January 1, 1980

Spin waves (continued). Recall the Hilbert space is the exterior algebra with basis e_n where e_n is the spin assignment with spin down at n and spin up elsewhere. Better

$$\mathcal{H} = \bigotimes_n \mathbb{C}^2$$

This identification is not compatible with signs (see page 513)

and we think of $\mathbb{C}^2$ as $\Lambda\mathbb{C}$ with $1 = |+\rangle$ and $|-\rangle =$ generator of $\Lambda^1\mathbb{C}$. The Hamiltonian is

$$H = \sum_\sigma \tfrac{1}{2}(1 - P_\sigma^{ex})$$

where σ runs over pairs of nearest neighbor sites.

I want to write H as $H_0 + H'$ where H_0 is the derivation of the exterior algebra extending H on $\mathcal{H}^1$ and where H' is ~~as~~ 2-particle interaction extended canonically to several particles by summing over pairs. It's enough to do this for the operator

$$\tfrac{1}{2}(1 - P_\sigma^{ex})$$

Suppose $\sigma = \{1,2\}$. Then

$$\mathcal{H} = \Lambda\{e_1, e_2\} \otimes \Lambda\{e_n\}_{n \neq 2}$$

and the operator $\frac{1}{2}(1 - P_\sigma^{ex})$ is of the form $A \otimes 1$. So we need only compute A on $\Lambda\{e_1, e_2\}$.

$$\begin{aligned}
1 &\longmapsto 0 \\
e_1 &\longmapsto \tfrac{1}{2}(e_1 - e_2) \\
e_2 &\longmapsto \tfrac{1}{2}(e_2 - e_1) \\
e_1 \wedge e_2 &\longmapsto 0
\end{aligned}$$

Let H_0 be the derivation extending A on Λ^1. Then

$$H_0(e_1 \wedge e_2) = \tfrac{1}{2}(e_1 - e_2) \wedge e_2 + e_1 \wedge \tfrac{1}{2}(e_2 - e_1) = e_1 \wedge e_2$$

so we have $A = H_0 + H'$ where

$$H_0 = \tfrac{1}{2}(a_1^* - a_2^*)a_1 + \tfrac{1}{2}(a_2^* - a_1^*)a_2 = \tfrac{1}{2}(a_1^* - a_2^*)(a_1 - a_2)$$

$$H' = -a_1^* a_2^* a_2 a_1$$

Hence the formula for the Heisenberg chain is

$$\boxed{H = \sum_n \tfrac{1}{2}(a_n^* - a_{n+1}^*)(a_n - a_{n+1}) - a_n^* a_{n+1}^* a_{n+1} a_n}$$

Let's now pass to the momentum representation. We suppose that our set of sites is $\mathbb{Z}/N\mathbb{Z}$ and we want to change from the orth. basis e_n to the orth basis

$$u_k = \frac{1}{\sqrt{N}} \sum_k e^{ikn} e_n \qquad k \in \frac{2\pi}{N}\mathbb{Z}/2\pi\mathbb{Z}$$

$$e_n = \frac{1}{\sqrt{N}} \sum_n e^{-ikn} u_k$$

ext. mult of

$$\begin{cases} a_n^* = e(e_n) = \dfrac{1}{\sqrt{N}} \displaystyle\sum_k e^{-ikn} a_k^* & a_k^* = e(u_k) \\ a_n = \dfrac{1}{\sqrt{N}} \displaystyle\sum_k e^{ikn} a_k \end{cases}$$

$$a_n - a_{n+1} = \frac{1}{\sqrt{N}} \sum_k (1 - e^{ik}) e^{ikn} a_k$$

$$a_n^* - a_{n+1}^* = \frac{1}{\sqrt{N}} \sum_{k'} (1 - e^{-ik'}) e^{-ik'n} a_{k'}^*$$

$$H_0 = \frac{1}{2} \sum_k (1 - e^{ik})(1 - e^{-ik}) a_k^* a_k = \sum_k (1 - \cos k) a_k^* a_k$$

Now if the lattice spacing were a we would get

$$H_0 = \sum_k (1-\cos ak)\, a_k^* a_k \qquad k \in \frac{2\pi}{Na}\mathbb{Z} \Big/ \frac{2\pi}{a}\mathbb{Z}$$

so the energy of the "particle" u_k of momentum k is

$$\varepsilon_k = 1-\cos ak \approx \tfrac{1}{2} a^2 k^2$$

Consider $H' = +\sum_n a_n^* a_{n+1}^* a_{n+1} a_n$

$$= +\frac{1}{2}\sum_{m,n} a_m^* a_n^* V(m-n)\, a_n a_m$$

where $$V(m-n) = \begin{cases} -1 & |m-n| = a \\ 0 & \text{otherwise} \end{cases}$$

Then

$$H' = +\frac{1}{2}\sum_{m,n}\left(\frac{1}{\sqrt{N}}\sum_{k_1} e^{-ik_1 m} a_{k_1}^*\right)\left(\frac{1}{\sqrt{N}}\sum_{k_2} e^{-ik_2 n} a_{k_2}^*\right)$$

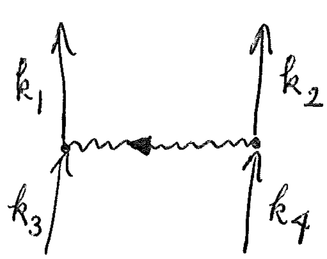
$$\times V(m-n)\left(\frac{1}{\sqrt{N}}\sum_{k_4} e^{ik_4 n} a_{k_4}\right)\left(\frac{1}{\sqrt{N}}\sum_{k_3} e^{-ik_3 m} a_{k_3}\right)$$

$$= +\frac{1}{2N}\sum_{k_1 \cdots k_4} a_{k_1}^* a_{k_2}^* a_{k_4} a_{k_3} \underbrace{\sum_{m,n} e^{im(-k_1+k_3)+in(-k_2+k_4)}\, V(m-n)\,\frac{1}{N}}$$

$$\underbrace{\frac{1}{N}\sum_{m,n} e^{i(m+n)(k_3-k_1)+in(k_4-k_2)}\, V(m)}$$

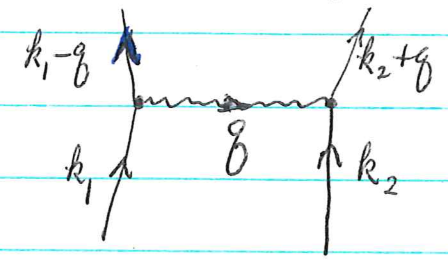
$$= \delta(k_3+k_4-k_1-k_2) \underbrace{\sum_m e^{im(k_3-k_1)} V(m)}_{\hat{V}(k_1-k_3)}$$

Picture



Thus

$$H' = +\frac{1}{2N} \sum_{k_1, k_2, q} a^*_{k_1-q}\, a^*_{k_2+q}\, \hat{V}(q)\, a_{k_2}\, a_{k_1}$$



In the situation considered above

$$\hat{V}(q) = -\left(e^{iaq} + e^{-iaq}\right) = -2\cos(aq)$$

Notice that in the expression

$$H' = \frac{1}{2} \sum_{m,n}{}' a^*_m\, a^*_n\, V(m-n)\, a_n\, a_m$$

the value $V(0)$ doesn't appear, and hence a constant can be added to $\hat{V}(q)$ without affecting the expression at the top of this page for H'.

Also we can pass to a continuum limit if we redefine

$$\hat{V}(q) = a \sum_n e^{-inq} V(n) \longrightarrow \int e^{-ixq} V(x)\, dx$$

Then we have

$$H' = \frac{1}{\underbrace{2Na}_{\text{Volume } V}} \sum a^*_{k_1-q}\, a^*_{k_2+q}\, \hat{V}(q)\, a_{k_2}\, a_{k_1}$$

It seems that for the Heisenberg chain this limit is useless, because then

$$\hat{V}(q) = -a\left(e^{iaq} + e^{-iaq} + \underbrace{\text{const.}}_{\text{take } = -2}\right) = 2a\left(1 - \cos(aq)\right) \approx a^3 q^2$$

and I want to divide H by a^2 in order that H_0 have a good limit as $a \to 0$.

A more serious problem is that the identification of $\mathcal{H}$ with an exterior algebra is faulty. The idea was to use an ordering of the sites in order to go from

$$e_{i_1} \cdots e_{i_p} \longleftrightarrow e_{i_1} \wedge \cdots \wedge e_{i_p} \qquad \text{if } i_1 < \cdots < i_p$$

But this gets in trouble with the Hamiltonian. Recall we wanted $\frac{1}{2}(1 - P^{ex}_{mn}) \longleftrightarrow \frac{1}{2}(a_m - a_m)^*(a_m - a_n) - a_m^* a_n^* a_n a_m$
Take $m = 1$, $n = 3$ and see what happens to $e_1 e_2$

$$\frac{1}{2}(1 - P^{ex}_{13})\, e_1 e_2 = \frac{1}{2} e_1 e_2 - \frac{1}{2} e_3 e_2 \longleftrightarrow \frac{1}{2}(e_1 \wedge e_2 \blacksquare\, e_2 \wedge e_3)$$

$$\left(\frac{1}{2}(a_1 - a_3)^*(a_1 - a_3) - a_1^* a_3^* a_3 a_1\right)(e_1 \wedge e_2) = \frac{1}{2}(a_1^* - a_3^*)\, e_2$$

$$= \frac{1}{2}(e_1 \wedge e_2 - e_3 \wedge e_2) = \frac{1}{2}(e_1 \wedge e_2 + e_2 \wedge e_3)$$

But maybe this problem doesn't occur with the linear chain. There [redacted] doesn't seem to be any trouble when the set of sites is $\mathbb{Z}$, but clearly the same problem arises in the periodic case for P^{ex}_{1N}.

Anyway it seems to be more important to understand

$$H = \sum_k \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1, k_2, q} a^*_{k_1 - q} a^*_{k_2 + q} \hat{V}(q) a_{k_2} a_{k_1}$$

Here V is the volume so that $k \in \frac{2\pi}{V}\mathbb{Z}$. Adding a constant to $\hat{V}(q)$ is irrelevant.

January 4, 1979

Schwinger's anti-commuting c-numbers.

Let's consider $\mathcal{H} = \Lambda V$ with creation-and-annihilation operators $a_n^* = e(e_n)$, $a_n = i(e_n^*)$ defined by some orthonormal basis e_n of V. Consider $H_0 = \sum \omega_n a_n^* a_n$. The problem will be to compute expectation values of products of the operators $a_n(t)$, $a_n^*(t)$ at different times. In particular, I want to see if there is some analogue of the time-ordered products encountered before.

Let's first review the boson situation. We have $H_0 = \omega a^* a$ and we form $H = \omega a^* a + J a + \tilde{J} a^*$ where $J, \tilde{J}$ are functions of t with compact support. Then let $U(t,t')$ be the propagator for

$$\frac{\partial \psi}{\partial t} = -H\psi$$

and let's compute $\operatorname{tr} U(\beta, 0)$ by Schwinger's method.

$$\delta U(\beta,0) = -\int_0^\beta dt \; U(\beta,t)\{\delta J\, a + \delta\tilde{J}\, a^*\}\, U(t,0)$$

$$\frac{d}{dt} U(\beta,t)\, a\, U(t,0) = U(\beta,t)\underbrace{[H,a]}_{-\omega a - \tilde{J}}U(t,0)$$

$$\therefore \left(\frac{d}{dt} + \omega\right) \operatorname{tr}\left(U(\beta,t)\, a\, U(t,0)\right) = -\tilde{J}\operatorname{tr}\left(U(\beta,0)\right)$$

$$\left(\frac{d}{dt} - \omega\right) \operatorname{tr}\left(U(\beta,t)\, a^*\, U(t,0)\right) = J\operatorname{tr}\left(U(\beta,0)\right)$$

$$\therefore \quad \frac{\operatorname{tr}\left(U(\beta,t)\, a\, U(t,0)\right)}{\operatorname{tr} U(\beta,0)} = -\left(\frac{d}{dt} + \omega\right)^{-1}\tilde{J}$$

$$\frac{\operatorname{tr}\left(U(\beta,t)\,a^*\,U(t,0)\right)}{\operatorname{tr} U(\beta,0)} = -\left(-\frac{d}{dt}+\omega\right)^{-1} J$$

where the inverses $\left(\pm\frac{d}{dt}+\omega\right)^{-1}$ are computed using periodic boundary conditions on $[0,\beta]$. Thus

$$\frac{\delta \operatorname{tr} U(\beta,0)}{\operatorname{tr} U(\beta,0)} = \int_0^\beta \left(\delta J\left(\frac{d}{dt}+\omega\right)^{-1}\tilde{J} + \delta\tilde{J}\left(-\frac{d}{dt}+\omega\right)^{-1} J\right) dt$$

$$= \int_0^\beta \left(\delta J\left(\frac{d}{dt}+\omega\right)^{-1}\tilde{J} + J\left(\frac{d}{dt}+\omega\right)^{-1}\delta\tilde{J}\right) dt$$

Thus integrating we get

$$\frac{\operatorname{tr} U(\beta,0)}{\operatorname{tr} U_0(\beta,0)} = \exp \int_0^\beta \left(J\left(\frac{d}{dt}+\omega\right)^{-1}\tilde{J}\right) dt$$

Notice this isn't symmetric in $J, \tilde{J}$. As a check you should let $\beta \to +\infty$ and replace the lower limit 0 by $\beta' \to -\infty$. Then

$$\operatorname{tr} U(\beta,\beta') \cong \langle 0 | U(\beta,\beta') | 0 \rangle \qquad \beta \gg 0 \gg \beta'.$$

Also $\left(\frac{d}{dt}+\omega\right)^{-1}$ has the kernel $\begin{cases} e^{-\omega(t-t')} & t>t' \\ 0 & t<t' \end{cases}$

so we get

$$\langle 0 | U(\beta,\beta') | 0 \rangle = \exp \iint_{t>t'} J(t)\, e^{-\omega(t-t')}\, \tilde{J}(t')\, dt\, dt'$$

What kinds of Green's functions arise? Use Dyson's formula

$$U(\beta,0) = U_0(\beta,0) - \int dt_1\, U_0(\beta,t_1)\, H_1'(t_1)\, U_0(t_1,0) + \cdots$$

Then
$$\frac{tr\, U(\beta,0)}{tr\, U_0(\beta,0)} = \sum_{m,n} \frac{(-1)^{m+n}}{m!\,n!} \int dt_1 \cdots dt_m\, dt_1' \cdots dt_n'$$ [redacted]
$$\times J(t_1) \cdots J(t_m)\, \tilde{J}(t_1') \cdots \tilde{J}(t_n')$$
$$\times \langle T[a(t_1) \cdots a(t_m)\, a^*(t_1') \cdots a^*(t_n')] \rangle$$

where $a(t) = e^{tH_0} a\, e^{-tH_0}$ and $\langle A \rangle = \dfrac{tr(e^{-\beta H_0} A)}{tr(e^{-\beta H_0})}$

So it's clear the Green's functions are [redacted] just

$$\langle T[a(t_1) \cdots a(t_m)\, a^*(t_1') \cdots\cdots a^*(t_n')] \rangle$$

and one has Wick's theorem evaluating them.

<u>Question:</u> Does there exist [redacted] a path integral description for $tr(U(\beta,0))$?

Possible approach: $Z(J,\tilde{J}) = tr\,(U(\beta,0)_{J,\tilde{J}})$ is a function of the pair $J, \tilde{J}$ which we might try to represent as the Fourier (or Laplace) transform of a measure on the dual space. So the problem becomes describing this measure. A first problem is whether $\tilde{J}$ should be required to be $\bar{J}$.

We can work backwards from the formula

[redacted] $$\int e^{-\frac{1}{2}X^t A X + J^t X} (dX) = \frac{e^{\frac{1}{2} J A^{-1} J}}{(\det A)^{1/2}}$$

which is valid for $Re(A) > 0$. So look at

$$\frac{tr(U(\beta,0))}{tr(U_0(\beta,0))} = \exp \int J \left(\frac{d}{dt} + \omega\right)^{-1} \tilde{J}\,.$$

Is the quadratic function

$$J, \tilde{J} \longmapsto \int J \left(\frac{d}{dt}+\omega\right)^{-1} \tilde{J}$$

positive-definite? Obviously not since it changes sign when J does. Note that if $\tilde{J} = \bar{J}$, then

$$\int J\left(\frac{d}{dt}+\omega\right)^{-1}\bar{J} = \int\left(\left(-\frac{d}{dt}+\omega\right)^{-1} J\right)\cdot \bar{J}$$

but it's simpler to work with Fourier series:

$$J(t) = \sum_{n \in \frac{2\pi}{\beta}\mathbb{Z}}' J_n \, e^{int}$$

$$\bar{J} = \sum \bar{J}_n \, e^{-int}$$

$$\left(\frac{d}{dt}+\omega\right)^{-1}\bar{J} = \sum_n \frac{\bar{J}_n \, e^{-int}}{-in+\omega}$$

$$\int J\left(\frac{d}{dt}+\omega\right)^{-1}\bar{J} = \sum_n{}' \frac{|J_n|^2}{-in+\omega}$$

Now

$$\operatorname{Re}\left(\frac{1}{-in+\omega}\right) = \frac{1}{2}\left(\frac{1}{-in+\omega} + \frac{1}{in+\omega}\right) = \frac{\omega}{n^2+\omega^2} > 0$$

So we can conclude that there should be a Gaussian measure on the dual space to the set of all $J, \tilde{J} = \bar{J}$ periodic on $[0,\beta]$. This space is the space of periodic functions (really distributions) $a(t)$. Recall the formula

$$\int e^{-a|z|^2 + Jz + \bar{J}\bar{z}} \, dz\, d\bar{z} = e^{\frac{|J|^2}{a}} \cdot \text{const}$$

Consequently for the form

$$\sum_n \frac{|J_n|^2}{-in+\omega}$$

we are going to want to use something like

$$(*) \quad \int \mathcal{D}a\,\mathcal{D}\bar{a} \quad e^{-\int\left(a\frac{d\bar{a}}{dt} + \omega a\bar{a} + Ja + \bar{J}\bar{a}\right)dt}$$

Let's evaluate by Fourier series: $J = \sum J_n e^{int}$.
$a = \sum a_n e^{-int}$

$$\int\left(a\frac{d\bar{a}}{dt} + \omega a\bar{a}\right)dt = \sum a_n(-in)\bar{a}_n + \omega a_n\bar{a}_n$$

$$\int Ja \; dt = \sum_n J_{-n}a_n$$

and so

$$(*) = \int \mathcal{D}a\,\mathcal{D}\bar{a} \quad e^{-\sum_n (-in+\omega)|a_n|^2 + J_{-n}a_n + \bar{J}_{-n}\bar{a}_n}$$

$$= \text{const.} \quad e^{\sum \frac{|J_n|^2}{-in+\omega}}$$

It seems like we do get a path-integral representation for the Schwinger generating function

$$\frac{\operatorname{tr} U(\beta,0)}{\operatorname{tr} U_0(\beta,0)}$$

however, the "measure" on the space of paths is a Gaussian with imaginary part.

January 5, 1980:

Yesterday we computed $\mathrm{tr}\,(U(\beta,0))$ for $H = \omega a^* a + Ja + \tilde{J}a^*$ using Schwinger's variational method. However a simpler approach uses

$$e^{+\beta H_0}\, U(\beta,0) = T\, e^{-\int_0^\beta (Ja+\tilde{J}a^*)(t)\,dt}$$

where T is time-ordering. Also one uses

$$e^A e^B = e^{[A,B]}\, e^B e^A$$

when $[A,B]$ commutes with A, B. Then one has

$$\cancel{e^{\gamma a^*} e^{\gamma' a} = e^{-\gamma\gamma'} e^{\gamma' a} e^{\gamma a^*}}$$

$$e^{\gamma a}\, e^{\gamma' a^*} = e^{\gamma\gamma'}\, e^{\gamma' a^*}\, e^{\gamma a}$$

since

$$a(t) = e^{+tH_0} a\, e^{-tH_0} = e^{-\omega t} a$$
$$a^*(t) = e^{\omega t} a^*$$

We therefore can evaluate the time-ordering by shoving $\tilde{J}(t)\, a^*(t)\, dt$ thru the previous a-part

$$e^{+\beta H_0}\, U(\beta,0) = e^{\int_{t_1>t_2} J(t_1) e^{-\omega t_1+\omega t_2} \tilde{J}(t_2)}\; e^{-\left(\int_0^\beta \tilde{J}(t) e^{\omega t} dt\right) a^*}\; e^{-\left(\int_0^\beta J(t) e^{-\omega t} dt\right) a}.$$

What we have to do is multiply by $e^{-\beta H_0} = e^{-\beta\omega a^* a}$ and take the traces. Somehow this doesn't look easy. Here are two ways of doing it:

Put $\langle A \rangle = \mathrm{tr}(e^{-\beta H_0}A)/\mathrm{tr}(e^{-\beta H_0})$. Then

$$\langle e^{\gamma' a^*} e^{\gamma a} \rangle = \sum_{m,n} \frac{(\gamma')^m}{m!} \frac{(\gamma)^n}{n!} \langle a^{*m} a^n \rangle$$

Wick's thm says

$$\langle a^{*m} a^n \rangle = \begin{cases} 0 & m \neq n \\ n! \langle a^* a \rangle^n & \text{if } m = n. \end{cases}$$

and

$$\langle a^* a \rangle = \frac{\sum n e^{-\beta\omega n}}{\sum e^{-\beta\omega n}} = -\frac{d}{d(\beta\omega)} \log \frac{1}{1-e^{-\beta\omega}} = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}$$

Therefore

$$\langle e^{\gamma' a^*} e^{\gamma a} \rangle = e^{\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \gamma' \gamma}$$

Use holomorphic representation: Want trace of

$$f(z) \xmapsto{e^{\gamma a}} f(z+\gamma) \xmapsto{e^{\gamma' a^*}} e^{\gamma' z} f(z+\gamma) \mapsto$$

$$\xrightarrow{e^{-\beta\omega a^* a}} e^{\gamma' e^{-\beta\omega} z} f(e^{-\beta\omega} z + \gamma)$$

Let's compute the trace using as basis the powers of $z-\alpha$ where α is the fixpoint for $z \longmapsto e^{-\beta\omega} z + \gamma$, i.e.

$$\alpha = \frac{\gamma}{1-e^{-\beta\omega}}$$

Multiplication by $e^{\gamma' e^{-\beta\omega} z}$ relative to this basis is triagonal with diagonal entries equal to

$$e^{\gamma' e^{-\beta\omega} \alpha} = e^{\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \gamma \gamma'}$$

The operator $f(z) \longrightarrow f(e^{-\beta\omega} z + \gamma)$ relative to this basis is diagonal with eigenvalues $e^{-\beta\omega n}$ for $(z-\alpha)^n$, hence one sees that

$$= \left(\eta\{a,a^*\} - \{\eta,a^*\}a\right)\tilde{\eta} + a^*\left(\eta\cancel{\{a,\tilde{\eta}\}} - \cancel{\{\eta,\tilde{\eta}\}}a\right)$$

$$= \eta\tilde{\eta} \quad \text{commutes with other operators.}$$

Thus we can use the $e^A e^B = e^{[A,B]} e^B e^A$ formula to evaluate the time-ordered product. We get

$$e^{\beta H_0} U(\beta,0) =$$

$$e^{\int_{t_1>t_2} \eta(t_1) e^{-\omega(t_1-t_2)} \tilde{\eta}(t_2)} \; e^{-a^* \int_0^\beta \tilde{\eta} e^{\omega t} dt} \; e^{-\left(\int_0^\beta \eta(t) e^{-\omega t} dt\right) a}$$

Now
$$\langle e^{-a^*\gamma'} e^{-\gamma a} \rangle = \langle (1 - a^*\gamma')(1-\gamma a) \rangle$$

$$= \langle 1 + \gamma' a^* + a\gamma + \gamma' a^* a \gamma \rangle$$

$$= 1 + \gamma' \langle a^* a \rangle \gamma$$

$$\langle a^* a \rangle = \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}}$$

So we get

$$\frac{\operatorname{tr} U(\beta,0)}{\operatorname{tr} U_0(\beta,0)} = \langle e^{\beta H_0} U(\beta,0) \rangle = e^{\int_{t_1>t_2} \eta(t_1) e^{-\omega(t_1-t_2)} \tilde{\eta}(t_2)} \left(1 + \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}} \gamma'\gamma\right)$$

$$\text{where } \gamma' = \int_0^\beta \tilde{\eta}(t) e^{+\omega t} dt \qquad \gamma = \int_0^\beta \eta(t) e^{-\omega t} dt$$

Recall that $\eta(t)$, $\tilde{\eta}(t)$ are elements of degree 1 of some Grassman algebra, hence the same is true for γ' and γ, so that we have $\gamma^2 = \gamma'^2 = 0$ and so I can write

$$\langle e^{-a^*\gamma'} e^{-\gamma a} \rangle = 1 + \gamma'\gamma \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}} = e^{\frac{e^{-\beta\omega}}{1+e^{-\beta\omega}} \underbrace{\gamma'\gamma}_{-\gamma\gamma'}}$$

It follows that we have

$$\operatorname{tr}\left(e^{-\beta\omega a^* a}\, e^{\gamma' a^*} e^{\gamma a}\right) = \frac{1}{1-e^{-\beta\omega}}\, e^{\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}\gamma\gamma'}$$

leading to the same formula for $\langle e^{\gamma' a^*} e^{\gamma a}\rangle$.

It follows from the above formulas that

$$\frac{\operatorname{tr} U(\beta,0)}{\operatorname{tr} U_0(\beta,0)} = \langle e^{\beta H_0} U(\beta,0)\rangle$$

$$= e^{\int_0^\beta J(t_1)\, G(t_1,t_2)\, \tilde{J}(t_2)}$$

where

$$G(t_1,t_2) = e^{-\omega(t_1-t_2)}\,\theta(t_1-t_2) + e^{-\omega(t_1-t_2)}\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}$$

Notice that $\quad \left(\frac{d}{dt_1}+\omega\right) G(t_1,t_2) = \delta(t_1-t_2)$

and

$$G(\beta,t_2) = e^{-\omega(\beta-t_2)}\left[1+\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}\right]$$

$$\Vert \quad G(0,t_2) = e^{\omega t_2}\left[\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}\right]$$

so G is the Green's function for $\left(\frac{d}{dt}+\omega\right)^{-1}$ with periodic boundary conditions.

Next let's go over the fermion case. We introduce "sources" $\eta, \tilde{\eta}$ which are odd degree elements in a Grassman algebra, and consider
1-dimensional so squares are zero

$$H = \omega a^* a + \eta a + a^* \tilde{\eta}$$

where $\{a,a^*\} = 1, \quad \{a,a\} = \{a^*,a^*\} = 0.$ Consider

$$e^{\beta H_0} U(\beta,0) = T\, e^{-\int_0^\beta \left(\eta(t) e^{-\omega t} a + e^{\omega t} a^* \tilde{\eta}(t)\right) dt}$$

and then note that $\quad [\eta a, a^*\tilde{\eta}] = [\eta a, a^*]\tilde{\eta} + a^*[\eta a, \tilde{\eta}]$

$$* \begin{cases} \dfrac{\mathrm{tr}\, U(\beta,0)}{\mathrm{tr}\, U_0(\beta,0)} = e^{\int_{t_1>t_2} \eta(t_1) e^{-\omega(t_1-t_2)} \tilde{\eta}(t_2) - \int \eta(t_1) e^{-\omega(t_1-t_2)} \tilde{\eta}(t_2) \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}}} \\ = e^{\int \eta(t_1) G(t_1,t_2) \tilde{\eta}(t_2)} \end{cases}$$

where
$$G(t_1,t_2) = e^{-\omega(t_1-t_2)} \left\{ \theta(t_1-t_2) - \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}} \right\}$$

Then
$$G(\beta,t_2) = e^{-\omega(\beta-t_2)} \left\{ 1 - \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}} \right\} = \frac{e^{-\omega\beta+\omega t_2}}{1+e^{-\beta\omega}}$$

$$G(0,t_2) = e^{\omega t_2} \left\{ -\frac{e^{-\beta\omega}}{1+e^{-\beta\omega}} \right\} = -G(\beta,t_2)$$

so that G is anti-periodic:
$$G = \left(\frac{d}{dt}+\omega\right)^{-1} \quad \text{on anti-periodic functions.}$$

We can check things a bit

$$G(t_1,t_2) = \langle T[a(t_1)\, a^*(t_2)] \rangle = e^{-\omega(t_1-t_2)} \begin{cases} \langle aa^* \rangle = \frac{1}{1+e^{-\beta\omega}} & t_1>t_2 \\ \langle a^*a \rangle = \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}} & t_1<t_2 \end{cases}$$

$$= e^{-\omega(t_1-t_2)} \left\{ \theta(t_1-t_2) - \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}} \right\}.$$

Notice also that

$$e^{\int \eta(t_1) G(t_1,t_2) \tilde{\eta}(t_2)} = \prod_{(t_1,t_2)} e^{\eta(t_1) G(t_1,t_2) \tilde{\eta}(t_2)\, dt_1 dt_2}$$

$$= \prod_{(t_1,t_2)} \left(1 + \eta(t_1) G(t_1,t_2) \tilde{\eta}(t_2)\, dt_1 dt_2 \right)$$

Schwinger derivation of $*$ on 522 goes as follows.

$$\delta\, tr(U(\beta,0)) = -\int_0^\beta dt \; tr\left(U(\beta,t)(\eta(t)a + a^*\tilde{\eta}(t))U(t,0)\right)$$

or

$$\delta \log tr(U(\beta,0)) = -\int_0^\beta \left(\eta(t)\langle a(t)\rangle + \langle a^*(t)\rangle \tilde{\eta}(t)\right) dt$$

Again one shows

$$\left(\frac{d}{dt}+\omega\right)\langle a(t)\rangle = -\tilde{\eta}(t)$$

$$\left(\frac{d}{dt}-\omega\right)\langle a^*(t)\rangle = \eta(t)$$

so that

$$\langle a(t)\rangle = -\left(\frac{d}{dt}+\omega\right)^{-1}\tilde{\eta}(t)$$

But the mistake to avoid goes as follows:

$$\langle a(t)\rangle = tr\left(U(\beta,t)\, a\, U(t,0)\right) / tr\left(U(\beta,0)\right)$$

This looks periodic in t because one would think

$$tr\left(a\, U(\beta,0)\right) = tr\left(U(\beta,0)\, a\right)$$

However the trace picks up only the even part of $aU(\beta,0)$, and hence depends on the odd part of $U(\beta,0)$, hence there is a minus sign. Anyway the conclusion is that $\langle a(t)\rangle$ is anti-periodic, so we get the anti-periodic Green's function, and the rest of the argument proceeds without change.

Two questions: What is the analogue of the path integral in the fermion case? How do things look when we Fourier transform $\eta, \tilde{\eta}$?

January 6, 1980

Yesterday we considered ΛV, $V = \mathbb{C}e$, $a = i(e^*)$ $a^* = e(e)$, $H_0 = \omega a^* a$. We computed the generating function for the Green's functions and found it to be

$$e^{\int \eta G \tilde{\eta}}$$

To fix the ideas suppose we want the finite temperature situation: The generating function is

$$\frac{\operatorname{tr}(\quad U(\beta,0))}{\operatorname{tr}(e^{\beta H_0})} = e^{\int \eta G \tilde{\eta}}$$

where G is the kernel for $\left(\frac{d}{dt}+\omega\right)^{-1}$ with anti-periodic boundary conditions.

$$e^{\int \eta G \tilde{\eta}} = \sum_n \frac{1}{n!} \int dt_1 dt_1' \cdots dt_n dt_n' \left(\eta(t_1) G(t_1,t_1') \tilde{\eta}(t_1')\right) \cdots (\quad)$$

$$= \sum_n \frac{1}{n!} \int dt_1 \cdots dt_n' \; \eta(t_1) \cdots \eta(t_n) \tilde{\eta}(t_n') \cdots \tilde{\eta}(t_1') \prod G(t_i, t_i')$$

$$= \sum_n \frac{1}{n!\,n!} \int \text{------------------} \sum_\sigma \operatorname{sgn}(\sigma) \prod G(t_i, t_{\sigma i}')$$

$$= \sum_n \frac{1}{n!\,n!} \int dt_1 \cdots dt_n' \; \eta(t_1) \cdots \tilde{\eta}(t_1') \; \det G(t_i, t_j')$$

I guess that one can conclude from this that

$$\langle T[a(t_1) \cdots a(t_n) a^*(t_n') \cdots a^*(t_1')] \rangle = \det G(t_i, t_j')$$

where $G(t,t') = \langle T[a(t) a^*(t')] \rangle$

The Pfaffian: Let $A = [a_{ij}]$ be a skew-symmetric $n \times n$ matrix with n even. In the exterior algebra with generators $\eta_1, \dots, \eta_n$ we can form $\omega = \frac{1}{2} \sum a_{ij} \eta_i \eta_j$. Then

$$\frac{\omega^{n/2}}{(n/2)!} = \mathrm{Pfaff}(A) \cdot \eta_1 \cdots \eta_n$$

Why should this be true? First of all, the above construction should be invariant under the action of $SO(n)$. Second for the matrix

$$A = \begin{pmatrix} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & \lambda_{n/2} \\ -\lambda_{n/2} & 0 \end{pmatrix} \end{pmatrix} \quad \text{it gives} \quad \omega = \sum_j \lambda_j \eta_{2j-1} \eta_{2j}$$

hence $\mathrm{Pfaff}(A) = \lambda_1 \cdots \lambda_{n/2}$.

These characterize the Pfaffian.

Notice that the Pfaffian is given as a sum over the $(2\frac{n}{2}-1)!!$ ways of partitioning $1, 2, \dots, n$ into pairs.

I want now to do the Fourier transform. We have this generating function

$$Z(\eta, \tilde{\eta}) = e^{\int \eta G \tilde{\eta}}$$

and we've seen that $G(t,t')$ is anti-periodic, in fact

$$G(t,t') = \sum_{k \in \frac{2\pi}{\beta}(\mathbb{Z}+\frac{1}{2})} \frac{e^{ik(t_1 - t_2)}}{ik + \omega}$$

So we should regard $\eta, \tilde{\eta}$ as anti-periodic on $[0, \beta]$.
Suppose

$$\eta = \frac{1}{\sqrt{\beta}} \sum \eta_k e^{ikt} \qquad \tilde{\eta} = \frac{1}{\sqrt{\beta}} \sum \tilde{\eta}_k e^{-ikt}$$

Then
$$G\tilde{\eta} = \frac{1}{\sqrt{\beta}} \sum \frac{1}{ik+\omega} \tilde{\eta}_k e^{-ikt}$$
and
$$\int \eta G \tilde{\eta} = \sum_k \frac{\eta_{-k} \tilde{\eta}_k}{ik+\omega}.$$

Here $\eta_k, \tilde{\eta}_k$ are anti-commuting variables

What are we after? I [redacted] eventually want to be able to handle Green's functions for interacting fermions, and I want to [redacted] get the perturbation expansion from a kind of path integral. The path integral should look like

$$\int \text{something} \; e^{\int \eta a \, dt \text{ [redacted]}} e^{\int a^* \tilde{\eta} \, dt} \quad \mathcal{D}a(t) \, \mathcal{D}a^*(t)$$

and hence

$$\int \text{something} \; \mathcal{D}a \, \mathcal{D}a^* \quad a(t_1) \ldots a(t_m) \, a^*(t_1') \ldots a^*(t_n')$$

should be a Green's function like the one obtained. The "something" brings in the Green's function.

Let's generalize [redacted] slightly so that instead of having [redacted] variables a, a^* we have something that goes with the Pfaffian, namely, a set of anti-commuting variables $\eta_1, \eta_2, \ldots, \eta_n$. These are like the variables q_i. I need a quadratic form to go with them, i.e. take
$$\omega = \tfrac{1}{2} \sum a_{ij} \eta_i \eta_j$$
where $[a_{ij}]$ is a skew-symmetric matrix. Now there has

to be someway to make sense out of

$$\int e^{\omega}\, \eta_{i_1} \cdots \eta_{i_j}$$

so that Wick's theorem holds. The obvious thing to try is take cap product with $\eta_1 \wedge \cdots \wedge \eta_n$. Then

$$\int e^{\omega} = \int \frac{\omega^{n/2}}{(n/2)!} = \mathrm{Pfaff}(A)$$

Recall that $Pf(A) = (\det A)^{1/2}$ which looks good in comparison with

$$\int e^{-\frac{1}{2}x^t A x}\, dx = \text{const } (\det A)^{-1/2}$$