January 11, 1980 — Analysis of KdV flow in terms of scattering data $e^{t(D^2+q)}$ for small t.

We made a digression to learn about KdV.

Consider $L = D^2 + q$. We want to construct isospectral deformations of L, that is, families $L(t) = D^2 + q(t)$ such that [redacted] $L(t) = U(t,t')\,L(t')\,U(t,t')^{-1}$ for a unitary operator $U(t,t')$. Then if $B(t) = \frac{d}{dt}(U(t,0))\cdot U(t,0)^{-1}$ one has

$$\frac{d}{dt}U(t,t') = \frac{d}{dt}U(t,0)\,U(0,t') = B(t)\,U(t,t')$$

and so $\frac{d}{dt}L(t) = [B(t), L(t)]$. We seek [redacted] B in the form of a skew-adjoint differential operator.

Easy example: $B = D$. Then $[B,L] = [D, D^2+q] = q'$ so we get the flow with $\partial_t q = \partial_x q$, which is simply translation.

Interesting example: $B = D^3 + fD + Df = D^3 + 2fD + f'$

Then

$$[B,L] = [D^3 + 2fD + f',\ D^2 + q]$$

$$= 3q'D^2 + 3q''D + q''' + 2\underbrace{[f,D^2]}_{-2f'D - f''}D + 2f\underbrace{[D,q]}_{q'} + \underbrace{[f',D^2]}_{-2f''D - f'''}$$

[redacted]

$$= (3q' - 4f')D^2 + (3q'' - 4f'')D + \underbrace{(q''' + 2fq' - f''')}$$

So if we put $f = \frac{3}{4}q$ we get $\frac{1}{4}q''' + \frac{3}{2}qq'$ so we get the flow

$$\boxed{\dot q = \frac{1}{4}q''' + \frac{3}{2}qq'} \qquad \boxed{B = D^3 + \frac{3}{2}qD + \frac{3}{4}q'}$$

Next suppose we have a solution of the KdV equation, such that $q(x,t) \longrightarrow 0$ fast as $x \longrightarrow +\infty$. Consider the solution $\psi(x,t,k)$ of $(D^2+q)\psi = -k^2\psi$

with the asymptotic behavior $\psi \approx e^{ikx}$ as $x \to +\infty$

Put $\lambda = -k^2$ and differentiate

$$(L-\lambda)\psi = 0$$

with respect to time to get

$$(\dot{L}\,[redacted])\psi + (L-\lambda)\dot{\psi} = 0$$

$$\dot{L}\psi = [B,L]\psi = \lambda B\psi - LB\psi$$

or $$(\lambda - L)(B\psi - \dot{\psi})\,[redacted] = 0.$$ As $x \to +\infty$

one has
$$B\psi \approx D^3 e^{ikx} = (ik)^3 e^{ikx}$$
$$\dot{\psi} \approx o(1)\, e^{ikx}$$

hence since $B\psi - \dot{\psi}$ must be an eigenfunction we have

$$B\psi - \dot{\psi} = (ik)^3\psi$$

From this we can read off what time evolution does to the scattering data. First take the case of a bound state: $k = i\beta$ with $\beta > 0$. Take the inner product with ψ, which is in L^2, and use skew-adjointness of B to get

$$\underbrace{-(\psi,\dot{\psi})}_{= \frac{1}{2}(\psi,\psi)^{\cdot}} = [redacted]\,(-\beta)^3(\psi,\psi) \qquad \psi = \psi(x, i\beta)$$

$$\frac{d}{dt}\log\|\psi\| = \beta^3 \qquad \text{or}$$

$$\|\psi(\,,t,i\beta)\| = e^{t\beta^3}\|\psi(\,,0,i\beta)\|$$

Next compute what happens to the transmission and reflection coefficients:

$$T(k)e^{-ikx} \longleftrightarrow e^{-ikx} + R(k)e^{ikx}$$

$$T(k)\phi(x,k) = \psi(x,-k) + R(k)\psi(x,k)$$

$$(B-(-ik)^3)\times \quad T\phi = \psi^- + R\psi$$

$$\mathring{T}\phi + T\underbrace{\mathring{\phi}}_{(B-(-ik)^3)\phi} = \underbrace{\mathring{\psi}^-}_{(B-(-ik)^3)\psi^-} + \mathring{R}\psi + R\underbrace{\mathring{\psi}}_{(B-(ik)^3)\psi}$$

$$\mathring{T}\phi = \mathring{R}\psi - 2(ik)^3 R\psi$$

so we conclude

$$\mathring{T} = 0 \qquad \mathring{R} = 2(ik)^3 R = -i2k^3 R$$

So $T(k)$ is constant in time and

$$R(k,t) = e^{-2ik^3 t}R(k,0)$$

Work out an example. Recall the factorization method of adding a bound state. We have a factorization

$$-\frac{d^2}{dx^2} - q + \beta^2 = \left(\frac{d}{dx}+p\right)\left(-\frac{d}{dx}+p\right)$$

given by $p = \frac{u'}{u}$ where u is killed by $-D^2 - q + \beta^2$. We want u to be non-vanishing on $\mathbb{R}$ which will be the case if $-\beta^2 <$ spectrum $-D^2 - q$ and we take

$$u = c_1\underbrace{\psi(x,+i\beta)}_{\sim e^{-\beta x} \text{ as } x\to+\infty} + c_2\underbrace{\phi(x,+i\beta)}_{\sim e^{\beta x} \text{ as } x\to-\infty}$$

with $c_1, c_2 > 0$. Then the new potential $\tilde{q}$ is given by

$$-\frac{d^2}{dx^2} - \tilde{q} + \beta^2 = \left(-\frac{d}{dx} + p\right)\left(\frac{d}{dx} + p\right)$$

so
$$-\tilde{q} + \beta^2 = p^2 - p' = -q + \beta^2 - 2p'$$

or
$$\tilde{q} = q + 2p'$$

Since $u \sim$ const $e^{\beta x}$ at $+\infty$, $\sim$ const $e^{-\beta x}$ at $-\infty$, $\frac{1}{u}$ is a bound state for $D^2 + \tilde{q}$ with eigenvalue $-\beta^2$. Also $p \sim \beta$ at ∞, $p \sim -\beta$ at $-\infty$ and $p' \to 0$ fast as $|x| \longrightarrow \infty$, so the same is true for $\tilde{q}$.

Scattering data:

$$\left(-D^2 - \tilde{q} + \cancel{\beta^2}\right)\left(-\frac{d}{dx} + p\right) = \left(-\frac{d}{dx} + p\right)\left(-D^2 - q + \cancel{\beta^2}\right)$$

so that $-D + p$ carries $\psi(x,k)$ into a multiple of $\tilde{\psi}(x,k)$ in fact

$$(-D+p)\,\overset{\sim e^{ikx} \text{ at } \infty}{\psi(x,k)} = (-ik+\beta)\,\tilde{\psi}(x,k)$$
$$(-D+p)\,\underset{\sim e^{-ikx} \text{ at } -\infty}{\phi(x,k)} = (ik-\beta)\,\tilde{\phi}(x,k)$$

Hence from

$$(-D+p) \qquad \phi(x,k) = A(k)\psi(x,-k) + B(k)\psi(x,k)$$
$$(ik-\beta)\tilde{\phi} = A(k)(ik+\beta)\tilde{\psi}(-k) + B(k)(-ik+\beta)\tilde{\psi}(k)$$

so that
$$\tilde{A}(k) = \frac{ik+\beta}{ik-\beta} A(k) \qquad \tilde{B}(k) = -B(k)$$

$\tilde{A}$ has new zero at $i\beta$

and hence
$$\tilde{R}(k) = \frac{\beta - ik}{\beta + ik} R(k)$$

Note that we get varying $\tilde{q}$ by altering the constants in $u = c_1\psi(x, i\beta) + c_2\phi(x, +i\beta)$. Such changes do not affect $\tilde{A}$ or $\tilde{R}$.

Let's start with $q = 0$ and take for solution of $(D^2+\beta^2)u=0$ the function

$$u = \tfrac{1}{2}\left(e^{\beta(x-\alpha)} + e^{-\beta(x-\alpha)}\right) = \cosh \beta(x-\alpha).$$

$$p = \frac{u'}{u} = \beta \tanh \beta(x-\alpha)$$

$$\tilde{q} = 2p' = 2\beta^2 \operatorname{sech}^2 \beta(x-\alpha)$$

The bound state for $D^2+\tilde{q}$ is $\frac{1}{u}$

$$\frac{1}{u} = \frac{2}{e^{-\beta\alpha}e^{\beta x} + e^{\beta\alpha}e^{-\beta x}} \approx 2e^{\beta\alpha}e^{-\beta x} \qquad x \to +\infty$$

so that

$$\tilde{\psi}(x, i\beta) = \frac{1}{2e^{\beta\alpha}}\,\frac{1}{u}$$

and the norming constant is

$$C = \|\tilde{\psi}(x, i\beta)\|^2 = \frac{1}{4e^{2\beta\alpha}\beta}\int \frac{\beta\,dx}{\cosh^2 \beta(x-\alpha)}$$

$$= \frac{1}{4e^{2\beta\alpha}\beta}\Big[\tanh \beta(x-\alpha)\Big]_{-\infty}^{\infty}$$

$$= \frac{1}{2\beta e^{2\beta\alpha}}$$

Under the KdV motion C gets multiplied by $e^{2t\beta^3}$ which is the same as $\alpha \longrightarrow \alpha - t\beta^2$. Hence

$$\tilde{q}(x,t) = 2\beta^2 \operatorname{sech}^2 \beta(x+t\beta^2)$$

should satisfy KdV.

January 12, 1980:

$$\left(-\frac{d^2}{dx^2} - q\right)\psi = k^2\psi$$ gives rise to the basic scattering relation

(1) $$T(k)\phi(x,k) = \psi(x,-k) + R(k)\psi(x,k)$$

I want to understand the key example $R=0$. In this case $|T(k)|=1$ for k real and we have

$$T(k) = \prod_{j=1}^{b} \frac{k+i\beta_j}{k-i\beta_j}$$

where the bound energies are $-\beta_1^2, \dots, -\beta_b^2$. Now

$$\psi(x,k) = e^{ikx} + \int_{y>x} dy\, T(x,y)e^{iky} \quad \in e^{ikx}(1+H^+)$$

where H^+ is the Hardy space. Similarly

$$\phi(x,k) = e^{-ikx} + \int_{y<x} dy\, T_1(x,y)e^{-iky} \quad \in e^{-ikx}(1+H^+).$$

So when $R=0$, we have

$$T(k)\, e^{ikx}\phi(x,k) = e^{ikx}\psi(x,-k)$$

$$\in T(k)(1+H^+) \cap (1+H^-)$$

When one is working with rational functions an $f \in H^+$ if it has no poles in the closed UHP and if it vanishes at ∞. I think it's more or less clear that

$$(1+H^+) \cap \left(\prod \frac{k-i\beta_j}{k+i\beta_j}\right)(1+H^-)$$

consists of rational functions of the form

$$1 + \sum_{j=1}^{b} \frac{\alpha_j}{k+i\beta_j}$$

Hence we have $$\psi(x,k) = e^{ikx} + \sum_{j=1}^{b} \frac{\alpha_j(x)}{k+i\beta_j}e^{ikx}$$

To determine the $\alpha_j(x)$ one has to use Marchenko's equation:

$$F(x+y) + T(x,y) + \int_{z>x} dz\, T(x,z)\, F(z+y) = 0 \qquad y > x$$

where

$$F(x) = \sum C_j e^{-\beta_j x} + \underbrace{\int \frac{dk}{2\pi} R(k) e^{ikx}}_{=0}$$

and the $C_j = \| \psi(x, i\beta_j) \|^{-2}$.

$$\sum C_j e^{-\beta_j(x+y)} + T(x,y) + \int_{z>x} dz\, T(x,z) \sum_k C_k e^{-\beta_k(z+y)} = 0$$

This shows that as a function of y we have

$$T(x,y) = \sum h_j(x)\, e^{-\beta_j y}$$

so that

$$\psi(x,k) = e^{ikx} + \int_{y>x} dy \sum h_j(x) e^{-\beta_j y} e^{iky}$$

$$= e^{ikx} + \sum_j h_j(x) \frac{e^{-\beta_j x + ikx}}{\beta_j - ik}$$

consistent with the above.

$$\sum C_j e^{-\beta_j(x+y)} + \sum h_j(x) e^{-\beta_j y} + \sum_{k,j} C_k\, h_j(x) \underbrace{\int_{z>x} e^{-\beta_j z - \beta_k(z+y)} dz}_{\frac{e^{-(\beta_j+\beta_k)x - \beta_k y}}{(\beta_j+\beta_k)}} = 0$$

Marchenko eqn.

$$F(x+y) + T(x,y) + \int_{z>x} dz\, T(x,z)\, F(z+y) = 0 \qquad y>x$$

$$F(x) = \sum C_j e^{-\beta_j x}$$

$$\sum C_j e^{-\beta_j (x+y)} + T(x,y) + \int_{z>x} dz\, T(x,z) \sum_j C_j e^{-\beta_j (z+y)} = 0 \qquad y>x$$

implies $\quad T(x,y) = \sum h_j(x) e^{-\beta_j y} \quad$ where

$$C_j e^{-\beta_j x} + h_j(x) + C_j \int_{z>x} dz \left(\sum_k h_k(x) e^{-\beta_k z}\right) e^{-\beta_j z} = 0$$

$$C_j e^{-\beta_j x} + h_j(x) + C_j \sum_k h_k(x) \frac{e^{-(\beta_k+\beta_j)x}}{\beta_k+\beta_j} = 0$$

$$\boxed{C_j + [e^{\beta_j x} h_j(x)] + C_j \sum_k [e^{\beta_k x} h_k(x)] \frac{e^{-2\beta_k x}}{\beta_k+\beta_j} = 0}$$

Also

$$\psi(x,k) = e^{ikx} + \int_{z>x} dz\, T(x,z)\, e^{ikz}$$

$$= e^{ikx} + \int_{z>x} dz \sum h_j(x) e^{-\beta_j z + ikz}$$

$$= e^{ikx} + \sum_j h_j(x) \frac{e^{-\beta_j x + ikx}}{\beta_j - ik}$$

Since $\beta_k > 0$ we have $\boxed{e^{\beta_j x} h_j(x) \longrightarrow -C_j \quad \text{as } x \to +\infty}$

$$C_j + \underbrace{e^{2\beta_j x}}_{\downarrow\ 0 \text{ as } x \to -\infty}\left[e^{-\beta_j x} h_j(x)\right] + C_j \sum_k \left[e^{-\beta_k x} h_k(x)\right] \frac{1}{\beta_k + \beta_j} = 0$$

hence one has

$$e^{-\beta_j x} h_j(x) \xrightarrow{x \to -\infty} a_k$$

$$\text{where} \quad 1 + \sum_k{}' a_k \frac{1}{\beta_k + \beta_j} = 0$$

assuming the limit exists.

Before we go on let us make explicit what we need to the equations for the h_j.

$$C_j + \theta_j' + C_j \sum{}' \theta_k' \frac{e^{-2\beta_k x}}{\beta_k + \beta_j} = 0 \qquad \theta_j' = e^{+\beta_j x} h_j(x)$$

or better if $\theta_j = e^{-\beta_j x} h_j(x)$ we want unique solutions for

$$(*) \qquad \boxed{1 + \frac{e^{2\beta_j x}}{C_j} \theta_j + \sum \theta_k \frac{1}{\beta_k + \beta_j} = 0 \qquad \theta_j = e^{-\beta_j x} h_j(x)}$$

Thus we want to know that

$$\det\left(\lambda_j \delta_{jk} + \frac{1}{\beta_j + \beta_k}\right) \neq 0$$

for any $\lambda_1, \ldots, \lambda_b > 0$. But this is clear because the matrix $\frac{1}{\beta_j + \beta_k}$ is positive-definite:

$$\sum \frac{u_j u_k}{\beta_j + \beta_k} = \sum u_j u_k \int_0^\infty e^{-(\beta_j + \beta_k)t}\, dt = \int_0^\infty \left(\sum u_j e^{-\beta_j t}\right)^2 dt$$

$$\boxed{\begin{array}{l} \theta_j = e^{-\beta_j x} h_j(x) \longrightarrow a_k \qquad \text{as } x \to -\infty \\ \text{where } a_k \text{ is the unique solution of} \\ \qquad 1 + \sum_k a_k \frac{1}{\beta_j + \beta_k} = 0 \end{array}}$$

This is also clear from

$$\psi(x,k) = e^{-ikx}\left(1 + \sum_j \theta_j(x) \frac{1}{\beta_j - ik}\right)$$

since on putting $k = i\beta_k$ we have

$$\psi(x, i\beta_k) = e^{-\beta_k x}\left(1 + \sum_j \theta_j(x) \frac{1}{\beta_j + \beta_k}\right)$$

blows up as $x \to -\infty$

is a bound state, hence proportional to $e^{\beta_k x}$ as $x \to -\infty$.

How to get the potential.

$$\left(-\frac{d^2}{dx^2} - q\right)\psi = k^2 \psi$$

$$(k^2 + D^2)\psi = -q\psi$$

We need $G(x,x')$ supported in $x' > x$.

$$G(x,x') = \begin{cases} 0 & x > x' \\ -\dfrac{\sin k(x-x')}{k} & x < x' \end{cases}$$

$$\psi(x,k) = e^{ikx} + \int_x^\infty dx' \, \frac{\sin k(x-x')}{k} q(x') \psi(x';k)$$

$$= e^{ikx} + \int_x^\infty dx' \, \frac{e^{ik(x-x')} - e^{ik(x'-x)}}{2ik} q(x') e^{ikx'} + \cdots$$

$$= e^{ikx}\left(1 + \int_x^\infty \frac{q(x')}{2ik} dx' \left(1 - e^{2ik(x'-x)}\right)\right) + \cdots$$

$$\boxed{\psi(x,k) = e^{ikx}\left(1 + \frac{1}{2ik}\int_x^\infty q(x')\,dx' + O\left(\frac{1}{k^2}\right)\right)}$$

Also $\quad \psi(x,k) = e^{ikx} + \int_x^\infty T(x,y)\,e^{iky}\,dy$

$$= e^{ikx}\left(1 + \int_0^\infty T(x,x+u)\,e^{iku}\,du\right)$$

$$= e^{ikx}\left(1 + T(x,x)\,\frac{-1}{ik} - \int_0^\infty \frac{d}{dx}T(x,x+u)\,\frac{e^{iku}}{ik}\,du\right)$$

$$= e^{ikx}\left(1 - \frac{1}{ik}T(x,x) + O\left(\frac{1}{k^2}\right)\right)$$

Thus $\quad \boxed{T(x,x) = -\frac{1}{2}\int_x^\infty q(x')\,dx'}$ $\quad$ and

$$\boxed{q(x) = +2\,\frac{d}{dx}T(x,x)}\,.$$

So in our case $\quad T(x,y) = \sum \underbrace{h_j(x)\,e^{-\beta_j x}}_{\theta_j(x)}\; e^{\beta_j(x-y)}$

or

$$T(x,x) = \sum_j \theta_j(x)$$

and so

$$\boxed{q(x) = +2\sum_j \frac{d\theta_j}{dx}(x)}$$

Here's a nice way to write the formulas:

Let $\quad \theta = (\theta_j) \qquad \mathbb{1} = (1,1,\dots,1)$ [redacted]

Then $\quad T(x,x) = \mathbb{1}\cdot\theta(x)$ $\quad$ and

$$\mathbb{1} + \frac{e^{2\beta x}}{c}\,\theta + \frac{1}{\beta+\beta^*}\,\theta = 0$$

where $\frac{1}{\beta+\beta^*}$ is the matrix $\frac{1}{\beta_j+\beta_k}$, $\frac{e^{2\beta x}}{C}$ = the diagonal matrix $\quad \dfrac{e^{2\beta_j x}}{c_j}\,\delta_{jk}$. $\quad$ Thus

$$\Theta(x) = -\left(\frac{e^{2\beta x}}{c} + \frac{1}{\beta+\beta^*}\right)^{-1} \mathbb{1}$$

so

$$\boxed{\begin{aligned} T(x,x) &= -\,\mathbb{1}\cdot\left(\frac{e^{2\beta x}}{c} + \frac{1}{\beta+\beta^*}\right)^{-1}\mathbb{1} \\ q(x) &= 2\,\frac{d}{dx}\,T(x,x) \end{aligned}}$$

<u>Example:</u> Take a single bound state.

$$T(x,x) = \Theta(x) = -\,\frac{1}{\dfrac{e^{2\beta x}}{c} + \dfrac{1}{2\beta}}$$

$$q(x) = 2\frac{d}{dx}T(x,x) = 2\,\frac{1}{\left(\dfrac{e^{2\beta x}}{c} + \dfrac{1}{2\beta}\right)^2}\left(\frac{2\beta}{c}\,e^{2\beta x}\right)$$

$$= 2\,\frac{1}{\left(\dfrac{e^{\beta x}\sqrt{2\beta}}{\sqrt{c}} + \dfrac{\sqrt{c}\,e^{-\beta x}}{\sqrt{2\beta}}\right)^2}\;\frac{\left(\dfrac{2\beta}{\cancel{c}}\,\cancel{e^{2\beta x}}\right)}{\dfrac{\cancel{e^{2\beta x}}}{2\beta}\,\cancel{c}}$$

$$= 2\beta^2\,\mathrm{sech}^2\,\beta(x-\alpha) \qquad \text{where } e^{\beta\alpha} = \sqrt{\frac{2\beta}{c}}$$

$$\text{or}\quad c = \frac{1}{2\beta}\,e^{-2\beta\alpha}$$

which agrees with p. 533.

January 14, 1980

The next project is to compute various trace invariants for D^2+q which will be invariants under the KdV flow. Recall that

$$\det\left(1+G_k^0 q\right) = A(k)$$

where $G_k^0 = \dfrac{e^{ik|x-x'|}}{2ik}$ is the Green's function for k^2+D^2 which is the L^2-inverse for $k \in UHP$. We want to find an asymptotic expansion for this determinant as $k \to \infty$. Actually [redacted] it might be a good idea to think of $k \to i\infty$ along the imaginary axis.

Recall $A(k)$ is defined by

$$e^{-ikx} \longleftrightarrow A(k)e^{-ikx} + B(k)e^{ikx}$$

hence we can get at it by WKB, namely find an asymptotic solution

$$u = e^{-ikx}\, e^{v}$$

v [redacted] $= v_0 + \frac{1}{k}v_1 + \frac{1}{k^2}v_2 + \cdots$

of the Schroedinger equation. The v_n will involve [redacted] integrals of q and its derivatives, and will be independent of x [redacted] for $x > \mathrm{Supp}(q)$.

Put $k = iK$, $\quad u = e^{\tilde{v}}$.

$$D^2 e^{\tilde v} = D(e^{\tilde v} D\tilde v) = e^{\tilde v}\left((D\tilde v)^2 + D^2\tilde v\right)$$

$$0 = (-K^2 + D^2 + q)u = (-K^2 + (D\tilde v)^2 + D^2\tilde v + q)e^{\tilde v}.$$

$$-K^2 + (D\tilde{v})^2 + D^2\tilde{v} + q = 0$$

$$\tilde{v} = Kx + v \qquad (D\tilde{v})^2 = (K+v')^2 = K^2 + 2Kv' + (v')^2$$

$$2Kv' + v'^2 + v'' + q = 0 \qquad v = \frac{v_1}{K} + \frac{v_2}{K^2} + \cdots$$

$$2K\left(\frac{v_1'}{K} + \frac{v_2'}{K^2} + \frac{v_3'}{K^3}\right) + \left(\frac{v_1'^2}{K^2} + \frac{2v_1'v_2'}{K^3} \cdots\right) + \left(\frac{v_1''}{K} + \frac{v_2''}{K^2} + \frac{v_3''}{K^3} \cdots\right) + q = 0$$

$$2v_1' + q = 0 \qquad v_1' = -\frac{1}{2}q$$

$$2v_2' + v_1'' = 0 \qquad v_2' = \frac{1}{4}q'$$

$$2v_3' + v_1'^2 + v_2'' = 0 \qquad v_3' = -\frac{1}{2}\left[\frac{q^2}{4} + \frac{q''}{4}\right] = -\frac{q^2}{8} - \frac{q''}{8}$$

$$2v_4' + 2v_1'v_2' + v_3'' = 0 \qquad v_4' = \left(-\frac{1}{2}\right)\left[2\left(-\frac{q}{2}\right)\left(\frac{q'}{4}\right) \text{[redacted]} - \frac{(q^2)'}{8} - \frac{q'''}{8}\right]$$

$$= \frac{(q^2)'}{16} + \frac{qq'}{8} + \frac{q'''}{16}$$

so

$$\log A(k) = -\frac{1}{2K}\int q + \frac{1}{K^2}\int\frac{q'}{4} + \frac{1}{K^3}\int\left(-\frac{q^2}{8} - \frac{q''}{8}\right) + \frac{1}{K^4}\int\left(\frac{(q^2)'}{16} + \frac{qq'}{8} + \frac{q'''}{16}\right)$$

$$= \frac{1}{K}\int\left(-\frac{q}{2}\right) + \frac{1}{K^3}\int\left(-\frac{q^2}{8}\right) + \text{[redacted]} + O\left(\frac{1}{K^5}\right)$$

Another method

$$\log A(k) = \log\det \text{[redacted]} (1 + G_k q)$$

$$= \operatorname{tr}\log(1 + G_k^0 q) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\operatorname{tr}(G_k^0 q)^n$$

$$\operatorname{tr}(G_k^0 q) = \int G(1,1)\,q(1) = \frac{1}{2ik}\int q = -\frac{1}{2K}\int q$$

$$\operatorname{tr}(G_k^0 q)^2 = \int G(1,2)q(2)G(2,1)q(1)$$

$$= \frac{1}{(2ik)^2}\int q(x_1)\,q(x_2)\,e^{2ik|x_1-x_2|}\,dx_1dx_2$$

$$= \frac{2}{(2K)^2}\int q(x)\,dx\int_0^\infty q(x+u)\,du\,e^{-2Ku}$$

$$\int_0^\infty \left(g(x) + g'(x)u + \frac{g''(x)}{2!}u^2\right) e^{-2Ku}\,du = \sum_{n\geq 0} \frac{g^{(n)}(x)}{\cancel{n!}} \frac{\cancel{\Gamma(n+1)}}{(2K)^{n+1}}$$

Hence, $$\operatorname{tr}(G_k^0 g)^2 = \frac{2}{(2K)^2} \sum_{n\geq 0} \frac{1}{(2K)^{n+1}} \int g(x) g^{(n)}(x)$$

$$= \frac{1}{K^3}\int \frac{g^2}{4} + \cdots\cdots$$

Notice that

$$\operatorname{tr}(G_k^0 g)^n = \left(\frac{1}{2ik}\right)^n \int dx_1 \ldots dx_n \, g(x_1)\ldots g(x_n)\, e^{ik\sum_{j=1}^n |x_j - x_{j+1}|}$$

and so for $k = iK$, the contribution to the asymptotic expansion comes from $x_1 = \cdots = x_n$. It would be nice if it were possible to [redacted] replace the ~~discontinuous~~ non-smooth exponential $e^{-K\sum|x_j - x_{j+1}|}$ by a Gaussian so as to be able to evaluate things via diagrams.

January 15, [redacted] 1980

Suppose $H = \frac{p^2}{2} + V$. Let's compute the diagonal [redacted] part of the kernel for e^{-TH} for small T using the path integral:

$$\langle x_0 | e^{-TH} | x_0 \rangle = \int_{\substack{x(0)=x_0 \\ x(T)=x_0}} \mathcal{D}x \; e^{-\int_0^T [\frac{1}{2}\dot{x}^2 + V(x)]dt}$$

To simplify suppose $x_0 = 0$ and rescale the time interval so that $[0,T] = T\cdot[0,1]$. Then the exponent becomes

$$\int_0^1 \left[\frac{1}{2}\left(\frac{dx}{d(Tt)}\right)^2 + V(x)\right] T\,dt = \int_0^1 \left[\frac{1}{T}\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + TV(x)\right] dt$$

Next change x into $\sqrt{T}x$, and then we get the path integral

$$\int_{x(0)=x(1)=0} \mathcal{D}x \; e^{-\int_0^1 [\frac{1}{2}\dot{x}^2 + TV(\sqrt{T}x)] dt}$$

where the $\mathcal{D}x$ is appropriately normalized. Now for small T we use the Taylor expansion

$$TV(\sqrt{T}x) = T\left[V(0) + V'(0)x\sqrt{T} + \frac{V''(0)}{2!}x^2 T + \cdots\cdots\right]$$

When this is plugged into the path integral we get an expansion in powers of T. ~~Starting with~~ The normalization makes

$$\int_{x(0)=x(1)=0} \mathcal{D}x \; e^{-\int_0^1 \frac{1}{2}\dot{x}^2 dt} = \langle 0 | e^{-T\frac{p^2}{2}} | 0 \rangle = \frac{1}{\sqrt{2\pi T}}$$

Now we want a diagram expansion, so we first remove the part $e^{-TV(0)}$

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$H = \frac{p^2}{2m} + V(x)$. We want to compute the amplitude

$$\langle x=0 | e^{-TH} | x=0 \rangle = \int_{x(0)=x(T)=0} \mathcal{D}x \; e^{-\int_0^T \left(\frac{m}{2}\dot{x}^2 + V(x)\right)dt}$$

for small T, so we rescale $t \mapsto tT$, $x \mapsto \sqrt{T}\,x$ to get

$$N \int_{x(0)=x(1)=0} \mathcal{D}x \; e^{-\int_0^1 \left[\frac{m}{2}\dot{x}^2 + V(\sqrt{T}x)T\right]dt}$$

where N is determined to give the right answer for $V=0$, which is

$$\langle x | e^{-T\frac{p^2}{2m}} | x' \rangle = \frac{1}{\sqrt{2\pi(T/m)}} \, e^{-\frac{m(x-x')^2}{2T}} = \frac{1}{\sqrt{2\pi(T/m)}} \text{ at } 0,0$$

Next use the Taylor expansion

$$V(\sqrt{T}x)T = TV(0) + T^{3/2}V'(0)x + T^2\frac{V''(0)}{2!}x^2 + \cdots\cdots$$

This gives

$$\langle x=0 | e^{-TH} | x=0 \rangle = \frac{1}{\sqrt{2\pi T/m}} \, e^{-TV(0)} \, e^{\text{connected diagram terms.}}$$

We have vertices of mult. $p = 1, 2, \ldots$. Suppose k_p = number of vertices of mult. p. Then

$$e = \tfrac{1}{2}(k_1 + 2k_2 + 3k_3 + \cdots)$$
$$v = k_1 + k_2 + k_3 + \cdots$$

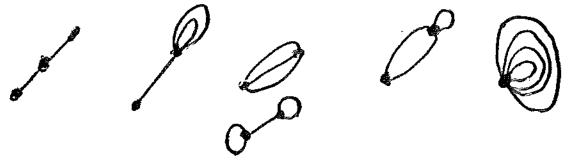
So $e + v = \frac{3}{2}k_1 + 2k_2 + \frac{5}{2}k_3 + 3k_4$ is the power of T belonging to a diagram:

The lower diagrams are:

T^2 : 

T^3 : 

T^4 : 

T^5 : 

Note that  which contributes $T^2 \frac{1}{2} V''(0) \int G(t,t)$ when integrated with o replaced by x will give 0. So the interesting terms are

 $T^3 \frac{1}{2} V'(0)^2 \int G(t_1, t_2)$

 $T^4 \frac{1}{4} V''(0)^2 \int G(t_1,t_2) G(t_2,t_1)$

 $T^4 \frac{1}{2} V'(0) V'''(0) \int G(t_1,t_2) G(t_2,t_2)$

Actually we have to consider terms like

 $T^2 \frac{1}{2} \left(-V''(0)\right) \int G(t,t)$

because these go in the exponential before being integrated.

I need the Green's fn for $-\frac{m}{2}\partial_t^2$ on $[0,1]$ with Dirichlet boundary conditions

$$G(t,t') = -\frac{2}{m} \frac{t_<(t_>-1)}{\begin{vmatrix} t & t-1 \\ 1 & 1 \end{vmatrix}} = +\frac{2}{m} t_< (1-t_>)$$

$$\int_0^1 G(t,t)\,dt = \frac{2}{m} \int_0^1 t(1-t)\,dt = \frac{2}{m}\left(\frac{1}{2}-\frac{1}{3}\right) = \left(\frac{2}{m}\right)\frac{1}{6}$$

$$\int G(t,t)^2\,dt = \left(\frac{2}{m}\right)^2 \int_0^1 t^2(1-t)^2\,dt = \left(\frac{2}{m}\right)^2 \frac{\Gamma(3)\Gamma(3)}{\Gamma(6)} = \left(\frac{2}{m}\right)^2 \frac{2\cdot 2}{120}$$

$$= \left(\frac{2}{m}\right)^2 \frac{1}{30}$$

$$\int G(t_1,t_2)\,dt_1\,dt_2 = \left(\frac{2}{m}\right) 2 \int_0^1 dt_1 \int_0^{t_1} dt_2\; t_2(1-t_1)$$

$$= \left(\frac{2}{m}\right) \cancel{2} \int_0^1 dt_1 (1-t_1)\frac{t_1^2}{\cancel{2}} = \left(\frac{2}{m}\right)\left(\frac{1}{3}-\frac{1}{4}\right) = \left(\frac{2}{m}\right)\frac{1}{12}$$

So

$$\langle 0|e^{-TH}|0\rangle = \frac{1}{\sqrt{2\pi T/m}}\, e^{-TV(0)}\, e^{\quad + \quad + \quad + O(T^4)}$$

$$= \frac{1}{\sqrt{2\pi T/m}}\, e^{-TV(0) + T^2\frac{1}{2}(-V''(0))\left(\frac{2}{m}\frac{1}{6}\right) + T^3\frac{1}{2}(V'(0))^2\left(\frac{2}{m}\frac{1}{12}\right) + T^3\frac{1}{8}(-V''''(0))\left(\left(\frac{2}{m}\right)^2\frac{1}{30}\right) + O(T^4)}$$

Now we want to apply this to $H = -\frac{\partial^2}{\partial x^2}$ [redacted] $- q$ so we want $m=\frac{1}{2}$, $q=-V$. This gives

$$\langle x|e^{-TH}|x\rangle = \frac{1}{\sqrt{4\pi T}}\, e^{Tq + T^2\frac{q''}{3} + T^3\left(\frac{q'^2}{6} - \frac{q''''}{60}\right) + O(T^4)}$$

$$= \frac{1}{\sqrt{4\pi T}}\left(1 + Tq + T^2\left(\frac{q^2}{2} + \frac{q''}{3}\right) + T^3\left(\frac{q^3}{6} + \frac{qq''}{3} + \frac{q'^2}{6} - \frac{q''''}{60}\right) + \cdots\right)$$

It follows that

$$\mathrm{tr}\left(e^{-TH}\right) = \frac{1}{\sqrt{4\pi T}}\left(\mathrm{vol} + T\int q + T^2\int\frac{q^2}{2} + T^3\int\left(\frac{q^3}{6} - \frac{q'^2}{6}\right) + \cdots\right)$$

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Still trying to understand $e^{t(D^2+g)}$ for small t, especially the Minak...-Pleyel result on the diagonal part of the Schwarz kernel.

Put $H = \frac{p^2}{2m} + V$ and see what path integral gives for

$$\langle x_f | e^{-TH} | x_i \rangle = \int_{\substack{x(0)=x_f \\ x(T)=x_i}} \mathcal{D}x \; e^{-\int_0^T [\frac{m}{2}\dot{x}^2 + V(x)]dt}$$

To simplify set $x_i = 0$. Rescale time for the paths $t \mapsto tT$

$$N \int_{\substack{x(0)=0 \\ x(1)=x_f}} \mathcal{D}x \; e^{-\int_0^1 [\frac{m}{2}\frac{\dot{x}^2}{T} + V(x)T]dt}$$

Next we shift by letting $x(t) = tx_f + y(t)$

$$N \int_{y(0)=y(1)=0} \mathcal{D}y \; e^{-\int_0^1 [\frac{m}{2T}(x_f + \dot{y})^2 + V(tx_f + y)] dt}$$

$$\int_0^1 \frac{m}{2T}(x_f^2 + 2x_f\dot{y} + \dot{y})^2 = \frac{m}{2T}x_f^2 + \int_0^1 \frac{m}{2T}\dot{y}^2 dt$$

Finally replace y by $\sqrt{T}y$ so as to get

$$e^{-\frac{m}{2T}x_f^2} N' \int_{y(0)=y(1)=0} \mathcal{D}y \; e^{-\int_0^1 [\frac{m}{2}\dot{y}^2 + V(tx_f + \sqrt{T}y)T] dt}$$

Now one can evaluate N' by letting $V=0$ and it seems one gets 1, because

$$\langle x_f | e^{-TH_0} | x_0 \rangle = \frac{e^{-\frac{m}{2T}x_f^2}}{\sqrt{2\pi T/m}}$$

Next we use Taylor for V:

$$V(tx_f+\sqrt{T}y)T = \sum_{n\geq 0} \frac{V^{(n)}(tx_f)}{n!}\, y^n\, T^{n/2+1}$$

This will lead to a diagram expansion where the coefficients belonging to a vertex depend on t and x_f. I want to work up to T^2 which means that we have only the diagram .

$$e^{-\frac{m}{2T}x_f^2}\frac{1}{\sqrt{2\pi T/m}}\; e^{-T\int_0^1 V(tx_f)dt} \int\limits_{y(0)=y(1)=0} \mathcal{D}y\; e^{-\int_0^1 \frac{m}{2}\dot{y}^2 + V'(tx_f)yT^{3/2} + V''(tx_f)\frac{y^2}{2}T^2}$$

 $\quad T^2\frac{1}{2}\int_0^1 \left(-V''(tx_f)\right) G(t,t)\, dt$

So we seem to be getting

$$e^{-\frac{m}{2T}x_f^2}\frac{1}{\sqrt{2\pi T/m}}\left(1 - T\int_0^1 V(tx_f)dt + \frac{T^2}{2}\left(\left(\int_0^1 V(tx_f)dt\right)^2 - \int_0^1 V''(tx_f)\,G(t,t)dt\right)\right.$$

But $\displaystyle \int_0^1 V(tx)dt = \frac{1}{x}\int_0^1 V(tx)x\,dt = \frac{1}{x}\int_0^x V(x')dx'$

hence

$$\langle x_f|e^{-TH}|0\rangle = e^{-\frac{m}{2T}x_f^2}\frac{1}{\sqrt{2\pi T/m}}\left(1 - \frac{T}{x_f}\int_0^{x_f} V(x')dx' + O(T^2)\right)$$

Another approach: Try to get at $e^{T(D^2+g)}$ using Mellin transform

$$\Gamma(s)\,(D^2+g)^{-s} = \int_0^\infty e^{T(D^2+g)}\, T^s \frac{dT}{T}$$

together with the fact that $(-(D^2+g))^{-s}$ has a ψDO

expansion.

$$(D^2+q)^{-s} = \left(\sum a_n(x,s)\, D^{-n}\right) D^{-2s}$$

$$(D^2+q)^{-s+1} = \sum_{n\geq 0} (D^2+q)\, a_n(s)\, D^{-n-2s}$$

$$= \sum_{n\geq 0} \left[a_n(s) D^2 + 2\partial_x a_n(s) D + \partial_x^2 a_n(s) + q\, a_n(s)\right] D^{-n-2s}$$

$$= \sum_{n\geq 0} \left[a_n(s) D^{-n} + 2\partial_x a_n(s) D^{1-n} + (\partial_x^2+q)\, a_n D^{-2-n}\right] D^{2-2s}$$

$$= \sum_{n\geq 0} \left[a_n(s) + 2\partial_x a_{n-1}(s) + (\partial_x^2+q)\, a_{n-2}(s)\right] D^{-n-2s+2}$$

gives recursion formula

$$a_n(s-1) = a_n(s) + 2\partial_x a_{n-1}(s) + (\partial_x^2+q)\, a_{n-2}(s)$$

which allows us to grind these out starting from $(D^2+q)^0 = 1$

$$a_0(s-1) = a_0(s) = 1$$

$$a_1(s-1) = a_1(s) + \cancel{2\partial_x a_0} = 0$$

$$a_2(s-1) = a_2(s) + q \qquad a_2(s) = -sq$$

$$a_3(s-1) = a_3(s) - 2sq' \qquad a_3(s) = s(s+1)q'$$

$$a_4(s-1) = a_4(s) + 2s(s+1)q'' + (\partial_x^2+q)(-sq)$$

$$= a_4(s) + q''[2s^2+s] + q^2[-s]$$

$$(D^2+q)^{-s} = \left(1 - sq D^{-2} + s(s+1)q' D^{-3} + \cdots\right) D^{-2s}$$

$$H^{-s} = \left(1 - sq D^{-2} + s(s+1)q' D^{-3} + \cdots\right) H_0^{-s}$$

$$H^{-s} = \left(1 + sq H_0^{-1} + s(s+1)q' D H_0^{-2} + \cdots\right) H_0^{-s}$$

$$\Gamma(s) H^{-s} = \Gamma(s) H_0^{-s} + q\,\Gamma(s+1) H_0^{-s-1} + q' D\,\Gamma(s+2) H_0^{-s-2} + \cdots$$

which on taking the inverse Mellin transform leads to

$$e^{-tH} = e^{-tH_0} + gte^{-tH_0} + g'Dt^2e^{-tH_0} +$$

However such an expansion should come out of the Campbell-Hausdorff formula:

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$$

$$e^{t(D^2+g)}e^{-tD^2} = e^{t(g)+t^2\frac{1}{2}[D^2+g,-D^2]}$$

$$= e^{tg+\frac{t^2}{2}[D^2,g]+\cdots}$$

$$= e^{tg+t^2(g'D+g''/2)+\cdots}$$

$$= 1 + tg + t^2\left(\frac{g^2}{2}+g'D+\frac{g''}{2}\right) + O(t^3)$$

Here's a rapid way to get this expansion

$$\frac{d}{dt}\, e^{+t(D^2+g)}e^{-tD^2} = e^{+t(D^2+g)}\, g\, e^{-tD^2}$$

$$e^{t(D^2+g)}e^{-tD^2} = 1+\int_0^t dt'\, e^{t'(D^2+g)}e^{-t'D^2}\left(e^{t'D^2}ge^{-t'D^2}\right)$$

$$e^{T(D^2+g)}e^{-TD^2} = 1+\int_0^T dt_1\, e^{t_1D^2}ge^{-t_1D^2} + \iint_{t_1>t_2} e^{t_2D^2}ge^{-t_2D^2}e^{t_1D^2}ge^{-t_1D^2}+\cdots$$

$$= 1+\int_0^T \sum \frac{t_1^n}{n!}(\mathrm{ad}\,D^2)^n g\, dt_1 + \sum_{n_1,n_2}\frac{(\mathrm{ad}\,D^2)^{n_2}g\,(\mathrm{ad}\,D^2)^{n_1}g}{n_2!\,n_1!}\iint_{T\geq t_1>t_2\geq 0} t_2^{n_2}t_1^{n_1}+\cdots$$

$$= 1+Tg+\frac{T^2}{2}[D^2,g]+\frac{T^3}{6}[D^2,[D^2,g]] + O(T^4)$$

$$+\frac{T^2}{2}g^2 + \left([D^2,g]g\iint_{t_1>t_2} t_2 + g[D^2,g]\iint_{t_1>t_2} t_1\right)$$

$$+\frac{T^3}{6}g^3$$

Now

$$\iint_{t_1>t_2} t_2^{n_2} t_1^{n_1} = \int_0^T dt_1\, t_1^{n_1} \int_0^{t_1} dt_2\, t_2^{n_2} = \int_0^T dt_1 \frac{t_1^{n_1+n_2+1}}{n_2+1} = \frac{T^{n_1+n_2+2}}{(n_2+1)(n_1+n_2+1)}$$

$$\therefore \iint_{t_1>t_2} (t_2+t_1) = T^3\left(\frac{1}{2\cdot 2}+\frac{1}{2\cdot 2}\right) = \frac{T^3}{2}$$

So

$$e^{T(D^2+g)}e^{-TD^2} = 1 + Tg + \frac{T^2}{2}\left(g^2 + [D^2,g]\right) + \frac{T^3}{6}[D^2,[D^2,g]] + T^3 g[D^2,g] + O(T^4)$$

$$[D^2,g] = 2g'D + g''$$

$$[D^2,[D^2,g]] = 2[D^2,g']D + [D^2,g'']$$

$$= 2(2g''D + g''')D + 2g'''D + g''''$$

$$= 4g''D^2 + 4g'''D + g''''$$

so

$$e^{T(D^2+g)}e^{-TD^2} = \left(1 + Tg + T^2\left(\frac{g^2}{2} + g'D + \frac{g''}{2}\right)\right.$$

$$\left. + T^3\left(\frac{2}{3}g''D^2 + \left(\frac{2}{3}g''' + 2gg'\right)D + \frac{1}{6}g'''' + gg'' + \frac{g^3}{6}\right) + O(T^4)\right)$$

I guess it's clear that

$$\langle x|e^{T(D^2+g)}|x'\rangle = \int \frac{dp}{2\pi} e^{ip(x-x')} \underbrace{\left(1 + Tg + T^2\left(\frac{g^2}{2} + g'D + \frac{g''}{2}\right)\right)}_{\text{evaluated at } x} e^{TD^2}$$

where $D = ip$.

Thus

$$\langle x|e^{T(D^2+g)}|x\rangle = \left(\int \frac{dp}{2\pi} e^{-Tp^2}\right)\left(1 + Tg + T^2\left(\frac{g^2}{2} + g'ip + \frac{g''}{2}\right) + \cdots\right)(x)$$

$$= \frac{1}{\sqrt{4\pi T}}\left(1 + Tg + \frac{T^2}{2}(g^2 + g'') + \cdots\cdots\right)$$

However one has to be careful because

$$\int \frac{dp}{2\pi}\, p^2 e^{-Tp^2} = \frac{1}{\sqrt{4\pi T}}\,\frac{1}{2T}$$

lowers the degree of T. Hence we get $-\frac{2}{3} q'' \frac{1}{2T} T^3$ addition to T^2. So assuming no more stuff in T^2 from above

$$\operatorname{tr} e^{T(D^2+q)} = \frac{1}{\sqrt{4\pi T}}\left(\mathrm{vol} + T\int q + \frac{T^2}{2}\int q^2 + O(T^3)\right)$$

It appears this method is not as easy as the one based on diagrams for obtaining $\langle x | e^{-TH} | x' \rangle$.

Here seems to be the Minak...-Pleyel way to obtain the heat kernel. Look for an expansion

$$e^{t(D^2+q)} = \sum_{n\geq 0}{}' a_n(x,D)\, t^n e^{tD^2}$$

Then

$$\partial_t e^{t(D^2+q)} = \sum_{n\geq 0}{}' \left(n a_n t^{n-1} + a_n D^2 t^n\right) e^{tD^2}$$

$$= \sum{}' \left((n+1)a_{n+1} + a_n D^2\right) t^n e^{tD^2}$$

$$(D^2+q)\, e^{t(D^2+q)} = \sum (D^2+q)\, a_n\, t^n e^{tD^2}$$

Thus

$$(D^2+q)\,a_n = (n+1)\,a_{n+1} + a_n D^2$$

or

$$(n+1)\,a_{n+1} = [D^2, a_n] + q a_n$$

$$= 2a_n' D + a_n'' + q a_n .$$

Here $a_n = \sum a_{nk}(x) D^k$ and

$$[D^2, a_n] = \sum{}' [D^2, a_{nk}] D^k = \underbrace{\sum 2a_{nk}' D^k D}_{2a_n' D} + \underbrace{\sum a_{nk}'' D^k}_{a_n''}$$

Thus we get a recursion relation from which the $a_n(x, D)$ can be found:

$$a_{n+1} = \frac{1}{n+1}\left((\partial_x^2 + q)a_n + 2\partial_x(a_n)D \right)$$

$$a_0 = 1$$

$$a_1 = q$$

$$a_2 = \frac{1}{2}\left[(\partial_x^2+q)q + 2\partial_x q\, D\right] = \frac{1}{2}\left[q'' + q^2 + 2q'D\right]$$

It clear from this formula that a_n is a differential operator of order $\leq n-1$. ∎

From this expansion one can, at least formally, obtain the heat kernel

$$\langle x | e^{t(D^2+q)} | x' \rangle = \int \frac{dp}{2\pi} e^{ip(x-x')} \left(\sum_{n\geq 0} a_n(x, ip) t^n \right) e^{-tp^2}$$

[redacted] Observe that when one does the Gaussian integrals one introduces factor $\frac{1}{2t}$ for each contraction. However $a_n = \sum_{k<n} a_{nk}(x)$ [redacted] D^k, so [redacted] that the term $a_{nk} D^k t^n$ upon contraction becomes proportional to $a_{nk} t^{n-k/2}$; this holds for $x = x'$ at least. For $x \neq x'$ use

$$\int \frac{dp}{2\pi} e^{ip(x-x')} p^k e^{-tp^2} = \int \frac{dp}{2\pi} e^{-t\left(p - i\frac{x-x'}{2t}\right)^2 - \frac{(x-x')^2}{4t}} p^k$$

$$= e^{-\frac{(x-x')^2}{4t}} \int \frac{dp}{2\pi} e^{-tp^2} \left(p + i\frac{x-x'}{2t}\right)^k$$

When expanded out one gets $p^a \frac{1}{t^b}$ with $a+b=k$ and the p^a contracts to $t^{-a/2}$, so one gets $t^{-a/2-b}$ where $a+b = k < n$. Since $b =$ [redacted] $n-1$ is possible for each n it is not clear that one gets an expansion

$$\langle x | e^{t(D^2+q)} | x' \rangle = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x')^2}{4t}} \sum_m \alpha_m(x, x') t^m$$

although this seems to be true by p. 548

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$$H_0 = -D^2 \qquad \Gamma(s)A^{-s} = \int_0^\infty e^{-tA}\, t^s \frac{dt}{t}$$

$$e^{-tH_0} = e^{tD^2}$$

$$\Gamma(s)\langle x | H_0^{-s} | x' \rangle = \int_0^\infty \langle x | e^{tD^2} | x' \rangle t^s \frac{dt}{t}$$

$$= \int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x')^2}{4t}} t^s \frac{dt}{t} \qquad t^{s-\frac{1}{2}}$$

$$= \int_0^\infty e^{-\frac{(x-x')^2}{4}u} \frac{1}{\sqrt{4\pi}} u^{-s+\frac{1}{2}} \frac{du}{u}$$

$$= \frac{1}{\sqrt{4\pi}} \Gamma(-s+\tfrac{1}{2}) \left[\left(\frac{x-x'}{2}\right)^2\right]^{+s-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{4\pi}} \Gamma(-s+\tfrac{1}{2}) \, |x-x'|^{2s-1} / 2^{2s-1}$$

Also

$$\langle x | H_0^{-s} | x' \rangle = \int \frac{dp}{2\pi} |p|^{-2s} e^{ip(x-x')}$$

$$\int_0^\infty \frac{dp}{2\pi} p^{-2s} e^{ipx} = \frac{1}{2\pi}\int_0^\infty e^{-(-ix)p} p^{-2s+1} \frac{dp}{p}$$

$$= \frac{1}{2\pi} \Gamma(1-2s) (-ix)^{2s-1}$$ x comes from UHP

$$\int_{-\infty}^0 \frac{dp}{2\pi} |p|^{-2s} e^{ipx} = \int_0^\infty \frac{dp}{2\pi} |p|^{-2s} e^{-ipx}$$

$$= \frac{1}{2\pi} \Gamma(1-2s) (ix)^{2s-1}$$ x comes from LHP

$$\therefore \int_{-\infty}^\infty \frac{dp}{2\pi} |p|^{-2s} e^{ipx} = \frac{1}{2\pi} \Gamma(1-2s) |x|^{2s-1} \underbrace{\left(e^{-i\frac{\pi}{2}(2s-1)} + e^{i\frac{\pi}{2}(2s-1)} \right)}_{2\cos\left(\pi s - \frac{\pi}{2}\right)}$$

$$= \frac{\sin \pi s}{\pi} \Gamma(1-2s) |x|^{2s-1}$$

Thus $\langle x | H_0^{-s} | x' \rangle = \frac{\sin(\pi s)}{\pi} \Gamma(1-2s) |x-x'|^{2s-1}$

which agrees with what one gets from e^{tD^2} since

$$\frac{1}{\sqrt{4\pi}} \frac{\Gamma(-s+\frac{1}{2})}{\Gamma(s)} 2^{-2s+1} = \frac{1}{\sqrt{\pi}} \underbrace{\Gamma(-s+\tfrac{1}{2})\Gamma(-s+1)}\frac{\sin \pi s}{\pi} 2^{-2s}$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi}} \Gamma(-2s+1)\, 2^{-[2(-s+\frac{1}{2})-1]} \frac{\sin \pi s}{\pi} 2^{-2s}$$

$$= \Gamma(1-2s) \frac{\sin \pi s}{\pi}$$

using Legendre's duplication formula

We have

$$e^{t(D^2+q)} = \left(1 + tq + t^2\left(\frac{q''+q^2}{2} + q'D\right) + \cdots\right) e^{tD^2}$$

hence

$$\Gamma(s) H^{-s} = \Gamma(s) H_0^{-s} + q\,\Gamma(s+1) H_0^{-(s+1)} + \left(\frac{q''+q^2}{2} + q'D\right) \Gamma(s+2) H_0^{-(s+2)} + \cdots$$

$$H^{-s} = H_0^{-s} + q\, s\, H_0^{-(s+1)} + (\qquad\qquad) s(s+1) H_0^{-(s+2)} + \cdots$$

Now the singularities of H_0^{-s} occur at $s = \frac{1}{2}, \frac{3}{2}, \ldots$ where it has simple poles. Actually I am confused here because I want to be able to take the trace, and setting $x = x'$ is valid when $|x|^{2s-1}$ is integrable i.e. $2s-1 > -1$.

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There's some confusion over the operator H_0^{-s} where $H_0 = -D^2$. Over $\mathbb{R}/L\mathbb{Z}$ we have to omit the 0 eigenvalue and define

$$\langle x | H_0^{-s} | x' \rangle = \sum_{k \neq 0} e^{ik(x-x')} |k|^{-2s} \qquad k \in \frac{2\pi}{L}\mathbb{Z}$$

One has

$$\langle x | e^{-tH_0} | x' \rangle = \sum_k e^{ik(x-x')} e^{-tk^2}$$

so the Mellin transform is

$$\frac{1}{\Gamma(s)} \int_0^\infty \sum_k e^{ik(x-x')} e^{-tk^2} t^s \frac{dt}{t}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^s \frac{dt}{t} + \sum_{k \neq 0} e^{ik(x-x')} |k|^{-2s}$$

I guess you interpret the first term as 0 or $2\pi\delta(s)$.

What functions have Mellin transforms? Analogous to what sequences a_n $n \in \mathbb{Z}$ have Laurent series $\sum a_n z^n$? If one splits the series into $\sum_{n \geq Q} a_n z^n$ which converges for $|z| < R_1$ and $\sum_{n<Q} a_n z^n$ which converges for $|z| > R_2$, then provided $R_1 > R_2$ one has convergence in an annulus $R_2 < |z| < R_1$. Suppose we restrict attention to rational functions, so that we can always analytically continue past R_1, R_2, then we can assign a sum to the series $\sum a_n z^n$, which is a rational function. Finally the rational function doesn't determine the Laurent series but only up to the invariant subspace defined by the Euler series:

$$\sum \zeta^n z^n \qquad \zeta \in \mathbb{C}^*.$$

Thus it appears that taking a Mellin transform is usually a valid operation, but that inverting it depends on choosing an annulus.

So next look at what happens over $\mathbb{R}$

$$\langle x | H_0^{-s} | x' \rangle = \int \frac{dk}{2\pi} e^{-ik(x-x')} |k|^{-2s}$$

$$= \frac{1}{\sqrt{4\pi}} \frac{\Gamma(\frac{1}{2}-s)}{\Gamma(s)\, 2^{2s-1}} |x-x'|^{2s-1}$$

This has simple poles when $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\ldots\ldots$. Somehow these poles are due to the fact that we allow $k \to 0$, and they don't appear for the box.