Fermion quantum mechanics: Clifford alg.

What I want is an analogue of a coupled system of [redacted] oscillators. In the boson case the classical equation of motion is

$$\ddot{q} = -\gamma^2 q$$

where γ^2 is a real positive-def matrix. One forms a real symplectic [redacted] vector space V with coords q_i, p_i and symplectic form $\Omega = \sum_i p_i \wedge q_i$. The above DE gives a symplectic flow on V. On the other hand to (V, Ω) belong a Weyl algebra which has one irreducible representation $\mathcal{H}$. The quantum mechanical situation is given by the operators of V on $\mathcal{H}$ and the induced flow.

For the fermion analogue we take a DE

$$\frac{\partial \psi}{\partial t} = \gamma \psi$$

where γ is a skew-symmetric real matrix. Let V be the [redacted] vector space of real solutions; then because γ is skew-symmetric time-evolution preserves the inner product Q on V. To (V, Q) belongs a Clifford algebra $C(Q)$ which has a unique irreducible representation $\mathcal{H}$ (here $C(Q)$ is taken over $\mathbb{C}$ and V is even-dimensional). The quantum situation is given by the operations of V on $\mathcal{H}$ and the induced flow.

(It seems to be bad to think of V as solutions of the DE. [redacted] Instead V is spanned by functions on [redacted] the space of solns.)

Let W be a complex vector space with inner product. If $w \in W$, let w^* denote the linear functional $w^*(w_1) = \langle w | w_1 \rangle = (w_1, w)$. We have operators $e(w), i(w^*)$ on ΛW, and these are adjoint for the natural inner product. Hence $e(w) + i(w^*)$ is hermitian. We have commutation relations

$$\{ e(w) + i(w^*),\ e(w') + i(w'^*) \} = w^* w' + w'^* w = 2 \operatorname{Re}(w, w')$$

or more simply
$$\left(e(w) + i(w^*) \right)^2 = |w|^2$$

which means that $\operatorname{End}(\Lambda W)$ is the Clifford algebra (over $\mathbb{C}$) of the underlying real vector space of W equipped with the quadratic form $w \mapsto |w|^2$, (this assumes W fin. dim.).

Let H be an endo. of W; it extends uniquely to a degree 0 derivation of W, which is hermitian if H is. If w_j is an orthonormal basis of W and $h_{ij} = w_i^* H w_j$, then one has

$$H \text{ on } \Lambda W = \sum_{i,j} h_{ij}\, e(w_i)\, i(w_j^*)$$

(Actually you might write this $\sum e(w_i)\, h_{ij}\, i(w_j^*)$ in the spirit of

$$H = \sum_{i,j} w_i \rangle \langle w_i | H | w_j^* \rangle \langle w_j^*$$

At this stage we have constructed the Clifford module for the Clifford algebra belonging to the real quadratic space $(W, |\ |)$. Now the program is to take a time-evolution in W and carry it over to the Clifford module. What this means is that to a DE in W

$$\frac{d}{dt} w = \gamma(t) w$$

preserving the quadratic form (this means $\gamma(t)$ is skew-symmetric wrt $\mathrm{Re}(\ ,\ )$, I should be able to associate a path of autos. in $C(W, |\ |)$, which should be inner autos. (Skolem–Noether). Thus there should exist a hermitian $H(t) \in C(W, |\ |) = \mathrm{End}(\Lambda W)$ such that

$$[iH(t), w] = \gamma(t) w$$

where $w \in W$ is identified with $e(w) + i(w^*) \in C(W, |\ |)$.

So if W is a complex vector space with inner product $(\ ,\ )$, then ΛW is the Clifford module for the real quadratic space defined by $(W, |\ |)$, with $w \in W$ acting as $e(w) + i(w^*)$. Thus if w_j is an orth. basis for W, then we get annihilation operators

$$a_j = i(w_j^*)$$

and creation operators

$$a_j^* = e(w_j)$$

satisfying

$$\{a_j, a_{j'}^*\} = \delta_{jj'} \qquad \{a_j, a_{j'}\} = \{a_j^*, a_{j'}^*\} = 0$$

Notice that if $\tilde{W}$ is the subspace of the Clifford algebra $\mathrm{End}(\Lambda W)$ spanned by the creation + annihilation operators, then $\tilde{W}$ has the conjugation $*$, and W is the subspace of hermitian operators. Thus $\tilde{W} \cong W \otimes_{\mathbb{R}} \mathbb{C}$.

Example: Suppose W is a 2-dimensional real Euclidean vector space with norm $|\ |$. Then to get a [redacted] complex structure on W all we have to do is to give an orientation. (Does $\exists$ a relation between picking positive energy solutions and picking a system of positive roots?).

What I want to do is to proceed by analogy with what I did for the harmonic oscillator. [redacted] I want a Hamiltonian H of the form $H_0 + V(t)$ where H_0 is fixed, but $V(t)$ is a time-dependent perturbation of compact support in time. This Hamiltonian is essentially a skew-adjoint operator in the space W.

In the boson case we had interesting behavior [redacted] with $\dim_{\mathbb{R}}(W) = 2$ because $U(1) < SL_2(\mathbb{R})$, but there won't be interesting fermion behavior because $U(1) =$ connected component of O_2.

So we have to work with W at least 4-diml over $\mathbb{R}$. H_0 will generate a 1-parameter group of orthogonal transformations in W. Assume H_0 non-singular [redacted] as usual. Then under the action of $\exp(tH_0)$, W decomposes into an orthogonal direct sum of 2 planes, and we get a complex structure on W by requiring the eigenvalues of H_0 in each 2 plane to be positive imaginary.

The above notation has to be changed because we want H_0 to be self-adjoint. So suppose $W = \mathbb{C}^n$, and the Hamiltonian H_0 is diagonalized. Except for ground state energy, we want the Hamiltonian on ΛW to be [redacted]

$$H_0 = \sum \lambda_j a_j^* a_j$$

March 14, 1979

Let W be a real Euclidean vector space of even dimension [redacted] and $C = C(W)$ its Clifford algebra over $\mathbb{C}$ with respect to $|\;|$. Then we know C has a unique irreducible module $\mathcal{H}$. To construct $\mathcal{H}$ we can choose a complex structure on W such that $|\;|$ comes from a hermitian inner product. Then put $\mathcal{H} = \Lambda W$ and associate to $w \in W$, the operator $e(w) + i(w^*)$. One has

$$\left(e(w) + i(w^*)\right)^2 = w^*(w) = |w|^2$$

so that $C(W)$ acts on $\mathcal{H}$, and it is easy to show that $C(W) \xrightarrow{\sim} \mathrm{End}(\mathcal{H})$.

Let's choose an orth. basis w_j for W and form the creation and annihilation operators

$$a_j^* = e(w_j) \quad , \quad a_j = i(w_j^*)$$

satisfying the commutation relations

$$a_j^* a_{j'}^* = a_j a_{j'} = 0 \qquad \{a_j, a_{j'}^*\} = \delta_{jj'}$$

Any element of $C(W)$ can be written as as linear combination of normal products of these operators.

Now the interesting point is that any orthogonal transformation θ of W induces an autom. of $C(W)$, and hence there is an automorphism of $\mathcal{H}$ compatible with θ which is unique up to a non-zero scalar. This leads to $\mathcal{H}$ being a representation of a double covering of the orthogonal group.

How to do this infinitesimally: Let $\tilde{W} \subset C^{odd}(W)$ be the subspace spanned by the a_j, a_j^*. It is [redacted] closed under $*$, and the self-adjoint elements of $\tilde{W}$ are of the form $e(w) + i(w^*)$ with $w \in W$, hence $\tilde{W} \cong W \otimes \mathbb{C}$.

The normal products of length 2 form a subspace of [redacted] spanned by

$$a_j a_k \,,\; a_j^* a_k^* \qquad j<k$$

$$a_j^* a_k \,.$$

$$\left.\right\} \text{total no. } 2\frac{n(n-1)}{2} + n^2 = 2n^2 - n$$

It is complementary to $\mathbb{C}1$ in $F_2 C^{ev}(W)$ and it has dimension $\frac{1}{2}(2n)(2n-1) = n(2n-1)$, so the above elements form a basis. [redacted]

Clearly by the Jacobi identity in the super-algebra sense

$$[\mathfrak{g}, \tilde{W}] \subset \tilde{W} \qquad [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$$

and so it is more or less clear that $\mathfrak{g}$ is the Lie algebra of the orthogonal group of $\tilde{W}$. (Check: dim of orth group $= \frac{2n(2n-1)}{2} = 2n^2 - n$).

Finally note that the skew-adjoint elements of $\mathfrak{g}$ preserve the ~~self~~-adjoint elements of $\tilde{W}$ since

$$[g,h]^* = [h^*,g^*] = -[g^*,h^*] = [g,h]$$

if $g^* = -g$ and $h^* = h$. Thus the skew-adjoint elements of h should be the orthogonal Lie algebra of W.

Of especial interest is the subspace of $\mathfrak{g}$ generated by the elements $a_j^* a_k$. This is just the Lie algebra of endomorphisms of W as a complex vector space extended to derivations on ΛW. The skew-adjoint elements of this subspace form the Lie algebra of the unitary group of W.

Next we do the analogue of the [redacted] finite perturbation of an oscillator. The free Hamiltonian will be

$$H_0 = \sum_{j=1}^{n} \lambda_j a_j^* a_j$$

with $\lambda_j > 0$. It generates a one [redacted]-parameter unitary group of automorphism of $\mathcal{H}$. Applied to the "r-particle" state

$$a_{j_1}^* \cdots a_{j_r}^* |0\rangle = e(w_1)\cdots e(w_j)\cdot 1 = w_1 \wedge \cdots \wedge w_j$$

H_0 gives the eigenvalue $\lambda_{j_1} + \cdots + \lambda_{j_r}$.

Next I need a perturbation of H_0 which will be a self-adjoint element of $\mathfrak{g}$. Recall is the [redacted] boson case the interesting Hamiltonians [redacted] for 1-degree of freedom were $p^2, \frac{1}{2}(pq+qp), q^2$ and that I looked at just q^2. [redacted] In this fermion situation I need 2 degrees of freedom and then I have available the space of operators spanned by $a_1 a_2$, $a_1^* a_2^*$ which are outside the unitary subgroups. $\dim O(4) = 6$, $\dim U(2) = 4$

Let's choose

$$H = H_0 - \varepsilon(t)(a_1 a_2 + a_2^* a_1^*)$$

Note: Recall that we are interested in computing $\langle 0|S|0\rangle$, where $|0\rangle$ denotes the unit [redacted] in ΛW. If the Hamiltonian H lies in the subspace $\mathrm{End}(W) \subset \mathrm{End}(\Lambda W)$, then the flow preserves the grading on ΛW, so there is no interesting mixing of particles, e.g. $\langle 0|S|0\rangle = 1$. Thus I want to perturb H_0 [redacted] out of the space $\mathrm{End}(W)$.

March 15, 1979

Review: In the fermion situation there doesn't seem to be a "coordinate representation" analogous to the one used to write the Schroedinger equation, so I have to start out using the complex representation, that is, the one where the creation + annihilation operators are obvious. Thus let W be a complex vector space with inner product, and let $\mathcal{H}$ be ΛW with the induced inner product, and with the operators

$$\text{creation:} \quad e(w) \qquad\qquad e(w)^* = i(w^*)$$
$$\text{annihilation:} \quad i(w^*)$$

The algebra of operators on ΛW generated by these is the Clifford algebra of the underlying real Euclidean space to W.

Let's fix an orthonormal basis w_j for W and put $a_j = i(w_j^*)$, $a_j^* = e(w_j)$. We consider the motion on $\mathcal{H}$ produced by a Hamiltonian

$$H = H_0 - V.$$

This means that we solve the DE

$$i\frac{\partial \psi}{\partial t} = H\psi$$

in the space $\mathcal{H} = \Lambda W$. We take $H_0 = \sum \lambda_j a_j^* a_j$ so that the motion under H_0 is known:

$$e^{-itH_0} a_{j_1}^* \cdots a_{j_r}^* |0\rangle = e^{-it(\lambda_{j_1} + \cdots + \lambda_{j_r})} a_{j_1}^* \cdots a_{j_r}^* |0\rangle.$$

The problem is now to compute the scattering matrix for the perturbed motion. Recall the scattering formulas:

$$i \frac{\partial}{\partial t} U(t,t') = (H_0 - V)\, U(t,t')$$

$$U(t',t') = I$$

$$\left(\frac{\partial}{\partial t} + iH_0\right) U(t,t') = iV(t)\, U(t,t')$$

Green's fn: for $\frac{\partial}{\partial t} + iH_0$ is

$$G_0(t,t') = \begin{cases} e^{-iH_0(t-t')} & t > t' \\ 0 & t < t' \end{cases}$$

So

$$U(t,t') = \int_{-\infty}^{t} e^{-iH_0(t-t_1)}\, iV(t_1)\, U(t_1,t')\, dt_1 \; + \; \text{solution of homog. DE}$$

If $t' <$ Supp $V(t)$, then $U(t,t') = e^{-iH_0(t-t')}$ for $t <$ Supp $V(t)$, so we get

$$U(t,t') = e^{-iH_0(t-t')} + \int_{-\infty}^{t} e^{-iH_0(t-t_1)}\, iV(t_1)\, U(t_1,t')\, dt_1$$

or better

$$\underbrace{e^{iH_0 t}\, U(t,t')\, e^{-iH_0 t'}}_{U_I(t,t')} = I + \int_{-\infty}^{t} dt_1 \; e^{iH_0 t_1}\, iV(t_1)\, e^{-iH_0 t_1} \times e^{iH_0 t_1}\, U(t_1,t')\, e^{-iH_0 t'}$$

So if we put $V_I(t_1) = e^{iH_0 t_1} V(t_1) e^{-iH_0 t_1}$, then

$$U_I(t,t') = I + \int_{-\infty}^{t} dt_1\, iV_I(t_1)\, U_I(t_1,t')$$

and

$$S = I + \int_{-\infty}^{\infty} dt_1\, iV_I(t_1) + \frac{1}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, P\{iV_I(t_1)\, iV_I(t_2)\} + \cdots\cdots$$

So now let's compute in the case where

$$V(t) = \varepsilon(t)(a_1 a_2 + a_2^* a_1^*)$$

[redacted]

$$H_0 = \lambda_1 a_1^* a_1 + \lambda_2 a_2^* a_2$$

Since

$$[H_0, a_1] = \lambda_1 [a_1^* a_1, a_1] = -\lambda_1 a_1$$

$$[H_0, a_1^*] = \lambda_1 [a_1^* a_1, a_1^*] = \lambda_1 a_1^* \{a_1, a_1^*\} = \lambda_1 a_1^*$$

$\therefore$

$$e^{itH_0} a_j e^{-itH_0} = e^{-i\lambda_j t} a_j$$

$$e^{itH_0} a_j^* e^{-itH_0} = e^{+i\lambda_j t} a_j^*$$

so

$V_I(t) =$ [redacted] $e^{itH_0} V(t) e^{-itH_0}$

$$= \varepsilon(t)\left(e^{-i(\lambda_1+\lambda_2)t} a_1 a_2 + e^{i(\lambda_1+\lambda_2)t} a_2^* a_1^*\right)$$

We want to compute

$\langle 0|S^{(2)}|0\rangle = \frac{(i)^2}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, \varepsilon(t_1)\varepsilon(t_2)\, \langle 0|\, P\{(e^{-i\lambda t_1}$ [redacted] $a_1 a_2 + e^{i\lambda t_1} a_2^* a_1^*)\cdot$

$$\cdot (e^{-i\lambda t_2} a_1 a_2 + e^{i\lambda t_2} a_2^* a_1^*)\}\,|0\rangle$$

[redacted] Suppose $t_1 > t_2$. Then

$$\langle 0|P \cdots |0\rangle = \langle 0|a_1 a_2 a_2^* a_1^*|0\rangle e^{-i\lambda t_1 + i\lambda t_2}$$

$$= e^{-i\lambda(t_1 - t_2)}$$

Thus
$$\langle 0|S^{(2)}|0\rangle = \frac{(i)^2}{2!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\ \varepsilon(t_1)\varepsilon(t_2)\,e^{-i\lambda|t_1-t_2|}$$

Notice that $\langle 0|S^{(1)}|0\rangle = 0$ because $\langle 0|\,{a_1 a_2 \atop a_2^* a_1^*}\,|0\rangle = 0$.
Also $\langle 0|S^{(3)}|0\rangle = 0$ because all 8 possibilities for

$$\langle 0|\begin{Bmatrix} a_1 a_2 \\ a_2^* a_1^* \end{Bmatrix}\cdot\begin{Bmatrix} a_1 a_2 \\ a_2^* a_1^* \end{Bmatrix}\cdot\begin{Bmatrix} a_1 a_2 \\ a_2^* a_1^* \end{Bmatrix}|0\rangle$$

give zero, since there are never the same number of creation & annihilation operators. In general all the odd terms vanish.

Fourth order: Since $\dim W = 2$ we can create two particles in different states, but no more, so we must annihilate immediately afterward. Therefore the only non-zero contribution to the time-ordered product with $t_1 > t_2 > t_3 > t_4$ is

$$\underbrace{\langle 0|a_1 a_2\, a_2^* a_1^*\, \underbrace{a_1 a_2\, a_2^* a_1^*|0\rangle}_{|0\rangle}}_{1}\ e^{-i\lambda t_1}e^{i\lambda t_2}e^{-i\lambda t_3}e^{+i\lambda t_4}$$

So
$$\langle 0|S^{(4)}|0\rangle = i^4 \iiiint_{t_1>t_2>t_3>t_4} \varepsilon(t_1)\varepsilon(t_2)\varepsilon(t_3)\varepsilon(t_4)\ e^{-i\lambda(t_1-t_2+t_3-t_4)}\,dt_1\,dt_2\,dt_3\,dt_4$$

$$\langle 0|S^{(6)}|0\rangle = i^6 \int_{t_1>\cdots>t_6}\!\!\cdots\!\int \varepsilon(t_1)\cdots\varepsilon(t_6)\ e^{-i\lambda(t_1-t_2+t_3-t_4+t_5-t_6)}\,dt_1\cdots dt_6$$

The problem now is to decipher these integrals, which probably have something to do with the roots of the orthogonal group.

Consider next the variational approach. Recall that if the Hamiltonian is subjected to an infinitesimal variation δH then the time evolution operator changes by

$$\delta U(t,t') = \int_{t'}^{t} dt_1 \, U(t,t_1) \frac{1}{i} \delta H(t_1) U(t_1,t')$$

In the situation at hand $\delta H(t) = -\delta\varepsilon(t)(a_1 a_2 + a_2^* a_1^*)$, and if we choose $t_1 < \mathrm{Supp}\,\varepsilon < t_2$, then $\langle 0|S|0\rangle = \langle 0|U(t_2,t_1)|0\rangle$, since $H_0|0\rangle = 0$; thus one has

$$\delta\langle 0|S|0\rangle = \langle 0|\delta U(t_2,t_1)|0\rangle$$

$$= \int_{t_1}^{t_2} dt \, \langle 0|U(t_2,t)(a_1 a_2 + a_2^* a_1^*) U(t,t_1)|0\rangle \, i\delta\varepsilon(t)$$

Recall that $\tilde{W}$ = subspace of Clifford algebra spanned by a_j, a_j^*. Given $w_1, w_2 \in W$ consider

$$f(t,w_1;t',w_2) = \langle 0|U(t_2,t)w_1 U(t,t')w_2 U(t';t_1)|0\rangle$$

Then

$$\frac{d}{dt} f(t,w_1;t',w_2) = \langle 0|U(t_2,t)i[H(t),w_1] U(t,t')w_2 U(t',t_1)|0\rangle$$

$$= f(t, i[H(t),w_1]; t', w_2)$$

Better:

Let's choose a basis b_k for $\tilde{W}$ and consider the matrix.

$$g_{jk}(t,t') = \langle 0 | U(t_2, t_{\text{[redacted]}})\, b_j\, U(t,t')\, b_k\, U(t', t_1) | 0 \rangle .$$

Let us introduce the matrix for the action of $iH(t)$ on $\widetilde{W}$.

$$[iH(t), b_j] = \sum_k \theta_{kj}(t)\, b_k$$

Then we have

$$\frac{d}{dt} g_{jk}(t,t') = \sum_l \text{[redacted]}\ \theta_{lj}(t)\, g_{lk}(t,t')$$

$$\frac{d}{dt'} g_{jk}(t,t') = \sum_l \theta_{lk}(t)\, g_{jl}(t,t')$$

Now I want to show that $g(t,t')$ is some sort of Green's matrix, [redacted] so I need to know how it jumps as t passes thru t'. It doesn't jumps so I don't yet have the correct gadget.

March 16, 1979:

Let W be a complex vector space with basis w_j, dual basis $w_j^* \in W^*$, and let $\mathcal{H} = \Lambda W$ with operators

$$a_j = i(w_j^*), \qquad a_j^* = e(w_j)$$

Then $W \oplus W^* \subset \mathrm{End}(\Lambda W)$ and one has

$$\left(e(w) + i(w^*)\right)^2 = w^*(w)$$

so that (for $\dim W < \infty$), $\mathrm{End}(\Lambda W)$ can be identified with the Clifford algebra of the space $W \oplus W^*$ with the "hyperbolic" quadratic form $w + w^* \longmapsto w^*(w)$. We get a basis for $C = \mathrm{End}(\Lambda W)$ using normal products in the a_j, a_j^*.

Let $\tilde{\mathfrak{g}}$ be the subspace of C^{ev} spanned by the 2 fold products of a_j, a_j^*; $\tilde{\mathfrak{g}}$ has the basis consisting of normal products of degree 2 together with 1. $\tilde{\mathfrak{g}}$ is a Lie algebra under $[\ ,\ ]$, and it acts on $\tilde{W} = W \oplus W^*$ by $[\ ,\ ]$. $\tilde{\mathfrak{g}}$ is clearly spanned by products $(e(w) + i(w^*)) \cdot (e(w') + i(w'^*))$, so it contains a subspace $\mathfrak{g}$ spanned by brackets

$$[e(w_1) + i(\lambda_1), e(w_2) + i(\lambda_2)]$$

(Note, not $\{\ \ \}$ brackets). Then

$$\begin{aligned}[\mathfrak{g}, \mathfrak{g}] &= [\mathfrak{g}, [\tilde{W}, \tilde{W}]] \\ &\subset [[\mathfrak{g}, \tilde{W}], \tilde{W}] + [\tilde{W}, [\mathfrak{g}, \tilde{W}]] \\ &\subset [\tilde{W}, \tilde{W}] = \mathfrak{g}\end{aligned}$$

so $\mathfrak{g}$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$.

The idea is that $\tilde{\mathfrak{g}}$ maps into the orthogonal Lie algebra of $\tilde{W}$ since

$$[H, \{\rho, \rho'\}] = \{[H,\rho],\rho'\} + \{\rho, [H,\rho']\}$$

and

$$\{e(w_1)+i(\lambda_1), e(w_2)+i(\lambda_2)\} = \lambda_2(w_1) + \lambda_1(w_2)$$

$$= \text{bilinear form on } \tilde{W}.$$

The map should be onto with kernel generated by multiples of $1 \in C$. Thus

$$0 \longrightarrow \mathbb{C} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{o}(\tilde{W}) \longrightarrow 0$$

is an extension of Lie algebras. By Levi it splits, and the actual splitting should be given by $\mathfrak{g} = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. Elements of $\mathfrak{g}$ are

$$\tfrac{1}{2}[e(w_1), e(w_2)] = e(w_1)e(w_2)$$

$$\tfrac{1}{2}[i(\lambda_1), i(\lambda_2)] = i(\lambda_1)i(\lambda_2)$$

$$\tfrac{1}{2}[e(w), i(\lambda)] = \tfrac{1}{2}\big(e(w)i(\lambda) - i(\lambda)e(w) + e(w)i(\lambda) + i(\lambda)e(w) - \lambda(w)$$

$$= e(w)i(\lambda) - \tfrac{1}{2}\lambda(w)$$

In terms of the a_j, a_j^*, the Lie algebra $\mathfrak{g}$ is spanned by

$$\begin{cases} a_j a_k, \; a_j^* a_k^* & j<k \\ a_j^* a_k - \tfrac{1}{2}\delta_{jk} \end{cases}$$

So what seems to be happening is that there is a ground state energy being associated to a Hamiltonian?

Suppose we return to the case where W is a real vector space with inner product of even dimension. Given a Hamiltonian, i.e. skew-adjoint transf. on W, we can choose a complex structure on W so that the skew-adjoint transformation [redacted] is $-iH$ where H is ≥ 0. Assume $H > 0$ and diagonalize it. Then I can lift it invariantly into $\mathfrak{g}$ by putting

$$H = \sum_j \lambda_j (a_j^* a_j - \tfrac{1}{2})$$

The ground state energy is then $-\frac{1}{2}\sum_j \lambda_j$, and the largest energy is $\frac{1}{2}\sum_j \lambda_j$.

Thus if I start with a skew-adjoint transformation on W and lift it to a Hamiltonian in $\mathfrak{g}$, I get a [redacted] vacuum $|0\rangle$ by looking at the minimum energy, and the elements of $\tilde{W}$ killing this vacuum form an isotropic subspace of $\tilde{W}$, whence one gets a complex structure on W, etc.

So return to $H(t) = H_0 - V(t)$, where $V(t)$ has compact support contained inside (t_i, t_f). (i=initial f=final). Let $U(t,t')$ be the propagator

$$\frac{d}{dt} U(t,t') = \frac{1}{i} H(t)\, U(t,t')$$
$$U(t',t') = I$$

and let $V(t)$ be subjected to an infinitesimal alteration $\delta V(t)$. Then we want to compute

$$\langle 0|\, \delta U(t_f, t_i) |0\rangle = \int_{t_i}^{t_f} dt \, \langle 0 |\, U(t_f, t)\, i\delta V(t)\, U(t, t_i) | 0 \rangle$$

as a kind of trace. Recall $\delta V(t) \in \tilde{g}$. Let b_i be an orthonormal basis for $\tilde{W}$ and consider the matrix

$$g(t,t')_{ij} = \begin{cases} \langle 0 |\, U(t_f, t)\, b_i\, U(t, t')\, b_j\, U(t', t_i) | 0 \rangle & t > t' \\ -\langle 0 |\, U(t_f, t')\, b_j\, U(t', t)\, b_i\, U(t, t_i) | 0 \rangle & t < t' \end{cases}$$

The discontinuity is

$$g(t,t')_{ij} \Big]_{t=t'-}^{t=t'+} = \langle 0 |\, U(t_f, t') (b_i b_j + b_j b_i) U(t', t_i) | 0 \rangle$$

$$= \langle 0 |\, U(t_f, t_i) | 0 \rangle \, \delta_{ij}$$

March 18, 1979

Classical action and quantum mechanical propagator.

Consider a 1-dimensional system with Lagrangian

$$L(x,\dot{x},t) = \frac{m}{2}\dot{x}^2 - V(x,t)$$

Hamilton's principle says that the trajectories of the system which go from (x',t') to (x,t) are extremal curves $\hat{t} \longmapsto x(\hat{t})$ with these endpoints for the action integral $\int L\,dt$. In nice cases we can take x, x' to be independent variables completely specifying a trajectory of the system and define [redacted]

$$S(x,t,x',t') = \int_{t'}^{t} L(x(\hat{t}), \dot{x}(\hat{t}), \hat{t})\,d\hat{t}$$

where $x(\hat{t})$ is the actual trajectory. Then if one varies [redacted] the endpoints by amounts $\delta x, \delta x'$ and adjusts the trajectory $x(\hat{t})$ accordingly one gets

$$\delta S = \int_{t'}^{t}\left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x}\right)d\hat{t} = \left[\frac{\partial L}{\partial \dot{x}}\delta x\right]_{t'}^{t} + \int_{t'}^{t}\underbrace{\left\{\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right\}}_{=0}d\hat{t}$$

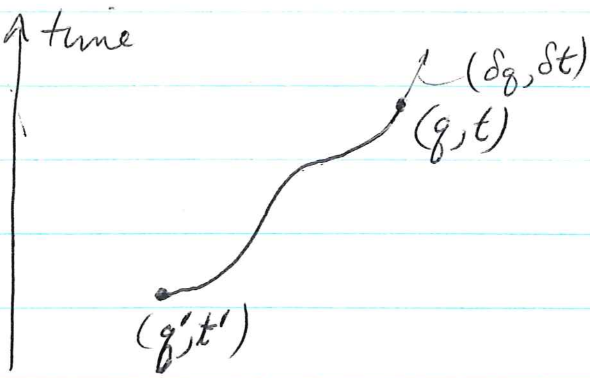
or

$$\frac{\partial S}{\partial x}(x,t,x',t') = \frac{\partial L}{\partial \dot{x}}(x,t) = \text{[redacted] momentum of trajectory at [redacted] } (x,t)$$

$$\frac{\partial S}{\partial x'}(x,t,x',t') = -\frac{\partial L}{\partial \dot{x}}(x',t') = -\text{ momentum of trajectory at } (x',t')$$

If [redacted] this [redacted] action function S is known, then via the implicit function thm. we can solve for x[redacted] as a function of x' and p' and so we know the motion of the system.

Let us fix q', t' and consider S as a function of the independent variables q, t. Let q, t undergo infinitesimal displacements $\delta q, \delta t$. Then I want δS.



Suppose $\delta q = \dot{q}\,\delta t$ where $\dot{q}$ is the velocity at the end of the trajectory. Then in the course of making the variation I don't change the path I use from t' to t, so that

$$\delta S = \delta \int_{t'}^{t} L(x(t), \dot{x}(t), t)\,dt$$

$$= L(q, \dot{q}, t)\,\delta t$$

But

$$\delta S = \frac{\partial S}{\partial q}\delta q + \frac{\partial S}{\partial t}\delta t = \left(\frac{\partial S}{\partial q}\dot{q} + \frac{\partial S}{\partial t}\right)\delta t$$

Thus

$$\frac{\partial S}{\partial q}\dot{q} + \frac{\partial S}{\partial t} = L$$

or

$$\frac{\partial S}{\partial t} = L - \frac{\partial S}{\partial q}\dot{q} = L - p\dot{q} = -H$$

and so S satisfies the PDE of first order

$$\boxed{\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0}$$

and this is true for any q', t'.

Now look at the quantum situation which is governed by Schroedinger's equation:

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} H\psi \qquad H = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 + V(x,t)$$

or

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right)\psi$$

Define $U(t,t')$ as usual. It is given by a kernel $K(x,t,x';t')$ in the coordinate representation:

$$\psi(x,t) = \int K(x,t,x';t')\,\psi(x',t')\,dx'$$

or

$$K(x,t,x';t') = \langle x \,|\, U(t,t') \,|\, x' \rangle$$

The basic idea is that $K(x,t,x';t')$ corresponds to the classical quantity $\exp\left(+\frac{i}{\hbar}S(x,t,x';t')\right)$. Dirac's way of viewing this is as follows:

$$\frac{\hbar}{i}\frac{\partial}{\partial x} K(x,t,x';t') = \underbrace{\frac{\hbar}{i}\frac{\partial}{\partial x}}_{p \text{ operator}} \underbrace{\langle x \,|U(t,t')|\, x' \rangle}_{\substack{\text{function of } x \text{ associated} \\ \text{to the state } U(t,t')|x'\rangle}}$$

$$= \langle x \,| p\, U(t,t') |\, x']$$

$$= \int \langle x | p | x'' \rangle dx'' \langle x'' | U(t,t') | x' \rangle$$

Thus one has

$$\frac{\partial}{\partial x} K(x,t,x';t') = \int \frac{i}{\hbar} \langle x | p | x'' \rangle\, dx''\, K(x'',t,x';t')$$

which is analogous to $\frac{\partial}{\partial x} e^{i/\hbar S} = \frac{i}{\hbar}\frac{\partial S}{\partial x} e^{i/\hbar S} = \frac{i}{\hbar} p\, e^{i/\hbar S}$.

Recall that if ψ is the state vector for the system, then $\bar{x} = \langle \psi | x | \psi \rangle$, $\bar{p} = \langle \psi | p | \psi \rangle$ are the measured values for position and angular momentum. These satisfy

$$\frac{d}{dt}\bar{x} = \langle -\tfrac{i}{\hbar} H\psi | x | \psi \rangle + \langle \psi | x | -\tfrac{i}{\hbar} H\psi \rangle$$

$$= \langle \psi | \tfrac{i}{\hbar} H^* x | \psi \rangle + \langle \psi | x(-\tfrac{i}{\hbar}) H | \psi \rangle$$

$$= \langle \psi | [\tfrac{i}{\hbar} H, x] | \psi \rangle$$

$$= \langle \psi | [\tfrac{i}{\hbar}(\tfrac{p^2}{2m} + V), x] | \psi \rangle$$

$$= \langle \psi | \tfrac{i}{\hbar} \tfrac{1}{2m} 2p \underbrace{[p, x]}_{\frac{\hbar}{i}} | \psi \rangle = \tfrac{1}{m} \langle \psi | p | \psi \rangle$$

$$= \frac{1}{m}\bar{p}$$

and

$$\frac{d}{dt}\bar{p} = \langle \psi | [\tfrac{i}{\hbar} H, p] | \psi \rangle$$

$$= \langle \psi | [\tfrac{i}{\hbar} V, \tfrac{\hbar}{i}\tfrac{\partial}{\partial x} | \psi \rangle$$

$$= -\langle \psi | \frac{\partial V}{\partial x} | \psi \rangle$$

$$= \overline{\left(-\frac{\partial V}{\partial x}\right)}$$

If ψ is a wave packet, i.e. with support in a small nbd. of $\bar{x}$, then the latter is $-\frac{\partial V}{\partial x}(\bar{x})$ nearly:

$$\frac{d}{dt}\bar{p} \approx -\frac{\partial V}{\partial x}(\bar{x})$$

So we get the usual Newtonian motion.

Notice that the last approximation is exact when V is quadratic. For then $\frac{\partial V}{\partial x} = ax+b$ is linear, hence

$$\overline{\frac{\partial V}{\partial x}} = \langle \psi | ax+b | \psi \rangle = a\bar{x}+b = \frac{\partial V}{\partial x}(\bar{x})$$

But the above argument would work equally well with an 'inner product' $\langle \phi | x | \psi \rangle$, and so these quantities satisfy the classical equations of motion when the Hamiltonian is quadratic.

So next let us return to fermion situation. There appears to be a slight inconsistency in the physicists notation. If one has a classical field one wants to quantize, e.g.

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} - m^2 \phi$$

then quantization associates operators to functions on the space of classical fields. To each coordinate and momentum one has operators. An example of a coordinate function is evaluation at x followed by some linear function on the space to which $\phi(x)$ belongs.

Now think of ϕ as being a section of a vector bundle, then each point of the dual bundle will give a coordinate operator in the quantum field theory.

So from this viewpoint when I work with the Clifford algebra, I should think of W^* as being the operators, where

$$W^* = \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{C}).$$

Then when a complex structure is given on the Euclidean space

W, the complex linear functions are the annihilators, while the anti-linear ones are the creators. Thus the model becomes

$$\mathcal{H} = \Lambda(W^\vee) \qquad W^\vee = \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$$

with annihilators $i(w)$ and creators $e(w^*)$.

Next consider the Schroedinger equation on $\mathcal{H}$ given by a Hamiltonian $H(t)$ which is quadratic, i.e. a self-adjoint element in the space of 2-fold products of operators in W^*. Suppose $H(t) = H_0 + V(t)$ where $V(t)$ has compact support in $[t_i, t_f]$. Recall

$$\delta \langle 0 | U(t_f, t_i) | 0 \rangle = \int_{t_i}^{t_f} \langle 0 | U(t_f, t)\, \delta H(t)\, U(t, t_i) | 0 \rangle \, dt$$

and that in order to interpret this as a trace we considered

$$g(t,t') = \begin{cases} \langle 0 | U(t_f, t)\, \lambda_1\, U(t, t')\, \lambda_2\, U(t', t_i) | 0 \rangle & t > t' \\ -\langle 0 | U(t_f, t')\, \lambda_2\, U(t', t)\, \lambda_1\, U(t, t_i) | 0 \rangle & t < t' \end{cases}$$

Here $g(t,t')$ is a bilinear function of $\lambda_1, \lambda_2 \in W^*$ and hence it can be interpreted as an element of $W_{\mathbb{C}} \otimes W_{\mathbb{C}}$.

We have seen that the time flow in W^* is given by

$$\frac{d}{dt}\lambda = [iH, \lambda]. \qquad [\ ,\ ] \text{ is action of quadratic elements in } W^*.$$

Since there is a sign because of the contragredient repn, it follows that this corresponds to the equation

$$\frac{d}{dt} w = \frac{1}{i} H \cdot w$$

So it is easy to see that as an element of $W_{\mathbb{C}} \otimes W_{\mathbb{C}}$, $g(t,t')$

[redacted] satisfies

$$\left(\frac{d}{dt} - \frac{1}{i} H(t)\right) g(t,t') = 0 \quad \text{for} \quad t \neq t'.$$

The jump of g at t' is the linear function sending

$$(\lambda_1, \lambda_2) \longmapsto \langle 0 | U(t_f, t_i) | 0 \rangle \cdot (\lambda_1, \lambda_2)$$

so

$$g(t,t')\Big]_{t=t'-}^{t=t'+} = \left(\begin{array}{l}\text{element of } W_{\mathbb{C}} \otimes W_{\mathbb{C}} \text{ defined} \\ \text{by inner product.}\end{array}\right) \langle 0 | U(t_f,t_i) | 0 \rangle .$$

So therefore $g(t,t')$ is a Green's function or matrix [redacted] for the actual flow which takes place within the space W.