June 29, 1980

Hamilton-Jacobi for quadratic Ham.: 884-893
WKB formula 898
Wheeler's const. 874

It is necessary to understand linearized versions of Hamilton's equations, action, etc. Consider a system described by a Hamiltonian $H(t, q, p)$. Stationary curves for the form $\eta = p\,dq - H\,dt$ are trajectories for Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

Consider the trajectories as forming a manifold on which one has coords q_t, p_t for any time t. If $t' < t$, let $S_{t,t'}$ be the function on this manifold giving the action from t' to t:

$$S_{t,t'}(\text{trajectory}) = \int_{t'}^{t} \eta$$

Then the first variation formula gives

$$dS_{t,t'} = p_t\,dq_t - p_{t'}\,dq_{t'}$$

This shows perhaps that $S_{t,t'}$ is most naturally a function of $q_t, q_{t'}$, the endpoints of the trajectory, so we define

$$S(t,q;t',q') = S_{t,t'} \text{ of trajectory with } q_{t'} = q', q_t = q.$$

$$= \text{action of trajectory from } x' = t'q' \text{ to } x = tq$$

If $x' = t'q'$ is held fixed, then we have for $S = S(tq, t'q')$

$$\frac{\partial S}{\partial q} = p \quad \text{for trajectory ending at } tq$$

$$\frac{dS}{dt} = p\frac{dq}{dt} - H \qquad \text{defn. of } S$$

$$= \frac{\partial S}{\partial q}\frac{dq}{dt} + \frac{\partial S}{\partial t} \qquad \text{(general fmla)}$$

and so we see that S satisfies Hamilton-Jacobi

[redacted] $$\frac{\partial S}{\partial t} + H\left(t, q, \frac{\partial S}{\partial q}\right) = 0$$

Let us now suppose given a trajectory $(q^\circ(t), p^\circ(t))$, and let us consider nearby trajectories

$$\left(q^\circ(t) + \delta q(t),\ p^\circ(t) + \delta p(t)\right).$$

Then to the first order $\delta q(t), \delta p(t)$ satisfy the linear DE

$$\dot{\delta q} = \frac{\partial^2 H}{\partial p \partial q}\delta q + \frac{\partial^2 H}{\partial p^2}\delta p$$

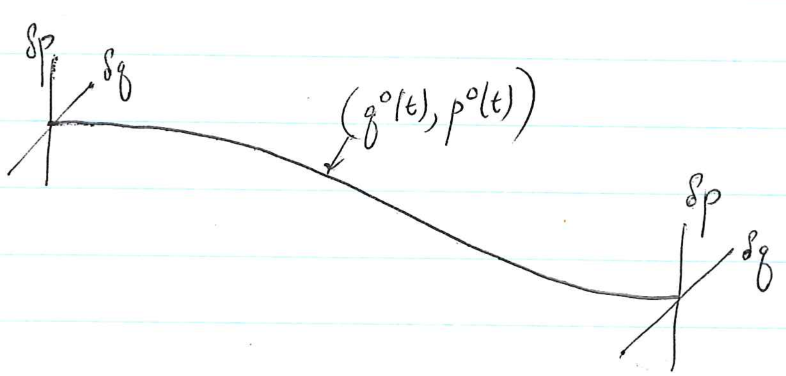
$$\dot{\delta p} = -\frac{\partial^2 H}{\partial q^2}\delta q - \frac{\partial^2 H}{\partial q \partial p}\delta p$$

where the 2nd partial derivatives of $H(t,q,p)$ are [redacted] evaluated at $t, q^\circ(t), p^\circ(t)$. These equations are Hamilton's equations for the [redacted] quadratic Hamiltonian

$$\tilde{H}(t, \delta q, \delta p) = \frac{1}{2}\frac{\partial^2 H}{\partial q^2}(t, q^\circ(t), p^\circ(t))\,\delta q^2 + \frac{\partial^2 H}{\partial q \partial p}(t, q^\circ(t), p^\circ(t))\,\delta q\,\delta p$$

$$+ \frac{1}{2}\frac{\partial^2 H}{\partial p^2}(t, q^\circ(t), p^\circ(t))\,\delta p^2$$

Picture:



I want to understand what is going on in a nbd of the given trajectory.

What I am doing ~~is~~ to introduce new coordinates into (t, q, p)-space by the formulas

$$\begin{cases} t = t' \\ q = q^\circ(t') + q' \\ p = p^\circ(t') + p' \end{cases}$$

[redacted] What happens to the form $\eta = p\,dq - H(t,q,p)\,dt$?

$$p\,dq - H\,dt = (p^\circ + p')\,d(q^\circ + q') - H(t', q^\circ + q', p^\circ + p')\,dt'$$

$$= (p^\circ \dot{q}^\circ + p' \cancel{\dot{q}^\circ})\,dt' + p^\circ dq' + p' dq'$$

$$-\Big(H(t', q^\circ, p^\circ)\,\text{[redacted]} + \frac{\partial H}{\partial q}(t', q^\circ, p^\circ)\,q' + \frac{\partial H}{\partial p}\cancel{(t', q^\circ, p^\circ)}\,p'$$

$$+ \text{[redacted]}\; \underbrace{\underbrace{\tilde{H}(t', q', p')}_{\text{quadratic terms}} + \cdots \Big)\,dt'}_{H'}$$

$$= \underbrace{(p^\circ \dot{q}^\circ - H^\circ)\,dt'}_{\text{function of } t'} + \underbrace{p^\circ dq' + \dot{p}^\circ q'\,dt'}_{d(p^\circ(t')\,\text{[redacted]}\,q')} + \underbrace{p' dq' - \tilde{H}\,dt' + \cdots}_{H'(t', q', p')\,dt'}$$

The first two forms are closed, hence the stationary curves for $p\,dq - H\,dt$ and $p'dq' - H'dt'$ are the same. The action along a trajectory is the action for q°, p° plus linear terms in $q'(t)$, t' = beginning and end, [redacted] plus the action for $p'dq' - H'dt'$.

[redacted] Put $t' = t$ and consider the action between t_2, t_1

$$S(t_1, q_1; t_2, q_2) = S(t_1, q_1^\circ(t_1); t_2, q_2^\circ(t_2)) + p^\circ(t_1)\,q_1' - p^\circ(t_2)\,q_2'$$

$$+ S'(t_1, q_1'; t_2, q_2')$$

It is obviously necessary to understand the linear case first. Consider linearized Hamilton's eqns.

$$\delta\dot{q} = A\delta q + B\delta p \qquad A = \frac{\partial^2 H}{\partial p \partial q} \quad B = \frac{\partial^2 H}{\partial p^2}$$

$$\delta\dot{p} = C\delta q + D\delta p \qquad C = -\frac{\partial^2 H}{\partial q^2} \quad D = -\frac{\partial^2 H}{\partial q \partial p}$$

Is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ an infinitesimal symplectic matrix?

This has to be true [redacted] since the above DE is Hamiltonian for a quadratic fn. of $\delta q, \delta p$. Better question: What are infinitesimal symplectic matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} = 0$$

$$\Longleftrightarrow \quad \begin{pmatrix} -B & A \\ -D & C \end{pmatrix} + \begin{pmatrix} B^t & D^t \\ -A^t & -C^t \end{pmatrix} = 0$$

$$\Longleftrightarrow \quad B = B^t, \; C = C^t, \; D = -A^t$$

This means

$$\dim Sp(2n) = \underset{\substack{\uparrow \\ \text{possible } A}}{n^2} + \underbrace{2\frac{n(n+1)}{2}}_{\text{possible } B, C} = 2n^2 + n = n(2n+1)$$

Notice that this is the same as the dimension of the space of quadratic functions of $2n$ variables which is $\frac{2n(2n+1)}{2} = n(2n+1)$.

$$S = \frac{1}{2}aQ^2 + bQq + \frac{1}{2}cq^2$$

Let's compute the symplectic transformation associated

to S:
$$P = \frac{\partial S}{\partial Q} = aQ + bq \implies q = -\frac{a}{b}Q + \frac{1}{b}P$$
$$p = -\frac{\partial S}{\partial q} = -bQ - cq \implies p = -bQ - c\left(-\frac{a}{b}Q + \frac{1}{b}P\right)$$
$$= \left(\frac{ac}{b} - c\right)Q - \frac{c}{b}P$$

Thus
$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -\frac{a}{b} & \frac{1}{b} \\ \frac{ac}{b} - b & -\frac{c}{b} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$$

and the determinant is $\quad \frac{ac}{b^2} - \left(\frac{ac}{b} - b\right)\frac{1}{b} = 1$
as it should be.

Question: Is there any relation of this with the scattering matrix for $(-\Delta + q)\psi = E\psi$ on the line?

July 2, 1980

more units. Recall that Coulomb's law gives a way to define a unit of charge in terms of units of length, time, mass. Thus if e is the charge of the electron

$$\frac{e^2}{1\,cm^2} = (\text{const})\, gr\, \frac{cm}{sec^2}$$

where (const) is dimensionless. Thus

$$[\text{charge}]^2 = gr\,\frac{cm^3}{sec^2}$$

Also

$$[\hbar] = gr\,\frac{cm^2}{sec^2}\,sec = \frac{gr\,cm^2}{sec}$$

$$[c] = \frac{cm}{sec}$$

Therefore

$$[\text{charge}]^2 = [\hbar c]$$

and so

$$\frac{e^2}{\hbar c} \quad \text{is a dimension-less constant}$$

Next the law of gravitation gives

$$\frac{gr\,cm}{sec^2} = G\,\frac{gr^2}{cm^2}$$

so

$$[G] = \frac{cm^3}{gr\,sec^2} = \frac{cm^3}{[\hbar]\frac{sec}{cm^2}\cdot sec^2}$$

$$\cancel{\frac{G}{c^2} = \frac{cm}{gr}} \quad = \frac{cm^2}{[\hbar]}[c]^3$$

Thus

$$\left[\frac{G\hbar}{c^3}\right] = cm^2$$

and so we get an absolute unit of length, namely

$$\boxed{\sqrt{\frac{G\hbar}{c^3}} \ \text{cm}}$$

This should be Wheeler's constant: Compute:

$$G = 6.672 \times 10^{-11} \ \underbrace{\underset{\frac{kg\ m}{sec^2}}{N}\text{-}m^2/kg^2}_{\frac{m^3}{kg\ sec^2}}$$

$$\hbar = \frac{6.626}{2\pi} \times 10^{-34} \ \underbrace{J\text{-}sec}_{\frac{kg\ m^2}{sec}}$$

$$c = 2.998 \times 10^{8} \ m/sec.$$

$$\frac{G\hbar}{c^3} = \frac{\cancel{6.7} \times 6.6}{\underbrace{\cancel{2\pi}\,(2.998)^3}_{\frac{2}{9}}} \ 10^{-11-34-24} \approx 2 \times 10^{-70} \ m^2$$

Thus

$$\sqrt{\frac{G\hbar}{c^3}}\text{-}m \approx 10^{-35} \ m \approx 10^{-33} \ cm.$$

July 3, 1980

The program now is to understand first and second variations associated to least action. Let's consider two times $t_1 < t_2$ and paths γ: $q(t), p(t)$ defined on $[t_1, t_2]$ and the action function on these paths

$$F(\gamma) = \int_\gamma p\,dq - H\,dt$$

Then if we have a variation $\delta q, \delta p$, we find

forgot →
$$\delta F = \int_{t_1}^{t_2} \Big[\delta p\, \dot{q} + p\, \delta\dot{q} + \delta p\, \delta\dot{q} - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p - \Big(\frac{1}{2}\frac{\partial^2 H}{\partial q^2}(\delta q)^2 + \frac{\partial^2 H}{\partial q \partial p}(\delta q)(\delta p) + \frac{1}{2}\frac{\partial^2 H}{\partial p^2}(\delta p)^2 \Big) \Big] dt$$

$$= \Big[p\,\delta q \Big]_{t_1}^{t_2} + \int_{t_1}^{t_2} \Big[\delta p \Big(\dot{q} - \frac{\partial H}{\partial p} \Big) - \delta q \Big(\dot{p} + \frac{\partial H}{\partial q} \Big) \Big] dt + \int \delta p\, \delta \dot{q}\, dt$$

$$- \int_{t_1}^{t_2} \Big\{ \frac{1}{2}\frac{\partial^2 H}{\partial q^2}(\delta q)^2 + \frac{\partial^2 H}{\partial q \partial p}(\delta q)(\delta p) + \frac{1}{2}\frac{\partial^2 H}{\partial p^2}(\delta p)^2 \Big\} dt$$

$$+ O(\delta q, \delta p)^3$$

Thus if we [redacted] fix q at the ends, [redacted] γ is a critical value of F when γ is a solution of Hamilton's equations. Moreover the Hessian of F is given by the quadratic terms in δF, hence γ is a local minimum when [redacted]

$$\int \Big[\delta p\, \delta\dot{q} - \Big(\frac{1}{2}\frac{\partial^2 H}{\partial q^2}(\delta q)^2 + \cdots + \frac{1}{2}\frac{\partial^2 H}{\partial p^2}(\delta p)^2 \Big) \Big] dt$$

[redacted] evaluated along γ is [redacted] ≤ 0 [redacted]

~~For example suppose~~

$$H = \frac{p^2}{2m} + V(q\text{[redacted]},t)$$

~~then the 2nd variation is~~

$$-\int_{t_1}^{t_2} \left(\frac{1}{2m}(\delta p)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial q^2}(\delta q)^2 \right) dt$$

~~Thus when $\frac{\partial^2 V}{\partial q^2} \geq 0$ along the trajectory, one has a local minimum for the action. This includes harmonic oscillator, uniform gravitational field, repulsive Coulomb potential, but not an attractive Coulomb potential.~~

Next I want to see if the corresponding Lagrangian variational [redacted] problem has a similar first and second variation.

$$\delta \int L(t,q,\dot{q})\,dt = \int \left\{ \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{1}{2}\frac{\partial^2 L}{\partial q^2}\delta q^2 + \cdots \right\} dt$$

$$= \left[\frac{\partial L}{\partial \dot{q}}\delta q \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right) \delta q \, dt$$

$$+ \text{[redacted]} \int_{t_1}^{t_2} \left\{ \frac{1}{2}\frac{\partial^2 L}{\partial q^2}(\delta q)^2 + \frac{\partial^2 L}{\partial q \partial \dot{q}}(\delta q)(\delta \dot{q}) + \frac{1}{2}\frac{\partial^2 L}{\partial \dot{q}^2}(\delta \dot{q})^2 \right\} dt$$

[redacted] In the example

$$H = \frac{p^2}{2m} + V(q,t)$$

$$L = \frac{m}{2}\dot{q}^2 - V(q,t)$$

the quadratic term is

$$\int_{t_1}^{t_2} \left\{ \frac{m}{2}(\delta \dot{q})^2 - \frac{1}{2}\left(\frac{\partial^2 V}{\partial q^2}\right)(\delta q)^2 \right\} dt \, .$$

Because δq is supposed to vanish at the ends this can be written

$$\int_{t_1}^{t_2} \frac{1}{2}\,\delta q\left(-m\,\delta\ddot{q} - \frac{\partial^2 V}{\partial q^2}\,\delta q\right) dt$$

which is the quadratic form associated to the operator

$$-m\frac{d^2}{dt^2} - \frac{\partial^2 V}{\partial q^2}$$

with Dirichlet boundary conditions on $[t_1, t_2]$. Now I believe one knows that the number of negative eigenvalues for this quadratic form is the number of conjugate points (counted properly) as one goes from t_1 to t_2.

To fix the ideas let us work with the simple harmonic oscillator $H = \frac{p^2}{2m} + \frac{k}{2}q^2$. Then the quadratic part above is

$$\int_{t_1}^{t_2} \left\{\frac{m}{2}(\delta\dot{q})^2 - \frac{k}{2}(\delta q)^2\right\} dt$$

corresponding to the operator m times

$$-\left(\frac{d^2}{dt^2} + \omega^2\right) \qquad\qquad \omega^2 = \frac{k}{m}$$

Now to simplify suppose $[t_1, t_2] = [0, L]$. Then Dirichlet eigenfunctions are

$$\sin\left(\frac{n\pi t}{L}\right) \qquad\qquad n = 1, 2, \ldots$$

and the eigenvalues are $\qquad \left(\frac{n\pi}{L}\right)^2 - \omega^2$

Thus the number of negative eigenvalues is the number of integers n with

$$1 \le n < \frac{\omega L}{\pi} \qquad \text{or} \qquad 0 < n\pi < \omega L$$

i.e. the number of zeroes of $\sin(\omega t)$ on $(0, L)$.

But one can associate to any path $q(t)$ a path $\gamma: q(t), p(t)$ in (t, q, p)-space by defining

$$p(t) = \frac{\partial L}{\partial \dot{q}}(t, q(t), \dot{q}(t))$$

Then

$$\int_\gamma p\,dq - H\,dt = \int [p\dot{q} - (p\dot{q} - L)]\,dt = \int L\,dt$$

so something is wrong. Error on 876.

The thing to look at is the quadratic part

$$\int_0^T \delta p\, d(\delta q) - \left(\frac{1}{2}\frac{\partial^2 H}{\partial q^2}(\delta q)^2 + \cdots\right) dt .$$

and to understand its eigenvalues. This is just the action integral for a Hamiltonian quadratic in q, p.

July 4, 1980

Consider $H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$ and let's compare the two actions:

$$\int_{\tilde{\gamma}} p\,dq - H\,dt \qquad \tilde{\gamma} \text{ is curve } q(t), p(t) \text{ over } [0,T] \text{ with } q(0) = q(T) = 0.$$

$$\int_{\gamma} L\,dt \qquad \gamma \text{ is the curve } q(t) \text{ over } [0,T] \text{ with } q(0) = q(T) = 0$$

Then

$$\int p\,dq - H\,dt = \int_0^T \left(p\dot{q} - \frac{p^2}{2} - \frac{\omega^2 q^2}{2}\right) dt$$

$$= \underbrace{\int_0^T -\frac{1}{2}(p - \dot{q})^2\,dt}_{\text{always } \leq 0} + \underbrace{\int_0^T \left(\frac{\dot{q}^2}{2} - \frac{\omega^2 q^2}{2}\right) dt}_{\int L\,dt}$$

Now we have seen that

$$\int L\,dt = \int_0^T \frac{1}{2} q\left[\left(-\frac{d^2}{dt^2} - \omega^2\right) q\right] dt$$

is a quadratic form on the space of paths with $q(0) = q(T) = 0$ having a finite number of negative eigenvalues and most of them positive. Hence we conclude that the action functional

$$F(\tilde{\gamma}) = \int_{\tilde{\gamma}} p\,dq - H\,dt$$

even for small paths does not have a local minimum, even when the Lagrange action $\int L\,dt$ does.

Notice that in (t, q, p) space the submanifold given by fixing t, q is such that the canonical form $\eta = p\,dq - H\,dt$ restricts to zero on it, hence is Lagrangian for $d\eta$. We have seen that [redacted] Lagrangian submanifolds of (t, q, p) space for $d\eta$ which project non-singularly on (t, q) space are [redacted] given by solutions of the HJ equation. So one should ask whether given two Lagrangian submanifolds, is it possible to assign [redacted] an action between them, so as to get a generalization of $S(tq, t'q')$?

Not really. Two Lagrangian submanifolds under nice [redacted] conditions can be expected to intersect along a trajectory. Now if one has specified S_i on each manifold [redacted] so that each satisfies $dS_i = \eta$, then $S_1 - S_2$ will be a constant. When one defines $S(tq, t'q')$ one chooses the action to be zero [redacted] on [redacted] each fibre over tq space.

July 5, 1980

Consider the Schroed. equation for a scalar wave fn.

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi \qquad \Delta = \nabla^2,\ \nabla = \frac{\partial}{\partial q}$$

Put $\psi = e^{\frac{i}{\hbar}S}$. Then S must satisfy

$$-\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m}\left(\left(\frac{i}{\hbar}\nabla S\right)^2 + \frac{i}{\hbar}\nabla^2 S\right) + V$$

or

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V = \frac{i\hbar}{2m}\nabla^2 S$$

If we look for an expansion $S = S_0 + \hbar S_1 + \cdots$, then

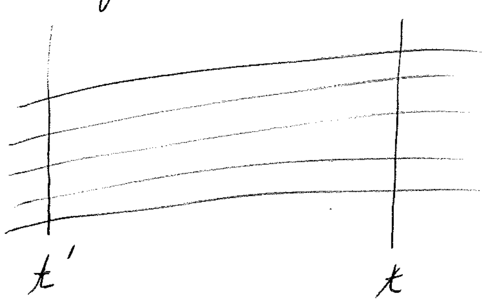
$$\frac{\partial S_0}{\partial t} + \frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V = 0 \qquad \text{Hamilton-Jacobi PDE}$$

$$\left(\frac{\partial}{\partial t} + \frac{1}{m}\frac{\partial S_0}{\partial q}\cdot\nabla\right)S_1 = \frac{i}{2m}\nabla^2 S_0$$

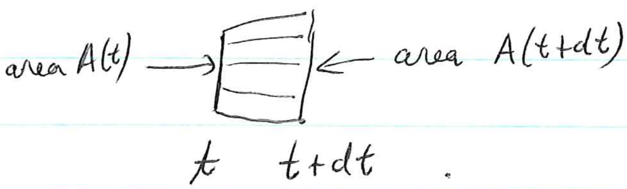
We already know what solns. S_0 of the HJ equation look like. They can be identified with a family of trajectories of Hamilton's equations, in this case

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -\nabla V \qquad \text{or } m\ddot{q} = -\nabla V,$$

such that [redacted] [redacted] $\nabla S_0(tq)$ is the momentum of the trajectory passing thru tq. (This is sort of like saying the flow is irrotational.)



Let us suppose that $S_0(tq)$ is given. To simplify suppose $m=1$ so that $\dot{q} = p = \nabla S$. Then $\nabla^2 S = \nabla \cdot \nabla S$ is the divergence of the flow. Recall how the divergence can be computed: You take a small volume, the divergence is the flux thru its surface divided by the volume. Therefore I take a little flow tube



First it is necessary to work in tq space, so notice that the tangent vector in tq space to the flow is $(1, \dot{q})$, and hence the divergence of $(1,\dot{q})$ is just $\nabla^2 S_0$. Also the flux thru a piece of $t =$ constant is just its area. Thus we get

$$\nabla^2 S_0 \cdot dt \cdot A(t) = A(t+dt) - A(t)$$

or

$$\nabla^2 S_0 = \frac{1}{A(t)} \frac{dA(t)}{dt}$$

So what we want to understand is how to compute the right side.

July 6, 1980

It is important to understand the linear case first. Consider the Hamiltonian

$$H = \frac{p^2}{2} + \frac{1}{2} q \cdot K q$$

where $K(t)$ is a symmetric matrix depending on t. The equation of motion is

$$\ddot{q} = -Kq$$

Let's work on the interval $[0,T]$ and denote by q, p (resp Q, P) the position + momentum at the beginning (resp. end). Given q, p the position + momentum at time t is given by

$$\begin{pmatrix} q_t \\ p_t \end{pmatrix} = U(t,0) \begin{pmatrix} q \\ p \end{pmatrix}$$

where $U(t,0)$ satisfies

$$\frac{d}{dt} U(t,0) = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} U(t,0)$$

$$U(0,0) = I$$

The action of the trajectory (q_t, p_t) over $[0,T]$ is

$$\int_0^T \left[\frac{1}{2} p_t^2 - \frac{1}{2} q_t K(t) q_t \right] dt$$

and it is evidently a quadratic function of the initial position + momentum. Better: it is a quadratic function on the $2n$ dimensional space of trajectories.

Let's suppose that $t = 0, T$ are good points so that (Q, q) are coordinates on the space of trajectories. Then we can write

$$S = \tfrac{1}{2} q \cdot a q + q \cdot b Q + \tfrac{1}{2} Q \cdot c Q$$

From the general relation $\quad dS = P\,dQ - p\,dq$

we get

$$\begin{cases} P = \dfrac{\partial S}{\partial Q} = b^t q + cQ \\[2ex] p = -\dfrac{\partial S}{\partial q} = -aq - bQ \end{cases} \qquad \left(\begin{aligned} q \cdot bQ &= q^t b Q \\ &= Q^t b^t q \end{aligned} \right)$$

$$bQ = -aq - p$$

$$Q = -(b^{-1}a)q - (b^{-1})p$$

$$P = b^t q + c\left[-(b^{-1}a)q - b^{-1}p\right]$$

$$= (b^t - cb^{-1}a)q - (cb^{-1})p$$

This becomes when $n = 1$

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{a}{b} & -\frac{1}{b} \\ b - \frac{ca}{b} & -\frac{c}{b} \end{pmatrix}}_{\det = 1} \begin{pmatrix} q \\ p \end{pmatrix}$$

<u>Example:</u> Take $H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$ with $\omega > 0$.

We know that

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \cos \omega T & \frac{\sin \omega T}{\omega} \\ -\omega \sin \omega T & \cos \omega T \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

hence $\quad -\dfrac{1}{b} = \dfrac{\sin \omega T}{\omega} \qquad -\dfrac{a}{b} = -\dfrac{c}{b} = \cos \omega T$

$$\therefore \ a = c = \frac{\omega \cos \omega T}{\sin \omega T} \qquad b = -\frac{\omega}{\sin \omega T}$$

hence the action for the simple harmonic oscillator over $[0,T]$ is given by

$$\boxed{S(Q,q) = \frac{1}{2}\frac{\omega\cos\omega T}{\sin\omega T}q^2 - \frac{\omega}{\sin\omega T}qQ + \frac{1}{2}\frac{\omega\cos\omega T}{\sin\omega T}Q^2}$$

see p 333 Oct. 79

Next I should understand the solutions of the Hamilton-Jacobi equation which are quadratic in q and hence which for fix t correspond to a linear Lagrangian subspace. For the simple harmonic oscillator such we consider the Lagrangian subspace of $(q,p,0)$-space given by $S(q0) = p_0 q$. Thus each particle has initial momentum p_0. The trajectory with initial position q is

$$q_t = (\cos\omega t)q + \left(\frac{\sin\omega t}{\omega}\right)p_0 .$$

this gives an affine subspace. You want $S(q0) = a\frac{q^2}{2}$ see p. 888

Take case $p_0 = 0$

$$S = \int\left(\frac{1}{2}\dot{q}_t^2 - \frac{1}{2}\omega^2 q_t^2\right)dt$$

$$= \frac{\omega^2}{2}q^2\int_0^t \underbrace{(\sin^2\omega t - \cos^2\omega t)}_{-\cos 2\omega t}dt$$

$$= \frac{\omega^2}{2}q^2\left(-\frac{\sin 2\omega t}{2\omega}\right) = -\frac{\omega}{2}q^2 \sin\omega t\frac{\cos^2\omega t}{\cos}$$

$$\boxed{S(tq) = -\left(\frac{\omega\sin\omega t}{\cos\omega t}\right)\frac{q^2}{2}}$$ satisfies $$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + V = 0$$

for $\frac{\partial S}{\partial t} = \frac{-\omega^2}{\cos^2\omega t}\frac{q^2}{2}$

$\frac{\partial S}{\partial q} = \frac{-\omega\sin\omega t}{\cos\omega t}q$

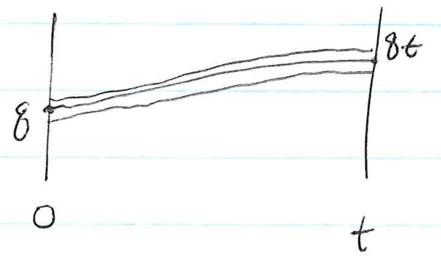
$$\therefore \frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 = \frac{\omega^2 q^2}{2}\left[\frac{-1}{\cos^2} + \frac{\sin^2}{\cos^2}\right]$$

$$= -\frac{\omega^2 q^2}{2} \quad ✓$$

Recall from yesterday we saw for a solution of the Hamilton - Jacobi equation $S(tq)$ that

$$\nabla^2 S = \frac{d}{dt} \log A(t)$$

where $A(t) = \det\left(\frac{\partial q_t}{\partial q}\right)$ is the Jacobian of the map $q \mapsto q_t$



Let's check this. $A(t) = \dfrac{\partial q_t}{\partial q} = \cos\omega t$

$$\nabla^2 S = \frac{\partial^2}{\partial q^2}\left(-\frac{\omega \sin\omega t}{\cos\omega t}\,\frac{q^2}{2}\right) = \frac{-\omega \sin\omega t}{\cos\omega t} \quad \checkmark$$

General quadratic solution of HJ for the simple harmonic oscillator.

$$S(tq) = a(t)\frac{q^2}{2} + b(t)q + c(t)$$

$$\frac{\partial S}{\partial t} = a'\frac{q^2}{2} + b'q + c'$$

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 = \frac{1}{2}(aq+b)^2 = \frac{1}{2}a^2q^2 + abq + \frac{1}{2}b^2$$

$$= -\frac{1}{2}\omega^2 q^2 \quad \text{yields} \quad \begin{aligned} a^2 + a' &= -\omega^2 \\ b' + ab &= 0 \qquad c' + \tfrac{1}{2}b^2 = 0 \end{aligned}$$

The equation for a is a Riccati eqn. assoc. to $u'' + \omega^2 u = 0$.

Solutions are $\quad a = \dfrac{u'}{u} \qquad a' = \dfrac{u''}{u} - a^2 = -\omega^2 - a^2$.

Then

$$\frac{b'}{b} + \frac{u'}{u} = 0 \quad \text{so} \quad b = \frac{\text{const}}{u}$$

But now you see the error on 886. In order to get a linear subspace (as opposed to affine) of (q,p) space you want $S(tq)$ to be [redacted] a quadratic form in q, that is, [redacted] for the simple oscillator

$$S(tq) = a(t)\frac{q^2}{2}$$

Then we have seen that a satisfies the Riccati equation, and hence we get the solutions

$$S(tq) = \frac{u'(t)}{u(t)}\frac{q^2}{2} \qquad u = A\cos\omega t + B\sin\omega t$$

of the Hamilton-Jacobi equation.

General case: $q'' + K(t)q = 0$. We want

$$S(tq) = \tfrac{1}{2}q\cdot a(t)q$$

to be a solution of the HJ equation. Here a is a symmetric matrix.

$$\frac{\partial S}{\partial t} = \tfrac{1}{2}q\cdot a'q \qquad \frac{\partial S}{\partial q} = aq \qquad \left(\frac{\partial S}{\partial q}\right)^2 = q\cdot \underbrace{a^t a}_{a^2} q$$

so the HJ equation is

$$\boxed{a' + \text{[redacted]}\,a^2 + K = 0}$$

July 7, 1980

We are considering a Hamiltonian

$$H = \frac{p^2}{2} + \frac{1}{2} q \cdot Kq$$

where K is a symmetric matrix depending on t, and the associated Schroedinger equation

$$\hbar i \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2} \Delta + \frac{1}{2} q \cdot Kq \right) \psi .$$

Feynman expressed the propagator $\langle tq | t'q' \rangle$ for the Schroedinger equation as a path integral

$$\langle tq | t'q' \rangle = \int e^{\frac{i}{\hbar} \int L\,dt} \, \mathcal{D}q .$$

In this case the Lagrangian is quadratic

$$L = \frac{p^2}{2} - \frac{1}{2} q \cdot Kq$$

and ~~so~~ the path integral is Gaussian and can be evaluated

$$\langle tq | t'q' \rangle = e^{\frac{i}{\hbar} S(tq, t'q')} \times (\text{det factor})^{-1/2}$$

Because of the quadratic character of L in q we know $S(tq, t'q')$ is quadratic in q, q', and also that the determinant factor is independent of q, q'; essentially it is the determinant of

$$\int_{t'}^{t} \left(-\frac{1}{2} \dot{q}^2 + \frac{1}{2} q \cdot Kq \right) dt$$

with $q = 0$ at the ends.

Therefore it makes sense to look for solutions of the Schroedinger equation of the form

$$\psi = e^{\frac{i}{\hbar}\tilde{S}} \qquad \tilde{S} = S + \hbar S_1$$

where $S = \frac{1}{2} q \cdot a(t) q$ is quadratic and S_1 depends only on t. To solve

$$\frac{\partial \tilde{S}}{\partial t} + \frac{1}{2}(\nabla \tilde{S})^2 + \frac{1}{2} q \cdot K q = \frac{i\hbar}{2} \nabla^2 \tilde{S}$$

or

$$\begin{cases} \dfrac{\partial S}{\partial t} + \dfrac{1}{2}(\nabla S)^2 + \dfrac{1}{2} q \cdot K q = 0 \\ \hbar \dfrac{\partial S_1}{\partial t} = \dfrac{i\hbar}{2} \nabla^2 S \end{cases}$$

Thus $a(t)$ has to satisfy the Riccati style eqn

$$a' + a^2 + K = 0$$

and

$$\frac{\partial S_1}{\partial t} = \frac{i}{2} \nabla^2 S = \frac{i}{2} \operatorname{tr} a$$

Consequently

$$\psi(tq) = e^{\frac{i}{\hbar} \frac{1}{2} q \cdot a(t) q - \frac{1}{2} \int^t (\operatorname{tr} a)\, dt}$$

Consider the problem of computing the determinant of the operator

$$\frac{d^2}{dt^2} + K$$

on $[0, T]$ with Dirichlet boundary conditions. One method might be to choose a basis $u_1, \dots, u_n$ for the solutions of $\ddot{u} + Ku = 0$ which vanish at $t = 0$. Recalling that u_i is a column vector $(u_{ij})_{j=1}^n$ we can form

$$\det(u_{ij}(t)).$$

This vanishes when the operator has an eigenfunction with eigenvalue $= 0$. Changing the basis $\{u_i\}$ is the same as multiplying the matrix (u_{ij}) on the right by a constant invertible matrix, hence we obtain a well-defined function [redacted] up to a mult. constant, which behaves like the desired determinant.

Let $A(t)$ be the matrix (u_{ij}) whose columns are [redacted] n-independent solutions of $\ddot{u} + Ku = 0$. Then

$$\ddot{A} + KA = 0$$

so if we put $\quad a = \dot{A}A^{-1} \quad$ then

$$\dot{a} = \dot{A}(-A^{-1}\dot{A}A^{-1}) + \underbrace{\ddot{A}}_{-KA}A^{-1}$$

so $$\dot{a} = -a^2 - K$$

and a satisfies the Riccati equation. Since K is symmetric this means that a is symmetric provided it is so at one point. Notice also that multiplying A on the right by a constant matrix doesn't change a. Notice that if

$$\dot{a} = -a^2 - K$$

and u [redacted] is a solution of $\quad \dot{u} = au$, then

$$\ddot{u} = \dot{a}u + a\dot{u} = (-a^2 - K)u + a(au) = -Ku$$

So [redacted] it seems to be clear that a solution of the Riccati equation can be identified with an n-diml subspace of solutions of $\ddot{u} + Ku = 0$, the Lagrangian subspaces corresponding to symmetric Riccati matrices.

Next recall that if $A = (u_{ij})$ is [redacted] a matrix of independent solutions of $\ddot{u} + Ku = 0$, then

$$\frac{d}{dt}\log\det A = tr\left(\frac{dA}{dt}A^{-1}\right) = tr(a)$$

so that

$$\det(A) = \text{const}\; e^{\int^t tr(a)\,dt}$$

Suppose we consider solutions of the HJ equation of the form

$$S(tq) = \tfrac{1}{2}q\cdot aq + b\cdot q + c$$

where a, b, c are functions of t. Then

$$\tfrac{1}{2}q\cdot\dot{a}q + \dot{b}\cdot q + \dot{c} + \tfrac{1}{2}(aq+b)^2 + \tfrac{1}{2}q\cdot Kq = 0$$

so we must have

$$\dot{a} + a^2 + K = 0$$

$$\dot{b} + ab = 0$$

$$\dot{c} + \tfrac{1}{2}b^2 = 0$$

Thus b is a solution of the first order linear DE with the matrix $-a$.

Hence if we consider

$$S(tq, t'q') = \tfrac{1}{2}q\cdot aq + q'\cdot bq + \tfrac{1}{2}q'\cdot cq'$$

then b will be a $n\times n$ matrix satisfying $\dot{b}+ab=0$ and hence

$$\det(b) = e^{-\int^t tr\, a}$$

This somewhat explains why the determinant factor for $\langle tq|t'q'\rangle$ can be written

$$\det\left(\frac{\partial^2 S}{\partial q'\partial q}\right)^{1/2}.$$

July 9, 1980:

Canonical transformations: Suppose a system is described by $H(t,q,p)$. The trajectories are those curves in (t,q,p) space which are stationary for the 1-form $p\,dq - H\,dt$. A canonical transformation is a family of $2n$ functions Q, P on (t,q,p)-space and a function $K(t,Q,P)$ such that

$$p\,dq - H\,dt = P\,dQ - K\,dt + dV$$

for some fn. V on (t,q,p) space. In good cases the functions (t,q,Q) form a system of coords on (t,q,p)-space and so $V = V(t,q,Q)$. Then

$$p = \frac{\partial V}{\partial q} \qquad P = -\frac{\partial V}{\partial Q} \qquad K = \frac{\partial V}{\partial t} + H$$

(The word "canonical" means that when one makes the change $(t,q,p) \longrightarrow (t,Q,P)$, the Hamilton equations go into equations of the same form; equations in Hamiltonian form are called "canonical.")

The interesting case is when $K=0$, i.e. when $V(t,q,Q)$ is a solution of the Hamilton-Jacobi equation depending on the n-constants $Q_1, \ldots, Q_n$. The trajectories are then given by $Q = \text{const}$, $P = \text{const}$.

Consider again the quadratic Hamiltonian

$$H = \frac{p^2}{2} + \frac{1}{2} q \cdot Kq \qquad K = K(t).$$

I want to find a formula which expresses the fundamental solution $\langle tq | t'q' \rangle$ in terms of the classical action $S(tq, t'q')$. We have seen that $S(tq, t'q')$ is

a homogeneous quadratic function of q, q'.
Moreover $\psi(tq) = e^{\frac{i}{\hbar}\tilde{S}(tq)}$ satisfies the Schroedinger equation

$$\hbar i \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2}\Delta + \frac{1}{2} q \cdot Kq \right) \psi$$

when $\tilde{S}$ satisfies

$$\frac{\partial \tilde{S}}{\partial t} + \frac{1}{2}\left(\frac{\partial \tilde{S}}{\partial q}\right)^2 + \frac{1}{2} q \cdot Kq = \frac{\hbar i}{2} \nabla^2 \tilde{S}$$

We can take $\tilde{S} = S + \hbar S_1$ where S is quadratic

$$S(tq) = \frac{1}{2} q \cdot aq + b \cdot q + c \qquad a, b, c \text{ fns of } t$$

and $S_1 = S_1(t)$ depends only on t. Then S, S_1 must satisfy

$$\frac{1}{2} q \cdot \dot{a} q + \dot{b} \cdot q + \dot{c} + \frac{1}{2}(aq + b)^2 + \frac{1}{2} q \cdot Kq = 0$$

$$(iS_1)^{\bullet} = -\frac{1}{2} \operatorname{tr}(a)$$

Thus we find

$$\dot{a} + a^2 + K = 0$$
$$\dot{b} + ab = 0$$
$$\dot{c} + \frac{1}{2} b^2 = 0$$

Begin again: Suppose that

$$S(tq, t'q') = \frac{1}{2} q \cdot Aq + q' \cdot Bq + \frac{1}{2} q' \cdot Cq'$$

where A, B, C depend on t, t'. Then we have

$$p = \frac{\partial S}{\partial q} = Aq + B^t q' \qquad p = A(-B^{-1}Cq' - B^{-1}p') + B^t q'$$

$$p' = -\frac{\partial S}{\partial q'} = -Bq - Cq' \qquad \Rightarrow q = -B^{-1}Cq' - B^{-1}p'$$

so
$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -B^{-1}C & -B^{-1} \\ B^t - AB^{-1}C & -AB^{-1} \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$$

Similarly we have
$$q' = -(B^t)^{-1}Aq + (B^t)^{-1}p$$
$$p' = -Bq - C\left(-(B^t)^{-1}Aq + (B^t)^{-1}p\right)$$

or
$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} -(B^t)^{-1}A & (B^t)^{-1} \\ -B + C(B^t)^{-1}A & -C(B^t)^{-1} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

Notice a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is symplectic when
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^t & \gamma^t \\ \beta^t & \delta^t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

or when
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \delta^t & -\beta^t \\ -\gamma^t & \alpha^t \end{pmatrix}$$

so comparing the above one sees that we have symplectic matrices. One can check that if $U(t,t')$ is the matrix giving $\binom{q}{p}$ in terms of $\binom{q'}{p'}$ then the equation
$$\dot{U} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} U$$
is equivalent to S satisfying HJ.

Here seems to be the missing argument. Let's begin with the Schroedinger equation
$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2}\Delta + V(t q) \right) \psi$$

and look for an asymptotic solution

$$\psi = e^{\frac{i}{\hbar}\tilde{S}} \qquad \text{where} \quad \tilde{S} = S + \hbar S_1 + \cdots$$

Then we want to satisfy the equations

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + V = 0$$

$$\frac{\partial S_1}{\partial t} + \frac{\partial S}{\partial q}\cdot\frac{\partial S_1}{\partial q} = \frac{i}{2}\sum_j \frac{\partial^2 S}{\partial q_j^2}$$

Now we are given a solution $S(tq,\alpha)$ of the HJ equation depending on n independent constants $\alpha_1,\ldots,\alpha_n$; for example $S(tq,t'q')$ with t' fixed. Differentiating

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + V = 0$$

wrt α gives

$$\frac{\partial^2 S}{\partial t\,\partial\alpha} + \sum_i \frac{\partial S}{\partial q_i}\frac{\partial^2 S}{\partial q_i\,\partial\alpha} = 0$$

and then with respect to q_j gives

$$\underbrace{\frac{\partial^3 S}{\partial t\,\partial q_j\,\partial\alpha} + \sum_i \frac{\partial S}{\partial q_i}\frac{\partial^3 S}{\partial q_i\,\partial q_j\,\partial\alpha}}_{\frac{d}{dt}\left(\frac{\partial^2 S}{\partial q_j\,\partial\alpha}\right)} + \sum_i \frac{\partial^2 S}{\partial q_j\,\partial q_i}\frac{\partial^2 S}{\partial q_i\,\partial\alpha} = 0$$

(here $\frac{d}{dt}$ refers to applying a vector field)

Consequently the matrix $b = \dfrac{\partial^2 S}{\partial q_j\,\partial\alpha}$ satisfies

$$\dot{b} + ab = 0$$

where $a = \dfrac{\partial^2 S}{\partial q_i\,\partial q_j}$, hence we can conclude that

$$\frac{d}{dt} \log \det b = -\operatorname{tr} a$$

and since $\frac{d}{dt}(iS_1) = -\frac{1}{2}\operatorname{tr}(a)$, we see that

$$iS_1 = \frac{1}{2}\log\det b + \text{const}$$

or that

$$\boxed{\psi(tq) = e^{\frac{i}{\hbar}S(tq,\alpha)}\left(\det\frac{\partial^2 S}{\partial q\,\partial\alpha}\right)^{1/2}}$$

is an approximate solution of the Schroedinger equation. It is exact when $V(tq)$ is quadratic in q: $\frac{1}{2}q \cdot K(t) q$ and S is also quadratic in q.

July 10, 1980

Consider free motion $\quad H = \frac{p^2}{2m}$

$$\langle tq | t'q' \rangle = \langle q | e^{-\frac{i}{\hbar}(t-t')\frac{p^2}{2m}} | q' \rangle$$

$$= \int \frac{dp}{2\pi\hbar} \underbrace{\langle q|p\rangle}_{e^{\frac{i}{\hbar}pq}} e^{-\frac{i}{\hbar}\Delta t \frac{p^2}{2m}} \langle p | q' \rangle$$

$$= \int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar}p\Delta q - \frac{i}{\hbar}\Delta t \frac{p^2}{2m}} = e^{-\frac{1}{2}\frac{m}{i\Delta t \hbar}(\Delta q)^2} \frac{\sqrt{2\pi}}{2\pi \sqrt{i\Delta t \hbar/m}}$$

$$= \frac{1}{\sqrt{2\pi i \hbar \Delta t/m}} e^{\frac{i}{\hbar}\frac{m}{2}\frac{(\Delta q)^2}{\Delta t}}$$

Thus $\qquad S(tq, t'q') = \frac{m}{2}\frac{(\Delta q)^2}{\Delta t} \qquad$ which checks

$$= \frac{1}{2}\frac{m}{\Delta t}q^2 - \frac{m}{\Delta t}qq' + \frac{1}{2}\frac{m}{\Delta t}q'^2$$

$$\therefore \quad \frac{\partial^2 S}{\partial q \partial q'} = -\frac{m}{\Delta t}$$

It would therefore seem that

$$\boxed{\langle tq | t'q' \rangle \doteq e^{\frac{i}{\hbar}S(tq, t'q')} \left(\det \frac{i}{2\pi\hbar} \frac{\partial^2 S}{\partial q \partial q'} \right)^{1/2}}$$

is the general WKB formula. (Previous work on this subject - see October 79).

Next project: Recall the classical picture of the motion of a particle: One is given a 4-manifold called space-time and in the cotangent bundle one is given a hypersurface e.g.

$$E - e\varphi = \sqrt{(p - eA)^2 + m^2}$$

One then looks at [redacted] curves in the hypersurface which are stationary with respect to the canonical form $\eta = p\,dq - E\,dt$ on the cotangent bundle.

There are two angles I want to test, both of which seem to introduce a new parameter.

A: Somehow replace the hypersurface by a conical hypersurface. This makes our particle a candidate for a wave singularity for some wave theory.

B: Somehow replace the constraint that curves lie in the hypersurface by introducing a Lagrange multiplier like the chemical potential.

July 12, 1980

It is possible to derive the FD and BE distributions using dominant term instead of the grand [redacted] partition function. Suppose we have a 1-particle system with energy levels E_α. In order to apply this method we assume these levels are highly degenerate, and group them together. Thus suppose we have levels E_s of multiplicity g_s where g_s is large. This means the 1-particle space is

$$V = \bigoplus_s V_s \qquad \dim V_s = g_s \qquad H = E_s \text{ on } V_s.$$

Then the N-particle space is

$$\Lambda^N V = \bigoplus_{\substack{\text{[redacted]} \\ \sum n_s = N}} \bigotimes_s \Lambda^{n_s} V_s$$

For each choice $\{n_s\}$ with

$$\sum n_s = N \qquad N \text{ particles present}$$

$$\sum n_s E_s = E \qquad \text{Total energy } E$$

we have

$$W = \dim \bigotimes_s \Lambda^{n_s} V_s = \prod_s \frac{g_s!}{n_s!(g-n_s)!}$$

possible states. Now maximize W subject to the constraints and one gets

$$-\log n_s + \log(g_s - n_s) - \alpha - \beta E_s = 0$$

$$\frac{g_s}{n_s} - 1 = e^{\alpha + \beta E_s}$$

$$\frac{n_s}{g_s} = \frac{1}{e^{\alpha+\beta E_s} + 1}$$

which is the FD distribution.

[redacted]

Curious point: suppose there is exactly one s with $E_s = 0$, whence [redacted] the constraint $\sum n_s = N$ says $n_s = N$. But apply Lagrange to this problem: To maximize

$$\log \frac{g!}{n!\,(g-n)!} \qquad \text{subject to} \qquad n = N$$

one forms

$$\log\left(\frac{g!}{n!\,(g-n)!}\right) - \alpha(n-N)$$

and differentiates to get

$$-\log n + \log(g-n) - \alpha = 0$$

$$\text{or} \qquad \frac{n}{g} = \frac{1}{e^{\alpha}+1}$$

[redacted] It works, <u>but</u> is silly.

Next take up BE statistics. Here you need

$$\dim S^n(V) = [redacted] \binom{n+g-1}{g-1} = \frac{(n+g-1)!}{n!\,(g-1)!}$$

and you get

$$-\log n_s + \log(n_s + g_s - 1) - \alpha - \beta E_s = 0$$

$$\text{negligible} \qquad 1 + \frac{g_s - 1}{n_s} = e^{\alpha + \beta E_s}$$

$$\frac{n_s}{g_s} = \frac{1}{e^{\alpha+\beta E_s} - 1}$$

Finally to get MB in the same way you don't want to use

$$\dim (V^{\otimes n}) = g^n$$

but rather you want to divide by $n!$ for ~~some~~ reason. Thus you maximize

$$(+) \qquad \prod_s \frac{g_s^{n_s}}{n_s!} \qquad \text{subject to constraints}$$

and get the equations

$$\log g_s - \log n_s - \alpha - \beta E_s = 0$$

or

$$\frac{n_s}{g_s} = e^{-\alpha - \beta E_s}$$

The way one sees that (+) gives essentially the number of states with the occupation numbers n_s is as follows. Fix N particles; this gives $V^{\otimes N} = (\oplus V_s)^{\otimes N}$. Then Σ_N acts on the subspace with occupation numbers $\{n_s\}$ and the dimension of this subspace is

$$\frac{N!}{\prod n_s!} \prod_s g_s^{n_s} \qquad g_s^{n_s} = \dim (V_s^{\otimes n_s}).$$

Thus except for the $N!$ one gets (+).

Let us consider now the least action variational principle again. Suppose given $H(t,q,p)$. Least action singles out a family of curves in (t,q,p)-space, namely the ones which are stationary for the form

$$pdq - Hdt.$$

We can also describe these curves as curves in the hypersurface $H(t,q,p) = E$ in the cotangent bundle to (t,q) space which are stationary for the canonical 1-form $p\,dq - E\,dt$.

The question arises as to whether the constraint condition $H(t,q,p) = E$ can be replaced by [redacted] a Lagrange multiplier[redacted] condition.

Let's change the notation slightly and suppose we work in a cotangent bundle with canonical form $p\,dq$, and we are given a hypersurface $H =$ constant E. We suppose given a critical curve and we are going to consider variations around it. Since we want to use Lagrange multipliers, it is clear that we have to be given the function H so that we can [redacted] write down the condition $\lambda(H-E)$. Therefore we [redacted] are given the Hamilton vector field X_H, and so along our trajectory we have a natural time parameter.