July 13, 1980

Return to ferromagnetism model. Suppose we have N-spin variables s_i and the energy of a configuration $s = \{s_i\}$ is

$$E(s) = -H \sum s_i - \frac{g}{2} \sum_{i \neq j}' s_i s_j$$

$$= -H \sum s_i - \frac{g}{2}\left(\sum s_i\right)^2 + \underset{\text{set } = 0.}{\underset{\uparrow}{\text{const}}}$$

The partition function is

$$Z = \sum_s e^{-\beta E(s)}$$

$$= \sum_{n=0}^{N} \binom{N}{n} e^{\beta H \Sigma + \frac{\beta g}{2}(\Sigma)^2} \qquad \Sigma = 2n - N$$

Let's determine the most likely n by finding dominant term

$$\frac{d}{dn}\left(\log \frac{N!}{n!\,(N-n)!} + \beta H(2n-N) + \frac{\beta g}{2}(2n-N)^2\right)$$

$$= -\log n + \log(N-n) + 2\beta H + \frac{\beta g}{\cancel{2}}\cancel{2}(2n-N)2 = 0$$

$$\log\left(\frac{n}{N-n}\right) = 2\beta\left(H + gN\frac{\Sigma}{N}\right)$$

Let's put $\bar{s} = \frac{\Sigma}{N}$ for the average spin and put

$$B = H + gN\bar{s} \qquad \text{for the average magnetic}$$

field. Finally

$$\frac{n}{N-n} = \frac{\frac{1}{2}(\Sigma + N)}{\frac{1}{2}(N - \Sigma)} = \frac{1+\bar{s}}{1-\bar{s}}$$

so the equation becomes

$$\frac{1+\bar{s}}{1-\bar{s}} = e^{2\beta B} \qquad \text{or}$$

Thus we get the Weiss equations:

$$\boxed{\begin{aligned} \bar{s} &= \frac{e^{\beta B} - e^{-\beta B}}{e^{\beta B} + e^{-\beta B}} \\ B &= H + gN\bar{s} \end{aligned}}$$

The previous approach to these equations proceeded by using a Gaussian transform

$$e^{\frac{1}{2}\beta g \Sigma^2 + \beta H \Sigma} = \frac{1}{\sqrt{2\pi\beta g}} \int_{-\infty}^{\infty} e^{\Sigma x}\, e^{-\frac{x^2}{2\beta g} + \frac{H}{g}x - \frac{\beta H^2}{2g}}\, dx$$

Plug this into the partition function, use

$$\sum_{S} e^{(\sum s_i)x} = (e^x + e^{-x})^N$$

and the partition function becomes

$$Z = \frac{1}{\sqrt{2\pi\beta g}} \int_{-\infty}^{\infty} (e^x + e^{-x})^N\, e^{-\frac{x^2}{2\beta g} + \frac{H}{g}x - \frac{\beta H^2}{2g}}\, dx$$

Now look for dominant part of integral and get

$$N\,\frac{e^x - e^{-x}}{e^x + e^{-x}} - \frac{x}{\beta g} + \frac{H}{g} = 0$$

which is the Weiss equation with $x = \beta B$.

July 16, 1980

Time to understand a little about the RG = renormalization group. Let's follow Jona-Lasinio* and think in probability terms. (*Adv. in Phys. Vol 27)

First example: Ising model with N independent spins $s_i = \mp 1$, $i = 1, \dots, N$. Put $\Sigma = \sum s_i$. Then Σ is a random variable and one has a probability distribution $p_N(\Sigma)$ which one can work out as follows. The partition fn. is

$$Z_N = \sum_{S=\{s_i\}} e^{-\beta E_S} \qquad E_S = -H\sum s_i$$

hence

$$p_N(\Sigma) = \sum_n \frac{1}{Z_N}\binom{N}{n}\left(e^{+\beta H}\right)^{\Sigma}\delta(\Sigma-(2n-N)) \quad \text{where } \Sigma = 2n-N$$

Next consider $2N$ independent spins as 2 groups of N spins. We then get

$$(*) \qquad p_{2N}(\Sigma) = \sum_{\Sigma',\Sigma''} p_N(\Sigma')p_N(\Sigma'')\,\delta(\Sigma-\Sigma'-\Sigma'')$$

because Σ', Σ'' are independent random variables. The idea of the renormalization grp is that, after suitable scaling one should be able make the ~~...~~ distributions $p_N(\Sigma)$, $N = 2^k$ converge to some limit distribution p_∞ which should be a fixpt. for the above transformation $(*)$ which gives p_{2N} from p_N.

In this case things are easily handled by the method of characteristic fns. Put

$$f_N(\xi) = \int e^{i\xi\Sigma}\, p_N(\Sigma)\, d\Sigma$$

Then (*) becomes

$$f_{2N}(\xi) = \left(f_N(\xi)\right)^2$$

In fact we know that

$$f_N(\xi) = \left(f_1(\xi)\right)^N$$

where

$$f_1(\xi) = \frac{1}{e^{\beta H}+e^{-\beta H}}\left(e^{\beta H + i\xi} + e^{-\beta H - i\xi}\right)$$

$$= pe^{i\xi} + qe^{-i\xi}$$

$p = \frac{e^{\beta H}}{e^{\beta H}+e^{-\beta H}}$

$p+q=1.$

Look more generally at the characteristic function of a probability distribution

$$f(\xi) = \int e^{-ix\xi}\, d\mu(x)$$

Then

$$|f(\xi)| \leq f(0) = 1$$

for all real ξ. (Suppose $|f(\xi)| = 1$, i.e. $f(\xi) = e^{i\alpha}$. Then

$$1 = f(\xi)e^{-i\alpha} = \int e^{ix\xi - i\alpha}\, d\mu$$

$$1 = \int d\mu$$

$$\Longrightarrow \quad 0 = \int \left(1 - e^{ix\xi - i\alpha}\right) d\mu$$

Take real parts

$$0 = \int \underbrace{\left(1 - \cos(x\xi - \alpha)\right)}_{\geq 0}\, d\mu$$

which shows $d\mu$ is supported on $\{x \mid x\xi - \alpha \in 2\pi\mathbb{Z}\}$.

Thus $|f(\xi)|$ has $\xi = 0$ for its unique maximum unless $d\mu$ is supported on a set of the form $a + b\mathbb{Z}$ in which case f is periodic. (more generally $f(\xi + \xi_0) = e^{i\alpha\xi_0} f(\xi)$)

So now we look at the function

$$f(\xi)^N$$

for large N and conclude it is small away from $\xi = 0$ (or $\xi \in \mathbb{Z}\xi_0$). It has no limit as $N \to \infty$ as a characteristic function, so it is necessary to rescale.

Let's suppose $d\mu$ has first + 2nd moments whence $f(\xi)$ is of class C^2 and we have

$$f(\xi) = 1 + i\xi c_1 - \frac{\xi^2}{2} c_2 + \cdots$$

near $\xi = 0$. Here $c_\nu = \int x^\nu \, d\mu$. Also

$$\log f(\xi) = i\xi c_1 - \frac{\xi^2}{2}(c_2 - c_1^2) + \cdots$$

So

$$f(\xi)^N = e^{iN\xi c_1 - \frac{N\xi^2}{2}(c_2 - c_1^2) + \cdots}.$$

The first thing to do is to replace ξ by ξ/N since

$$f(\xi)^N = \int e^{i\xi \sum_1^N x_i} \, d\mu_N$$

it follows $f(\xi/N)^N$ is the characteristic function of $\frac{1}{N}\sum_1^N x_i$. Since

$$f(\xi/N)^N = e^{i\xi c_1 - \frac{\xi^2}{2N}(c_2 - c_1^2) + \cdots}$$

$$\longrightarrow e^{i\xi c_1} \quad \text{as } N \to \infty$$

it follows the distribution of the r.v.

$\frac{1}{N}\sum_1^N x_i$ approaches $\delta(x-c_1)$. Finally if $c_1=0$, or one [redacted] centers the distribution $d\mu$ at c_1, then

$$f(\xi/\sqrt{N})^N \longrightarrow e^{-\frac{\xi^2}{2}c_2}$$

which means that $\frac{1}{\sqrt{N}}\sum_1^N x_i$ has its distribution approaching a normal distribution.

Next I want to consider the Weiss model which has partition fn.

$$\sum_{S=\{S_i\}} e^{-\beta E_S} \qquad E_S = -\left(H\sum s_i + \frac{g}{2}\left(\sum s_i\right)^2\right)$$

July 17, 1980

Mark Reynolds problem: Consider the space $\mathbb{R}^{\mathbb{N}}$ of infinite sequences [redacted] of real numbers (x_n), $n \geq 0$. On this space we can put the Gaussean measure

$$d\mu_0 = \bigotimes_n e^{-x_n^2/2} \frac{dx_n}{\sqrt{2\pi}}$$

Let a_n be a sequence of positive numbers. The function

$$e^{-\frac{1}{2}\sum a_n x_n^2} = \prod_n e^{-\frac{1}{2}a_n x_n^2}$$

is a monotone limit of the functions

$$\prod_{n \leq N} e^{-\frac{1}{2}a_n x_n^2}$$

and one has

$$\int \prod_{n\leq N} e^{-\frac{1}{2}a_n x_n^2} d\mu_0 = \prod_{n \leq N} (1+a_n)^{-1/2}$$

hence by monotone convergence (or dominated convergence) one knows that

$$\int e^{-\sum \frac{1}{2}a_n x_n^2} d\mu_0 = \left[\prod (1+a_n)\right]^{-1/2}.$$

In more suggestive language this says

$$\int e^{-\frac{1}{2}(\phi, A\phi)} d\mu_0 = \det(1+A)^{-1/2}$$

This result isn't interesting unless $tr(A) = \sum a_n < \infty$, for otherwise it says that $e^{-\frac{1}{2}\sum a_n x_n^2}$ is 0.

almost everywhere.

According to Simon (book on fnl. integration) if A is Hilbert-Schmidt: $\mathrm{tr}(A^2) = \sum a_n^2 < \infty$, then one can do the following. Put

$$f_N' = \prod_{n \le N} \left(e^{-\frac{1}{2} a_n x_n^2} \, e^{\frac{1}{2} a_n} \right)$$

so that

$$\int f_N' \, d\mu_0 = \left[\prod_{n \le N} (1 + a_n) e^{-a_n} \right]^{-1/2}$$

Now

$$\log\left[(1+a)e^{-a}\right] = -\frac{a^2}{2} + \frac{a^3}{3} - \cdots = O(a^2)$$

so that when $\sum a_n^2 < \infty$, the infinite product

$$\prod (1 + a_n) e^{-a_n}$$

converges. It is called $\det_2(1+A)$. Similarly one can define

$$\det_3(1+A) = e^{\mathrm{tr} \log(1+A) - \mathrm{tr} A + \frac{1}{2} \mathrm{tr}(A^2)}$$

Simon claims that for A Hilbert-Schmidt the sequence f_N converges in $L^1(d\mu_0)$. Reynolds wants to extend this one step further. Define

$$f_N'' = \prod_{n \le N} e^{-\frac{1}{2} a_n x_n^2} \, e^{\frac{1}{2} a_n - \frac{1}{4} a_n^2}$$

so that

$$\int f_N'' \, d\mu_0 = \underbrace{\left[\prod_{n \le N} (1 + a_n) e^{-a_n + \frac{1}{2} a_n^2} \right]^{-1/2}}_{\longrightarrow \det_3(1+A)}$$

He wants to show f_N'' converges in $L^1(d\mu_0)$ when $tr(A^3) < \infty$. More generally one could ask about the functions

$$\tilde{f}_N = \prod_{n \le N} e^{-\frac{1}{2} a_n x_n^2} (1+a_n)^{1/2}$$

which have $\int \tilde{f}_N \, d\mu_0 = 1.$

If one knows $\tilde{f}_N$ converges in $L^1(d\mu_0)$, then $\lim \tilde{f}_N d\mu_0$ is a probability measure absolutely continuous wrt $d\mu_0$.

All of the functions $f_N', f_N'', \tilde{f}_N$ are of the form

$$\prod_{n \le N} g_n$$

where g_n is a function of x_n alone and $g_n > 0$. Thus if we denote by f_N the above product we have for $N > M$

$$\int |f_N - f_M| \, d\mu_0 = \int \Big| \prod_{M+1}^{N} g_n - 1 \Big| f_M \, d\mu_0 .$$

Now f_M depends upon $x_1, \dots, x_M$ and $\prod_{M+1}^{N} g_n$ depends on $x_{M+1}, \dots, x_N$ and $d\mu_0$ is a product measure, so the above integral is

$$\int \Big| \prod_{M+1}^{N} g_n - 1 \Big| \, \overbrace{d\mu_0(x_{M+1}, \dots, x_N)}^{\text{Gaussian measure with indicated variables}} \cdot \underbrace{\int f_M \, d\mu_0(x_1, \dots, x_M)}_{\text{converges by construction as } M \to \infty}$$

g_n has the form

$$e^{-\frac{1}{2} a_n (x_n^2 - c_n)}$$

$c_n = 1$ Simon
$c_n = 1 - \frac{1}{2} a_n$ Reynolds
$c_n = \frac{1}{a_n} \log(1+a_n)$ for $\tilde{f}$.

where $c_n \to 1$ as $a_n \to 0$.
So we have to estimate

$$\int \left| e^{-\frac{1}{2}\sum_{M+1}^{N} a_n (x_n^2 - c_n)} - 1 \right| e^{-\frac{1}{2}\sum_{M+1}^{N} x_n^2} \prod_{M+1}^{N} \frac{dx_n}{\sqrt{2\pi}}$$

Clearly if we keep $N-M$ fixed this will go to 0 since the a_n are approaching zero. Hence the only way to get a counterexample is to let $N-M \to \infty$. Let's try to construct a counterexample where the sequence a_n has large constant blocks of increasing size, e.g.

$$1 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}$$

Then
$$\sum a_n^2 = 1 + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{3}\right)^2 + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$$

$$\sum a_n^3 = 1 + 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{3}\right)^3 + \cdots = \zeta(2) < \infty.$$

Thus if we have N a's all equal to a we want to evaluate

$$\int \left| e^{-\frac{1}{2} a (|x|^2 - Nc)} - 1 \right| e^{-\frac{1}{2}|x|^2} \prod_{1}^{N} \frac{dx_n}{\sqrt{2\pi}}$$

which is
$$> \boxed{\int_{|x|^2 \leq Nc} \left(e^{-\frac{1}{2} a (|x|^2 - Nc)} - 1 \right) e^{-\frac{1}{2}|x|^2} \prod_{1}^{N} \frac{dx_n}{\sqrt{2\pi}}}$$

In order to evaluate this we obviously want to compute

$$\int_{|x|^2 \le Nc} e^{-\frac{1}{2}|x|^2} \prod_1^N \frac{dx_n}{\sqrt{2\pi}}$$

Actually it's interesting to ask how big a ball in $\mathbb{R}^N$ is needed to get most of the Gaussian measure.

$$(*) \quad \int_{|x| \le R} e^{-\frac{1}{2}|x|^2} \prod_1^N \frac{dx_n}{\sqrt{2\pi}} = \int_0^R e^{-\frac{1}{2}r^2} \frac{vol(S^{N-1})\, r^{N-1} dr}{(\sqrt{2\pi})^N}$$

Since

$$1 = \underbrace{\int_0^\infty e^{-\frac{1}{2}r^2} r^N \frac{dr}{r}}_{} \; \frac{vol(S^{N-1})}{(2\pi)^{N/2}}$$

$u = r^2$

$\frac{du}{u} = 2\frac{dr}{r}$

$$\int_0^\infty e^{-\frac{1}{2}u} u^{N/2} \frac{du}{2u} = \frac{1}{2} \frac{\Gamma(N/2)}{(\frac{1}{2})^{N/2}}$$

$$\therefore \quad 1 = \frac{\Gamma(N/2)}{2} \frac{vol(S^{N-1})}{\pi^{N/2}}$$

$$\boxed{vol(S^{N-1}) = 2\frac{\pi^{N/2}}{\Gamma(N/2)}}$$

check:

$vol(S^0) = 2\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2})} = 2$

$vol(S^1) = 2\pi$

$vol(S^2) = 2\frac{\pi^{3/2}}{\frac{1}{2}\sqrt{\pi}} = 4\pi$

Thus

$$(*) = \frac{\int_0^R e^{-\frac{1}{2}r^2} r^{N-1} dr}{\int_0^\infty \quad "}$$

and for large N the integrand has a peak (recall derivation of Stirling's formula) around where

$$0 = \frac{d}{dr}\left(-\frac{1}{2}r^2 + N\log r\right) = -r + \frac{N}{r} = 0 \qquad \text{or } r = \sqrt{N}$$

Hence if $R=\sqrt{Nc}$ with $c \uparrow 1$ with N, it's clear we have to be careful about the estimates. On the other hand the estimates can perhaps be done by the steepest descent method.

First ~~[illegible]~~ rescale

$$\int_{|x|^2\le Nc} e^{-\frac{1}{2}a(|x|^2-Nc)}\, e^{-\frac{1}{2}|x|^2}\prod_{1}^{N}\frac{dx_n}{\sqrt{2\pi}} = e^{\frac{N}{2}ac}\int_{|x|^2\le Nc} e^{-\frac{1}{2}(1+a)|x|^2}\prod\frac{dx_n}{\sqrt{2\pi}}$$

Set $y=\sqrt{1+a}\,x$

$$= e^{\frac{N}{2}ac}\int_{|y|^2\le(1+a)Nc} e^{-\frac{1}{2}|y|^2}\prod\frac{dy_n}{\sqrt{2\pi}}\,\frac{1}{(\sqrt{1+a})^N}$$

$$= e^{\frac{N}{2}[ac-\log(1+a)]}\int_{|y|^2\le(1+a)Nc} e^{-\frac{1}{2}|y|^2}\prod_{1}^{N}\frac{dy_n}{\sqrt{2\pi}}$$

Thus the boxed quantity at the bottom of p. 913 is

Gaussian N-dim'l measure

$$\underbrace{e^{\frac{N}{2}[ac-\log(1+a)]}}\int_{|x|^2\le(1+a)Nc} dG_N \;-\; \int_{|x|^2\le Nc} dG_N$$

In my case: $c=\frac{1}{a}\log(1+a)$, this is 1. In Reynolds case

$$ac-\log(1+a) = a\left(1-\frac{a}{2}\right)-\left(a-\frac{a^2}{2}+\frac{a^3}{3}-\cdots\right)$$

$$= -\frac{a^3}{3}+\cdots = O(a^3)$$

Since we want $\sum a_n^3<\infty$ and since we have a block of N a_n all equal to a we must have as the blocks get bigger $Na^3 \to 0$. So in constructing a counterexample I can replace $e^{\frac{N}{2}[ac-\log(1+a)]}$ by 1.

It remains to estimate

$$\circledast \qquad \int\limits_{Nc \le |x|^2 \le (1+a)Nc} dG_N = \frac{\displaystyle\int_{Nc \le r^2 \le Nc(1+a)} e^{-\frac{1}{2}r^2} r^{N-1}\, dr}{\displaystyle\int_0^\infty e^{-\frac{1}{2}r^2} r^{N-1}\, dr}$$

as $N \to \infty$, where $Na^2 \to \infty$ but $Na^3 \to 0$ and $c = 1 - \frac{a}{2}$.

$$f(r) = [redacted] -\tfrac{1}{2}r^2 + N \log r \qquad = \tfrac{1}{2}(N\log N - N)$$
$$f'(r) = -r + \frac{N}{r} \qquad = 0 \qquad \text{where } r = \sqrt{N}$$
$$f''(r) = -1 + \frac{N}{-r^2} = -2 \qquad \text{"} \qquad \text{"}$$

Thus

$$e^{-\frac{1}{2}r^2} r^N \frac{dr}{r} \sim e^{\frac{1}{2}(N\log N - N)} \, e^{\frac{1}{2}(-2)(r-\sqrt{N})^2 + \cdots} \frac{dr}{r}$$

which means the peak occurs at $r = \sqrt{N}$ and has a width of about 1, that is, the width is constant as $N \to \infty$. We want the measure in the range

$$Nc \le r^2 \le Nc(1+a)$$

$$\sqrt{N} \underbrace{\sqrt{1 - \frac{a}{2}}}_{1 - \frac{a}{4} + O(a^2)} \le r \le \sqrt{N} \underbrace{\sqrt{\left(1-\frac{a}{2}\right)(1+a)}}_{1 + \frac{a}{4} + O(a^2)}$$

Thus if $\sqrt{N}\,a \to 0$, which would happen if we constructed a sequence of blocks with $\sum a_n^2 = \sum N a^2 < \infty$ the numerator of ⊛ becomes a vanishing fraction of the denominator, so ⊛ $\to 0$. However if $\sqrt{N}\,a \to$ constant $\neq 0$, then clearly ⊛ does not approach zero.

Let's take in the N-th block all $a = \frac{1}{\sqrt{N}}$ and take $N = 1, 2, 4, 8, \cdots$. Thus the sequence a_n is

$$1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \underbrace{\frac{1}{\sqrt{4}}, \cdots, \frac{1}{\sqrt{4}}}_{4 \text{ times}}, \underbrace{\frac{1}{\sqrt{8}}, \cdots, \frac{1}{\sqrt{8}}}_{8 \text{ times}}$$

Then

$$\sum a_n^{2+\varepsilon} = \sum_{N=2^k} N\left(\frac{1}{\sqrt{N}}\right)^{2+\varepsilon} = \sum_{k=0}^{\infty} \left(2^k\right)^{-\varepsilon/2}$$

$$= \sum_{k=0}^{\infty} \left(2^{-\varepsilon/2}\right)^k < \infty \qquad \forall \varepsilon > 0.$$

and because we have blocks with $\sqrt{N}\,a = 1$ it follows that ⊛ $\not\to 0$, and so we get a counter-example to Reynolds problem.

July 18, 1980

Let's see if we can make sense of the RG in the Weiss model. One has to first understand the basic thermodynamical quantities.

One gets a probability distribution $\{\pm 1\}^N$ which is invariant under the symmetric group Σ_N. Now $\{\pm 1\}^N / \Sigma_N$ can be identified with $\{n \mid 0 \leq n \leq N\}$ where n is the number of up spins.

The N spin sites give 2^N configurations. But the symmetry group Σ_N tells one that certain configurations are effectively the same. Thus one is ultimately interested in the quotient with the correct weighting of the orbits. (Something similar happens with gauge theories: One looks at all connections A with gauge group as the symmetry group. One has an amplitude which is an integral over the possible configurations, and one tries to write it as an integral over the quotient by the gauge group.)

At this point it will be necessary to review the physical quantities of interest. These are the magnetic susceptibility and the specific heat. First the specific heat. Start with partition fn

$$Z = \sum e^{-\beta E_s}$$

Then the energy at temp. T is

$$U = -\frac{\partial}{\partial\beta} \log Z = \sum_s E_s \underbrace{\frac{e^{-\beta E_s}}{Z}}_{p_s = \text{prob. of } E_s} = \langle E_s \rangle$$

and the specific heat is

$$C = \frac{\partial U}{\partial T} = \frac{\partial U}{\partial \beta}\frac{\partial}{\partial T}\left(\frac{1}{kT}\right) = -\frac{1}{kT^2}\frac{\partial U}{\partial \beta}$$

$$C = \frac{1}{kT^2}\frac{\partial^2}{\partial \beta^2}\log Z$$

Also

$$+\frac{\partial U}{\partial \beta} = \sum_s E_s\left[\frac{e^{-\beta E_s}(-E_s)}{Z} - \frac{e^{-\beta E_s}}{Z^2}\overbrace{\frac{\partial Z}{\partial \beta}}^{-Z\langle E_s\rangle}\right]$$

$$\therefore \quad C = \frac{1}{kT^2}\left(\langle E_s^2\rangle - \langle E_s\rangle^2\right)$$

Next suppose one has an Ising model with N spins, so that

$$E_s = -H\sum s_i - \frac{1}{2}\sum J_{ij}s_is_j$$

$$Z = \sum e^{\beta H\sum s_i + \frac{\beta}{2}\sum J_{ij}s_is_j}$$

The total magnetization is

$$M = \frac{1}{\beta}\frac{\partial}{\partial H}\log Z = \langle\sum s_i\rangle = \sum_s\left(\sum s_i\right)\frac{e^{-\beta E_s}}{Z}$$

and the magnetic susceptibility is

$$\chi = \frac{\partial M}{\partial H} = \frac{1}{\beta}\frac{\partial^2}{\partial H^2}\log Z = \sum_s\left(\sum s_i\right)\left\{\beta\left(\sum s_i\right)\frac{e^{-\beta E_s}}{Z} - e^{-\beta E_s}\frac{1}{Z^2}\underbrace{\frac{\partial Z}{\partial H}}_{\beta Z\langle\sum s_i\rangle}\right\}$$

$$= \beta\left[\left\langle\left(\sum s_i\right)^2\right\rangle - \left\langle\sum s_i\right\rangle^2\right]$$

Now we are interested in a system where N is large and where all sites are equivalent, so that $\langle s_i \rangle$ is independent of i. Consequently we perhaps would find the magnetization per site

$$\frac{M}{N} = \left\langle \frac{1}{N}\sum_i s_i \right\rangle = \langle s_i \rangle$$

more natural.

Notice that

$$\chi = \beta\left[\sum_{i,j} \langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle\right]$$

[redacted] In the Weiss model $\langle s_i\rangle = \bar{s}$ and $\langle s_i s_j\rangle = \langle s_1 s_2\rangle$ for $i \neq j$. Hence

$$\chi = \beta\left[N + N(N-1)\,\text{[redacted]}\,\langle s_1 s_2\rangle - N^2\bar{s}^2\right]$$
$$= \beta N\left[1 + (N-1)\langle s_1 s_2\rangle - N\bar{s}^2\right]$$

So

$$\frac{\chi}{N} = \beta\left[1 - \bar{s}^2 + (N-1)\left(\langle s_1 s_2\rangle - \text{[redacted]}\,\bar{s}^2\right)\right]$$

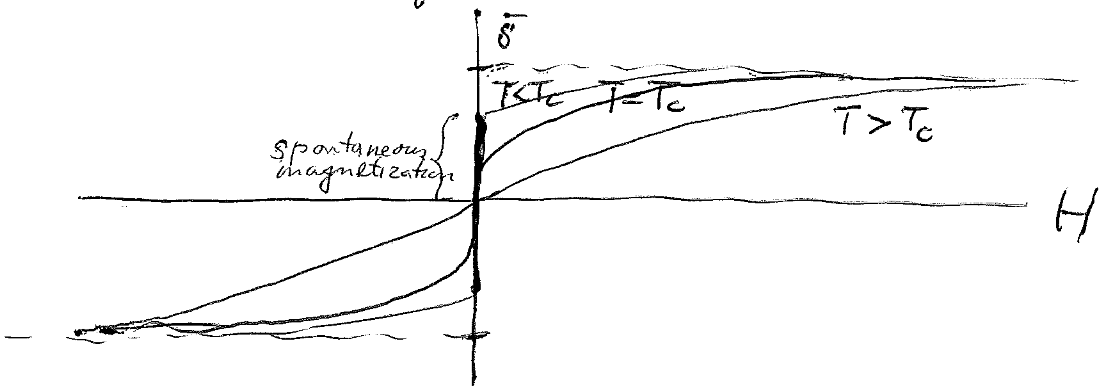
The goal will be to compute $\frac{M}{N} = \bar{s}$ and $\frac{\chi}{N}$ in the Weiss model, as $N \to \infty$.

Interesting point: $\frac{\chi}{\beta}$ = dispersion of $\sum_i s_i$

The fact that $\frac{\chi}{N}$ is expected to converge as $N \to \infty$ means the dispersion of $\sum s_i$ is proportional to $\sqrt{N}$ as in the case of the random walk.

But $\frac{\chi}{N}$ becomes infinite for $H = 0$ as $T \downarrow T_c$

To understand this clearly, put $\bar{s} = \frac{M}{N}$ for the magnetization per site. Recall that the [redacted] basic curves are as follows:



The slope $\frac{\partial \bar{s}}{\partial H}$ is $\frac{\chi}{N}$, the magnetic susceptibility / site.

We should recall the basic equation of state:

$$\bar{s} = \frac{e^{\beta B} - e^{-\beta B}}{e^{\beta B} + e^{-\beta B}} \qquad B = H + \bar{s} \qquad \text{assuming } gN \to 1$$

or

$$H = -\bar{s} + \frac{1}{2\beta} \log \frac{1+\bar{s}}{1-\bar{s}}$$

$$\cong \left(-1 + \frac{1}{\beta}\right)\bar{s} = (T-1)\,\bar{s} + O(\bar{s}^3) \qquad k=1$$

Thus for $T >$ critical temp $= 1$ we have

$$\frac{\chi}{N} = \frac{\partial \bar{s}}{\partial H} = \frac{1}{T-1} \qquad \text{at } H=0$$

July 20, 1980

Weiss model again with $gN = 1$, $\beta = \frac{1}{T}$. What is special about this model is that because of the symmetry group Σ_N the energy of a configuration is a function of the magn./site $\bar{s} = \frac{1}{N}\sum s_i$, and consequently the quantities of interest can all be computed from the probability distribution of the random variable $\bar{s}$.

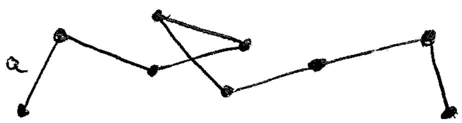
We have seen that for $\beta > 1$, i.e. $T < T_c = 1$ and $H = 0$, that this probability distribution $p_N(\bar{s})$ has two peaks

located at [redacted] ± the spontaneous magnetization. What sort of dynamical behavior belongs with this picture?

Recall that one [redacted] can think of [redacted] an Ising system as [redacted] running rapidly thru all its configurations, each individual configuration occurring with probability given by the Bolzmann law. Hence the Weiss model, viewed as a collection of scintillating lights, is alternately well-lit and well-dark a large fraction of the time.

It seems that there has to be some kind of random motion dynamics belonging to a given equilibrium system.

Rubber elasticity: Let us consider a polymer made up of N rigid segments of length a which are freely jointed



Ignoring the fact that the polymer can't fold back upon itself, we may identify a configuration of the polymer with a random walk of N steps of length a. Suppose the beginning of the chain fixed at the origin, and that we work at inverse temperature β. Then all configurations are equally likely so the end of the chain is found near the origin with high probability. If we want the end to be found at some other point, we have to apply a force.

Amazingly even though we neglect forces at the joints, and the mass of the atoms in the polymer, so that we have no way of storing energy in the polymer, it behaves like a spring, so it seems.

To calculate this force, let's take a one dimensional situation. Then a configuration is described by a sequence $s_1, \dots, s_N$ with each $|s_i| = a$ and the end of the chain is

$$x = \sum s_i$$

Let's apply a force g to the end of the chain, or better, regard a configuration as having potential energy $-gx$, where x is the end of the configuration. Then the partition function is

$$Z = \sum_{\{s_i\},\ |s_i| = a} e^{\beta g \sum_i s_i} = \left(e^{\beta g a} + e^{-\beta g a}\right)^N$$

and the expected position of the end is

$$\langle x \rangle = \frac{1}{\beta}\frac{\partial}{\partial g} \log Z = Na\,\frac{e^{\beta g a} - e^{-\beta g a}}{e^{\beta g a} + e^{-\beta g a}}$$

Let's take the limit as $a \to 0$, $N \to \infty$; β, g fixed.

$$\langle x \rangle = Na \underbrace{\frac{e^{\beta g a} - e^{-\beta g a}}{e^{\beta g a} + e^{-\beta g a}}}_{\approx \frac{2\beta g a}{2}} \approx (Na^2)\beta g$$

Thus the spring constant is $1/(Na^2)\beta$.
Notice that Na^2 is the mean square length of the chain, and that it is natural to assume $Na^2 \to$ non-zero limit as $N \to \infty$.

Next consider three dimensions and try to do the calculation in greater generality. We want a random walk in 3 dimensions:

$$\vec{x} = \sum_{i=1}^{N} \vec{x}_i$$

where we suppose the $\vec{x}_i$ are independent vector random variables with the same distribution, which is a probability measure $d\mu$ in $\mathbb{R}^3$. Again take a potential field $-\vec{g}\cdot\vec{x}$ representing the constant force $\vec{g}$ on the end of the chain. The partition fn is

$$Z = \int \underbrace{e^{\beta \vec{g}\cdot\vec{x}}}_{e^{\beta \sum \vec{g}\cdot\vec{x}_i}} \, d\mu(\vec{x}_1) \otimes \cdots \otimes d\mu(\vec{x}_N)$$

[redacted]

$$= f^N \qquad f = \int e^{\beta \vec{g}\cdot\vec{x}} \, d\mu(\vec{x})$$

The expected end position is

$$\langle \vec{x} \rangle = \frac{1}{\beta} \frac{\vec{\partial}}{\partial g} \log Z$$

$$= N \frac{\int \vec{x} \, e^{\beta \vec{g}\cdot\vec{x}} \, d\mu(\vec{x})}{\int e^{\beta \vec{g}\cdot\vec{x}} \, d\mu}$$

Assume $d\mu$ has mean zero and is spherically symmetric. Then

$$\int e^{\beta \vec{g}\cdot\vec{x}} \, d\mu = \int \left(1 + \cancel{\beta \vec{g}\cdot\vec{x}} + \frac{\beta^2}{2}(\vec{g}\cdot\vec{x})^2 + \cdots \right) d\mu$$

$$= 1 + \frac{\beta^2 g^2}{2} \alpha + \cdots$$

where α is determined as follows. Let $d\mu = f(r)\, d^3\vec{x}$

Then

$$\int (\vec{g}\cdot\vec{x})^2 \, d\mu = \iiint (g r \cos\varphi)^2 f(r) r^2 \sin\varphi \, dr \, d\varphi \, d\theta$$

$\vec{g}$ points in $\hat{z}$ direction

$$= g^2 \int_0^\infty r^2 f(r) r^2 \, dr \underbrace{\int_0^\pi \cos^2\varphi \sin\varphi \, d\varphi}_{\left[-\frac{\cos^3\varphi}{3}\right]_0^\pi = \frac{2}{3}} \cdot 2\pi$$

$$= g^2 \frac{4\pi}{3} \int_0^\infty f(r) r^4 \, dr$$

Now $\iiint f(r) r^2 \sin\varphi \, dr \, d\varphi \, d\theta = \int_0^\infty f(r) r^2 dr \underbrace{\int_0^\pi \sin\varphi \, d\varphi}_{[-\cos\varphi]_0^\pi = 2} \cdot 2\pi = 4\pi \int_0^\infty f(r) r^2 dr$

This is 1, so that

$$\int (\vec{g}\cdot\vec{x})^2\, d\mu = \frac{g^2}{3} \underbrace{\frac{\int_0^\infty f(r) r^4 dr}{\int_0^\infty f(r) r^2 dr}}_{\text{mean square distance from } O.}$$

| call this a^2 so that a is the length of the rod in rigid case.

Consequently

$$\int e^{\beta \vec{g}\cdot\vec{x}}\, d\mu = 1 + \frac{\beta^2 g^2}{2}\frac{a^2}{3} + \cdots$$

$$\therefore \quad \frac{\int \vec{x}\, e^{\beta\vec{g}\cdot\vec{x}}\, d\mu}{\int e^{\beta\vec{g}\cdot\vec{x}}\, d\mu} = \frac{1}{\beta} \nabla_{\vec{g}} \left(1 + \frac{\beta^2 g^2}{2}\frac{a^2}{3} + \cdots\right)$$

$$= \beta \vec{g}\,\frac{a^2}{3} + \cdots$$

Therefore in the limit $N \to \infty$, $a \to 0$ but $Na^2 \to$ non-zero constant we get for the mean end position

$$\langle \vec{x} \rangle = \beta \frac{Na^2}{3} \vec{g}$$

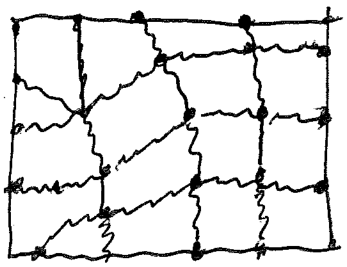
showing the spring constant is $1/\beta \frac{Na^2}{3}$. It's the same formula as before if you use that $\frac{a^2}{3}$ = mean square of $\vec{x}\cdot\vec{u}$ for any unit vector $\vec{u}$. (This $\frac{1}{3}$ is easy to see: $\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle$ and $\langle x^2+y^2+z^2\rangle = a^2$.)

July 21, 1980

Rubber elasticity. The key idea is that a freely jointed chain of N rods of length a behaves like a spring. The force f required to hold the ends at a distance L apart is given by

$$L = \beta f \frac{Na^2}{3} \qquad \text{or} \qquad \vec{f} = \left(\frac{3}{Na^2} kT\right) \vec{L}$$

Actual rubber is a network of such chains



cross-connected by the process of vulcanization. We can think of it as a networks of springs. Let $\vec{v}_\alpha$ denote the vector for the α-th spring with some orientation, and let k_α be the spring constant, so that $\vec{f}_\alpha = k_\alpha \vec{v}_\alpha$. The vertices of the network achieve an equilibrum position governed by equations

$$\sum_{\substack{\alpha \text{ spring} \\ \text{with vertex } i}} \vec{f}_\alpha = \sum k_\alpha \vec{v}_\alpha = 0$$

for each vertex. (It is necessary to suppose the boundary of the rubber held fixed by some mysterious force, otherwise the network would collapse to a point.) Since all k_α are proportional to T the equilibrum is not affected by temperature change.

Next let us make a bulk deformation of the rubber described by a linear transformation A. Then the new spring vectors are $A\vec{v}_\alpha$, the new forces are $\vec{f}_\alpha' = k_\alpha A \vec{v}_\alpha$ and

$$\sum_{\substack{\alpha \in [redacted] \\ \text{vertex } i}} f_\alpha' = A \sum k_\alpha \vec{v}_\alpha = 0$$

so the equilibrium positions change according to the linear transf. A.

Let's compute what happens to the potential energy

$$U_0 = \sum_{\text{all } \alpha} \frac{1}{2} k_\alpha |\vec{v}_\alpha|^2 \longmapsto \sum_\alpha \frac{1}{2} k_\alpha |A\vec{v}_\alpha|^2 = U$$

when A = diagonal matrix $\begin{pmatrix} \lambda_x & & \\ & \lambda_y & \\ & & \lambda_z \end{pmatrix}$. Then

$$\sum_\alpha \frac{1}{2} k_\alpha |A\vec{v}_\alpha|^2 = \sum_\alpha \frac{1}{2} k_\alpha \left(\lambda_x^2 (v_\alpha^x)^2 + \lambda_y^2 (v_\alpha^y)^2 + \lambda_z^2 (v_\alpha^z)^2 \right)$$

Assuming the network is isotropic in the undeformed state, one has

$$\frac{1}{2} \sum k_\alpha (v_\alpha^x)^2 = \frac{1}{2} \sum k_\alpha (v_\alpha^y)^2 = \frac{1}{2} \sum k_\alpha (v_\alpha^z)^2$$

$$= \frac{1}{3} U_0$$

hence

$$U = \frac{1}{3} (\lambda_x^2 + \lambda_y^2 + \lambda_z^2) U_0$$

When we stretch the rubber along [redacted] the x axis from [redacted] length L_0 to L, one has $\lambda_x = \frac{L}{L_0}$ and since the volume is known to stay fixed one has

$$\lambda_y = \lambda_z = \frac{1}{\sqrt{\lambda_x}}$$

Thus

$$U = \frac{1}{3} \left(\left(\frac{L}{L_0} \right)^2 + 2 \left(\frac{L_0}{L} \right) \right) U_0$$

and so the force is given by

$$f = \frac{dU}{dL} = \frac{1}{3}\left(2\frac{L}{L_0^2}\ \text{[redacted]} - 2\frac{L_0}{L^2}\right)U_0$$

$$= \frac{2}{3}\left(\left(\frac{L}{L_0}\right) - \left(\frac{L_0}{L}\right)^2\right)\frac{U_0}{L_0}$$

Put in that U_0 is proportional to T, and put $\lambda = \frac{L}{L_0}$ for the amount of stretching and we get

$$\boxed{f = (\text{const})\, T\left(\lambda - \frac{1}{\lambda^2}\right)}$$

[redacted]

Actually Nash says something about the constant. Recall

$$U_0 = \sum \frac{1}{2} k_\alpha |v_\alpha|^2$$

$$k_\alpha = \frac{3}{N_\alpha a^2 \beta} = \frac{3kT}{N_\alpha a^2}$$

$$\therefore\ U_0 = \frac{3}{2} kT \sum \frac{|v_\alpha|^2}{N_\alpha a^2}.$$

He has someway of seeing that in the unstressed state $N_\alpha a^2 =$ [redacted] mean square length of the α-th chain, should equal $|v_\alpha|^2$. Consequently

$$U_0 = \frac{3}{2} kT N \qquad N = \text{number of edges in our network.}$$

and so

$$f = kT\frac{N}{L_0}\left(\lambda - \frac{1}{\lambda^2}\right)$$

The above calculations show that the key to rubber elasticity is contained in the fact that a freely jointed chain at a given temperature requires a force to keep its ends apart. The usual way of determining the force is to argue thru thermo and the Boltzmann entropy formula. Thermo. says

$$1) \qquad dU = TdS + f\,dL$$

where U is the internal energy. But the internal energy is zero for all configurations, hence we get

$$2) \qquad f = -T\frac{dS}{dL}$$

Finally one computes $S = k \log W(L)$, where $W(L)$ is the probability or number of configurations of length L. By binomial distribution we have

$$W(L) = \frac{1}{\sqrt{2\pi Na^2}} e^{-\frac{L^2}{2Na^2}}$$

hence

$$S(L) = -k\frac{L^2}{2Na^2}$$

so

$$f = \frac{kT}{Na^2}L \qquad \text{or spring constant of } \frac{kT}{Na^2}$$

The problem with this method is how to establish 1), 2) when one doesn't have a partition function. Actually one does have a partition function – it just doesn't depend on temperature.

First review the formulas: Suppose one has $\Omega(E,V,N)$ states of energy E, volume V, N particles. Then one puts

$$Z(T,V,N) = \int e^{-\beta E}\,\Omega(E,V,N)\,dE$$

so that
$$U = -\frac{\partial}{\partial \beta} \log Z \qquad S = k(\beta U + \log Z).$$

$$\begin{aligned} T\,dS &= kT\, d(\beta U + \log Z) \\ &= dU + \frac{1}{\beta}\left(\cancel{U\,d\beta} + \cancel{\frac{\partial \log Z}{\partial \beta}d\beta} + \frac{\partial \log Z}{\partial V}dV + \frac{\partial \log Z}{\partial N}dN\right) \\ &= dU + p\,dV - \mu\, dN \end{aligned}$$

Now if it should happen that $\Omega(E,V,N) = \delta(E)\cdot W(V,N)$ then we get
$$Z(T,V,N) = W(V,N)$$
so $U = 0$ and $S = k \log W(V,N)$.

In the jointed chain instead of V we have the length L, and
$$Z(T,L) = W(L) = W_0\, e^{-\frac{1}{2}\frac{L^2}{Na^2}}$$

Hence
$$S = k \log W(L) = \text{const} - \frac{k}{2Na^2}L^2 .$$

We want
$$dU = T\,dS + f\,dL$$
and by the above derivation this will follow if we define
$$f = -\frac{1}{\beta}\frac{\partial}{\partial L}\log Z .$$

Thus
$$f = \frac{1}{\beta N a^2} L = -T\frac{\partial S}{\partial L} \quad ✓$$

So it works, but the philosophy of Laplace- Legendre transforming wrt L ~~is~~ somehow esthetically better.

July 22, 1980

To understand the Einstein theory of Brownian motion. He considers motion on a line. The path $x(t)$ of the particle satisfies

$$m\ddot{x} + \mu\dot{x} = F(t)$$

where $F(t)$ is the force at time t on the particle. The analysis proceeds as follows (Feynman lectures) Start the particle at $x=0$ when $t=0$ and try to compute $\langle x^2\rangle$ the mean square distance.

$$\frac{d}{dt}\langle x^2\rangle = 2\langle x\dot{x}\rangle$$

$$\underbrace{\langle m x\ddot{x}\rangle} + \mu\langle x\dot{x}\rangle = \langle Fx\rangle$$

$$\frac{d}{dt} m\langle x\dot{x}\rangle - m\langle \dot{x}^2\rangle$$

Now one argues that $\langle Fx\rangle = 0$ and that $\langle x\dot{x}\rangle$ is constant. If these are valid, then

$$-m\langle\dot{x}^2\rangle + \frac{\mu}{2}\frac{d}{dt}\langle x^2\rangle = 0$$

and by equipartition of energy one knows

$$\langle \tfrac{1}{2}m\dot{x}^2\rangle = \tfrac{1}{2}kT$$

Thus $$\frac{\mu}{2}\frac{d}{dt}\langle x^2\rangle = kT \quad \text{or} \quad \boxed{\langle x^2\rangle = \frac{2kT}{\mu}t}$$

The goal will be to make sense of this analysis using the machinery of random variables. We first have to interpret F. Imagine an ensemble consisting of different particles on a line all with different force functions. So for any time t we have a random variable $F(t)$.

To simplify the discussion, one supposes that the process $t \mapsto F(t)$ is Gaussian and stationary. This means we can think of $F(t)$ as being a path in a Hilbert space such that the inner products $\langle F(t) | F(t') \rangle$ depend on $t-t'$. The Hilbert space has a 1-parameter unitary group $U(t)F(t') = F(t+t')$ and the cyclic vector $F(0)$, hence is isomorphic to $L^2(R, d\mu)$ with $U(t) = e^{ixt}$ and $F(0) = 1$, $d\mu$ having finite measure: $\|F(0)\|^2 = \int d\mu$.

Wait: We must first understand the Wiener process which describes the random walk on the line. The discrete case: Here $t = n = 0, 1, 2, \ldots$ and

$$x(n) = \sum_{i=1}^{n} x_i$$

where the x_i are independent Gaussian r.v.'s with mean 0 and variance a^2. $\therefore$ $\langle x_i^2 \rangle = a^2$ and

$$\langle x(n)^2 \rangle = n a^2$$

$x(n)$ is a Gaussian r.v. with mean 0 and variance na^2. Thus we see that $x(m)$, $x(n) - x(m)$ are independent Gaussian (mean 0) random variables. In the continuous case one has Gaussian r.v. $x(t)$ with mean 0 and variance ta^2.

There seems to be a Wiener measure defined on all continuous functions $x(t)$ defined on the line, but it is not a probability measure. One fixes a time, say $t=0$, and puts Lebesgue measure on the coordinate $x(0)$. On paths with a fixed value of $x(0)$, one then gets a Gaussian probability measure defined in the usual way.