May 10, 1980

Curie-Weiss model 792
angular momentum + Pauli matrices 786
magnetic dipoles 781
Debye spec. heat 779
Planck law

The program is to understand an interacting fermi system. It should be a perturbation of the free system, so there should be a vacuum state, and 1-particle states indexed by a wave vector, and 2-particle states. What intrigues me is the nature of these one-particle states – somehow they are waves travelling thru the gas. It seems profitable to understand the analogue of phonons in a crystal.

Previous study of phonons is June 26 & July 6, 1979. Picture a crystal as a lattice Λ in $\mathbb{R}^3$, the points i of Λ denoting the equilibrium positions R_i of the atoms. Let q_i be the displacement of the i-th atom and p_i its momentum. Then assuming small displacements

$$H = \sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} q_i \cdot A_{ij}\, q_j$$

where A_{ij} depends only on $i-j$ in Λ. The Einstein approximation consists in regarding the i-th atom as moving in the potential where all the other atoms are fixed at equilibrium. This means we retain only the diagonal terms A_{ii} which do not depend on i, and so each atom is an independent 3-dim'l oscillator.

Let's derive the Einstein and Debye formulas for the specific heat of the crystal. Recall for a single oscillator

$$Z = \sum_{n\geq 0} e^{-\beta(n+\frac{1}{2})\hbar\omega} = \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)}$$

$$U = -\frac{\partial}{\partial\beta}\log Z = \frac{\cosh\left(\frac{\beta\hbar\omega}{2}\right)}{\sinh\left(\frac{\beta\hbar\omega}{2}\right)}\frac{\hbar\omega}{2} = \hbar\omega\left[\frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{+\beta\hbar\omega}-1}$$

Thus for a crystal with vibration modes α we have

$$U = \sum_\alpha \left(\frac{1}{2}\hbar\omega_\alpha + \frac{\hbar\omega_\alpha}{e^{+\beta\hbar\omega_\alpha}-1}\right).$$

In the Einstein approximation all the ω_α are assumed to have the same value ω. This gives

$$U = 3N\left(\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\right)$$

if the crystal has N atoms. To get the specific heat one can forget the ground energy and look at

$$\frac{3N\hbar\omega}{e^{\beta\hbar\omega}-1} \doteq \begin{cases} 3N\hbar\omega\, e^{-\beta\hbar\omega} & \text{exponentially small as [redacted] } T\to 0 \\ \dfrac{3N\hbar\omega}{\beta\hbar\omega} = \dfrac{3N}{\beta} = 3Nk_BT & \text{as } T\to\infty. \end{cases}$$

The low temperature behavior is wrong - it should be $\sim T^4$ so as to get a specific heat $\sim T^3$.

In the Debye version one [redacted] gets a better idea of the modes. First note that the modes α are described by pairs (k,λ) where k is a wave vector in the Brillouin zone (better $k \in \mathrm{Hom}(\Lambda, S^1)$) and λ describes the three possible polarizations. Debye's idea is to identify such modes [redacted] with the modes of vibration of the crystal as an [redacted] elastic body, and to take the [redacted] frequency of the modes as calculated by elastic body theory. This has the effect of assuming the dispersion law

$$\omega_{k\lambda} = s_\lambda |k|$$

where s_λ is the [redacted] speed for transverse or longitudinal waves. To simplify suppose $\omega_{k\lambda} = s|k|$. Instead of the Brillouin zone let's use a sphere $|k| < k_0$ so as to get the correct number of modes. Thus we want for a crystal of volume V [redacted] with N atoms

$$N = [\text{redacted}] \sum_{|k|<k_0} 1 = [\text{redacted}]\, V \int_{|k|<k_0} \frac{dk^3}{(2\pi)^3} = [\text{redacted}]\, V \frac{4\pi}{(2\pi)^3}\frac{k_0^3}{3}$$

so $\hbar k_0$ is like the fermi momentum. Then

$$U = 3 \sum_{k<k_0} \left(\frac{1}{2} \hbar s|k| + \frac{\hbar s|k|}{e^{\beta\hbar s|k|}-1} \right)$$

$$= V \cdot 3 \cdot \frac{4\pi}{(2\pi)^3} \int_0^{k_0} k^2 dk \left(\frac{1}{2} \hbar s k + \frac{\hbar s k}{e^{\beta \hbar s k}-1} \right)$$

Ignoring the ground energy gives

$$U = U_0 + V \cdot 3 \cdot \frac{4\pi}{(2\pi)^3} \int_0^{k_0} \frac{\hbar s k^3 dk}{e^{\beta\hbar s k}-1}$$

$$U = U_0 + V \cdot 3 \cdot \frac{4\pi}{(2\pi)^3} \int_0^{k_0\beta\hbar s} \frac{x^3 dx}{e^x - 1} \frac{\hbar s}{(\beta\hbar s)^4}$$

For high temperature

$$U \doteq U_0 + V \cdot 3 \cdot \frac{4\pi}{(2\pi)^3} \underbrace{\int_0^{k_0} \frac{\hbar s k^3 dk}{\beta \hbar s k}}_{\frac{k_0^3}{3\beta}} \qquad \text{or} \qquad U - U_0 \doteq 3NkT$$

For low temperature

$$U - U_0 \doteq \text{const} \cdot T^4$$

so the specific heat is $\sim T^3$. In practice $k_0\beta\hbar s$ is put in the form Θ_D/T where Θ_D is the Debye temperature.

Finally Planck's radiation law is the same sort of thing because we can think of radiation in a cavity as vibrations of the ether. Light waves are transverse vibrations of the ether; they have speed c so that the relation between frequency and wave number is

$$\omega_k = c|k|. \qquad \text{There are 2 polarizations}$$

The formula for the interval energy (leaving off the ground energy which is infinite) is

$$U = 2 \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^3} \frac{\hbar \omega_k}{e^{\beta\hbar\omega_k}-1} = V \cdot 2 \cdot \frac{4\pi}{(2\pi)^3} \int_0^\infty k^2 dk \, \frac{\hbar \overbrace{ck}^{\omega}}{e^{\beta\hbar ck}-1}$$

so

$$\frac{U}{V} = \frac{8\pi\hbar}{(2\pi)^3 c^3} \int_0^\infty \frac{\omega^3 d\omega}{e^{\beta\hbar\omega}-1}$$

Thus the energy distribution is $\boxed{\dfrac{8\pi\omega^2 d\omega}{c^3(2\pi)^3} \cdot \dfrac{\hbar\omega}{e^{\beta\hbar\omega}-1} \quad \text{Planck's law}}$

and the total energy is

$$\frac{U}{V} = \frac{8\pi\hbar}{c^3(2\pi)^3} \frac{1}{(\beta\hbar)^4} \underbrace{\int_0^\infty \frac{x^3 dx}{e^x-1}}$$

$$\Gamma(4)\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right) = \int_0^\infty x^4 \frac{dx}{x}\left(e^{-x}+e^{-2x}+\cdots\right)$$

$= 6 \cdot \frac{\pi^4}{90} = \frac{\pi^4}{15}$ hence

$$\text{total energy density} = \frac{8\pi}{c^3(2\pi)^3\hbar^3}\frac{\pi^4}{15}(kT)^4 = \frac{8\pi^5 k^4}{15\hbar^3 c^3} T^4$$

which is the Stefan - Bolzman law.

May 17, 1980

Let's review spin waves as an example of an interacting Bose system. First we have to understand the spin Hamiltonian. Let's recall the non-relativistic Pauli equation with external field; it gives the Hamiltonian

$$H = e\varphi + \frac{1}{2m}\left(\sigma\cdot(p-eA)\right)^2$$

$$= e\varphi + \frac{1}{2m}(p-eA)^2 - \frac{e\hbar}{2m}\sigma\cdot B$$

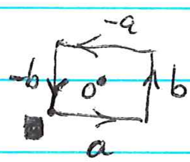
(March 14, 1980 p. 686)

where the Pauli spin matrices [redacted] $\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$ have square 1, anti-commute, and have $\sigma_x\sigma_y\sigma_z = i$. (Recall these matrices generate a group of order 16 which is an extension of a rank 3 2-torus by $\{\pm 1\}$; there are two dim. 2 irred. repns, one where $\sigma_x\sigma_y\sigma_z = i$ and the other with $\sigma_x\sigma_y\sigma_z = -i$.)

Thus due to the spin the electron has the extra energy

$$-\frac{e\hbar}{2m}\sigma\cdot B.$$

Here's the classical version: Suppose we have a small current loop



j = current thru loop.

in a uniform magnetic field B. Let's find the torque τ on the loop. The total force on the loop will be zero, since B is assumed not to change appreciably over the loop. Hence the torque will not depend on where the origin is taken (remember: $\tau = r\times F$), and so we take it in the center O of the loop. The force on the bottom is $j(\vec{a}\times B)$ and its torque is $-\frac{b}{2}\times j(a\times B)$. Thus

$$\tau = j\left[-\frac{b}{2}\times(a\times B) + \frac{a}{2}\times(b\times B) + \frac{b}{2}\times(-a\times B) + \frac{-a}{2}\times(-b\times B)\right]$$

$$= j\left[a\times(b\times B) - b\times(a\times B)\right] = j\,(a\times b)\times B$$

by Jacobi identity

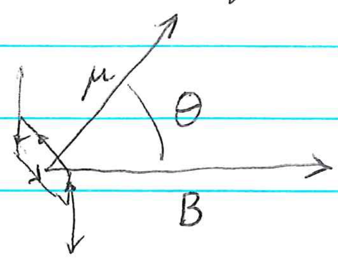
Hence the torque on the loop is

$$\tau = \mu \times B \qquad \mu = j\,(\vec{a} \times \vec{b})$$

μ is called the <u>magnetic</u> <u>moment</u> of the loop.

Next compute the work done in turning the loop against the field.

$$|\tau| = |\mu \times B| = \mu B \sin\theta$$ directed into the paper



$$\text{Work} = \int_0^{\pi} \mu B \sin\theta \, d\theta = \mu B(-\cos\theta)\Big]_0^{\pi} = 2\mu B$$

Thus the potential energy is $\mu B \cos\theta = \mu \cdot B$ up to sign. The stable position is $\theta = 0$, hence we have

$$\boxed{\text{magnetic energy of the current loop with moment } \vec{\mu} \;=\; -\vec{\mu} \cdot \vec{B}}$$

Next consider a charge e rotating circularly. The angular momentum is $J = mvr$. The magnetic moment is

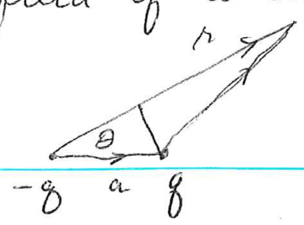
$$\mu = e \frac{v}{2\pi r} \cdot \pi r^2 = \frac{e}{2m} mvr = \frac{e}{2m} J$$

When quantized $J = \hbar j \qquad j = 1, 2, \dots$ and so

$$\mu = \frac{e\hbar}{2m} j.$$

Again we get the Bohr magneton $\frac{e\hbar}{2m}$. (Note the Dirac theory says for an electron at rest, when $j = \frac{1}{2}$, that $\mu = \frac{e\hbar}{2m}$, which is twice the above. This has to do with the Landé g-factor, see Feynman).

The field of a dipole:

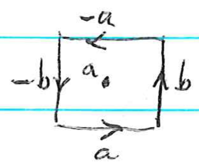


in the limit $a \to 0$, $qa \simeq$ const.

$$\varphi = \frac{-q}{|r|} + \frac{q}{|r-a|} = \frac{q}{|r||r-a|}\left(|r| - |r-a|\right) \doteq \frac{q}{r^2}(a\cos\theta)$$

$$\varphi \simeq \frac{q\vec{a}\cdot r}{r^3} = -(q\vec{a}\cdot\nabla)\left(\frac{1}{r}\right)$$

Next let's compute the field in a small current loop:



Recall Maxwell

current I

$$\begin{cases} \nabla\cdot B = 0 & \nabla\times E = -\frac{\partial B}{\partial t} \\ \nabla\cdot E = 4\pi\rho & c^2(\nabla\times B) = 4\pi j + \frac{\partial E}{\partial t} \end{cases}$$

In the magnetostatic situation $B = \nabla\times A$, where $\nabla\cdot A = 0$

must satisfy $c^2(\nabla\times B) = c^2(\underbrace{\nabla\times(\nabla\times A)}_{\nabla(\nabla\cdot A) - \Delta A}) = -c^2\Delta A = 4\pi j$

and so

$$c^2 A = \iiint \frac{j(r')\,d^3r'}{|r-r'|}$$

In other words, the vector potential belonging to a current element $\vec{j}\,d^3r'$ is $\frac{1}{c^2}\frac{\vec{j}\,d^3r'}{|r-r'|}$. Thus for the current loop above:

$$\frac{c^2\vec{A}}{I} = \left(\frac{\vec{b}}{|r-\frac{a}{2}|} + \frac{-\vec{a}}{|r-\frac{b}{2}|} + \frac{-\vec{b}}{|r+\frac{a}{2}|} + \frac{\vec{a}}{|r-\frac{b}{2}|}\right)$$

$$= \frac{\vec{b}(a\cdot r)}{r^3} - \frac{\vec{a}(b\cdot r)}{r^3} = \frac{(\vec{a}\times\vec{b})\times\vec{r}}{r^3}$$

Thus the vector potential for a small current loop is

$$\boxed{\vec{A} = \frac{1}{c^2}\frac{\vec{\mu}\times\vec{r}}{r^3} \qquad \vec{\mu} = I(\vec{a}\times\vec{b}) \quad \text{magnetic moment}}$$

Another way to write this is $A = -\frac{1}{c^2}\vec{\mu}\times\nabla\left(\frac{1}{r}\right)$. Thus

$$B = \nabla\times A = -\frac{1}{c^2}\underbrace{\nabla\times\left(\mu\times\nabla\left(\frac{1}{r}\right)\right)}_{\left(\nabla\cdot\nabla\left(\frac{1}{r}\right)\right)\mu - (\mu\cdot\nabla)\nabla\left(\frac{1}{r}\right)}$$

$$\vec{B} = \frac{1}{c^2}(\mu\cdot\nabla)\vec{\nabla}\left(\frac{1}{r}\right) = -\vec{\nabla}\underbrace{\left(-\frac{1}{c^2}\,\mu\cdot\nabla\left(\frac{1}{r}\right)\right)}_{\text{dipole potential.}}$$

const coeff vector field

Therefore one sees that the magnetic field of a current loop is the same as a magnetic dipole field.

Next rederive the energy of a magnetic dipole in a magnetic field B. The force on this dipole is

$$-m \xrightarrow[a]{} m$$

$$m\,B\left(\vec{r}+\frac{\vec{a}}{2}\right) - m\,B\left(\vec{r}-\frac{\vec{a}}{2}\right) = (m\vec{a}\cdot\nabla)\vec{B}$$

hence the work done in bring this dipole in from ∞ is

$$W = -\int_\infty (m\vec{a}\cdot\nabla)\vec{B}\cdot\vec{ds}$$

Put $\mu = m\vec{a}$ and use the identity

$$\mu\times(\nabla\times B) = \nabla(\mu\cdot B) - (\mu\cdot\nabla)B.$$

Some remarks are necessary. Here μ is constant as far as ∇ is concerned, because μ will depend on the path parameter, assuming the dipole rotates. Thus we have a situation like in

$$v\times(\nabla\times A) = \nabla(v\cdot A) - (v\cdot\nabla)A$$

encountered in deriving the Lagrangian for a charged particle.

Therefore

$$W = -\int_\infty (\mu\cdot\nabla)B\cdot\vec{ds} = -\int_\infty \nabla(\mu\cdot B)\,ds + \int_\infty \mu\times(\nabla\times B)\cdot ds$$

$$= -\mu\cdot B + \int_\infty \mu\times(4\pi j)\cdot\vec{ds}$$

So we get $\boxed{W = -\mu \cdot B}$ provided the path is in a current free region, or provided the path has $\vec{\mu} \parallel \vec{ds}$.
This is consistent with thinking of the dipole as two opposite poles; where the field B isn't conservative, one expects a closed curve will get work $\neq 0$, unless the two poles get treated the same, say $\mu \parallel ds$.

Suppose we have two dipoles with moments μ_1, μ_2 with the first at origin and the second at position r. Then field B at r due to μ_1 is

$$B = \frac{1}{c^2} \nabla\left(\mu_1 \cdot \nabla\left(\frac{1}{r}\right)\right)$$

and so the energy of the two dipoles is

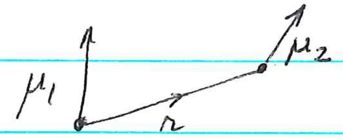
$$-\mu_2 \cdot B = -\frac{1}{c^2} (\mu_2 \cdot \nabla)(\mu_1 \cdot \nabla)\left(\frac{1}{r}\right)$$

Now
$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x_i}\left(-\frac{x_j}{r^3}\right) = 3\frac{x_j x_i}{r^5} - \frac{\delta_{ij}}{r^3}$$

So
$$\sum_{ij} \mu_i^1 \mu_j^2 \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{3(\mu_1 \cdot r)(\mu_2 \cdot r)}{r^5} - \frac{\mu_1 \cdot \mu_2}{r^3}$$

hence

$$\text{energy of two dipoles} = -\frac{1}{c^2}\,\frac{3(\mu_1 \cdot \hat{r})(\mu_2 \cdot \hat{r}) - \mu_1 \cdot \mu_2}{r^3}$$

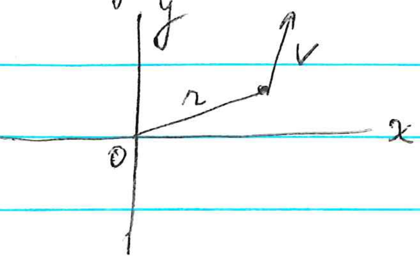


$$\hat{r} = \frac{\vec{r}}{r}$$

This classical formula doesn't seem to imply that the energy is minimum when the spins line up.

May 20, 1980

Angular momentum: In two dims:



$$J = r \times mv = r \times p$$

$$= \begin{vmatrix} i & j & k \\ x & y & \cancel{z}^{0} \\ p_x & p_y & \cancel{p_z}^{0} \end{vmatrix} = (xp_y - yp_x)\hat{k}$$

Thus the angular momentum about the origin is the operator

$$J = xp_y - yp_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

As

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\underbrace{\frac{\partial x}{\partial \theta}}_{-r\sin\theta = -y} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)u$$

we see that in polar coordinates

$$\boxed{J = xp_y - yp_x = \frac{\hbar}{i}\frac{\partial}{\partial \theta}}$$

Operating on functions on the plane, the eigenfunctions are $e^{in\theta}$ and the eigenvalues are $n\hbar$, $n \in \mathbb{Z}$.

In three dimensional angular momentum has 3 components

$$J_x = yp_z - zp_y, \quad J_y = zp_x - xp_z, \quad J_z = xp_y - yp_x$$

satisfying

$$[J_x, J_y] = [yp_z - zp_y, zp_x - xp_z] = \frac{\hbar}{i}yp_x + xp_y\, i\hbar$$

$$\boxed{[J_x, J_y] = i\hbar J_z \quad \text{, cyclic perm. for others}}$$

Recall that the Lie algebra of SO_3 can be identified with $\mathbb{R}^3$ by assigning to $\omega \in \mathbb{R}^3$ the infinitesimal rotation

$$r \mapsto \omega \times r$$

and that the bracket is given by the ordinary cross product of vectors. Hence a representation of $\mathrm{Lie}(SO_3)$ is given by operators (skew-adjoint) $\theta_x, \theta_y, \theta_z$ corresponding to the unit vectors i, j, k satisfying

$$[\theta_x, \theta_y] = \theta_z \qquad \text{and cyclic perms.}$$

Thus the angular momentum operators give a repn. of $\mathrm{Lie}(SO_3)$ if we put

$$J_x = i\hbar\,\theta_x \qquad \text{etc.}$$

May 21, 1980:

The key example is furnished by the Pauli matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

which anti-commute, have square 1, and satisfy $\sigma_x \sigma_y = i\sigma_z$ + cyclic permutations. Hence

$$[\sigma_x, \sigma_y] = 2i\sigma_z \qquad \text{and cyclic perms.}$$

which means we get a 2-diml. representation of the angular momentum operators provided we put

$$J_x = \frac{\hbar}{2}\sigma_x \qquad \text{etc.}$$

A more convenient basis is furnished by

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_+ = \tfrac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_- = \tfrac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which is the standard basis for sl_2, and satisfies

$$[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm \qquad [\sigma_+, \sigma_-] = \sigma_z$$

If V denotes the standard 2-dim'l representation of sl_2, the irreducible representations are S_nV for $n \geq 0$. On this space, σ_z has the eigenvalues $-n, -n+2, \dots, n-2, n$, hence it has maximal weight n. The Casimir element is a multiple of $\sigma_x^2 + \sigma_y^2 + \sigma_z^2 = (\sigma_+ + \sigma_-)^2 - (\sigma_+ - \sigma_-)^2 + \sigma_z^2$

$$= 2(\sigma_+\sigma_- + \sigma_-\sigma_+) + \sigma_z^2 = 4\sigma_-\sigma_+ + 2\sigma_z + \sigma_z^2$$

which has the value $2n + n^2 = (n+1)^2 - 1^2$. It seems that $\sigma_x^2 + \sigma_y^2 + \sigma_z^2$ is the Casimir element

Consider a system with [redacted] 2 spin sites. It is described by the space $V \otimes V$ with the operator $\vec{\sigma} \otimes \vec{\sigma} = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$. The Casimir operator in $V \otimes V$ is

$$\sum_{i=x,y,z} (\sigma_i \otimes 1 + 1 \otimes \sigma_i)^2 = \underbrace{\sigma^2}_{3} \otimes 1 + 2(\vec{\sigma} \otimes \vec{\sigma}) + 1 \otimes \underbrace{\sigma^2}_{3}$$

$V \otimes V \cong S_2V \oplus S_0V$, so Casimir has eigenvalues $8, 8, 8, 0$ on $V \otimes V$ and $3, 3$ on V hence

$$\begin{matrix} & & 2, 2, 2, -6 & \\ \vec{\sigma} \otimes \vec{\sigma} & \text{has eigenvalues} & 1, 1, 1, -3 & \text{on } V \otimes V \end{matrix}$$

as we've already seen (recall $\vec{\sigma} \otimes \vec{\sigma} = 2P_{ex} - 1$).

More generally suppose we have two sites of spins $\frac{m}{2}$ and $\frac{n}{2}$. This system is described by $S_mV \otimes S_nV$ with the operator $\vec{\sigma} \otimes \vec{\sigma}$. One way to motivate this is as follows. If $\varphi \in S_mV$ is a unit vector, then the value of $\vec{\sigma}$ in the state φ is the vector

$$\langle \varphi | \vec{\sigma} | \varphi \rangle$$

So if $\varphi \in S_mV$, $\psi \in S_nV$ are unit vectors, then the value of $\vec{\sigma} \otimes \vec{\sigma}$ in the state φ, ψ is

$$\langle \varphi \otimes \psi | \sum_a \sigma_a \otimes \sigma_a | \varphi \otimes \psi \rangle = \sum_a \langle \varphi | \sigma_a | \varphi \rangle \langle \psi | \sigma_a | \psi \rangle$$

$$= \langle \varphi|\vec{\sigma}|\varphi\rangle\cdot\langle\psi|\vec{\sigma}|\psi\rangle$$

which agrees with the feeling that the interaction energy should be proportional to the dot product of the spins.

Actually why do we have to keep the state space at a site irreducible, why not take the state space to be $S(V) = \bigoplus_{n\geq 0} S_n V$? Whatever we take, because of the identity

$$\sum (\sigma_a \otimes 1 + 1\otimes \sigma_a)^2 = \sigma^2\otimes 1 + 2\sum_a \sigma_a\otimes\sigma_a + 1\otimes\sigma^2$$

it follows that $\vec{\sigma}\otimes\vec{\sigma}$ is essentially the (non-primitive) part of the Casimir operator.

Now suppose we have several sites $i \in I$, and let $\vec{\sigma}^i = (\sigma_a^i)$ be the spin operators belonging to the i-th site space extended by the identity on the other site spaces. The interaction energy is

$$\frac{1}{2}\sum_{i\neq j} g_{ij}\, \vec{\sigma}^i\cdot\vec{\sigma}^j$$

where g_{ij} gives the coupling between the i-th and j-th sites. Suppose all $g_{ij}=1$. Then

$$\frac{1}{2}\sum_{i\neq j}\vec{\sigma}^i\cdot\vec{\sigma}^j = \frac{1}{2}\sum_a\left(\sum_{i,j}\sigma_a^i\sigma_a^j - \sum_i(\sigma_a^i)^2\right)$$

$$= \frac{1}{2}\sum_a\left(\sum_i \sigma_a^i\right)^2 - \frac{1}{2}\sum_i(\sigma^i)^2$$

Casimir on ith site space extended by identity

and $\sum_i \sigma_a^i$ is just σ_a on the big tensor products. Thus again the interaction is the non-primitive part of Casimir.

Digression: Compute Casimir on $S(V)$. We have that σ_x etc are derivations of $S(V)$, hence putting $a = a_\uparrow$, [redacted] $b = a_\downarrow$ we have

$$\sigma_x = a^*b + b^*a \qquad \sigma_+ = a^*b$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_y = \sum a_i^* \langle i | \sigma_y | j \rangle a_j \qquad \sigma_- = \text{[redacted]}\, b^*a$$

$$= i(b^*a - a^*b)$$

$$\sigma_z = a^*a - b^*b$$

Then Casimir is
$$4\sigma_-\sigma_+ + 2\sigma_z + \sigma_z = 4b^*a\,a^*b + 2(a^*a - b^*b) + (a^*a - b^*b)^2$$

$$= 4b^*b(a^*a+1) + 2(a^*a - b^*b) + (a^*a)^2 - 2a^*a\,b^*b + (b^*b)^2$$

$$= 2(a^*a + b^*b) + (a^*a + b^*b)^2 = 2N + N^2$$

where N is the number operator.

Recall the electron gas where one has an exterior algebra on a vector space with basis $|x\lambda\rangle$, where $\lambda = \uparrow$ or $\downarrow$. [redacted] This exterior algebra is

$$\Lambda\Big(\bigoplus_x V\Big) = \bigotimes_x \underbrace{\Lambda(V)}_{4\ \text{diml}}.$$

For each x one has the spin density operator $\psi(x)^* \vec{\sigma}\, \psi(x)$ which involves projecting on the x-th component and using the operator $\psi^* \vec{\sigma}\, \psi$ on $\Lambda(V)$. Here

$$\psi = \begin{pmatrix} a_\uparrow \\ a_\downarrow \end{pmatrix} \qquad \psi^* = \begin{pmatrix} a_\uparrow^* & a_\downarrow^* \end{pmatrix}$$

so
$$\psi^* \sigma_x \psi = \begin{pmatrix} a_\uparrow^* & a_\downarrow^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_\uparrow \\ a_\downarrow \end{pmatrix} = \text{[redacted]}\; a_\uparrow^* a_\downarrow + a_\downarrow^* a_\uparrow$$

Thus $\psi^* \sigma_x \psi$ is just the derivation of ΛV belonging to σ_x on V; similarly for σ_y, σ_z.

Here's how we can handle the spin wave situation: Think of an electron gas with the electrons held in place by the atoms. At each site one can have either no electrons, or 1 electron with either ↑ or ↓ spin, [redacted] or two electrons with opposite spin. In the first and last cases the net spin at the site is zero.

Let denote the sites by $i \in I$ and let a_i, b_i be the destruction operators for ↑ and ↓ states, so that

$$\sigma_x^i = a_i^* b_i + b_i^* a_i$$

$$\sigma_y^i = i(b_i^* a_i - a_i^* b_i)$$

$$\sigma_z^i = a_i^* a_i - b_i^* b_i$$

Then the interaction between i-th and j-th sites is

$$\sigma_x^i \sigma_x^j + \sigma_y^i \sigma_y^j + \sigma_z^i \sigma_z^j = (a_i^* b_i + b_i^* a_i)(a_j^* b_j + b_j^* a_j)$$
$$- (b_i^* a_i - a_i^* b_i)(b_j^* a_j - a_j^* b_j)$$
$$+ (a_i^* a_i - b_i^* b_i)(a_j^* a_j - b_j^* b_j)$$

$$= a_i^* b_i b_j^* a_j + b_i^* a_i a_j^* b_j + \sigma_z^i \sigma_z^j$$
$$+ a_i^* b_i b_j^* a_j + b_i^* a_i a_j^* b_j$$

$$= 2(\sigma_+^i \sigma_-^j + \sigma_-^i \sigma_+^j) + \sigma_z^i \sigma_z^j$$

which is some fourth order combination of creation and annihilation operators. Perhaps you can describe this interaction by diagrams.

May 22, 1980

Let's consider a "box" into which we can put particles which can assume two states: either spin up or spin down. Suppose there are N particles and that they are distinguishable, so that we 2^N possible states. Let's assume a standard Ising interaction energy:

$$E = -g\frac{1}{2}\sum_{i\neq j} \sigma^i\sigma^j - B\sum_i \sigma^i$$

The parameter g is going to play the role of 1/volume of the box, so that if the volume is small g is large.

Let's put

$$\sigma^i = 2s_i - 1 \qquad \text{so that} \qquad \begin{array}{l} s_i = 0 \implies \sigma^i = -1 \\ s_i = 1 \implies \sigma^i = 1 \end{array}$$

and write E in terms of the s_i.

$$\frac{1}{2}\sum_{i\neq j}\sigma^i\sigma^j = \frac{1}{2}\left[\left(\sum\sigma^i\right)^2 - \sum_i(\sigma^i)^2\right] = \frac{1}{2}\left[\left(\sum\sigma^i\right)^2 - N\right]$$

$$= \frac{1}{2}\left[\left(2\sum s_i - N\right)^2 - N\right]$$

hence

$$-E = \frac{g}{2}\left[\left(2\sum s_i - N\right)^2 - N\right] + B\left[2\sum_i s_i - N\right]$$

$$= 2g\left(\sum s_i\right)^2 + \left(\sum s_i\right)\underbrace{\left(2B - 2gN\right)}_{b} + \text{const.}$$

So if we put $n = \sum s_i$ the partition function is

$$Z = \sum_{n=0}^{N}\binom{N}{n} e^{\beta(2gn^2 + bn)}$$

What we want to do is to understand this for large N. Ideally what we want to do is take the low density case $(\rho \to 0)$ and increase the density until there is a phase transition (= singularity of some sort).

May 23, 1980 (Alice is 18)

I'm considering putting N particles with either spin up or down into a "box". The particles are distinguishable so the partition function is

$$Z = \sum_{n=0}^{N} \binom{N}{n} e^{-\beta E_n}$$

where n is the number of up spins, and the energy of a configuration with n-spins up is quadratic in n

$$-E_n = 2g\,n^2 + bn$$

What we would like to do is to evaluate this partition function, and then understand how it varies with N, β, g, b.

Example: suppose $g=0$. Then

$$Z = \sum_{n=0}^{N} \binom{N}{n} e^{\beta b n} = \left(1+e^{\beta b}\right)^N$$

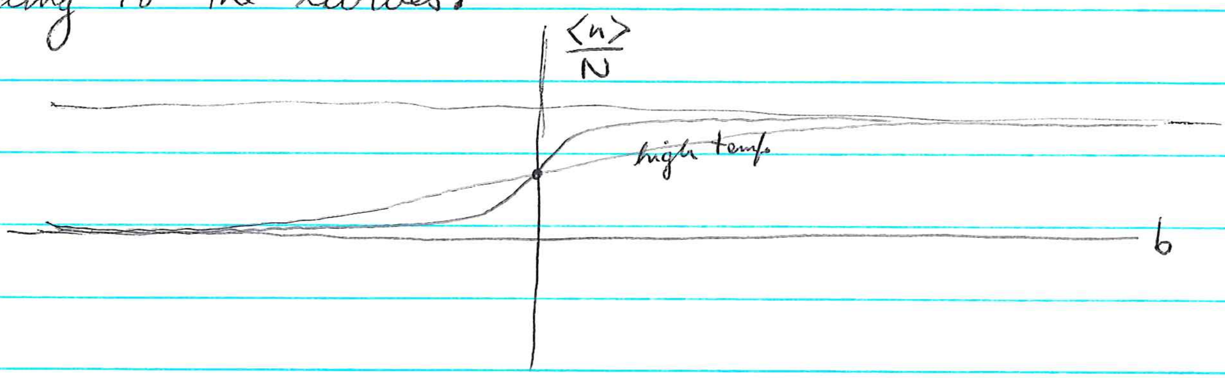
so

$$\frac{1}{N}\log Z = \log\left(1+e^{\beta b}\right)$$

and

$$\frac{\langle n \rangle}{N} = \frac{1}{\beta}\frac{\partial}{\partial b}\frac{\log Z}{N} = \frac{e^{\beta b}}{1+e^{\beta b}}$$

leading to the curves:



Recall that $-E_n = 2g\,n^2 + n(2B - 2gN)$ so that if $B=0$, one has $-E_n = 2g\,n(n-N)$ so E_n is maximum when $n = \frac{N}{2}$ and minimum where $n=0$ or N. Thus the effect of the interaction is to weight the center region $n \sim \frac{N}{2}$

less than the ends. The binomial factor $\binom{N}{n}$ [redacted] weights the center region heavily. So what I expect as g increases is a smooth change up to a point where the binomial factor is overcome, at which point we should have a phase transition.

Unfortunately we don't seem to be able to compute the partition function exactly, hence we resort to approximations.

$$\log\binom{N}{pN} \doteq \cancel{N\log N} - \cancel{N} + \cancel{\tfrac{1}{2}\log N} + \cancel{\tfrac{1}{2}\log 2\pi}$$
$$-pN(\log p + \cancel{\log N}) + \cancel{pN} - \tfrac{1}{2}(\log p + \cancel{\log N})$$
$$-qN(\log q + \cancel{\log N}) + \cancel{qN} - \tfrac{1}{2}(\log q + \log N) - \tfrac{1}{2}\log 2\pi$$

$$= -N(p\log p + q\log q) - \tfrac{1}{2}\log N - \tfrac{1}{2}\log(2\pi pq)$$

where $q = 1-p$. Let set $x = \frac{n}{N} = p$. Then our first approx is:

$$Z = \sum_{n=0}^{N}\binom{N}{n}e^{\beta(2gn^2+bn)} \doteq N\int_0^1 dx\, e^{-N(x\log x+(1-x)\log(1-x))+\beta(2gN^2x^2+bNx)}\frac{1}{\sqrt{2\pi N(x(1-x))}}$$

(This approximation ought to be good for large N provided one avoids weighting $x=0,1$ with the other factor). The next approximation is to uses Laplace's method on the integral, since the large factor N appears in the exponent.

$$f(x) = x\log x + (1-x)\log(1-x) - \beta 2gN\,x^2 - \beta b\,[redacted]\,x$$

$$f'(x) = \log x + x\cdot\frac{1}{x} - \log(1-x) + (1-x)\frac{-1}{1-x} - 2\beta 2gN\,x - \beta b\,[redacted]$$

$$= \log\frac{x}{1-x} - 2\beta(2gN)x - \beta b$$

$$f''(x) = \frac{1}{x} + \frac{1}{1-x} - 2\beta 2gN$$

$$= \frac{1}{x(1-x)} - 2\beta 2gN$$

It's clear at this stage that if I want to get anywhere I want to let $N \to \infty$ with gN constant. I am thinking of g as $\frac{1}{Vol}$, hence this amounts to a infinite volume limit with fixed density.

Recall we are interested in $\lim \frac{1}{N} \log Z$. By Laplace we have

$$\lim \frac{1}{N} \log Z = -f(\bar{x})$$

where $\bar{x}$ is chosen to be the minimum point of f, i.e.

$$f'(\bar{x}) = 0$$

Since $f''(x) = \frac{1}{x(1-x)} - 4\beta g N \geq 4 - 4\beta g N$, as long as $\beta(gN) < 1$ there is a unique minimum point, if it exists.

On the other hand we should think of the integral defining Z as a kind of partition function, hence the minimum f value $\bar{x}$ is actually the average $\langle x \rangle = \lim \frac{\langle n \rangle}{N}$. Hence

$$(*) \qquad f'(\bar{x}) = \log\left(\frac{\bar{x}}{1-\bar{x}}\right) - 4\beta(gN)\bar{x} - \beta b = 0$$

is the equation defining the fraction $\bar{x}$ of up spins in terms of the parameters gN, β, b. This is reminiscent of the Weiss theory.

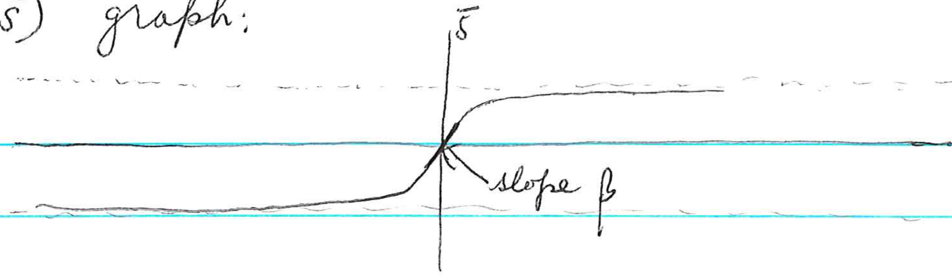
Let's change variables to $2\bar{x} - 1 = \bar{s}$, so that $\bar{s}$ is the average spin. Then $\bar{x} = \frac{1}{2}(1+\bar{s})$, $1-\bar{x} = \frac{1}{2}(1-\bar{s})$ and the equation $(*)$ becomes

$$\log\left(\frac{1+\bar{s}}{1-\bar{s}}\right) = 4\beta(gN)\tfrac{1}{2}(1+\bar{s}) + \beta(2B - 2gN)$$

$$\frac{1}{2\beta} \log\left(\frac{1+\bar{s}}{1-\bar{s}}\right) = (gN)\bar{s} + B \qquad \text{or}$$

$$\boxed{\bar{s} = \tanh \beta\left((gN)\bar{s} + B\right)}$$

Here's the way to plot this. Start with the inverse $\tanh(\bar{s})$ graph:

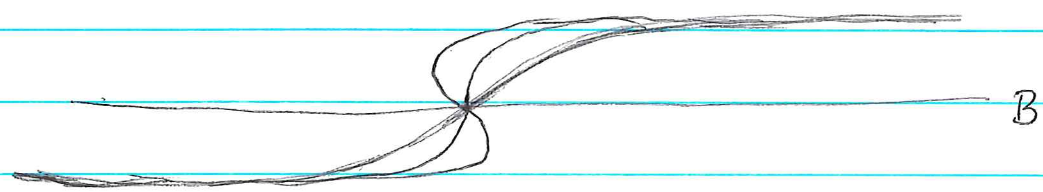


$$\bar{s} = \tanh \beta B \qquad \text{or} \qquad B = \frac{1}{\beta}\frac{1}{2}\log\left(\frac{1+\bar{s}}{1-\bar{s}}\right)$$

For small $\bar{s}, B$ this is $\bar{s} \approx \beta B$ so it has slope β. Now consider

$$B = \frac{1}{2\beta}\log\left(\frac{1+\bar{s}}{1-\bar{s}}\right) - (gN)\bar{s}$$

which will distort the curve



For small $\bar{s}$, $\quad B \approx \left(\frac{1}{\beta} - gN\right)\bar{s}$ so the curve has slope $\dfrac{1}{\frac{1}{\beta} - gN} = \dfrac{\beta}{1-\beta g N}$ and the phase transition occurs when $\beta(gN) = 1$.

Review the Weiss theory. Let the spin be s_i at the i-th site. [redacted] Assume that [redacted] the effect of the other spins on the i-th site can be replaced by their average. Thus the i-th site has two states $s_i = \pm 1$ with energy

$$E = -Bs_i - \sum_{j \neq i} g \underset{\substack{\uparrow \\ \text{replaced by } \bar{s}}}{s_j}\, s_i \qquad = -s_i\big(B + \underbrace{(N-1)g\bar{s}}_{\approx N g \bar{s}}\big)$$

Using this to compute $\langle s_i \rangle = \bar{s}$, we get

$$\bar{s} = \frac{e^{\beta(B+Ng\bar{s})} - e^{-\beta(B+Ng\bar{s})}}{e^{\beta(B+Ng\bar{s})} + e^{-\beta(B+Ng\bar{s})}} = \tanh \beta(B+Ng\bar{s})$$

which is the same as before.

So we now haves a model we [redacted] might be able to understand completely, and which [redacted] agrees with the Weiss theory above the critical temperatures.

May 24, 1980

Review: I'm considering an Ising model with N spins $s_1, \dots, s_N$ each coupled to the others the same way. The energy is

$$E(s) = -B\sum s_i - g\frac{1}{2}\sum_{i\neq j}' s_i s_j$$

$$= -B\left(\sum s_i\right) - \frac{g}{2}\left(\left(\sum' s_i\right)^2 - N\right)$$

Write this in terms of $\bar{s} = \frac{1}{N}\sum s_i$ the average spin

$$E(\vec{s}) = -N\left(B\bar{s} + \frac{gN}{2}\bar{s}^2\right) + \frac{Ng}{2}$$

We are going to let $N \to \infty$ in such a way that gN is constant. (I think of g as proportional to $\frac{1}{V}$, think of particles spreading out all over the box; then gN = constant is the same as constant density.)

The partition function is

$$Z = \sum_{n=0}^{N}{}' \binom{N}{n} e^{\beta E_n}$$

where n is the number of up spins: $\sum s_i = 2n - N$. Recall

$$\log\binom{N}{n} \doteq -N\left(x\log x + (1-x)\log(1-x)\right) - \frac{1}{2}\log\left(N2\pi\, x(1-x)\right)$$

by Stirling's formula, where $x = \frac{n}{N}$. Now

$$\bar{s} = \frac{1}{N}\sum s_i = 2x - 1 \qquad x = \frac{1}{2}(1+\bar{s}) \quad 1-x = \frac{1}{2}(1-\bar{s})$$

$$x\log x + (1-x)\log(1-x) = \frac{1}{2}(1+\bar{s})\log(1+\bar{s}) + \frac{1}{2}(1-\bar{s})\log(1-\bar{s}) - \log 2$$

Thus

$$Z \doteq \sum e^{-N\left(\frac{1}{2}(1+\bar{s})\log(1+\bar{s}) + \frac{1}{2}(1-\bar{s})\log(1-\bar{s}) - \log 2 - \beta B\bar{s} - \frac{\beta g N}{2}\bar{s}^2\right) + \frac{\beta N g}{2}}$$

$$\times \sqrt{N2\pi\,\frac{1-\bar{s}^2}{4}}$$

We can ignore constants independent of spin $\bar{s}$, and so we get

$$Z = C\int \frac{d\bar{s}}{\sqrt{1-\bar{s}^2}}\, e^{-Nf(\bar{s})}$$

C depends on N, β

where

$$f(\bar{s}) = \tfrac{1}{2}(1+\bar{s})\log(1+\bar{s}) + \tfrac{1}{2}(1-\bar{s})\log(1-\bar{s}) - (\beta B)\bar{s} - \frac{(\beta g N)}{2}\bar{s}^2$$

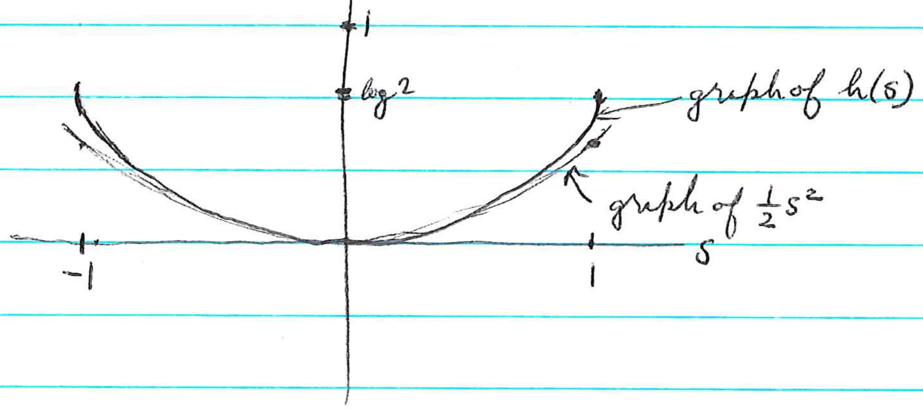
Let's graph. Put

$$h(s) = \tfrac{1}{2}(1+s)\log(1+s) + \tfrac{1}{2}(1-s)\log(1-s) \approx \frac{s^2}{2} + \frac{s^4}{12} + O(s^6)$$

$$h'(s) = \tfrac{1}{2}\log\left(\frac{1+s}{1-s}\right) \approx s + \frac{s^3}{3} + O(s^5)$$

$$h''(s) = \tfrac{1}{2}\left[\frac{1}{1+s} + \frac{1}{1-s}\right] = \frac{1}{1-s^2} \approx 1 + s^2 + O(s^4)$$

$h \longrightarrow \log 2 = .7$ as $s \to \pm 1$



Now we are interested in the function

$$f(s) = h(s) - bs - \frac{1+\varepsilon}{2}s^2 \approx -bs - \frac{\varepsilon}{2}s^2 + \frac{s^4}{12} + O(s^6)$$

where we have put $b = \beta B$, $1+\varepsilon = \beta g N$.

$$f'(s) = -b - \varepsilon s + \frac{s^3}{3} + O(s^5)$$

$$f''(s) = \frac{1}{1-s^2} - (1+\varepsilon) \approx -\varepsilon + s^2 + O(s^4)$$

If $\varepsilon < 0$, then $f(s)$ is concave upward, and so at least for small b it has a unique minimum near zero.

The point is that for large N, the partition function can be evaluated by steepest descent. One should find that

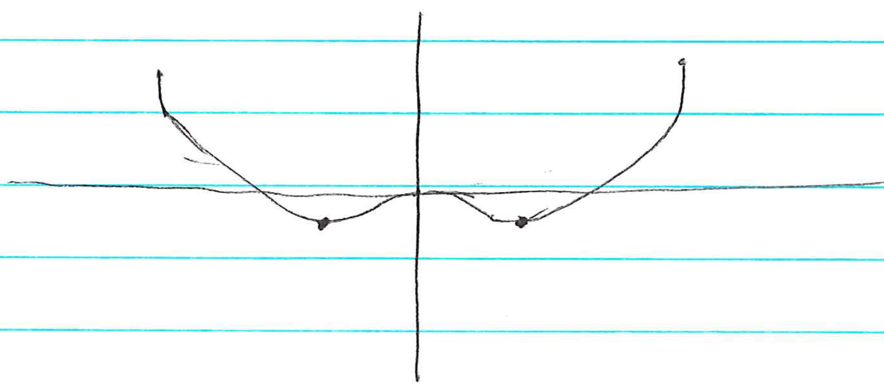
$$\lim_{N\to\infty} \frac{1}{N} \log Z = -f(\bar{s})$$

where $\bar{s}$ is the minimum [redacted] point for f. More generally if there are more than one minimum points one should take the sum of the f values. Now we want to work around $b, \varepsilon = 0$.

For ε slightly > 0 and $b = 0$ we get

$$f(s) \doteq -\frac{\varepsilon}{2} s^2 + \frac{s^4}{12}$$

$$f'(s) \doteq -\varepsilon s + \frac{s^3}{3} = 0 \quad \text{when} \quad s = \pm\sqrt{3\varepsilon},\ 0$$

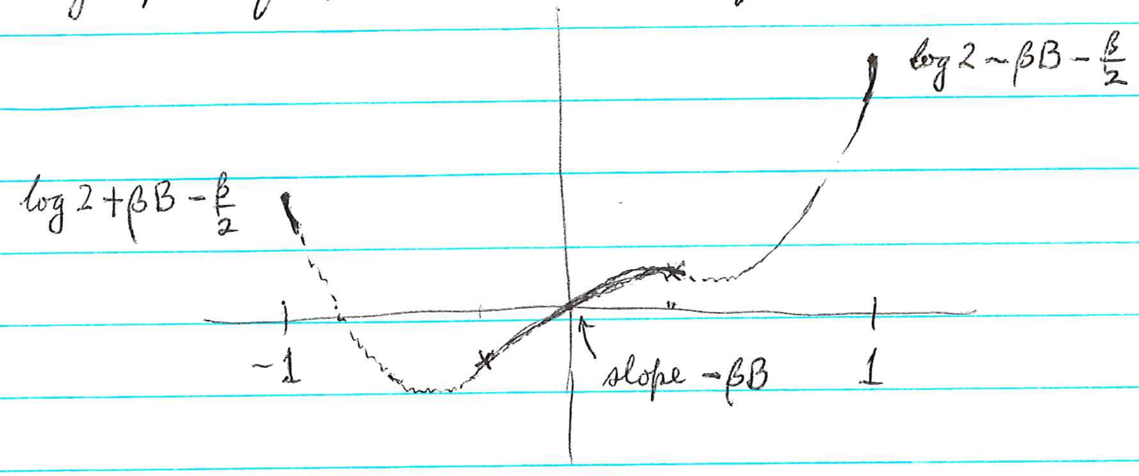


Now if $b \neq 0$ the two minima points should separate in f-value, so there should be only one minimum point, which is consistent with the Lee-Yang result that there is no phase transition with $b \neq 0$.

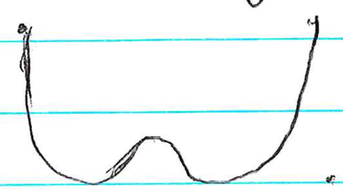
Let us fix β, [redacted] B, gN. Actually we might as well put $gN = 1$. I know that for large N the Stirling approximation for $\binom{N}{n}$ is good provided $\frac{n}{N}$ stays away from the ends. Look carefully at the exponent

$$f(s) = \frac{1}{2}(1+s)\log(1+s) + \frac{1}{2}(1-s)\log(1-s) - \beta B s - \beta \frac{s^2}{2}$$

Then $f'(s) = \frac{1}{2}\log\left(\frac{1+s}{1-s}\right) - \beta B - \beta s$ goes from $-\infty$ to ∞ as s goes from -1 [redacted] to 1. So there is at least one zero.
The graph of $f(s)$ looks as follows (when $B<0$



Also $f''(s) = \frac{1}{1-s^2} - \beta$. If $\beta<1$, the graph is concave upward and can have only one minimum point necessarily <0, (assuming $B<0$). Otherwise you [redacted] have two inflection points and possibly a local minimum for $s>0$ as drawn. In any case it's clear that we have a unique minimum point for $B\neq 0$ [redacted] no matter what B, β are. (Notice that even for β [redacted] >1 the graph is obtained by shearing the case $B=0$:

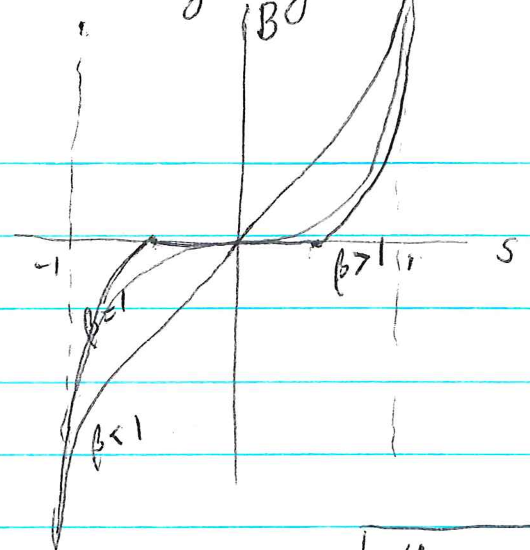


Since the minimum point occurs inside $-1<s<1$, we can be confident that [redacted] the Stirling approximation as well as replacing sum by integral are valid as $N\to\infty$ and we get

$$\frac{1}{N}\log Z \longrightarrow -f(\bar{s})$$ where $\bar{s}$ is the unique minimum point for $B\neq 0$.

When $B=0$ we have two symmetrical minimum points but they have the same f value.

It's clear now that this model gives the Weiss theory on the nose. The magnetization curves are



$$\boxed{\frac{1}{2\beta}\log\left(\frac{1+s}{1-s}\right) - s = B}$$

equation of state

The internal energy is $\boxed{\frac{U}{N} = -Bs \text{ [redacted] } - \frac{s^2}{2}}$ because the basic probability distribution peaks at the value of s given by the equation of state.

Note that $\frac{U}{N} = 0$ for $\beta < 1$ when $B = 0$. If $\beta > 1$ and $B = 0$ we need [redacted] to find $-\frac{s^2}{2}$, where s is the minimum point. This minimum point is the spontaneous magnetization and is found [redacted] by solving

$$\underbrace{\frac{1}{2\beta}\log\left(\frac{1+s}{1-s}\right)}_{\approx \frac{1}{\beta}\left(s + \frac{s^3}{3}\right)} - s = 0$$

$$1 + \frac{s^2}{3} = \beta \qquad s = \sqrt{3(\beta - 1)}$$

Thus for $\beta > 1$ but close to 1 we have

$$\text{spontaneous magnetization} \doteq \sqrt{3(\beta-1)} \qquad (T - T_c)^{1/2}$$

$$\text{internal energy} \quad \frac{U}{N} \doteq -\frac{3}{2}(\beta - 1)$$

<u>May 26, 1980</u>

We should go over the previous calculations for the symmetric Ising model with the idea that N is finite, but that we are calculating asymptotic expansions as $N \to \infty$. If we think of N being infinite, then because $gN = 1$, there is no coupling between the spins. To be precise, we have for a given choice of β, H (H = the external field, denoted previously by B) a probability measure on $\{\pm 1\}^N$ and hence for any finite subset s_i, $i \in I$ of spins we have an induced probability measure on $\{\pm 1\}^I$. If we let $N \to \infty$ keeping I fixed, then its clear that on $\{\pm 1\}^I$ we get the product measure for the measure on $\{\pm 1\}$ giving the correct average spin s determined by the [redacted] Curie-Weiss law

$$\frac{1}{2\beta} \log\left(\frac{1+s}{1-s}\right) - s = H$$

with suitable modifications for $\beta > 1$.

in terms of β, H. What we want to do is to compute the induced measure on $\{\pm 1\}^I$ as an asymptotic expansion in N.

Since the approximations (Stirling, $\sum \mapsto \int$) give $\frac{1}{N}$ errors, we want to express the partition function

$$\sum_n \binom{N}{n} e^{-\beta E_n}$$

directly as an integral. Recall

$$-E_n = H \sum s_i + \frac{g}{2} \sum_{i \neq j} s_i s_j$$

$$= H \sum s_i + \frac{g}{2}\left[\left(\sum s_i\right)^2 - \left(\sum s_i^2\right)\right]$$

This fouls up much

$$= g\frac{\sigma^2}{2} + \left(H - \cancel{\tfrac{g}{2}}\right)\sigma - \frac{Ng}{2} \qquad \sigma = \sum s_i = 2n - N$$

$$-\beta E_n = \beta g \frac{\sigma^2}{2} + \left(\beta H - \cancel{\tfrac{\beta g}{2}}\right)\sigma - \frac{\beta N g}{2}$$

$$\frac{1}{\sqrt{2\pi a}}\int_{-\infty}^{\infty} e^{-\frac{1}{2a}k^2+bk+\sigma k}\,dk = \frac{1}{\sqrt{2\pi a}}\int_{-\infty}^{\infty} e^{-\frac{1}{2a}(k-(b+\sigma)a)^2+(b+\sigma)^2\frac{a}{2}}\,dk$$

$$= e^{\frac{a}{2}(\sigma+b)^2} = e^{\frac{a}{2}\sigma^2+ab\sigma+\frac{ab^2}{2}}$$

$$\therefore \quad e^{\frac{a}{2}\sigma^2+(ab)\sigma} = \frac{1}{\sqrt{2\pi a}}\int_{-\infty}^{\infty} e^{\overbrace{-\frac{1}{2a}k^2+bk-\frac{a}{2}b^2}^{-\frac{1}{2a}(k-ab)^2}+k\sigma}\,dk$$

$$\sum\binom{N}{n}e^{\frac{\beta g}{2}\sigma^2+(\beta H-\cancel{\frac{\beta g}{2}})\sigma} = \frac{1}{\sqrt{2\pi\beta g}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\beta g}(k-\beta H+\cancel{\frac{\beta g}{2}})^2}\underbrace{\sum\binom{N}{n}e^{k\sigma}}\,dk$$

$$\left(e^{k}+e^{-k}\right)^N = e^{-kN}\left(1+e^{2k}\right)^N = \sum\binom{N}{n}e^{k(2n-N)}$$

Therefore

$$Z = \sum\binom{N}{n}e^{-\beta E_n} = \frac{1}{\sqrt{2\pi\beta g}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\beta g}(k-\beta H+\cancel{\frac{\beta g}{2}})^2+N\log(e^k+e^{-k})}\,dk$$

and this is an exact formulae. Now we want to put $g=\frac{1}{N}$ and evaluate this asymptotically in N. The exponent is

$$-\frac{N}{2\beta}\left(k-\beta H+\cancel{\frac{\beta}{2N}}\right)^2+N\log\left(e^k+e^{-k}\right)$$

and its critical points are given by

$$-\frac{1}{\beta}\left(k-\beta H+\cancel{\frac{\beta}{2N}}\right)+\frac{e^k-e^{-k}}{e^k+e^{-k}}=0\,.$$

If we put $s=\frac{e^k-e^{-k}}{e^k+e^{-k}}$, then $k=\frac{1}{2}\log\left(\frac{1+s}{1-s}\right)$ and the equation becomes

$$s+H-\cancel{\frac{1}{2N}} = \frac{k}{\beta} = \frac{1}{2\beta}\log\left(\frac{1+s}{1-s}\right)$$

which is the Weiss equation, at least when $N\to\infty$.

It doesn't seem to be possible to get the approximate integral for Z from the exact one by the ~~substitution~~ $s = \frac{e^k - e^{-k}}{e^k + e^{-k}}$.

The quantity [redacted] k/β seems to be interpretable as the magnetic field B. Let's go back to the energy

$$-E(\vec{s}) = H \sum_i s_i + \frac{g}{2}\left(\left(\sum_i s_i\right)^2 - \sum_i s_i\right).$$

Then $-\frac{\partial E}{\partial s_i} = H + g\left(\sum_j s_j\right) - \frac{g}{2}$ is something like the magnetic field B_i at the [redacted] i-th site. The only thing that's mysterious is the $g/2$, in fact it's absurd to differentiate with respect to something that only takes the values ± 1. The correct thing if you are thinking of continuous spins is

$$-E(s) = H \sum_i s_i + \frac{g}{2}\left(\left(\sum_i s_i\right)^2 - \sum s_i^2\right) \qquad ?$$

which gives $B_i = -\frac{\partial E}{\partial s_i} = H + g \sum_{j \neq i} s_j$. This is confusing, but somehow what's involved is whether you think of s_i as contributing to the [redacted] field at the i-th site. Let's tentatively define

$$B = H + g \sum_j s_j - \frac{g}{2} = H + gNs - \frac{g}{2}$$

where $s = \frac{1}{N}\sum s_i$ is the average spin. Then the Weiss equation becomes

$$\begin{cases} B = H + (Ng)s - \cancel{\frac{g}{2}} \\ s = \dfrac{e^{\beta B} - e^{-\beta B}}{e^{\beta B} + e^{-\beta B}} \end{cases} \qquad ?$$

$$Z = \frac{1}{2^N} \sum \binom{N}{n} e^{-\beta E_n} = \sqrt{\frac{\beta}{2\pi g}} \int_{-\infty}^{\infty} e^{-\frac{\beta}{2g}\left(B - H + \cancel{\frac{g}{2}}\right)^2 + N \log\left(\frac{e^{\beta B} + e^{-\beta B}}{2}\right)} dB$$

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It is necessary to correct yesterday's error. The energy is

$$-E(s) = H\sum s_i + g\,\tfrac{1}{2}\sum_{i\neq j} s_i s_j$$

$$= H\sum s_i + \frac{g}{2}\Big(\sum_{i,j} s_i s_j - \sum_{i=j} s_i s_j\Big)$$

$$= H\sum s_i + \frac{g}{2}\Big(\big(\sum s_i\big)^2 - N\Big)$$

$$= H\sigma + \frac{g}{2}\sigma^2 - \text{const} \qquad \text{where} \qquad \sigma = \sum s_i = 2n - N$$

However adding a constant to the energy doesn't affect the predictions, so put

$$-E(s) = H\sigma + \frac{g}{2}\sigma^2$$

$$-\beta E(s) = \beta H\sigma + \frac{\beta g}{2}\sigma^2$$

Then we use

$$e^{\frac{a}{2}\sigma^2} = \int e^{-\frac{k^2}{2a} + \sigma k \cancel{- \frac{a}{2}\sigma^2 + \frac{a}{2}\sigma^2}} \frac{dk}{\sqrt{2\pi a}}$$

$$e^{\frac{a}{2}\sigma^2 + b\sigma} = \int e^{-\frac{(k-b)^2}{2a} + \sigma k} \frac{dk}{\sqrt{2\pi a}}$$

$$e^{\frac{\beta g}{2}\sigma^2 + \beta H\sigma} = \int e^{-\frac{(k-\beta H)^2}{2\beta g} + \sigma k} \frac{dk}{\sqrt{2\pi\beta g}}$$

$$Z = \sum_{n=0}^{N} 2^{-N}\binom{N}{n} e^{\frac{\beta g}{2}\sigma^2 + \beta H\sigma} = \int e^{-\frac{(k-\beta H)^2}{2\beta g}} \underbrace{\sum \binom{N}{n} 2^{-N} e^{(2n-N)k}} \frac{dk}{\sqrt{2\pi\beta g}}$$

$$(\cosh k)^N = \left(\frac{e^k + e^{-k}}{2}\right)^N = 2^{-N}\left(e^{-k}\right)^N\left(1 + e^{2k}\right)^N$$

Thus
$$Z = \sum_{n=0}^{N} \frac{1}{2^N}\binom{N}{n} e^{\frac{\beta g}{2}\sigma^2 + \beta H \sigma} = \int_{-\infty}^{\infty} e^{-\frac{1}{2\beta g}(k-\beta H)^2 + N \log(\cosh k)} \frac{dk}{\sqrt{2\pi\beta g}}$$

and this gets evaluated by steepest descent. The maximum point for the integrand occurs at

$$-\frac{1}{\beta g}(k - \beta H) + N \frac{e^k - e^{-k}}{e^k + e^{-k}} = 0$$

Thus if I put $B = k/\beta$ I get the equations

$$\begin{cases} B = H + NgS \\ S = \dfrac{e^{\beta B} - e^{-\beta B}}{e^{\beta B} + e^{-\beta B}} \end{cases}$$

of the Weiss theory. B is the average magnetic field at any of the sites. Normally the field B_i at the i-th site would be

$$B_i = H + g\sum_{j \neq i} S_j \implies B = \langle B_i \rangle = H + (N-1)gS$$

which means that the theory "wants" a self-energy of $-\frac{g}{2}S_i^2$ of the i-th site with itself, so it seems.