0 \\ a & 1\end{pmatrix}\begin{pmatrix} \beta & \\ & (\beta^t)^{-1}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1\end{pmatrix}$$

$$= \begin{pmatrix} \beta c & \beta \\ a\beta c-(\beta^t)^{-1} & a\beta\end{pmatrix}.$$

Call this "transfer" matrix $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ so that

$$\begin{pmatrix} q \\ p\end{pmatrix} = \begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q' \\ p'\end{pmatrix}$$

$B=\beta \qquad A=\beta c \implies c = B^{-1}A$

$a\beta = D \implies a = DB^{-1}$

$C = a\beta c-(\beta^t)^{-1} \implies (\beta^t)^{-1} = DB^{-1}A - C$

Solving yields

$$\begin{pmatrix} p \\ p'\end{pmatrix} = \begin{pmatrix} DB^{-1} & C-DB^{-1}A \\ B^{-1} & -B^{-1}A\end{pmatrix}\begin{pmatrix} q \\ q'\end{pmatrix} = \begin{pmatrix} a & -(\beta^t)^{-1} \\ \beta^{-1} & -c\end{pmatrix}\begin{pmatrix} q \\ q'\end{pmatrix}$$

Notice that this is consistent with

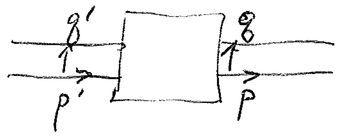
$$p = \frac{\partial S}{\partial q} = aq - (\beta^t)^{-1}q' \qquad\qquad S = \tfrac{1}{2}q\cdot aq - q'\cdot\beta^{-1}q + \tfrac{1}{2}q'\cdot cq'$$

$$-p' = \frac{\partial S}{\partial q'} = -\beta^{-1}q + cq'$$

Also we have

$$\tfrac{1}{2}(p\cdot q - p'\cdot q') = S$$

What I would like to understand is [redacted] whether there is a way of seeing S as a net power flow into a port. Picture:



q = voltage
p = current

Then $\quad pq$ = power out to right
$\quad -p'q'$ = power out to left

so $\quad pq - p'q'$ = net power loss.

Next project is to work out the formulas in the holomorphic representation. Consider the Hamiltonian

$$H = k a^* a + l \frac{a^{*2}}{2} + \bar{l} \frac{a^2}{2} \qquad k \text{ real}$$

Then the equations of motion are

$$\mathring{a} = [iH, a] = i\left[k a^* a + l \frac{a^{*2}}{2}, a\right]$$
$$= i(-ka - la^*)$$

or

$$\boxed{\begin{aligned} i\mathring{a} &= ka + la^* \\ -i\mathring{a}^* &= \bar{l}a + ka^* \end{aligned}}$$

or

$$\frac{d}{dt}\begin{pmatrix} a \\ a^* \end{pmatrix} = \begin{pmatrix} -ik & -il \\ i\bar{l} & ik \end{pmatrix}\begin{pmatrix} a \\ a^* \end{pmatrix}$$

~~In several dimensions a matrix in Lie(U(n,n)) is of the form~~ $\cancel{\begin{pmatrix} A & B \\ B^* & D \end{pmatrix}}$ ~~with~~ $\cancel{A = -A^*,\ D = -D^*}$

~~Recall that~~

$$\cancel{f(z) = \sum \frac{z^n}{n!}\langle a^n f\rangle = \int_u \sum \frac{z^n \bar u^n}{n!} e^{-|u|^2} f(u) = \langle e_{\bar z} | f \rangle}$$

~~where~~ $\cancel{e_{\bar z} = \sum \frac{\bar z^n}{n!} (a^*)^n |0\rangle}$

Recall that

$$f(\lambda) = \sum \frac{\lambda^n}{n!}(a^n f)(0) = \int_z \sum \frac{\lambda^n}{n!}\bar z^n f(z) e^{-|z|^2} = \langle e_{\bar\lambda} | f \rangle$$

where
$$e_{\bar\lambda} = e^{\bar\lambda z} \text{ as an analytic fn. of } z$$

Let $U(t)$ be the propagator for

$$(*) \qquad i\frac{\partial \psi}{\partial t} = H\psi$$

Then we feel that $U(t)e_\lambda$ has the form

$$(U(t)e_\lambda)(z) = \langle e_{\bar z} | U(t) e_\lambda \rangle = e^{\frac{1}{2}\alpha z^2 + \beta z\lambda + \frac{1}{2}\gamma\lambda^2 + \delta}$$

If ~~this~~ satisfies $(*)$ we get

$$i\left(\frac{1}{2}\dot\alpha z^2 + \dot\beta z\lambda + \frac{1}{2}\dot\gamma\lambda^2 + \dot\delta\right) = \left(kz\frac{d}{dz} + \ell\frac{z^2}{2} + \frac{\bar\ell}{2}\frac{d^2}{dz^2}\right)$$

so

$$i\left(\frac{1}{2}\dot\alpha z^2 + \dot\beta z\lambda + \frac{1}{2}\dot\gamma\lambda^2 + \dot\delta\right) = kz(\alpha z + \beta\lambda) + \ell\frac{z^2}{2} + \frac{\bar\ell}{2}\left[(\alpha z + \beta\lambda)^2 + \alpha\right]$$

yielding

$$\begin{cases} i\dot\alpha = 2k\alpha + \ell + \bar\ell\alpha^2 \\ i\dot\beta = k\beta + \bar\ell\alpha\beta \\ i\dot\gamma = \bar\ell\beta^2 \\ i\dot\delta = \frac{\bar\ell}{2}\alpha \end{cases}$$

which is not particularly illuminating

So try to compute the matrix belonging to $u(t)$:

$$u(t) = e^{\delta}\, e^{\frac{1}{2}\alpha a^{*2}}\, e^{(\log\beta)a^*a}\, e^{\frac{1}{2}\gamma a^2}$$

We need

$$e^{-\frac{1}{2}\gamma a^2}\begin{pmatrix} a \\ a^* \end{pmatrix} e^{\frac{1}{2}\gamma a^2} = \begin{pmatrix} a \\ a^* - \gamma a \end{pmatrix} = \begin{pmatrix} 1 & \\ -\gamma & 1 \end{pmatrix}\begin{pmatrix} a \\ a^* \end{pmatrix}$$

$$e^{-\frac{1}{2}\alpha a^{*2}}\begin{pmatrix} a \\ a^* \end{pmatrix} e^{\frac{1}{2}\alpha a^{*2}} = \begin{pmatrix} a + \alpha a^* \\ a^* \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix}\begin{pmatrix} a \\ a^* \end{pmatrix}$$

$$e^{-(\log\beta)a^*a}\begin{pmatrix} a \\ a^* \end{pmatrix} e^{+(\log\beta)a^*a} = \begin{pmatrix} \beta & \\ & \beta^{-1} \end{pmatrix}\begin{pmatrix} a \\ a^* \end{pmatrix}$$

So

$$u(t) \longleftrightarrow \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \beta & \\ & \beta^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \beta & 0 \\ -\beta^{-1}\gamma & \beta^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \beta - \alpha\beta^{-1}\gamma & \alpha\beta^{-1} \\ -\beta^{-1}\gamma & \beta^{-1} \end{pmatrix}$$

Notice that β^{-1} appears in the lower right corner which means that we can find β^{-1} by starting with a solution $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and seeing its a^* part at the exit. Thus β^{-1} is like the transmission coefficient.

All this is complicated, and since I have to proceed via the matrices it seems desirable to work out the formulas in n-dimensions.

Note a is a column vector $a = (a_i)$, a^* is a row vector (a_i^*). To avoid confusion put $\tilde{a} = (a^*)^t$ = column vector (a_i^*). Then

$$e^{-\frac{1}{2} a^t \gamma a} \begin{pmatrix} a \\ \tilde{a} \end{pmatrix} e^{\frac{1}{2} a^t \gamma a} = \begin{pmatrix} a \\ \tilde{a} - \gamma a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} \begin{pmatrix} a \\ \tilde{a} \end{pmatrix}$$

where $\gamma = \gamma^t$. Similarly

$$e^{\frac{1}{2} a^* \alpha \tilde{a}} \longleftrightarrow \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

Next find $e^{a^* b a}$, i.e. its matrix.

$$-\left[a^* b a, \begin{pmatrix} a \\ \tilde{a} \end{pmatrix} \right] = -\left[\sum_{i,j} a_i^* b_{ij} a_j, \begin{pmatrix} a_k \\ a_\ell^* \end{pmatrix} \right]$$

$$= \begin{pmatrix} \sum_j b_{kj} a_j \\ -\sum_i a_i^* b_{i\ell} \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & -b^t \end{pmatrix} \begin{pmatrix} a \\ \tilde{a} \end{pmatrix}$$

so that

$$e^{a^* b a} \longleftrightarrow \begin{pmatrix} e^b & 0 \\ 0 & (e^{-b})^t \end{pmatrix}$$

So now we can make the following calculation:

$$\begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} \begin{pmatrix} (\beta^t)^{-1} & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} = \begin{pmatrix} (\beta^t)^{-1} - \alpha\beta\gamma & \alpha\beta \\ -\beta\gamma & \beta \end{pmatrix}$$

This is the transfer matrix associated to the operator

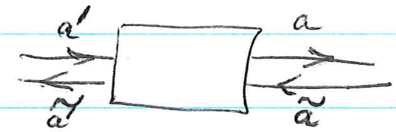
$$U = e^{\frac{1}{2} a^* \alpha \overline{a^*}} \, e^{a^*(-\log \beta^t) a} \, e^{\frac{1}{2} a^t \gamma a}$$

which belongs to the function $\langle e_{\bar{z}} | U | e_\lambda \rangle = e^{\frac{1}{2} z^t \alpha z + \lambda^t \frac{1}{\beta} z + \frac{1}{2} \lambda^t \gamma \lambda}$

Then
$$\begin{pmatrix} a \\ \tilde{a} \end{pmatrix} = \begin{pmatrix} (\beta^t)^{-1} - \alpha\beta\gamma & \alpha\beta \\ -\beta\gamma & \beta \end{pmatrix} \begin{pmatrix} a' \\ \tilde{a}' \end{pmatrix}$$

which can be solved to give
$$\begin{pmatrix} a \\ \tilde{a}' \end{pmatrix} = \begin{pmatrix} \alpha & (\beta^t)^{-1} \\ \beta^{-1} & \gamma \end{pmatrix} \begin{pmatrix} \tilde{a} \\ a' \end{pmatrix}$$

Notice the picture



so that $\begin{matrix} a \\ \tilde{a} \end{matrix}$ are incoming $\begin{matrix} a \\ \tilde{a}' \end{matrix}$ are outgoing.

Question: What is the intelligent way to get at these formulas?

The action is
$$S(z,\lambda) = \frac{1}{2} z^t \alpha z + \lambda^t \beta^{-1} z + \frac{1}{2} \lambda^t \gamma \lambda$$
where one thinks of λ as the (eigen)value of a' and z as the (eigen) value of $\tilde{a}$.