September 4, 1980

Review: We consider a transmission line

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}$$

of length l with periodic boundary conditions. The modes are

$$V = \mathrm{Re}\left(\hat{V}\, e^{-i(kx-\omega_k t)}\right) \qquad \omega_k = c|k| \qquad c = \frac{1}{\sqrt{LC}}$$

$$I = \mathrm{Re}\left(\hat{I}\, e^{-i(kx-\omega_k t)}\right) \qquad \frac{\hat{V}}{\hat{I}} = \sqrt{\frac{L}{C}}\,\frac{k}{|k|}$$

The energy of this mode is computed as follows. Put $t=0$

$$\int \tfrac{1}{2} C V^2\, dx = \tfrac{1}{2} C \int \left(\frac{\hat{V}e^{ikx} + \bar{\hat{V}}e^{-ikx}}{4}\right)^2 dx = \tfrac{1}{4} C |\hat{V}|^2 l$$

$$\int \tfrac{1}{2} L I^2\, dx = \tfrac{1}{4} L |\hat{I}|^2 l$$

so adding

$$E = \tfrac{1}{2} C |\hat{V}|^2 l = \tfrac{1}{2} L |\hat{I}|^2 l$$

More generally (see yesterday) the energy of

$$V = \mathrm{Re}\left(\sum_k \hat{V}_k\, e^{i(kx-\omega_k t)}\right)$$

is

$$E = \tfrac{1}{2} C l \sum_k |\hat{V}_k|^2$$

The curious thing is that the energy of a mode doesn't depend on its frequency.

Consider for contrast the 1-dimensional crystal:

$$m\ddot{q}_x = K(q_{x+a} - 2q_x + q_{x-a}) \qquad x \in \mathbb{Z}a/\mathbb{Z}\underbrace{Na}_{l}$$

The modes are $\quad q_x = e^{i(kx-\omega t)} \quad$ where

$$-m\omega^2 = K\left(e^{ika} - 2 + e^{-ika}\right) = K\,2(\cos ka - 1)$$

$$\omega^2 = \frac{K}{m}\, 4 \sin^2\!\left(\frac{ka}{2}\right) = \frac{Ka^2}{m}\left(\frac{\sin\left(\frac{ka}{2}\right)}{a/2}\right)^2$$

In the continuum limit $\frac{m}{a} \to \rho$, $Ka \to \gamma$, $a \to 0$ and

$$\omega^2 = \frac{\gamma}{\rho} k^2 \quad \text{or} \quad \omega = c|k| \qquad c = \sqrt{\frac{\gamma}{\rho}}$$

The energy of the mode $q_x = \mathrm{Re}\, A e^{i(kx-\omega t)}$ is found as follows. Put $t=0$.

$$K.E. = \sum_x a \frac{m}{2a}\left[\mathrm{Re}\left(-i\omega A e^{ikx}\right)\right]^2 \longrightarrow \int \frac{\rho}{2}\omega^2\left[\mathrm{Re}\left(-iAe^{ikx}\right)\right]^2 dx$$

$$P.E. = \sum_x a \frac{Ka}{2}\left[\mathrm{Re}\left(A\left(\frac{e^{ika}-1}{a}\right)e^{ikx}\right)\right]^2 \longrightarrow \int \frac{\gamma}{2} k^2\left[\mathrm{Re}\left(iAe^{ikx}\right)\right]^2 dx$$

$$K.E. = \text{[redacted]} \ \frac{1}{2}\rho\,\omega^2 |A|^2 \frac{\ell}{2}$$

$$P.E. = \frac{1}{2}\gamma k^2 |A|^2 \frac{\ell}{2}$$

Same

Thus the energy of the mode is $\frac{1}{2}\rho\,\omega^2|A|^2\ell$.
More generally for $q_x = \mathrm{Re}\sum A_k e^{i(kx-\omega_k t)}$
one has

$$E = \frac{1}{2}\rho \sum_k \omega_k^2 |A_k|^2 \ell,$$

so the energy of a [redacted] mode is proportional to the square of its frequency.

[redacted] However, whether the energy of a mode is proportional to a function of the frequency is probably not an intrinsic thing i.e. it depends on the choice of coordinates.

According to Planck the energy of an oscillator can be arbitrarily large, but the values for the energy are restricted to $(n+\frac{1}{2})\hbar\omega$, where ω is the frequency of the oscillator. The average energy of the oscillator at temperature T is

$$\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} \sim k_B T \qquad \text{if } \beta\hbar\omega \text{ is small.}$$

Now let us consider a state of the transmission line:

$$V(x,t) = \mathrm{Re}\Big(\sum_{k\in\frac{2\pi}{\ell}\mathbb{Z}} A_k\, e^{i(kx-\omega_k t)}\Big)$$

Here the A_k are (generalized) "numbers" which are to be consistent with the line being at the temperature T. This means that

$$\frac{1}{2}C\ell\,|A_k|^2 = \frac{1}{2}\hbar\omega + \frac{\hbar\omega_k}{e^{\beta\hbar\omega_k}-1}$$

(Generalized numbers are [redacted] "random variables" or "operators", but perhaps in the spirit of Heisenberg & Dirac one should think of them as abstract quantities with properties to be determined.)

Suppose we interpret A_k as a family of independent Gaussian random variables. Thus we have $A_k(\xi)$ a sequence of numbers for each ξ in our ensemble $\mathcal{E}$. Let's look at the voltage at $x=0$

$$V(0,t)_\xi = \mathrm{Re}\Big(\sum A_k(\xi)\, e^{-i\omega_k t}\Big)$$

and compute its mean square value. Put $t=0$

$$\cancel{\langle V(0,t)_\xi^2\rangle = \frac{1}{4}\int \sum_{k,k'} A_k(\xi)A_{k'}(\xi)}$$

$$\langle V(0,0)^2_{\text{[redacted]}} \rangle = \int_{\mathcal{E}} Re\left(\sum A_k(\xi)\right)^2 d\xi = \frac{1}{4}\int_{\xi} \left(\sum A_k + \bar{A}_k\right)^2 d\xi$$

$$= \frac{1}{4}\int \sum_{k,k'} \left(A_k A_{k'} + 2A_k \bar{A}_{k'} + \bar{A}_k \bar{A}_{k'}\right) d\xi$$

$$= \frac{1}{4}\sum_k \int_{\mathcal{E}} \left(A_k^2 + 2A_k\bar{A}_k + \bar{A}_k^2\right) d\xi$$

because the A_k are ind. Gaussian variables. Now A_k is complex-valued with indep. real + imag. parts of the same variance (we assume this), so

$$\int A_k^2 \, d\xi = \int (z^2)\, e^{-\frac{|z|^2}{a}} dx dy/\pi$$

$$= \int (x^2 - y^2 + 2ixy)\, e^{-\frac{|z|^2}{a}} dxdy/\pi = 0$$

Thus we have

$$\boxed{\langle V(0,t)^2_{\text{[redacted]}} \rangle = \frac{1}{2}\sum_k \langle |A_k|^2 \rangle}$$

for $t=0$ and hence all t.

Now what I want to do is to compute the part of $\langle V(0,t)^2 \rangle$ which comes from the frequency range $(\omega, \omega + d\omega)$. Each mode in this range has

$$\frac{1}{2} C\ell \langle |A_k|^2 \rangle = k_B T \qquad \text{assume } \beta\hbar\omega \ll 1$$

$$\text{or } \langle |A_k|^2 \rangle = \frac{2}{C\ell} k_B T$$

On the other hand $\omega = c|k|$ and $k = \frac{2\pi}{\ell} n$ with $n \in \mathbb{Z}$. Thus we want those $n \in \mathbb{Z}$ with

$$\omega < c\frac{2\pi}{\ell}|n| < \omega + d\omega$$

The number is $dn = 2\,\frac{1}{c}\,\frac{\ell}{2\pi}\,d\omega = \frac{\ell}{c\pi}\,d\omega.$

Thus the total contribution to $\langle V(0,t)^2\rangle$ from the range $(\omega, \omega+d\omega)$ is

$$\frac{1}{2}\langle |A_k|^2\rangle \cdot dn = \frac{1}{\cancel{2}}\frac{\cancel{2}}{C\cancel{\ell}}\,k_B T\,\frac{\cancel{\ell}}{\cancel{c}\pi}\,d\omega \qquad c = \frac{1}{\sqrt{LC}}$$

$$= \sqrt{\frac{L}{C}}\cdot k_B T \cdot \frac{d\omega}{\pi}$$

Next example: Take a line of length ℓ with open ends

$$V(x,t) = \mathrm{Re}\sum_{k\in\frac{\pi}{\ell}\mathbb{N}}{}' A_k \cos kx\, e^{-ickt}$$

$$I(x,t) = \mathrm{Re}\sum{}' iA_k\sqrt{\frac{C}{L}}\,\sin kx\, e^{-ickt} \qquad \text{vanishes at ends.}$$

$$\frac{1}{2}C\int_0^\ell V^2\,dx = \frac{1}{2}C\sum_k{}' \frac{\ell}{2}\left(\mathrm{Re}\, A_k e^{-ickt}\right)^2$$

$$\frac{1}{2}L\int_0^\ell I^2\,dx = \frac{1}{2}L\sum_k{}' \frac{\ell}{2}\,\frac{C}{L}\left(\mathrm{Re}\, iA_k e^{-ickt}\right)^2$$

$$\text{total energy} = \frac{1}{4}C\ell\sum_{k\in\frac{\pi}{\ell}\mathbb{N}}|A_k|^2$$

$$\langle V(x,t)^2 \rangle = \sum_k \cos^2 kx \underbrace{\left\langle \left[\mathrm{Re}(A_k e^{-ickt}) \right]^2 \right\rangle}_{\frac{1}{2}\langle |A_k|^2 \rangle}$$

$$\therefore \quad \langle V(x,t)^2 \rangle = \frac{1}{2} \sum_k \langle |A_k|^2 \rangle \cos^2 kx$$

$$\langle I(x,t)^2 \rangle = \frac{1}{2}\frac{C}{L} \sum_k \langle |A_k|^2 \rangle \sin^2 kx$$

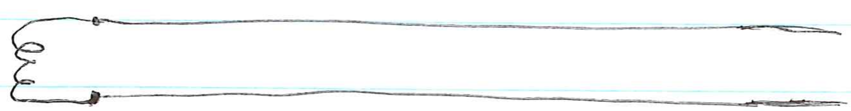
Let's compute the contribution to $\langle V(x,t)^2 \rangle$ due to modes in the frequency range $(\omega, \omega + d\omega)$. $\omega = ck = c\frac{\pi}{\ell} n$, $dn = \frac{\ell}{\pi c} d\omega$. [redacted] The k-th mode has total energy

$$\frac{1}{4} C\ell \langle |A_k|^2 \rangle = k_B T$$

Thus the part of $\langle V(x,t)^2 \rangle$ in $(\omega, \omega + d\omega)$ is

$$\frac{\ell}{\pi c} d\omega \, \frac{4 k_B T}{C\ell} \, \frac{1}{2} \cos^2 \frac{\omega x}{c} = \frac{2}{\pi} \sqrt{\frac{L}{C}} \, k_B T \, \cos^2\!\left(\frac{\omega x}{c}\right) d\omega$$

The next problem is to consider something like



and to determine the [redacted] mean square voltage across the inductance. More generally we should be able to handle more complicated circuits.

First we [redacted] know that the modes of the terminated line are of the form

$$V = \mathrm{Re}\left[A_k \left(e^{-ikx} + \frac{B(k)}{A(k)} e^{ikx} \right) e^{-ickt} \right] \qquad k > 0.$$

The $\frac{\text{energy}}{\text{length}}$ of this mode is

$$\underset{\substack{\uparrow \\ \text{electrostatic} \\ +\text{ magnetic}}}{2}\ \frac{1}{2} C \frac{1}{\ell} \int_0^\ell V^2 \, dx = \frac{1}{2} C \, |A_k|^2 \left(1 + \left|\frac{B(k)}{A(k)}\right|^2\right)$$

Next if

$$V(0,t) = \mathrm{Re}\left(\sum A_k \left(1 + \frac{B}{A}\right) e^{-ickt}\right)$$

where the A_k are ind. Gaussians, then

$$\langle V(0,t)^2 \rangle = \frac{1}{2} \sum_k \langle |A_k|^2 \rangle \left| 1 + \frac{B(k)}{A(k)} \right|^2$$

At the temp. T

$$\frac{1}{2} C \ell \langle |A_k|^2 \rangle \left(1 + \left|\frac{B}{A}\right|^2\right) = k_B T$$

so

$$\langle V(0,t)^2 \rangle = \sum_k \frac{k_B T}{C\ell} \frac{\left|1 + \frac{B}{A}\right|^2}{1 + \left|\frac{B}{A}\right|^2} .$$

We want the part of this in the frequency range $\omega, \omega + d\omega$, i.e. for ck in this range and with k satisfying the bdry condition

$$e^{-ik\ell} + \frac{B(k)}{A(k)} e^{ik\ell} = 0$$

If $d\omega$ is small the reflection coeff. doesn't change, so for large ℓ we get

$$e^{-2ik\ell} = \text{fixed point on } S^1 \qquad \text{say } e^{i\theta}$$

i.e. $2\ell k \in \theta + 2\pi\mathbb{Z}$

$$k \in \frac{\theta}{2\ell} + \frac{\pi}{\ell}\mathbb{Z}$$

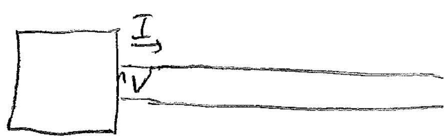
Thus the number of k with $ck \in (\omega, \omega + d\omega)$ is

$$dn = \frac{\ell}{c\pi}\, d\omega .$$

Hence the contribution to $\langle V(0,t)^2 \rangle$ due to this frequency range is

$$\frac{\ell}{c\pi} d\omega \cdot \frac{k_B T}{C\ell} \cdot \frac{\left|1+\frac{B}{A}\right|^2}{1+\left|\frac{B}{A}\right|^2} = \boxed{\underbrace{\sqrt{\frac{L}{C}}}_{R} \cdot k_B T \cdot \frac{\left|1+\frac{B}{A}\right|^2}{\underbrace{1+\left|\frac{B}{A}\right|^2}_{2}} \cdot \frac{d\omega}{\pi}}$$

In the above derivation I assumed $\left|\frac{B}{A}\right| = 1$ and maybe this is necessary. Now suppose the line is connected to a pure reactance



with impedance $Z(\omega)$. Then at $x=0$

$$V = \left(1+\frac{B}{A}\right) e^{-i\omega t}$$

$$I = \frac{1}{R}\left(-1+\frac{B}{A}\right) e^{-i\omega t}$$

$$R = \sqrt{\frac{L}{C}}$$

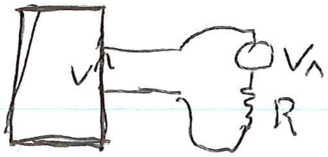
and

$$-Z = \frac{V}{I} = \frac{1+\frac{B}{A}}{-1+\frac{B}{A}} \qquad \frac{Z}{R} = \frac{1+\frac{B}{A}}{1-\frac{B}{A}} \qquad \frac{B}{A} = \frac{Z-R}{Z+R}$$

$$\left|1+\frac{B}{A}\right|^2 = \left|\frac{2Z}{Z+R}\right|^2 = \frac{4|Z|^2}{|Z|^2+R^2}$$

Thus $\underset{\omega,\, \omega+d\omega}{\text{part of in}} \langle V^2 \rangle = R\, k_B T \, \frac{4|Z|^2}{|Z|^2+R^2} \, \frac{d\omega}{2\pi}$

Finally if we think of the line as being a pure resistance R + noise V_n we have



$$\cancel{\hat{V} = \hat{V}_n + \hat{I}R}$$

$$\cancel{\text{so } \hat{V} =}$$

$$\hat{V}_n = \hat{V} + R\hat{I} = (Z+R)\hat{I} = \frac{Z+R}{Z}\hat{V}$$

or
$$\hat{V} = \frac{Z}{Z+R}\hat{V}_n \qquad \boxed{\hat{V}(\omega) = \frac{Z(\omega)}{Z(\omega)+R}\hat{V}_n(\omega)}$$

Now I need to relate

$$\underset{\text{in } \omega,\, \omega+d\omega}{\text{part of } \langle V^2 \rangle} \quad \text{to} \quad \left|\hat{V}(\omega)\right|^2 \frac{d\omega}{2\pi}$$

Lets suppose they are equal (this ought to be Wiener-Khinchin). Then

[redacted] $$R\, k_B T \frac{4|Z|^2}{|Z|^2+R^2}\frac{d\omega}{2\pi} = \left|\hat{V}(\omega)\right|^2 \frac{d\omega}{2\pi}$$

$$= \frac{|Z|^2}{|Z|^2+R^2}\left|\hat{V}_n(\omega)\right|^2 \frac{d\omega}{2\pi}$$

so we get

$$\boxed{\underbrace{\left|\hat{V}_n(\omega)\right|^2 \frac{d\omega}{2\pi}}_{\substack{\text{part of } \langle V_n^2 \rangle \\ \text{in range } \omega,\, \omega+d\omega}} = 4R\, k_B T \frac{d\omega}{2\pi}}$$

which is the Nyquist formula.

September 5, 1980

Wiener-Khinchin: Suppose given a signal $V(t)$ (a complex function of t). This signal ~~is assumed~~ is assumed to be "random" and "stationary". (These terms have an intuitive meaning - I can think of the output of a process which is unchanged in time and which looks random. But they are to be made more precise by what follows.)

First suppose the signal is periodic of period L, and expand in a F.S.

$$V(t) = \sum_{\omega \in \frac{2\pi}{L}\mathbb{Z}} \hat{V}(\omega)\, e^{-i\omega t}$$

$$\hat{V}(\omega) = \frac{1}{L}\int_{-L/2}^{L/2} e^{i\omega t}\, V(t)\, dt.$$

Then

$$\frac{1}{L}\int_{-L/2}^{L/2} \overline{V(t+s)}\, V(t)\, dt = \sum_{\omega,\omega'} \frac{1}{L}\, \overline{\hat{V}(\omega)}\, \hat{V}(\omega') \int_{-L/2}^{L/2} e^{i\omega(t+s)}\, e^{-i\omega' t}\, dt$$

$$= \sum_{\omega \in \frac{2\pi}{L}\mathbb{Z}} e^{i\omega s} \left|\hat{V}(\omega)\right|^2$$

Now I want to let $L \to \infty$. Because the signal is random + stationary I expect that

$$\frac{1}{L}\int_{\substack{\text{any interval}\\ \text{of length } L}} \overline{V(t+s)}\, V(t)\, dt \longrightarrow K(s)$$

as $L \to \infty$, where $K(s)$ decays as $|s| \to \infty$. Thus $K(s)$ will have a nice Fourier transform which will be $|\hat{V}(\omega)|^2$ provided modify the defn. by $\sqrt{L}$.

Thus we have

$$\boxed{\lim_{L\to\infty} \frac{1}{L}\int_{-L/2}^{L/2} \overline{V(t+s)}\, V(t)\,dt = \int \frac{d\omega}{2\pi}\, e^{i\omega s}\,|\hat{V}(\omega)|^2}$$

where

$$\boxed{\hat{V}(\omega) = \lim \frac{1}{\sqrt{L}}\int_{-L/2}^{L/2} e^{i\omega t}\, V(t)\,dt}$$

So one sees that these random-stationary signals are neither in the L^2 or almost-periodic categories. Perhaps it is difficult to produce examples which are mathematically rigorous.

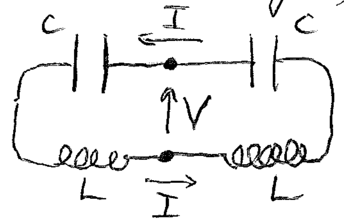
Crazy ideas:

1) Can the emission-absorption rules of quantum mechanics be explained by a f_*, f^* formalism?

2) Can the many Feynman diagrams be organized by formal group law formalisms.

September 6, 1980.

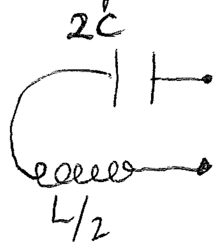
Suppose we have a 1-port which is made up of inductances + capacitances. Can it have a bound state? A bound state is a non-trivial motion such that $V=0, I=0$ at the terminals of the port. Answer: Yes, use symmetry



$$V = \left(\frac{1}{Cs} + Ls\right) I \qquad\qquad V = -\left(\frac{1}{Cs} + Ls\right) I$$

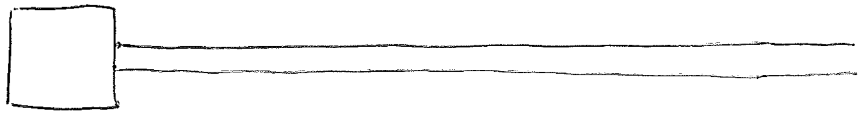
Thus for $s = i\omega$, $\omega = \frac{1}{\sqrt{LC}}$ we can have $V=0$, $I \neq 0$. Thus I could attach a resistor across the terminals and the circuit would oscillate indefinitely.

This port is equivalent to



because its impedance is $\frac{1}{2}\left(\frac{1}{Cs} + Ls\right)$.

Let's review the Nyquist business. I have a transmission line connected to a loss-less 1-port



all at the temperature T. I can compute the modes of vibration of this combined system:

$$V = \left(e^{-ikx} + R(k)\, e^{ikx}\right) e^{-i\omega t}$$

$$R_0 I = \left(-e^{-ikx} + R(k)\, e^{ikx}\right) e^{-i\omega t}$$

$k > 0$
$\omega = ck$
$\left(c = \frac{1}{\sqrt{LC}}\right.$
$R_0 = \sqrt{\frac{L}{C}}$

where the reflection coefficient is determined by the impedance $Z(\omega)$ of the 1-port.

$$-Z(\omega) = \left.\frac{V}{I}\right|_{x=0} = R_0 \frac{1+R(k)}{-1+R(k)}$$

Because the system is at temperature T I know that the modes have to be excited a certain amount on the average. How does one compute this?

Take

$$(*)\quad \begin{cases} V = \mathrm{Re}\left(A\left(e^{-ikx} + R e^{ikx}\right) e^{-i\omega t}\right) \\ I = \mathrm{Re}\left(\frac{A}{R_0}\left(-e^{-ikx} + R e^{ikx}\right) e^{-i\omega t}\right) \end{cases}$$

The energy in a given length dx of line is

$$\underbrace{\left(\tfrac{1}{2} C V^2 + \tfrac{1}{2} L I^2\right)}\, dx$$

This is of the form $\frac{1}{2}C\left\{\left[\mathrm{Re}\left(\tilde{A}\cos(kx-\alpha)\right)\right]^2 + \left[\mathrm{Re}\left(i\tilde{A}\sin(kx-\alpha)\right)\right]^2\right\}$ so it seems to be independent of x. NO Do carefully suppose $R(k) = e^{-2i\alpha}$. Then

$$V = \mathrm{Re}\Big(\underbrace{2Ae^{-i\alpha}e^{-i\omega t}}_{\tilde{A}}\Big(\frac{e^{-ikx+i\alpha} + e^{-i\alpha+ikx}}{2}\Big)\Big)$$

$$= \mathrm{Re}\left(\tilde{A}\cos(kx-\alpha)\right) = \left(\mathrm{Re}\,\tilde{A}\right)\cos(kx-\alpha)$$

$$I = \frac{1}{R_0}\mathrm{Re}\Big(2iAe^{-i\alpha}e^{-i\omega t}\Big(\frac{-e^{i\alpha-ikx} + e^{-i\alpha+ikx}}{2i}\Big)\Big)$$

$$= \frac{1}{R_0}\mathrm{Re}\left(i\tilde{A}\sin(kx-\alpha)\right) = \frac{1}{R_0}\left(-\mathrm{Im}\,\tilde{A}\right)\sin(kx-\alpha)$$

Thus $\frac{1}{2}CV^2 + \frac{1}{2}LI^2 = \frac{1}{2}C\left[(\mathrm{Re}\,\tilde{A})^2 \cos^2(kx-\alpha) + (\mathrm{Im}\,\tilde{A})^2 \sin^2(kx-\alpha)\right]$

is not independent of x. However the total energy is

$$\frac{1}{2} C \frac{\ell}{2} |\tilde{A}|^2 = C\ell |A|^2$$

at least when $k \in \frac{\pi}{\ell}\mathbb{Z}$. Notice that this is the total energy in the line segment of length ℓ and it does not include the energy stored in the 1-port nor the termination at $x = \ell$. However these edge energies disappear in influence as $\ell \to \infty$.

So now we consider a linear combination of the basic modes

$$V = \mathrm{Re}\left(\sum_k A_k \left(e^{-ikx} + R(k) e^{ikx}\right) e^{-i\omega t} \right)$$

where the A_k are ind. Gaussian random vbls., subject to the requirement that the system be at temperature T: This means

$$C\ell \langle |A_k|^2 \rangle = \underbrace{\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1}\right)\hbar\omega}_{\text{denote this } \varepsilon_\omega} \sim k_B T \quad \text{if } \beta\hbar\omega \ll 1.$$

the energy in a harm. oscillator at temp. T.

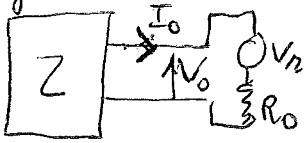
Let $V_0 = V(0,t)$. Then

$$\langle V_0^2 \rangle = \frac{1}{2} \sum_k \langle |A_k|^2 \rangle |1 + R(k)|^2$$

Now let $\ell \to \infty$ and use that $k \in \frac{\pi}{\ell}\mathbb{Z}$ approx and $\omega = ck$ so that $\sum_k \longmapsto \frac{\ell}{\pi}\int dk = \frac{\ell}{c}\int \frac{d\omega}{\pi}$.

$$\langle V_0^2 \rangle = \frac{1}{2} \int \frac{d\omega}{\pi} \frac{\ell}{c} \frac{1}{C\ell} \varepsilon_\omega |1 + R(k)|^2 = \sqrt{\frac{L}{C}} \int \frac{d\omega}{2\pi} \varepsilon_\omega |1 + R_\omega|^2.$$

Finally one interprets $|1+R_\omega|^2$, which is due to the fact that the impedance of the port affects the voltage distribution at the frequency ω. Think of the line as the resistance $R_0 = \sqrt{\frac{L}{C}}$ with a noise generator V_r.



Then

$$\hat{V}_0 = 1+R \qquad\qquad \therefore \quad Z = -\frac{V_0}{I_0} = R_0\,\frac{1+R}{1-R}$$

$$\hat{I}_0 = \frac{1}{R_0}(-1+R)$$

$$\therefore \; R(\omega) = \frac{Z(\omega)-R_0}{Z(\omega)+R_0}$$

Also

$$\hat{V}_r = \hat{V}_0 + R_0\hat{I}_0 = (Z+R_0)\hat{I}_0 = (Z+R_0)\frac{\hat{V}_0}{Z}$$

$$\therefore \quad \hat{V}_0 = \frac{Z}{Z+R_0}\hat{V}_r \qquad |\hat{V}_0|^2 = \frac{|Z|^2}{|Z|^2+R_0^2}|\hat{V}_r|^2$$

$$|1+R|^2 = \left|\frac{2Z}{Z+R_0}\right|^2 = \frac{4|Z|^2}{|Z|^2+R_0^2}$$

So what we find is that

$$\text{the contribution to } \langle V_0^2\rangle \text{ in the range } \omega, \omega+d\omega = \sqrt{\frac{L}{C}}\,\varepsilon_\omega\,|1+R_\omega|^2\,\frac{d\omega}{2\pi}$$

$$= \underbrace{\frac{|Z|^2}{|Z|^2+R_0^2}}_{\uparrow}\;\underbrace{4R_0\,\varepsilon_\omega\,\frac{d\omega}{2\pi}}_{\text{thermal contribution}}$$

conversion factor from a voltage in the line to V_0 (think of R_0 as an internal resistance)

September 7, 1980

Idea: From the study of noise on a transmission line we get some idea of what a "state" of the line must be (state is used in the sense of complete history). It's somehow a linear combination of modes whose amplitudes are "generalized numbers." It also appears that there is a notion of <u>generic point</u> for these amplitudes which gives one an <u>actual history</u> with the <u>ergodic property</u>, so that statistical averages can be computed as time averages.

Things to understand: 1) Feynman's version of a noise signal from a Geiger counter 2) Review Wiener prediction theory.

Pick a finite interval of time $[0, L]$. A history of a Geiger counter over this period consists of clicks, hence it is a positive divisor on this interval. We know how to make these histories into a probability space: The probability of having 1 click in intervals dt_j around t_j for $j = 1, \ldots, N$ is

$$\frac{\frac{z^N}{N!}\, dt_1 \cdots dt_N}{\underbrace{\sum_N \frac{z^N}{N!} \int_{[0,L]^N} dt_1 \cdots dt_N}_{e^{zL}}}$$

What we have is a 1-diml gas. z is adjusted so as to give the desired density

$$\langle N \rangle = z \frac{\partial}{\partial z} \log Z_{gr} = zL$$

$$\Longrightarrow \quad z = \frac{\langle N \rangle}{L} = \rho.$$

In the ideal classical grand-ensemble gas we saw that what one is doing is to chop up configuration space into small pieces dV and then put independent Poisson processes with $\langle N \rangle = \rho\, dV$ in each piece. There doesn't seem to be any problem with doing this over the whole line. Notice that the linear ordering of time is irrelevant so far.

So now we understand what a Geiger counter is. A history is a positive divisor on $\mathbb{R}$ with finitely many (counting mult.) points in any finite interval. Any finite interval gives us a cylinder set measure on the ensemble of histories.

Next suppose that a click of the Geiger counter at time τ produces a signal $g(t-\tau)$, where g has support in $[0, a]$. Then each history $n = \{n(\tau)\}$ gives me a signal

$$V(t) = \sum_{\tau \in \mathbb{R}}{}' \, n(\tau)\, g(t-\tau)$$

and I can ask about $\langle V(t) \rangle$, $\langle V(t)^2 \rangle$ etc. Notice that for t fixed $V(t)$ is an example of a 1-particle function on the gas

$$\tilde{f} : n \longmapsto \sum n(\tau) f(\tau)$$

and hence I have calculated its generating function.

$$\langle e^{J\tilde{f}} \rangle = \sum_N \frac{z^N}{N!} \int e^{J \sum_j f(t_j)}\, dt_1 \cdots dt_N \Big/ \sum \frac{z^N}{N!} \int dt_1 \cdots dt_N$$

$$= \exp\left\{ z \int \left(e^{Jf(t)} - 1\right) dt \right\} \qquad z = \text{density}$$

hence we know that

$$\langle \tilde{f} \rangle = z \int f(t)\,dt$$

$$\langle \tilde{f}^2 \rangle - \langle \tilde{f} \rangle^2 = z \int f(t)^2\,dt$$

Next let us consider the signal

$$V(t) = \sum n(\tau)\, g(t-\tau)$$

and compute its moments. Choose a $J(t)$ and consider

$$\int J(t) V(t)\,dt = \sum_\tau n(\tau) \int J(t) g(t-\tau)\,dt = \tilde{f}$$

where $f(\tau) = \int J(t)\, g(t-\tau)\,dt$. Then

$$\left\langle e^{\int J(t) V(t)\,dt} \right\rangle = \exp\left\{ z \int \left(e^{\int J(t) g(t-\tau)\,dt} - 1 \right) d\tau \right\}.$$

so

$$\langle V(t) \rangle = z \int g(t-\tau)\,d\tau = z \int g(\tau)\,d\tau.$$

independent of t.

$$\langle V(t_1) V(t_2) \rangle - \langle V \rangle^2 = z \int g(t_1-\tau)\, g(t_2-\tau)\,d\tau$$

Another proof:

$$V(t) = \int n(\tau)\, g(t-\tau)\,d\tau \qquad n(\tau) = \sum \delta(\tau - t_j)$$

now

$$\langle V(t_1) V(t_2) \rangle = \int g(t_1-\tau_1)\, g(t_2-\tau_2) \underbrace{\langle n(\tau_1)\, n(\tau_2) \rangle}_{z^2 + z\,\delta(\tau_1-\tau_2)} d\tau_1\, d\tau_2$$

In order to obtain a Gaussian process, Feynman take $z \to \infty$ so that the density of the pulses increases, but he lets the size of the pulse go to zero. So suppose that

$$\int g(t)\,dt = 0$$

and then replace g by $\frac{1}{\sqrt{z}}g$. Then in the limit as $z \to +\infty$ we have

$$\left\langle e^{\int J(t)V(t)\,dt} \right\rangle \longrightarrow \exp\left\{ \frac{1}{2} \int J(t_1)\, g(t_1-\tau)\, J(t_2)\, g(t_2-\tau)\, dt_1\, dt_2\, d\tau \right\}$$

$$= \exp\left\{ \frac{1}{2} \int J(t_1)\, K(t_1-t_2)\, J(t_2)\, dt_1\, dt_2 \right\}$$

where
$$K(t_1-t_2) = \int g(t_1-\tau)\, g(t_2-\tau)\, d\tau.$$

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Problem: Consider a transmission line

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}.$$

Can these equations be derived from a variational principle?

Note: The energy on $[a,b]$ is

$$E = \int_a^b \left(\tfrac{1}{2}LI^2 + \tfrac{1}{2}CV^2\right)dx$$

and

$$\frac{\partial E}{\partial t} = \int_a^b \left(LI\frac{\partial I}{\partial t} + CV\frac{\partial V}{\partial t}\right)dx$$

$$= \int_a^b \left(-I\frac{\partial V}{\partial x} - V\frac{\partial I}{\partial x}\right)dx = -\int_a^b \frac{\partial}{\partial x}(VI)\,dx$$

$$= [VI]_b^a$$

Hence the energy density is

$$\mathcal{E} = \tfrac{1}{2}LI^2 + \tfrac{1}{2}CV^2$$

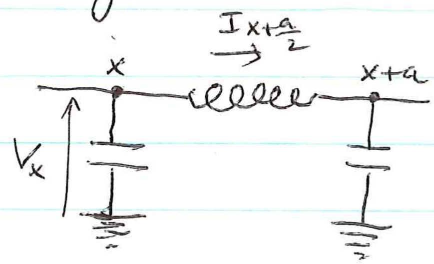
the power flow at a point is VI, and one has the equation of continuity

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x}(VI) = 0$$

The transmission line equations are first order in time, i.e. can be written

$$\frac{\partial}{\partial t}\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{C} \\ -\frac{1}{L} & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} V \\ I \end{pmatrix}$$

hence the solutions can be thought of a trajectories in a space having the coords $V(x)$, $I(x)$ for each x. This maybe is clearer if one takes the discrete versions



$$I_{x+\frac{a}{2}} - I_{x-\frac{a}{2}} = -Ca \cdot \frac{\partial V_x}{\partial t}$$

$$V_{x+a} - V_x = -La \frac{\partial I_{x+\frac{a}{2}}}{\partial t}$$

The question is whether this flow is Hamiltonian with respect to a 2-form on the space of (V_x, I_x). The total energy which should be the Hamiltonian is

$$\frac{1}{2} Ca \sum_x{}' V_x^2 + \frac{1}{2} La \sum_x{}' I^2_{x+\frac{a}{2}} \sim \int \left(\frac{1}{2} CV^2 + \frac{1}{2} LI^2\right) dx$$

Suppose the desired 2-form is

$$\omega = \sum_{x,y} W_{xy}\, dV_x\, dI_y \qquad \begin{array}{l} x \in \mathbb{Z}a \\ y \in (\mathbb{Z} + \frac{1}{2})a \end{array}$$

We want

$$i(X)\,\omega = dH = \sum_x Ca\, V_x\, dV_x + \sum_y La\, I_y\, dI_y$$

$$X = \sum a_x \frac{\partial}{\partial V_x} + \sum b_y \frac{\partial}{\partial I_y}$$

$$a_x = \dot{V}_x = -\frac{1}{Ca}\left(I_{x+\frac{a}{2}} - I_{x-\frac{a}{2}}\right)$$

$$b_y = \dot{I}_y = -\frac{1}{La}\left(V_{y+\frac{a}{2}} - V_{y-\frac{a}{2}}\right)$$

$$i(X)\omega = \sum a_x W_{xy}\, dI_y - W_{xy}\, b_y\, dV_x$$

$$\therefore \quad \sum_x a_x W_{xy} = \sum_x \frac{1}{Ca}\left(I_{x-\frac{a}{2}} - I_{x+\frac{a}{2}}\right) W_{xy} = La\, I_y$$

$$\sum_y W_{xy}\, b_y = \sum_y \frac{1}{La}\left(V_{y+\frac{a}{2}} - V_{y-\frac{a}{2}}\right) W_{xy} = Ca\, V_x$$

Thus if we take

$$W_{xy} = \begin{cases} LCa^2 & x > y \\ 0 & x < y \end{cases}$$

we can solve both equations. So W has to invert the difference operator, and there is going to be some funny business if we want the line periodic.

In the continuous case we get the form

$$\omega = LC \iint_{x>y} dV_x\, dI_y$$

It might be good to think of an actual skew-symm. bilinear form on the vector space of functions (V, I) of x. Then this form is

$$(V_1, I_1), (V_2, I_2) \longmapsto LC \iint_{x>y} dx\,dy \left(V_1(x) I_2(y) - V_2(x) I_1(y)\right)$$

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2dim'l QED. In one-space dimension defining E, B by the Lorentz formula $F = e(E + v \times B)$ shows $B=0$ and then Maxwell's equations become

$$\nabla \cdot E = \frac{\partial E}{\partial x} = 4\pi\rho \qquad \frac{\partial E}{\partial t} = -4\pi J.$$

Hence if $\rho = J = 0$, then E is a constant in space-time, hence $E=0$ if one asks for E to vanish as $x \to \infty$.

To get waves one has to work with the gauge potentials (ϕ, A). Recall (Oct 17, 1979) the canonical 1-form

$$\omega = \phi\, dt - A_i dx^i$$

has

$$d\omega = E_i\, dt\, dx^i + B_{ij} dx^i dx^j \qquad B_{ij} = \partial_j A_i - \partial_i A_j$$

or

$$E = -\nabla\phi - \frac{\partial A}{\partial t} \qquad B = \nabla \times A$$

So with one space dimension x we have

$$\omega = \phi\, dt - A\, dx$$

and

$$E = -\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t}$$

Normally ϕ, A is subject to the gauge condition which is like ~~the~~ equation of continuity

$$\frac{\partial \phi}{\partial t} + \frac{\partial A}{\partial x} = 0$$

So when there are no charges or currents present, so that $E=0$, we get that the gauge potential satisfies

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{\partial A}{\partial x} = 0 \\[2ex] \dfrac{\partial \phi}{\partial x} + \dfrac{\partial A}{\partial t} = 0 \end{cases}$$

These are the same as the transmission line equations and so they give waves of speed 1.

Therefore [redacted] in 2diml QED we have gauge waves rather than EM waves. What kind of energy belongs with these waves?

Let's consider plane wave[redacted] solutions of Maxwell's equations in space

$$\nabla \cdot B = 0 \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot E = 0 \qquad \nabla \times B = \frac{\partial E}{\partial t}$$

$$E = \hat{E}\, e^{i(kx-\omega t)} \qquad B = \hat{B}\, e^{i(kx-\omega t)}$$

$$k \cdot \hat{E} = k \cdot \hat{B} = 0$$

$$k \times \hat{E} = -\omega \hat{B} \qquad k \times \hat{B} = +\omega \hat{E}$$

$$\therefore \quad k \times (k \times \hat{E}) = k \times (-\omega \hat{B}) = -\omega^2 \hat{E}$$

$$\cancel{(k\cdot E)k} - |k|^2 \hat{E} \qquad \therefore \quad \omega^2 = |k|^2$$

Thus we have waves of speed 1 with E, B transversal to the direction of propagation. Also $E \perp B$.

Next I need to bring in (ϕ, A). Assume they satisfy

$$B = \nabla \times A \qquad E = -\nabla \phi - \frac{\partial A}{\partial t}$$

and that they are plane waves $A = \hat{A}\, e^{i(kx-\omega t)}$.

Then
$$\begin{cases} \hat{B} = ik \times \hat{A} \\ \hat{E} = -ik\hat{\phi} + i\omega \hat{A} \end{cases}$$

These E, B satisfy $\nabla \cdot B = 0$ $\nabla \times E = -\frac{\partial B}{\partial t}$. The other two Maxwell equations give

$$0 = ik\cdot\hat{E} = |k|^2\hat{\phi} - \omega(k\cdot\hat{A})$$

$$ik \times (ik \times \hat{A}) = -i\omega \hat{E} = -i\omega(-ik\hat{\phi} + i\omega\hat{A})$$

$$-(k\cdot A)k + |k|^2\hat{A} = -\omega k\hat{\phi} + \omega^2\hat{A}$$

or

$$k\cdot\hat{A} = \frac{|k|^2}{\omega}\hat{\phi}$$

$$(|k|^2 - \omega^2)\hat{A} = k(k\cdot\hat{A} - \omega\hat{\phi})$$

$$= \frac{k}{\omega}(|k|^2 - \omega^2)\hat{\phi}$$

If $\omega^2 \neq |k|^2$, then $\hat{A} = \frac{\hat{\phi}}{\omega}\hat{k}$ so $\hat{B} = 0$ and $\hat{E} = 0$ which is a valid choice for (ϕ, A); it means

$$(\phi, A) = \left(-\frac{\partial f}{\partial t}, \nabla f\right)$$

Excluding this case, we know $\omega^2 = |k|^2$ and hence

$$\hat{\phi} = \frac{\hat{k}\cdot\hat{A}}{\omega}$$

which implies that the gauge condition:

$$\frac{\partial \phi}{\partial t} + \nabla\cdot\hat{A} = 0$$

is satisfied.