October 1, 1980:

Review the forced harmonic oscillator.

$$\begin{cases} H_0 = \omega a^* a \\ H_{int} = J(t)\, a + \bar{J}(t)\, a^* \end{cases}$$

The S-matrix is

$$S = T\left\{ \prod_t e^{-i\,dt\,(J(t)\hat{a}(t) + \bar{J}(t)\hat{a}^*(t))} \right\}$$

where $\hat{a}(t) = e^{iH_0 t} a\, e^{-iH_0 t}$ and similarly for $\hat{a}^*$.

Recall $$e^A e^B = e^B e^A e^{[A,B]}$$

if $[A,B]$ commutes with A, B. We use this to write the above time-ordered product in normal form, by pushing $e^{-i\,dt\,J(t)\hat{a}(t)}$ thru $\prod_{t'<t} e^{-i\,dt'\bar{J}(t')\hat{a}^*(t')}$.

This gives

$$S = \prod_{t'<t} e^{[-i\,dt\,J(t)\hat{a}(t),\, -i\,dt'\bar{J}(t')\hat{a}^*(t')]} \cdot \prod_{t''} e^{-i\,dt''\bar{J}(t'')\hat{a}^*(t'')}$$

$$\times \prod_t e^{-i\,dt\,J(t)\hat{a}(t)}$$

In this 1-diml case one has $\hat{a}(t) = e^{-i\omega t} a$ etc, so that

$$S = \langle 0|S|0\rangle\, e^{-\bar{\gamma} a^*} e^{\gamma a}$$

where $\gamma = -i\int dt\, J(t)\, e^{-i\omega t}$ $\quad -\bar{\gamma} = -i\int dt\, \bar{J}(t)\, e^{i\omega t}$

and $$\langle 0|S|0\rangle = \exp\left\{ -\iint_{t>t'} dt\,dt'\; J(t)\, e^{-i\omega t} e^{i\omega t'} \bar{J}(t') \right\}$$

<u>Question:</u> Does the phase of $\langle 0|S|0\rangle$ have "metaplectic" (or quadratic-reciprocity) significance? Its abs. value is

$$|\langle 0|S|0\rangle|^2 = \exp\left\{-\iint dt\,dt'\ J(t)\, e^{-i\omega(t-t')}\,\bar{J}(t')\right\}$$

$$= \exp\left(-|\gamma|^2\right).$$

You can also see this because S has to be unitary and

$$S^*S = e^{\bar{\gamma}a^*}e^{-\gamma a}e^{-\bar{\gamma}a^*}e^{\gamma a}\,|\langle 0|S|0\rangle|^2$$

$$= \cancel{e^{\bar{\gamma}a^*}e^{-\bar{\gamma}a^*}}\,e^{[-\gamma a,\,-\bar{\gamma}a^*]}\,\underbrace{\cancel{e^{-\gamma a}e^{\gamma a}}}_{1} = e^{|\gamma|^2}|\langle 0|S|0\rangle|^2$$

Let's look at the classical equations.

$$H = \omega a^*a + Ja + \bar{J}a^*$$

$$\dot{a} = [iH, a] = -i\omega a - i\bar{J}$$

$$\dot{a}^* = [iH, a^*] = i\omega a^* + iJ$$

which when integrated gives

$$a(t) = \text{const.}\, e^{-i\omega t} - i\int_{-\infty}^{t} e^{-i\omega(t-t')}\,\bar{J}(t')\,dt'$$

The scattering operator takes the solution $Ae^{-i\omega t}$ $t \ll 0$ into the solution $(A-\bar{\gamma})e^{-i\omega t}$, where

$$+\bar{\gamma} = i\int e^{i\omega t'}\bar{J}(t')\,dt' \qquad \text{as above.}$$

Let's recall what translations look like in the holomorphic fn. repn.

$$\|f\|^2 = \int |f(z)|^2 e^{-|z|^2}\frac{dxdy}{\pi} = \int |f(z+\gamma)|^2 e^{-|z+\gamma|^2}\frac{dxdy}{\pi}$$

$$= \int |f(z+\gamma)|^2 e^{-|z|^2 - \bar{\gamma} z - \gamma \bar{z} - \frac{|\gamma|^2}{2}} \cdots$$

$$= \int \left| f(z+\gamma) e^{-\bar{\gamma} z - |\gamma|^2/2} \right|^2 e^{-|z|^2} \cdots$$

Thus $\quad f(z) \longmapsto f(z+\gamma)\, e^{-\bar{\gamma} z - |\gamma|^2/2}$

is a unitary operator. In terms of creation + ann. ops. it is

$$e^{-|\gamma|^2/2}\, e^{-\bar{\gamma} a^*}\, e^{\gamma a}$$

October 3, 1980

<u>forced oscillator</u>: Let's begin with a positive hermitian operator H_0 on a Hilbert space $\mathcal{H}$ and extend it to the boson Fock space:

$$H_0 = \sum a_\alpha^* \langle\alpha|H_0|\beta\rangle a_\beta$$

Here $a_\beta = a_{\langle\beta|}$ and $a_\alpha^* = a^*_{|\alpha\rangle}$ where $|\alpha\rangle$ is an orth. basis for $\mathcal{H}$. The interaction is

$$H_{int} = \sum \bar{J}_\alpha a_\alpha + J_\alpha a_\alpha^* \qquad = a_J^* + a_{(J^*)}$$

where $J = \sum |\alpha\rangle J_\alpha \qquad J^* = \sum \bar{J}_\alpha \langle\alpha|$. Here the vector J depends on t. Note the change in notation from J to $\bar{J}$.

The S-matrix is

$$S = T\left\{\prod e^{-i\,dt\, e^{iH_0t} a_J^* e^{-iH_0t}} e^{-i\,dt\, e^{iH_0t} a_{J^*} e^{-iH_0t}}\right\}$$

$$= \langle 0|S|0\rangle e^{-i a_\gamma^*} e^{-i a_{\gamma^*}}$$

where

$$\gamma = \int dt\, e^{iH_0t} J(t).$$

Note that $e^{-ia_\gamma^*}$ is multiplication by $e^{-i\gamma(z)}$ in the holom. repn. where $\gamma(z) = \sum \gamma_\alpha z_\alpha$ is a linear function of $\{z_\alpha\}$. Similarly $e^{-ia_{\gamma^*}}$ is translation

$$e^{-ia_{\gamma^*}} = e^{-i\sum \bar{\gamma}_\alpha \frac{\partial}{\partial z_\alpha}} \; : \; f(z) \longmapsto f(z - i\gamma^*)$$

Now I wanted to take the case where

$$J(t) = e^{-i\omega_0 t} v$$

and v is fixed. This represents a driving force of frequency ω_0.

Let's suppose that H_0 has a continuous spectrum of eigenfunctions:

$$H_0 = \int \frac{dk}{2\pi} \, |k\rangle \, \omega_k \, \langle k| ,$$

for example, suppose $\mathcal{H} = L^2(\mathbb{R})$ and $H_0 = -\Delta$ whence $|k\rangle = e^{ikx}$ and $\omega_k = k^2$. Then

$$\int dt \, e^{iH_0 t} J(t) = \int dt \int \frac{dk}{2\pi} \, |k\rangle \, e^{i(\omega_k - \omega_0)t} \langle k| v$$

$$= \int \frac{dk}{\text{[redacted]}} \, |k\rangle \, \delta(\omega_k - \omega_0) \langle k| v$$

$$= \int \frac{dk}{\text{[redacted]}} \, |k\rangle \, \delta(k^2 - k_0^2) \langle k| v \qquad k_0 = \sqrt{\omega_0}$$

Now $\delta(k^2 - k_0^2)\, dk = \delta(k^2 - k_0^2) \dfrac{dk^2}{2k} = \dfrac{\delta(k-k_0)\,dk}{2k_0} + \dfrac{\delta(k+k_0)\,dk}{2k_0}$

So in this example

$$\gamma = \int dt \, e^{iH_0 t} J(t) = \frac{1}{2k_0} \Big(|k_0\rangle\langle k_0| v + |-k_0\rangle\langle -k_0| v \Big).$$

$$= \frac{1}{2k_0} \Big(e^{ik_0 x} \hat{v}(k_0) + e^{-ik_0 x} \hat{v}(-k_0) \Big).$$

We see that γ is not in the Hilbert space, no matter how smooth v is. Thus the S-matrix does not exist in the strict sense.

Curious: Suppose H_0 not positive-definite, e.g. $H_0 |k\rangle = k|k\rangle$, $H_0 = \frac{1}{i}\frac{d}{dx}$ [redacted] on $L^2(\mathbb{R})$. Then I could consider $J(t) = v$ constant and I would get

$$\int dt \, e^{iH_0 t} J(t) = \delta(H_0) v = \int \frac{dk}{\text{[redacted]}} \, |k\rangle \, \delta(k) \langle k| v$$

$$= \hat{v}(0).$$

So it seems we get a constant perturbation without an S-matrix. It seems the same thing results if I take

$$H_0|k\rangle = |k|\cdot|k\rangle$$

so I can get an <u>example</u> with $H_0 \geq 0$.

Recall what happens for a constant source perturbation of a simple harmonic oscillator

$$H = \omega a^* a + J a^* + \bar{J} a$$

$$= \omega\left(a^* + \frac{\bar{J}}{\omega}\right)\left(a + \frac{J}{\omega}\right) - \frac{|J|^2}{\omega}.$$

One has a ground energy shift of $-\frac{|J|^2}{\omega}$; both wave operators are the same and given by translation thru something like $\frac{J}{\omega}$. For many degrees of freedom:

$$H = \sum \omega_k a_k^* a_k + J_k a_k^* + \bar{J}_k a_k$$

$$= \sum \omega_k \left(a_k^* + \frac{\bar{J}_k}{\omega_k}\right)\left(a_k + \frac{J_k}{\omega_k}\right) - \sum \bar{J}_k \frac{1}{\omega_k} J_k$$

so it should be the case that the ground energy shift is

$$-J^* \frac{1}{H_0} J$$

and both wave operators are the same and given essentially by translation thru $\left(\frac{1}{H_0} J\right)^*$. [redacted]

Something interesting: It seems that a time dependence $J = \tilde{J} e^{i\mu t}$ can be removed by shifting the ground point energy of H_0. For example we know that Schroedinger's eqn.

with $\qquad H(t) = \omega a^* a + J a^* + \bar{J} a$

is equivalent to Schroedinger's equation with

$$H'(t) = J e^{i\omega t} a^* + \bar{J} e^{-i\omega t} a$$

In effect if $\qquad i \frac{\partial}{\partial t} \Psi(t)$ [redacted] $= H(t)\Psi(t)$ [redacted]

then

$$i \frac{\partial}{\partial t} e^{-iH_0 t} \Psi(t) = -H_0 e^{iH_0 t} \Psi(t) + e^{iH_0 t} H(t) \Psi(t)$$

$$= e^{-iH_0 t} (J a^* + \bar{J} a) \Psi(t)$$

$$= \left(J e^{i\omega t} a^* + \bar{J} e^{-i\omega t} a\right) e^{-iH_0 t} \Psi(t)$$

Thus we see that

$$H_0(t) = (\omega + \mu) a^* a + J_0 a^* + \bar{J}_0 a$$

is equivalent to

$$H_1(t) = \text{[redacted]} \; J_0 e^{+i(\mu+\omega)t} a^* + \bar{J}_0 e^{-i(\mu+\omega)t} a$$

is equivalent to

$$H_2(t) = \omega a^* a + J_0 e^{+i\mu t} a^* + \bar{J}_0 e^{-i\mu t} a$$

Formulas: $\qquad \displaystyle\int_{-T/2}^{T/2} e^{i\omega t} \, dt = \frac{e^{i\omega T/2} - e^{-i\omega T/2}}{2i\omega/2} = \frac{\sin(\omega T/2)}{(\omega/2)}$

for large T one has $\qquad \boxed{\dfrac{\sin(\omega T/2)}{\omega/2} \sim 2\pi \delta(\omega)}$

(This comes from $\int e^{i\omega t} dt = 2\pi \delta(\omega)$.) Also Parseval implies

$$\int \left(\frac{\sin(\omega T/2)}{\omega/2}\right)^2 d\omega = 2\pi \int_{-T/2}^{T/2} 1^2 dt = 2\pi T$$

so that for large T one has

$$\boxed{\left(\frac{\sin(\omega T/2)}{\omega/2}\right)^2 \sim 2\pi T\,\delta(\omega)}$$

<u>Question</u>: Can I get an example of the Golden Rule from the forced oscillator?

I let the perturbation $J(t) = e^{i\omega_0 t}\,v$ act for the time interval $[-T/2, T/2]$. The S-matrix is

$$S = \langle 0|S|0\rangle\, e^{-i\, a_\gamma^*}\, e^{-i\, a_{\gamma^*}}$$

where
$$\gamma = \int_{-T/2}^{T/2} dt \; e^{i(H_0-\omega_0)t}\, v$$
$$= \int_{-T/2}^{T/2} dt \int dk\; |k\rangle\, e^{i(\omega_k-\omega_0)t}\,\langle k|v$$
$$= \int dk\; |k\rangle\, \frac{\sin\frac{(\omega_k-\omega_0)T}{2}}{\left(\frac{\omega_k-\omega_0}{2}\right)}\,\langle k|v$$

The intriguing thing is that if we consider transitions from the ground state $|0\rangle$, we get

$$S|0\rangle = \underbrace{\langle 0|S|0\rangle}_{\substack{\text{has abs. value}\\ e^{-|\gamma|^2/2}}}\;\underbrace{e^{-i\gamma z}}_{\sum_n \frac{(-i\gamma z)^n}{n!}}$$

and this yields a Poisson distribution.

October 5, 1980

Einstein's A, B: Suppose we have a container filled with a gas of molecules with internal energy levels of energies ε_n. If the gas has the temperature T, then the number N_m of molecules in the m-th ~~state~~ satisfies the Bolzman rule:

$$\frac{N_m}{N_n} = \frac{e^{-\beta \varepsilon_m}}{e^{-\beta \varepsilon_n}} .$$

Consider next the possibility of a molecule in state m changing to state n with emission or absorption of radiation. Suppose $\varepsilon_m > \varepsilon_n$. Classically we know that an accelerating electron radiates, and we will take the quantum analogue of this to be the fact that there a probability of spontaneous emission. Let

$$A^m_n \, dt = \begin{cases} \text{probability of the transition } m \to n \\ \text{in time } dt \text{ due to spontaneous emission} \\ \text{of a photon of energy } \varepsilon_m - \varepsilon_n \end{cases}$$

Classically we know that in the presense of radiation an oscillating electron will absorb or gain energy depending the relative phases. The quantum analogue will be assumed to be the possibility of induced emission or absorption of photons with probability proportional to the intensity of the radiation of the right frequency. Let

$$B^m_n I \, dt = \begin{cases} \text{probability of transition } m \to n \text{ in time} \\ dt \text{ due induced emission, where} \\ I = \text{intensity of radiation at energy } \varepsilon_m - \varepsilon_n . \end{cases}$$

$$B^n_m I \, dt = \begin{cases} \text{prob. of transition } n \to m \text{ in time } dt \\ \text{due to [redacted] absorption, where} \\ I = \text{intensity.} \end{cases}$$

Assuming detailed balance (the number of $m \to n$ is the same as the number $n \to m$) we get

$$B^n_m I \, dt \, N_n = \left(A^m_n \, dt + B^m_n I \, dt \right) N_m$$

$$\left(B^n_m N_n - B^m_n N_m \right) I = A^m_n N_m$$

$$I = \frac{A_n^m N_m}{B_m^n N_n - B_n^m N_m} = \frac{A_n^m}{B_m^n e^{\beta(\varepsilon_m - \varepsilon_n)} - B_n^m}$$

But I is given by Plank's formula

$$I = \frac{2 \cdot 4\pi}{(2\pi)^3} \frac{\hbar}{c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \qquad \omega = \varepsilon_m - \varepsilon_n$$

Assuming $I \to \infty$ as $\beta \to 0$ one sees that

$$B_m^n = B_n^m \qquad \frac{A_n^m}{B_n^m} = \frac{2 \cdot 4\pi}{(2\pi)^3} \frac{\hbar}{c^3} \omega^3$$

Question: How did Planck compute Planck's constant? Did he know Boltzmann's constant?

Presumably one can actually measure energy density of radiation in a cavity at a given temperature, and get

$$I = \frac{1}{\pi^2} \frac{\omega^2}{c^3} \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

For $\frac{\omega}{T} \ll 1$ this is given by Rayleigh's fmula:

$$I = \frac{1}{\pi^2} \frac{\omega^2}{c^3} kT$$

and so k could be computed, (once energy + temperature scales are fixed.) The total energy density over all frequencies is

$$\int_0^\infty \frac{1}{\pi^2} \frac{\hbar\, \omega^3 d\omega}{c^3 (e^{\beta\hbar\omega} - 1)} = \frac{\hbar}{\pi^2 c^3} \left(\frac{kT}{\hbar}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

hence is proportional to $\frac{k^4 T^4}{\hbar^3}$. Thus one can determine $\hbar$ from total intensity measurements. Also I is proportional to the function $\frac{x^3}{e^x - 1}$ where $x = \frac{\hbar\omega}{kT}$ so by finding the maximum of the intensity curve and comparing with the max of $\frac{x^3}{e^x-1}$ one gets $\hbar/k$, knowing ω, T.

Boltzmann's constant appears any time one can separate off a little system in equilibrium with the rest. One should look for the simplest situation which would allow one to determine k in terms of other physical constants.

October 6, 1980

Let's return to the forced oscillator

$$H = H_0 + a^*_J + a_{J^*}$$

where J acts for a finite time interval. I know the S-matrix which describes the transitions induced by the forcing term is

$$S = \langle 0|S|0\rangle \, e^{-ia^*_\gamma} e^{-ia_{\gamma^*}}$$

where

$$\gamma = \int e^{iH_0 t} J \, dt \qquad \text{and} \qquad |\langle 0|S|0\rangle| = e^{-|\gamma|^2/2}$$

Let's first understand what happens to first order in J. Then

$$S = id - i a^*_\gamma - i a_{\gamma^*}$$

which we could see more simply [redacted] from Dyson's formula:

$$S = 1 - i\int dt \, H_{int}(t) + \cdots$$

What are the transitions of first order? First suppose we have a simple harmonic oscillator. Then

$$S = 1 - i(\gamma a^* + \bar{\gamma} a)$$

and the Hilbert space has the basis $|n\rangle = \frac{(a^*)^n|0\rangle}{\sqrt{n!}}$. So

$$S|n\rangle = |n\rangle - i\gamma\sqrt{n+1}\,|n+1\rangle - i\bar{\gamma}\sqrt{n}\,|n-1\rangle$$

and the first order transition probabilities are

$$p_{n\to n+1} = (n+1)|\gamma|^2$$

$$p_{n\to n-1} = n\,|\gamma|^2$$

The error should be $O(\gamma^3)$. Maybe you should check this more carefully.

$$S = \Big(\big(1 - i(\gamma a^* + \bar{\gamma} a) - \tfrac{1}{2}(\gamma^2 a^{*2} + 2\gamma\bar{\gamma} a^* a + \bar{\gamma}^2 a^{*2}) + \cdots \big)$$

$$\times \Big(1 - \iint_{t>t'} \bar{J}(t) e^{-iH_0(t-t')} J(t')\, dt\, dt' + \cdots \Big)$$

Thus one sees that

$$\langle n+1|S|n\rangle = \underbrace{\langle 0|S|0\rangle}_{\text{has abs. val } 1+O(\gamma^2)} \big[(-i\gamma)\sqrt{n+1} + O(\gamma^3)\big].$$

hence $\quad |\langle n+1|S|n\rangle|^2 = |\gamma|^2(n+1)\,(1+O(\gamma^2)).$ Similarly

$$|\langle n-1|S|n\rangle|^2 = |\gamma|^2\, n\,(1+O(\gamma^2))$$

Also

$$\langle n|S|n\rangle = \big(1 - |\gamma|^2 \langle n|a^*a|n\rangle\big)\langle 0|S|0\rangle + O|\gamma|^4$$

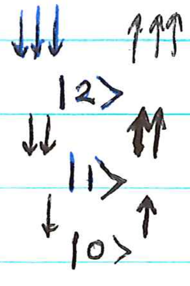
$$|\langle n|S|n\rangle|^2 = \big(1 - 2|\gamma|^2 n\big)\big(1 - |\gamma|^2\big) + O(|\gamma|^4)$$

$$= 1 - (2n+1)|\gamma|^2 + O(|\gamma|^4).$$

Summarizing:

$$\boxed{\begin{aligned} |\langle n+1|S|n\rangle|^2 &= (n+1)|\gamma|^2\,(1+O(|\gamma|^2)) \\ |\langle n-1|S|n\rangle|^2 &= n|\gamma|^2\,(\quad " \quad) \\ |\langle n|S|n\rangle|^2 &= (1-(2n+1)|\gamma|^2)\,(\quad " \quad) \end{aligned}}$$

Picture:



Think of $|n\rangle$ as having n particles each of which might decay to yield $n-1$ particles, hence the rate $n \to n-1$ is $n|\gamma|^2$ where $|\gamma|^2$ is the rate $1 \to 0$. On the other hand if $|\gamma|^2$ is the rate $0 \to 1$, then the transition $n \to n+1$ can be done in $n+1$ ways.

October 10, 1980

Let's try to understand the time-dependent approach to scattering to the first order in $V = H_{int}$. The time development in the Dirac picture is

$$U(t,t') = T\left\{ e^{-i\int_{t'}^{t} dt_1 \underbrace{e^{iH_0t_1} V e^{-iH_0t_1}}_{V(t_1)}} \right\}$$

$$= 1 - i\int_{t'}^{t} dt_1\, V(t_1) + O(V^2)$$

Let H_0 have continuous [redacted] spectrum

$$H_0 |a\rangle = E_a |a\rangle .$$

[redacted] I want to compute all the quantities occurring in the time-dependent theory. First let's try to get the wave operators.

$$U(t,t')\varphi_a = \overset{|a\rangle}{\cancel{\varphi_a}} - i\int_{t'}^{t} dt_1\, \underbrace{e^{iH_0t_1} V \underbrace{e^{-iH_0t_1} \overset{|a\rangle}{\cancel{\varphi_a}}}_{e^{-iE_at_1}\, \cancel{\varphi_a}|a\rangle}}_{e^{i(H_0-E_a)t_1}\, V \cancel{\varphi_a}|a\rangle}$$

$$= \overset{|a\rangle}{\cancel{\varphi_a}} - i\, \frac{e^{i(H_0-E_a)t} - e^{i(H_0-E_a)t'}}{i(H_0-E_a)}\, V\cancel{\varphi_a}|a\rangle$$

This does not converge ~~at~~ $t' \to -\infty$. There are various possibilities

a) adiabatic potential: Replace V by $e^{\varepsilon t}V$ and then take $t \to -\infty$ and then $\varepsilon \searrow 0$.

b) wave packet: Instead of $\cancel{\varphi_a}|a\rangle$ use [redacted] an L^2 [redacted] vector

$$\varphi = \int da\, \cancel{\varphi_a}\, |a\rangle \underbrace{\langle a|\varphi}_{\text{smooth with compact support}}$$

c) it might be possible to assume t' complex because H_0 is bounded below (?).

Lets examine b) closely

$$\varphi = \int da\ |a\rangle \langle a|\varphi$$

$$e^{-iH_0 t_1}\varphi = \int da\ |a\rangle e^{-iE_a t_1} \langle a|\varphi$$

$$-i\int_{t'}^{t} dt_1\ e^{iH_0 t_1} V e^{-iH_0 t_1}\varphi = -i\int_{t'}^{t} dt_1\ e^{iH_0 t_1} V \int da\ |a\rangle e^{-iE_a t_1}\langle a|\varphi$$

assume V nice operator i.e. finite rank

$$= \int da \int_{t'}^{t} (-i)dt_1\ e^{i(H_0 - E_a)t_1} V|a\rangle\langle a|\varphi$$

$$= \int da\ \frac{e^{-i(E_a - H_0)t} - e^{-i(E_a-H_0)t'}}{E_a - H_0}\ V|a\rangle\ \langle a|\varphi$$

Q: Can you see this converging as $t' \longrightarrow -\infty$?

Try taking inner product with $\tilde{\varphi} = \int db\ |b\rangle$ [redacted] $\langle b|\tilde{\varphi}$.

You get

$$-i\iint db\, da\ \langle\tilde{\varphi}|b\rangle\ \frac{e^{+iE_{ba}t} - e^{iE_{ba}t'}}{i(E_b - E_a)}\ \langle b|V|a\rangle\langle a|\varphi\rangle.$$

Here $\langle a|\varphi\rangle$, $\langle b|\tilde{\varphi}\rangle$ are smooth rapidly decreasing functions so the same should be true of $\langle\tilde{\varphi}|b\rangle\langle b|V|a\rangle\langle a|\varphi\rangle$; it should be a rapidly decreasing function on the product.

So [redacted] the problem is to understand the significance of the function

$$\frac{e^{iE_{ba}t} - e^{iE_{ba}t'}}{iE_{ba}} = \text{[redacted]} \int_{t'}^{t} dt_1 \, e^{+iE_{ba}t_1}$$

The point is that this, as $t \to +\infty$, $t' \to -\infty$, is

$$2\pi \, \delta(E_{ba})$$

This is what's behind the formula

$$\begin{aligned} \langle b | S | a \rangle &= \langle b | 1 - i \int dt_1 \, V(t_1) | a \rangle \\ &= \langle b | a \rangle - i 2\pi \, \delta(E_b - E_a) \, V_{ab} \\ &= \delta(b-a) - 2\pi i \, \delta(E_b - E_a) \, V_{ab} \end{aligned}$$

October 11, 1980

Still the problem of transition probability per unit time:
Suppose $H = H_0 + V$ where V is very gentle and H_0 has continuous spectrum: $H_0 |a\rangle = E_a |a\rangle$. In the Born approx. the ~~time~~-development operator in the interaction picture is

$$U(t,t') = 1 - i \int_{t'}^{t} dt_1\, e^{iH_0 t_1} V e^{-iH_0 t_1}$$

and its matrix elements are

$E_{ba} = E_b - E_a$
$V_{ba} = \langle b|V|a\rangle$

$$\langle b|U(t,t')|a\rangle = \langle b|a\rangle - i\, \frac{e^{iE_{ba}t} - e^{iE_{ba}t'}}{iE_{ba}}\, V_{ba}$$

This is a nice function of b, a off the diagonal $b = a$.
Its time derivative is

$$\frac{d}{dt}\langle b|U(t,t')|a\rangle = -i\langle b|e^{iH_0 t} V e^{-iH_0 t}|a\rangle$$
$$= -i\, e^{iE_{ba}t}\, V_{ba}$$

Then

$$\frac{d}{dt}\left|\langle b|U(t,t')|a\rangle\right|^2 = 2\,\mathrm{Re}\left\{\frac{d}{dt}\langle b|U(t,t')|a\rangle\ \overline{\langle b|U(t,t')|a\rangle}\right\}$$

$E_{ba} = 0$ when $b = a$

$$= 2\,\mathrm{Re}\left\{-ie^{iE_{ba}t}\, V_{ba}\, \overline{\langle b|a\rangle}\right\} + 2\,\mathrm{Re}\left\{|V_{ba}|^2\, \frac{1 - e^{iE_{ba}(t-t')}}{-iE_{ba}}\right\}.$$

$$= 2\,\mathrm{Im}(V_{ba})\, \delta(b-a) + 2|V_{ba}|^2\, \frac{\sin\left(E_{ba}(t-t')\right)}{E_{ba}}$$

So far I haven't used continuous spectrum. What I have done agrees with previous calculations:

(Note $\mathrm{Im}(V_{aa}) = 0$ if V is hermitian)

$$\left|\langle b|U(t,0)|a\rangle\right|^2 = |V_{ba}|^2\, \frac{\sin^2\left(\frac{E_{ba}t}{2}\right)}{\left(\frac{E_{ba}}{2}\right)^2}$$

In the continuous case we have as $t' \to -\infty$

$$\frac{\sin E_{ba}(t-t')}{E_{ba}} \longrightarrow$$ [redacted] $\pi\,\delta(E_b - E_a)$

as distributions on b, a space. Thus [redacted] in the Born approximation:

$$\lim_{t' \to -\infty} \frac{d}{dt} |\langle b | u(t,t') | a \rangle|^2 = 2\,\mathrm{Im}(V_{ba})\,\delta(b-a) + 2\pi |V_{ba}|^2 \delta(E_b - E_a)$$

Let's work to higher order

$$S = 1 - i \int_{-\infty}^{\infty} dt_1 V(t_1) + (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2\, V(t_1) V(t_2) + \cdots\cdots$$

Then

$$\langle b | S^{(2)} | a \rangle = (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \sum_c e^{iE_b t_1} V_{bc} e^{-iE_c t_1} e^{iE_c t_2} V_{ca} e^{-iE_a t_2}$$

$$= (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \sum_c e^{i(E_b - E_c)t_1} V_{bc}\, e^{i(E_c - E_a)t_2} V_{ca}$$

$$= (-i)^2 \int_{-\infty}^{\infty} dt_1 \sum_c e^{i(E_b - E_c)t_1} V_{bc} \frac{e^{i(E_c - E_a)t_1}}{i(E_c - E_a - i0^+)} V_{ca}$$

$$= (-i)^2 \int_{-\infty}^{\infty} dt_1\, e^{i(E_b - E_a)t_1} \sum_c V_{bc} \frac{1}{i(E_c - E_a - i0^+)} V_{ca}$$

$$= -2\pi i\, \delta(E_b - E_a) \sum_c V_{bc} \frac{1}{E_a + i0^+ - E_c} V_{ca}$$

Similarly

$$\langle b | S^{(3)} | a \rangle = -2\pi i\, \delta(E_b - E_a) \sum_{c,d}{}' V_{bc} \frac{1}{E_a + i0^+ - E_c} V_{cd} \frac{1}{E_a + i0^+ - E_d} V_{da}$$

and so we obtain the formula

$$\langle b|S|a\rangle = \delta(b-a) - 2\pi i\, \delta(E_b - E_a)\, T_{ba}$$

where $\quad T_{ba} = \langle b|T(E_a + i0^+)|a\rangle \quad$ and

$$T(E) = V + V\frac{1}{E-H_0}V + V\frac{1}{E-H_0}V\frac{1}{E-H_0}V + \cdots$$

Also

$$U(0,-\infty) = 1 - i\int_{-\infty}^{0} dt_1\, V(t_1) + (-i)^2\int_{-\infty}^{0} dt_1 \int_{-\infty}^{0} dt_2\, V(t_1)V(t_2) + \cdots$$

can be handled the same way to get

$$U(0,-\infty)|a\rangle = |a\rangle + \frac{1}{E_a + i0^+ - H_0}V|a\rangle + \left(\frac{1}{E_a + i0^+ - H_0}V\right)^2|a\rangle + \cdots$$

In order to understand what this means we have to think in terms of distributions. The operator $U(t,-\infty)$ has matrix "elements" $\langle b|U(t,-\infty)|a\rangle$ which should be thought of as a distribution on b,a-space. In good situations it should be possible to fix $|a\rangle$ and interpret $U(t,-\infty)|a\rangle$ as a distribution with "value" $\langle b|U(t,-\infty)|a\rangle$ at b.

Each term of

$$U(t,-\infty)|a\rangle = |a\rangle + (-i)\int_{-\infty}^{t} dt_1\, e^{iH_0t_1}Ve^{-iH_0t_1}|a\rangle$$

$$+ (-i)^2\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\, e^{iH_0t_1}Ve^{-iH_0t_1}e^{iH_0t_2}Ve^{-iH_0t_2}|a\rangle$$

$$+ \cdots$$

can perhaps be interpreted as a distribution. For example let's take the 2nd degree term and replace V by $Ve^{\varepsilon t}$

$$(-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \; e^{iH_0 t_1} V \overset{e^{\varepsilon t_1}}{\wedge} e^{-iH_0 t_1} \, e^{i(H_0-E_a)t_2+\varepsilon t_2} V|a\rangle$$

$$= (-i)^{[redacted]} \int_{-\infty}^{t} dt_1 \; e^{iH_0 t_1} V e^{\overset{2}{\downarrow}\varepsilon t_1 - iH_0 t_1} \, e^{i(H_0 - E_a - i\varepsilon)t_1} \frac{1}{E_a + i\varepsilon - H_0} V|a\rangle$$

$$= e^{i(H_0-E_a-2i\varepsilon)t} \frac{1}{E_a+2i\varepsilon-H_0} V \frac{1}{E_a+i\varepsilon-H_0} V|a\rangle$$

Thus it seems that as a distribution we get

$$U(t,-\infty)|a\rangle = e^{i(H_0-E_a)t}\left(1+G^0(E_a^{+i0^+})V + (G^0(E_a+i0^+)V)^2 + \dots\right)|a\rangle$$

or $\quad U(t,-\infty)|a\rangle = e^{i(H_0-E_a)t}\,\psi_a^+ \qquad G_0(E) = \frac{1}{E-H_0}$

It's probably useful to remember that Gelfand has a theory of "rigged" Hilbert spaces whose point is to make sense of [redacted] eigenfunctions like ψ_a^+.

Now [redacted]

$$\langle b|U(t,-\infty)|a\rangle = e^{i(E_b-E_a)t}\langle b|\psi_a^+\rangle$$

and the point is that $\langle b|\psi_a^+$ is a distribution of b with singular support contained in the set where $E_b = E_a$. (You can see this by

$$\psi_a^+ = |a\rangle + G_0(E_a+i0^+)\,T(E_a+i0^+)|a\rangle$$

$$\langle b|\psi_a^+\rangle = \delta(b-a) + \frac{1}{E_a+i0^+-E_b} T_{ba}$$

and the fact that T tends to be nice and smooth.)

Since $\langle b|\psi_a^+\rangle$ has singularities on $E_b = E_a$ it follows that $\quad e^{i(E_b-E_a)t}\langle b|\psi_a^+\rangle$

can grow with t in a certain sense. I can see why

$$|\langle b|U(t,-\infty)|a\rangle|^2$$

is meaningless.

However its derivative does have a meaning.

$$\langle b|U(t,-\infty)|a\rangle = e^{i(E_b-E_a)t}\underbrace{\langle b|\psi_a^+\rangle}_{\left\{\delta(b-a)+\frac{1}{E_a+i0^+-E_b}T_{ba}\right\}}$$

$$\frac{d}{dt}\langle b|U(t,-\infty)|a\rangle \text{ [redacted]}$$

$$= \text{[redacted] } e^{i(E_b-E_a)t}\left\{-iT_{ba}\right\}$$

[redacted] Since the latter is smooth we can form the product

$$2\,\mathrm{Re}\left\{\frac{d}{dt}\langle b|U(t,-\infty)|a\rangle\cdot\overline{\langle b|U(t,-\infty)|a\rangle}\right\}$$

$$\overset{2\mathrm{Re}}{=}\left(-iT_{ba}\right)\left(\delta(b-a)+\frac{1}{E_a-i0^+-E_b}\overline{T_{ba}}\right)$$

$$= 2\,\mathrm{Im}(T_{aa})\,\delta(b-a) + 2\,\mathrm{Im}\left(\frac{1}{E_a-i0^+-E_b}|T_{ba}|^2\right)$$

$$= 2\,\mathrm{Im}(T_{aa})\,\delta(b-a) + 2\pi\,\delta(E_a-E_b)|T_{ba}|^2$$

which is formally $\frac{d}{dt}|\langle b|U(t,-\infty)|a\rangle|^2$. [redacted]

[redacted] Schweber writes

$$\lim_{t'\to-\infty}\frac{d}{dt}|\langle b|U(t,t')|a\rangle|^2$$

but technically the quantity $|\langle b|U(t,t')|a\rangle|^2$ is undefined, for example, if $V=0$ one has difficulty with $\delta(b-a)^2$.

October 12, 1980 (Becky is 14)

I want to work out the S matrix using frequency rather than time integrals. Recall

$$S = 1 + (-i)\int dt_1\, \tilde{V}(t_1) + \frac{(-i)^2}{2!}\int dt_1 \int dt_2\, T\{\tilde{V}(t_1)\tilde{V}(t_2)\} + \cdots$$

where $\tilde{V}(t) = e^{iH_0 t} V(t) e^{-iH_0 t}$. Suppose

$$V(t) = g(t)\, V$$

where g is something like [redacted] $e^{-\varepsilon |t|}$. Then

$$\langle b | (-i)\int dt_1\, g(t_1)\, e^{iH_0 t_1} V e^{-iH_0 t_1} | a\rangle$$

$$= (-i)\int dt_1 \left(\int \frac{d\omega_1}{2\pi} e^{-i\omega_1 t_1}\hat{g}(\omega)\right) e^{iE_b t_1 - iE_a t_1} \langle b | V | a\rangle$$

$$= \int \frac{d\omega_1}{2\pi}\hat{g}(\omega_1)\,(-i)\,2\pi\,\delta(-\omega_1 + E_b - E_a)\langle b|V|a\rangle$$

$$= -i\,\hat{g}(E_b - E_a)\,\langle b|V|a\rangle \longrightarrow -2\pi i\,\delta(E_b - E_a)\langle b|V|a\rangle$$

[redacted] $(-i)^2 \int dt_1 \int^{t_1} dt_2 \int \frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi} e^{-i\omega_1 t_1 - i\omega_2 t_2}\hat{g}(\omega_1)\hat{g}(\omega_2)\, e^{iE_b t_1} V_{bc}$

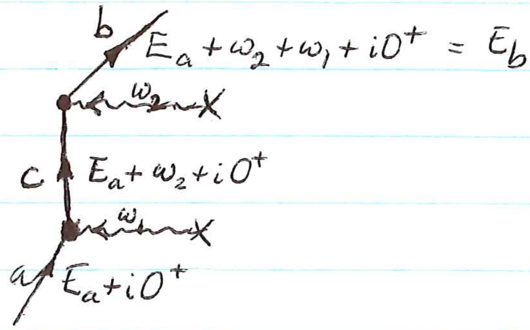
$$\times\, e^{-iE_c t_1} e^{iE_c t_2} V_{ca} e^{-iE_a t_2}$$

$$= (-i)^2 \int \frac{d\omega_1\, d\omega_2}{(2\pi)^2}\hat{g}(\omega_1)\hat{g}(\omega_2)\sum_c \int dt_1\, e^{-i\omega_1 t_1 + i(E_b - E_c)t_1} V_{bc} \frac{e^{i(-\omega_2 + E_c - E_a - i0^+)t_1}}{i(-\omega_2 + E_c - E_a - i0^+)} V_{ca}$$

$$= (-i)^2 \int \frac{d\omega_1\, d\omega_2}{(2\pi)^2}\hat{g}(\omega_1)\hat{g}(\omega_2)\sum_c \delta(-\omega_1 + E_b - E_a - \omega_2)\, V_{bc}\frac{1}{(-i)(E_a + \omega_2 + i0^+ - E_c)} V_{ca}$$

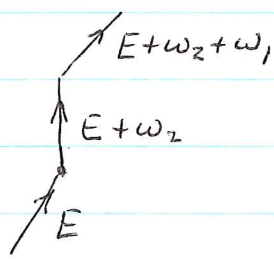
We can interpret this via diagrams as follows. One starts in the state $|a\rangle$, then hits a vertex contributing $\hat{g}(\omega_2)\, V_{ca}$. The edge to the next vertex contributes $\dfrac{1}{E_a + \omega_2 + i0^+ - E_c}$

and the next vertex contributes $\hat{g}(\omega_1) V_{bc}$. Finally one has a factor $\delta(E_b - (E_a + \omega_2 + \omega_1))$ because the final energy must be E_b.



It seems that one should first look at diagrams where the energy of the external lines isn't fixed.

(*)



This contributes the operator

$$\frac{1}{E + \omega_2 + \omega_1 - H_0} \hat{V}_{\omega_1} \frac{1}{E + \omega_2 - H_0} \hat{V}_{\omega_2} \frac{1}{E - H_0}$$

Here I am considering a general $V(t)$, not just $g(t)V$, and I have

$$V(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{V}_\omega$$

Notice that when V is constant, then $\hat{V}_\omega = 2\pi\delta(\omega) V$ and so we are getting the T-matrix.

The question is whether there is any significance to the general diagrams (*). For example where V is stationary then we get

$$\frac{1}{E - H_0} + \frac{1}{E - H_0} V \frac{1}{E - H_0} + \cdots = \frac{1}{E - H}$$

October 13, 1980.

[redacted] The following series

$$G(E) = \frac{1}{E-H_0} + \int \frac{d\omega_1}{2\pi} \frac{1}{\omega_1+E-H_0} \hat{V}(\omega_1) \frac{1}{E-H_0}$$

$$+ \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \frac{1}{\omega_1+\omega_2+E-H_0} \hat{V}(\omega_1) \frac{1}{\omega_2+E-H_0} \hat{V}(\omega_2) \frac{1}{E-H_0}$$

$$+ \cdots\cdots$$

looks extremely natural. It would be nice to find a good interpretation for it. Note that if V is independent of t, then it is the series

$$G(E) = \frac{1}{E-H_0} + \frac{1}{E-H_0} V \frac{1}{E-H_0} + \cdots\cdots$$

$$= \frac{1}{E-H}$$

for the resolvent of $H = H_0 + V$. Is it possible to obtain the S matrix from $G(E)$.

[redacted] The first order term in $\langle b|S|a\rangle$ is

$$(-i)\int dt \langle b| e^{iH_0t} \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{V}(\omega) e^{-iH_0t} |a\rangle$$

$$= \int \frac{d\omega}{2\pi} (-i) \int dt\, e^{-i(E_b-\omega-E_a)t} \langle b|\hat{V}(\omega)|a\rangle$$

$$= \text{[redacted]} \int \frac{d\omega}{2\pi} (-2\pi i)\, \delta(E_b-\omega-E_a) \langle b|\hat{V}(\omega)|a\rangle$$

So the problem of getting the S-matrix involves going from

$$\frac{1}{\omega_{\text{[redacted]}}+E-E_b} \frac{1}{E-E_a} \quad \text{to } (-2\pi i)\,\delta(\omega+E_a-E_b)$$

Here's a possible interpretation of $G(E)$.

$$U(0,t) = e^{iH_0 t} + (-i)\int_t^0 dt_1\, U(0,t_1)\, V(t_1)\, \underbrace{e^{-iH_0(t_1-t)}}_{e^{iH_0(t-t_1)}}$$

where U is the propagator for the Schroedinger equation. Here $t \leq 0$.

$$\int_{-\infty}^0 dt\, e^{iH_0 t} e^{-iEt} = \frac{1}{iH_0 - iE} = \frac{i}{E + i0^+ - H_0}$$

Thus if we put

$$G(E) = (-i)\int_{-\infty}^0 dt\, U(0,t)\, e^{-iEt} \qquad E \in UHP$$

we get the integral equation

$$iG(E) = \frac{i}{E-H_0} + (-i)\int \frac{d\omega}{2\pi}\, iG(E+\omega)\, \hat{V}(\omega)\, \frac{i}{E-H_0}$$

or

$$\boxed{G(E) = \frac{1}{E-H_0} + \int \frac{d\omega}{2\pi}\, G(E+\omega)\, \hat{V}(\omega)\, \frac{1}{E-H_0}}$$

October 20, 1980 (Cindy born Oct. 17)

Consider a forced oscillator

$$H = H_0 + a_J^* + a_{J^*}$$

where H_0 is a self-adjoint op. on a Hilbert space $\mathcal{H}$ which extended to the bosen space of $\mathcal{H}$, and where $J = J(t)$ is a vector in $\mathcal{H}$ depending on time. To the first order in J, the S-matrix is

$$S = 1 - i(a_\gamma^* + a_{\gamma^*})$$

where

$$\gamma = \int dt\, e^{iH_0 t} J(t).$$

In order to understand S we want to expand J in terms of frequency and the eigenvectors of H_0. Let $|\alpha\rangle$ be a orthonormal system of eigenvectors for H_0 and

$$H_0 |\alpha\rangle = \varepsilon_\alpha |\alpha\rangle \qquad \int d\alpha\ |\alpha\rangle\langle\alpha| = id$$

Then we can write $\quad \langle\alpha|J(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{J}(\alpha,\omega)$

$$\begin{aligned}\langle\alpha|\gamma &= \int dt\, e^{i\varepsilon_\alpha t} \langle\alpha|J(t) \\ &= \int dt \frac{d\omega}{2\pi} e^{i(\varepsilon_\alpha - \omega)t} \hat{J}(\alpha,\omega) \\ &= \hat{J}(\alpha, \varepsilon_\alpha)\end{aligned}$$

In other words if J is described by the distribution $\hat{J}(\alpha,\omega)$ via the formula:

$$J(t) = \int d\alpha \frac{d\omega}{2\pi} |\alpha\rangle e^{-i\omega t} \hat{J}(\alpha,\omega)$$

then $\quad \gamma = \int d\alpha\ |\alpha\rangle \hat{J}(\alpha, \varepsilon_\alpha).$

Thus the singularities in γ result from the singularities of $\hat{J}(\alpha, \omega)$ and pulling back via the section

$$\alpha \longmapsto (\alpha, \varepsilon_\alpha)$$

Let's consider some examples. Suppose $J(t) = e^{-i\omega_0 t} v$. Then

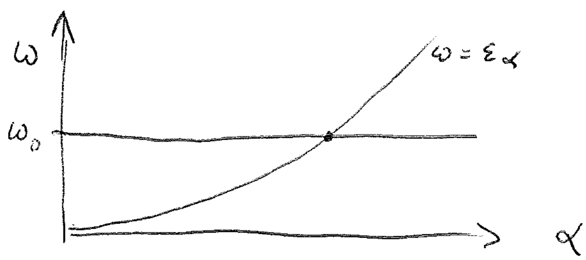
$$\hat{J}(\alpha, \omega) = \langle \alpha | v \rangle 2\pi \delta(\omega - \omega_0)$$

and ~~so~~

$$\gamma = \int d\alpha \; |\alpha\rangle \langle \alpha | v \rangle \; 2\pi \, \delta(\varepsilon_\alpha - \omega_0)$$

$$= 2\pi \, \delta(H_0 - \omega_0) \, v$$

which is not in L^2. Picture:



Another case is where $|\alpha\rangle$ is a discrete spectrum and $\hat{J}(\alpha, \omega)$ is smooth in the ω-direction. Then

$$\gamma = \sum_\alpha |\alpha\rangle \hat{J}(\alpha, \varepsilon_\alpha)$$

is a nice L^2-vector.

What I should really understand is the case where J is constant, say $J = v$, but where H_0 has 0 part of its continuous spectrum. One would hope that the S matrix would exist, because one has a constant perturbation. But it's clear that it doesn't.

October 22, 1980

$$H = a^*\omega a + a^*J + J^*a$$

where a is a column vector, a^* a row vector, and ω a matrix.

$$H = \left(a^* + J^*\frac{1}{\omega}\right)\omega\left(a + \frac{1}{\omega}J\right) - J^*\frac{1}{\omega}J$$

One would like H to be equivalent unitarily to $H_0 = a^*\omega a$. The unitary operator implementing this equivalence should be

$$\Omega^+ = U_D(0,-\infty) = \text{const}\cdot e^{-ia^*\gamma}\,e^{-i\gamma^* a}$$

where

$$i\gamma = \int_{-\infty}^{0} dt\, i e^{i\omega t} J = \frac{1}{\omega - i0^+}J$$

and the constant is

$$\langle 0|\Omega^+|0\rangle = \exp\left\{-\iint_{0\geq t>t'>-\infty} dt\,dt'\; J^* e^{-i\omega(t-t')}J\right\}.$$

Notice that

$$\int_{-\infty}^{t} dt'\; e^{-i\omega(t-t')}J = \int_{-\infty}^{0} du\, e^{+i\omega u}J = \frac{1}{i(\omega - i0^+)}J$$

and consequently

$$\int_{-\infty}^{0} dt \int_{-\infty}^{t} dt'\; J^* e^{-i\omega(t-t')}J = \int_{-\infty}^{0} dt\; J^*\frac{1}{i(\omega - i0^+)}J$$

We start with a self-adjoint operator ω on $\mathcal{H}$ and a vector J in $\mathcal{H}$. Then I form the Fock space of $\mathcal{H}$ and extend ω as a derivation to get

$$H_0 = a^* \omega a$$

Here I think of $\mathcal{H}$ as provided with an orthonormal basis $|\alpha\rangle$ and a as the column vector (a_α) and a^* as the row vector (a^*_α).

Next we consider the perturbation

$$H = a^*\omega a + a^* J + J^* a .$$

We want to apply scattering theory to this perturbation. So we look at whether the wave operator

$$\Omega^+ = U_D(0, -\infty)$$

exists. We use the adiabatic method which replaces J by $J(t) = e^{\varepsilon t} J$ and then lets $\varepsilon \downarrow 0$. We have

$$\Omega^+ = T\left\{ e^{-i\int_{-\infty}^{0} dt\,(a^*(t)J(t) + J(t)^* a(t))} \right\}$$

Computation shows

$$\boxed{\Omega^+ = e^{-w}\, e^{-a^*\gamma}\, e^{\gamma^* a}}$$

where

$$\gamma = \int_{-\infty}^{0} dt\, i e^{i\omega t} J(t) = \int_{-\infty}^{0} dt\, i\, e^{(i\omega+\varepsilon)t} J$$

$$\boxed{\gamma = \frac{1}{\omega - i\varepsilon} J}$$

and

$$w = + \int_{-\infty}^{0} dt \int_{-\infty}^{t} dt'\, J(t)^* e^{-i\omega(t-t')} J(t')$$

$$= + \int_{-\infty}^{0} dt \int_{-\infty}^{t} dt' \; J^* e^{(\varepsilon - i\omega)t} \; e^{(i\omega+\varepsilon)t'} J$$

$$= + \int_{-\infty}^{0} dt \; J^* \; e^{(\varepsilon - i\omega)t} \; e^{(i\omega+\varepsilon)t} \frac{1}{i\omega+\varepsilon} J$$

$$\boxed{w = + \frac{1}{2\varepsilon} J^* \frac{1}{i\omega + \varepsilon} J}$$

or

$$w = + \frac{1}{2\varepsilon} J^* \frac{\varepsilon - i\omega}{\omega^2 + \varepsilon^2} J$$

Thus

$$Re(w) = \frac{1}{2} J^* \frac{1}{\omega^2+\varepsilon^2} J = \frac{1}{2} \gamma^* \gamma$$

as it should be.

Next I want to let $\varepsilon \downarrow 0$. The first case to look at is where $\frac{1}{\omega} J \in \mathcal{H}$. To be more precise we want to assume that 0 is not in the discrete spectrum of ω and that J is in [redacted] $\omega(\mathcal{D}_{\mathcal{H}})$. Then

$$\lim_{\varepsilon \to 0} \gamma = \frac{1}{\omega} J$$

However the imaginary part of w is

$$Im(w) = - \frac{1}{2\varepsilon} J^* \frac{\omega}{\omega^2 + \varepsilon^2} J$$

and this becomes infinite unless $J^* \frac{1}{\omega} J = 0$.

Nevertheless

$$U = e^{-\frac{1}{2} J^* \frac{1}{\omega^2} J} \; e^{-a^* \frac{1}{\omega} J} \; e^{-J^* \frac{1}{\omega} a}$$

is a well-defined unitary operator which differs from the hypothetical Ω^+ by a scalar of modulus 1. We have

[redacted] $e^{ta} e^{a^* \gamma} = e^{a^* \gamma} e^{ta} \underbrace{e^{[ta, a^* \gamma]}}_{e^{t\gamma}}$

hence
$$a e^{a^*\gamma} = e^{a^*\gamma}(a+\gamma)$$

Thus
$$\mathcal{U} a \mathcal{U}^{-1} = a + \frac{1}{\omega} J$$
$$\mathcal{U} a^* \mathcal{U}^{-1} = a^* + J^* \frac{1}{\omega}$$

and so
$$\mathcal{U} H_0 \mathcal{U}^{-1} = \left(a^* + J^*\frac{1}{\omega}\right)\omega\left(a + \frac{1}{\omega}J\right)$$
$$= \underbrace{a^*\omega a + J^*a + a^*J}_{H_{\text{[redacted]}}} + J^*\frac{1}{\omega}J$$

Therefore when $\frac{1}{\omega}J$ exists in $\mathcal{H}$, we see that the perturbed operator H is unitarily equivalent to $H_0 + \Delta E_0$, where ΔE_0 is the ground energy shift:
$$\Delta E_0 = -J^*\frac{1}{\omega}J$$

However the wave operator Ω^+ doesn't exist when this $\Delta E_0 \neq 0$.

Suppose however that $\Delta E_0 = 0$. Then we have

$$w = \frac{1}{2\varepsilon} J^*\left[\frac{1}{i\omega+\varepsilon} - \frac{1}{i\omega}\right] J$$
$$= \frac{1}{2} J^*\left[\frac{-1}{(i\omega+\varepsilon)(i\omega)}\right] J \longrightarrow \frac{1}{2} J^*\frac{1}{\omega^2}J$$

and so Ω^+ exists (at least adiabatically) and is [redacted] equal to the operator $\mathcal{U}$. It seems then that in this case the standard scattering theory picture is valid, although the S-matrix is trivial. [redacted]

Next I should look at the S-matrix in the adiabatic approximation.

[redacted] $$U_D(\infty,0) = e^{-w'} e^{-a^*\gamma'} e^{+\gamma'^* a}$$

where $$\gamma' = i\int_0^\infty dt\, e^{i\omega t} e^{-\varepsilon t} J = i \frac{-1}{i\omega - \varepsilon} J = \frac{-1}{\omega + i\varepsilon} J$$

and $$w' = \underbrace{\int_0^\infty dt \int_0^{t'} dt'}_{\int_0^\infty dt' \int_{t'}^\infty dt} J^* e^{-\varepsilon t} e^{-i\omega t} e^{i\omega t'} e^{-\varepsilon t'} J$$

$$\gamma = \frac{1}{\omega - i\varepsilon} J$$

$$= J^* \int_0^\infty dt' \frac{e^{-(\varepsilon + i\omega)t'}}{\varepsilon + i\omega} e^{(i\omega - \varepsilon)t'} J$$

$$w' = \frac{1}{2\varepsilon} J^* \frac{1}{\varepsilon + i\omega} J \qquad \text{same as } w^-$$

Thus
$$S = U_D(\infty,0)\, U_D(0,-\infty) = e^{-w'} e^{-a^*\gamma'} e^{+\gamma'^* a} e^{-w^-} e^{-a^*\gamma} e^{+\gamma^* a}$$

$$= e^{-w' - w^- [redacted]\, \gamma'^*\gamma} e^{-a^*(\gamma' + \gamma)} e^{+(\gamma' + \gamma)^* a}$$

Now $$\gamma' + \gamma = \left(\frac{-1}{\omega + i\varepsilon} + \frac{1}{\omega - i\varepsilon}\right) J = \frac{2i\varepsilon}{\omega^2 + \varepsilon^2} J \longrightarrow 2\pi i \delta(\omega) J$$

[redacted] which is zero when $\frac{1}{\omega} J$ exists. [redacted] Also

$$w' + w^- [redacted] + \gamma'^*\gamma = \frac{1}{\varepsilon} J^* \overbrace{\frac{\varepsilon - i\omega}{\omega^2 + \varepsilon^2}}^{\frac{1}{\varepsilon + i\omega}} J \; [redacted] \; J^* \frac{1}{(\omega - i\varepsilon)^2} J$$

[redacted] $$= \frac{1}{\varepsilon} J^* \frac{1}{[redacted]\, i\omega} J + \frac{1}{\varepsilon} J^* \Big[\underbrace{\frac{1}{\varepsilon + i\omega} - \frac{1}{i\omega}}_{\frac{+\varepsilon}{\omega(\omega - i\varepsilon)}}\Big] J \; [redacted] \; J^* \frac{1}{(\omega - i\varepsilon)^2} J$$

$$= \frac{1}{\varepsilon} J^* \frac{1}{i\omega} J + \underbrace{J^* \Big[\frac{1}{\omega(\omega - i\varepsilon)} \; [redacted] \; \frac{1}{(\omega - i\varepsilon)^2}\Big] J}_{\text{goes to zero as } \varepsilon \downarrow 0 \text{ assuming } 1/\omega J \text{ exists.}}$$

Therefore we see that in the good case $\frac{1}{\omega}J \in \mathcal{H}$ $J^*\frac{1}{\omega}J = 0$, that the S-matrix is the identity.

Summary: We have been looking at the case where $\frac{1}{\omega}J \in \mathcal{H}$. If $\Delta E_0 = -J^*\frac{1}{\omega}J = 0$, then the good-potential-scattering situation holds, i.e. the wave operators exist:

$$\Omega^+ = e^{-\frac{1}{2} J^*\frac{1}{\omega^2}J}\, e^{-a^*\frac{1}{\omega}J}\, e^{J^*\frac{1}{\omega}a}$$

$$(\Omega^-)^* = e^{-\frac{1}{2} J^*\frac{1}{\omega^2}J}\, e^{a^*\frac{1}{\omega}J}\, e^{-J^*\frac{1}{\omega}a}$$

However the scattering is trivial, i.e. $S = id$. If $\Delta E_0 = -J^*\frac{1}{\omega}J \neq 0$, then the wave operators don't exist, but the operators on the right above do exist and intertwine H_0 with $H - \Delta E_0$. In some sense Ω^+ differs by $e^{\frac{1}{2}i(\Delta E_0)\infty}$ from the expression on the right. Similarly the S operator is $e^{i(\Delta E_0)\infty}$, so the scattering is trivial

Now the problem becomes to understand what is going on when $\frac{1}{\omega}J$ is not in $\mathcal{H}$.

(Remark: In the preceding I ignored rigorous problems like whether $\frac{1}{\omega - i\varepsilon}J \longrightarrow \frac{1}{\omega}J$. My feeling is that this sort of thing can be handled by restricting to the cyclic subspace for ω generated by J, whence one has a spectral measure on $\mathbb{R}$ to describe things.)