Conversation with B. Kostant

<u>On Clifford Algebra.</u> V vector space with $(\ ,\ )$ nondegenerate symmetric bilinear form. The orthogonal gp. of $(\ ,\ )$ acts on ΛV and to each $a \in$ Lie alg of orth gp there is a unique element $\varphi(a) \in \Lambda^2 V$ such that $a\omega = [\varphi(a), \omega]$ where $[\ ]$ is relative to Clifford multiplication. In fact ~~letting C^+ be the even part of the Cliff alg. then C^+ is generated by image of $\Lambda^2 V$. and we can define uniquely a Lie bracket on C^+~~ letting C^+ be the even part of the Clifford alg, the multiplication in C^+ is uniquely determined by this bracket. Now C^+ is a simple ass. algebra and has a unique irreducible modules. This module is the spinor representation of the L.A. of the orth gp. ~~Now recall that if W is a vector space with inner product then $\mathrm{Hom}(\Lambda W, \Lambda W)$ is the Clifford algebra. If we are in the complex case then can choose a basis of V $v_1, \dots, v_n$ with $(v_i, v_j) = \delta_{ij}$~~ If W a vector space, then $\mathrm{Hom}(\Lambda W, \Lambda W)$ is the Clifford algebra of the vector space $W^* \oplus W$ with the inner product $(f+\omega, f'+\omega') = f'(\omega) + f(\omega')$.

<u>Weyl algebra.</u> Let V be a real vector space with ω a non deg. skew symmetric bilinear form. ~~[illegible] [illegible] [illegible]~~ Then the abstract Weyl alg. is $T(V) / x \otimes y - y \otimes x = [x, y]$ and by usual way it gets represented as unbounded operators on L^2(q space) when we ~~change~~ choose p and q's. However it is possible to ~~consider bounded functions of the p's and q's~~ associate to each

~~continue~~ C^∞ fn. of p and q with compact support an operator in this L^2(q space) and then to consider the C^* algebra generated by these operators. This we call the concrete Weyl algebra. It can be realized as the C_0^∞ fns. on V with convolution product

$$f*g = \int \hat{f}(x)\, g(x+y)\, e^{\overset{?}{-}(x,y)}\, dy\, dx$$

where $$\hat{f}(x) = \int f(y)\, e^{(x,y)}\, dy$$

Now by thm. of Stone and von Neumann $W(V)$ has a unique irreducible representation. Furthermore this convolution satisfies

$$\{f,g\} = f*g - g*f$$ false

where $\{f,g\}$ is the Poisson bracket.

In both this case and in Clifford case one is presented with a classical system represented by the Poisson bracket and one endeavors to make an associative multiplication which yields this bracket. The resulting algebra has a unique irred. representation which then becomes the quantum mechanical system. Choice of δ's analogous to choice of ω for Cliff. case.

Results of Palais: M manifold with closed non deg. 2 form Ω (i.e. M has Hamiltonian structure). Then $\exists$ 1-1 correspondence between closed 1 forms and infinitesimal contact transformations given by $i(X)\Omega \leftrightarrow X$.

Then $i(X)\Omega$ is exact $\Longleftrightarrow X = [Y, Z]$ $\quad$ Y, Z cont. transf.

Therefore $\dfrac{\text{inf. contact transf}}{\text{brackets of inf. cent. transf}} \cong \mathfrak{g}_{ab}$ where $\mathfrak{g}$ = inf. contact transf. is $\simeq H^1(M)$ by de Rham.

Representations of Lie gps. Kostant's conjectured theory: $\mathfrak{g}$ Lie alg. $\mathfrak{g}'$ its dual. Let $f \in \mathfrak{g}'$ and Gf denote orbit which is a manifold namely G/H where H = isotropy gp. of f. Now $x, y \rightsquigarrow f([x,y])$ defines ~~an invariant~~ skew symm. form on G invariant under H and furthermore it is non-degenerate on $\mathfrak{g}/\mathfrak{h}$. Hence G/H has a G invariant Hamiltonian structure and so each orbit in $\mathfrak{g}'$ is even dim. Kostant can show that $\exists$ a subalg. $\mathfrak{b}$ $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{b}$ maximally isotropic for $f([x,y])$ and hence this gives a G invariant fibration of G/H over G/B corresponding to a choice of p's and q's. Now B isotropic $\Rightarrow f$ vanishes on ~~[illegible]~~ $[B,B]$ hence f is a character of B. Inducing this character up from B should give an irreducible representation of G and all irred. representations should come in this way. Known at present for semi simple gps by Gelfand-Naumark and for nilpotent gps by Dixmier-Kirilov.

In semi simple case if f is a regular element then H is a Cartan subalgebra of G and hence B comes from a positive set of roots and B is a Borel subgp. of G. Then G/B is a complex proj. variety - flag manifold. ~~[illegible]~~ To understand finite dim. representations we observe that to each

~~integer valued linear functional~~ character of H the Cartan subalgebra we get in the natural way a cx line bundle over G/B. If this character is in the positive Weyl chambre then can show this line bundle ~~has enough~~ is ample and we get a finite dimensional representation of G which is irreducible, using holomorphic sections. This is content of Borel-Weil theorem. To get Weyl character formula easiest method is to apply Bott Atiyah Lefschetz formula — in fact reproduce the homogeneous vector bundles paper.

Alternative method of getting irreducible representation — find the analogue of the Weyl algebra for the homogeneous symplectic manifold G/H and prove it has only one irreducible representation. In nilpotent case it turns out that G/H is flat and hence one can introduce the usual Weyl algebra. Kostant would like to have a Weyl algebra for an arbitrary ~~symp~~ Hamiltonian manifold thus getting a canonical quantization of a classical mechanical system. Observe that by use of Weyl algebra he can choose naturally a Hamiltonian operator for any smooth C^∞ function on p,q space.

Now we have two methods of getting from orbits in $\mathfrak{g}'$ to representations. Finally we mention a third relation between orbits in $\mathfrak{g}'$ and representations. ~~To~~ Each irred rep. of G ~~has~~

gives a character of the center Z of $U(\mathfrak{g})$. Under PBW isom.

$$S(\mathfrak{g}) \longleftrightarrow U(\mathfrak{g})$$

$$I \longleftrightarrow Z$$

where I = invariants of $S(\mathfrak{g})$. Now in nilpotent case Kirilov shows that $I \Leftrightarrow Z$ is an algebraic isomorphism and hence characters of Z correspond to points of I. Now ~~to each~~ to each orbit Gf we also get a point of I. Thus we have a diagram

$$\begin{tikzcd}[column sep=small]
\text{orbits of } \mathfrak{g}' \arrow[dr] & \approx & \text{irred reps of } G \arrow[dl] \\
& \text{Characters of } Z = \operatorname{Spec} I &
\end{tikzcd}$$

which commutes

and by ~~[illegible]~~ various theorems one has that the vertical maps are generically isomorphisms. ~~Thus lose info~~ I believe they are onto. Thus bad characters are exceptionelle in Dixmier. Thus lose info in going to characters.

In semi simple case the map

$$I \longleftrightarrow Z$$

is not an algebraic isom. but becomes one by use of Harish-Chandra shift involving e^{ρ} $\rho = \Sigma$ pos. roots. But one does have that points of I and char. of Z correspond 1-1. and further ~~by~~ use of $\mathfrak{h}$ one has that $S(\mathfrak{g}')$ is integral over I.

so that points of I ~~[illegible]~~ correspond to closed orbits of $\mathfrak{g}'$. The manifold G/B is not unimodular under G and its canonical class is e^{ρ} (ρ a pos. root), hence reason for $e^{\frac{1}{2}\rho}$ in formulas to get unitary representation. ~~[illegible]~~ Kostant knows now that Weyl algebra of semi simple case is more subtle than flat case because it must take into account ~~[illegible]~~ $e^{\frac{1}{2}\rho}$ and topology of G/B. He guesses that abstract Weyl alg. belonging to ~~[illegible]~~ ~~$f \in \mathfrak{g}'$~~ Orbit of f

is
$$\mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g})\, m$$
where m = max ideal in Z corr to point f in Spec I

because this has correct dimension since

$$\simeq \; S(\mathfrak{g}) / S(\mathfrak{g})\, m = \text{functions on orbit.}$$

as filtered rings to

(by a thm. of Kostant)

and because m must go into 0 under induced rep.

Extra fact: $H^1(\text{orbit}) = 0$ since it breaks down to solvable case which is flat and semi-simple case which has only even cohomology.

Conversation with Kostant

H compact group acting on a space V. Define a function on $V^* \times V$ by

$$F(f,x) = \int_H e^{i\langle hf, x\rangle}\, dh$$

so that F is constant on the orbits of H in V^* and in V. If $\Delta \in S(V)$ is invariant under the action of H, then

$$\Delta_x F(f,x) = \int \Delta(hf)\, e^{\langle hf, x\rangle} dh = \Delta(f)\, F(f,x)$$

and so $F(f,x)$ is an eigenfunction for the invariant operators on V. If $\varphi \in C_0^\infty(V)$ is constant on the H orbits then from the formula

$$\hat{\varphi}(f) = \frac{1}{(2\pi)^n} \int e^{-i\langle f, y\rangle} \varphi(y)\, dy$$

we see that $\hat{\varphi}$ is constant on H orbits and hence that

$$\varphi(x) = \int \hat{\varphi}(f) e^{i\langle f, x\rangle} df = \int_{V/H} d\rho \cdot \hat{\varphi}(Hf)\, F(f,x)$$

(modulo an i which Kostant omits)

where $d\rho$ is the measure on V/H such that $d\rho \times dh =$ Lebesgue measure. Thus the eigenfunctions $F(f,x)$ are complete for functions on the H orbits.

~~Examples: If~~

Examples. Let $H = SO(2)$ acting on R^2. Then the orbit space is R^+

if $g = \frac{1}{2}\sum_{\varphi>0} \varphi$, then g is an invariant of Weyl gp. and hence defines an invariant ~~central~~ function on $\mathfrak{g}$ and we have that

$$\frac{F(\lambda+g, x)}{F(g, x)}$$

is an eigenfunction for Z and that all eigenfunctions of Z are obtained in this way. Furthermore one can show that on restricting to Cartan subalgebra we have

$$F(\lambda, x) = \sum_{\sigma \in W} \operatorname{sgn}(\sigma)\, e^{\langle \sigma\lambda, x\rangle}$$

and so $\quad F(g, x) = \prod_{\varphi>0} \left(e^{\frac{\varphi}{2}} - e^{-\frac{\varphi}{2}}\right)$

Furthermore it is easy to see that the character of a representation is an eigenfunction for Z and it is a theorem that

$$\frac{F(\lambda+g, x)}{F(g, x)}$$

is the character of the representation ~~belong to λ~~, having highest weight λ. (Weyl character formula)

Other less coherent facts:

Eigenvalues of Casimir operator ~~on~~ irreducible rep of max. wgt λ is $\|\lambda+g\|^2 - \|g\|^2$.

Eigenvalue of a primitive element p in Z for λ is $p(\lambda+g) - p(g)$.

More Conversations with Kostant:

Only derivations of ΛT^* are linear combinations of $e(u)\, i(x)$ $x \in T$ $u \in T^*$

G Lie gp., H closed subgp. $I_G \subset S(\mathfrak{g}^*)$ $I_H \subset S(\mathfrak{g}^*)$ then we have

$$H^*(G/H, R) = \mathrm{Tor}_{I_H}(I_G, k)$$

~~Hence if H and G have same rank.~~

Chevalley's thm. W a finite gp. generated by reflections in V. Then $I_W \subset S(V^*)$ is a poly ring and $S(V^*) = I_W \otimes_{\mathbb{R}} \tilde{V}$.

About Harish-Chandra shifts. Have

$$\begin{array}{cc} I & Z \\ \cap & \cap \\ S(\mathfrak{g}^*) & U(\mathfrak{g}) \end{array}$$

and $I \simeq Z$ but not by BW map. ($\exists$ Formula of Dynkin for eigenvalues of Casimir operator.) The isomorphism of I with Z given on the primitive elements by $p(x) \mapsto p(x+g) - p(g)$ where enough to know for $x \in \mathfrak{h}$ since p invariant and where $g = \frac{1}{2}$ sum of positive roots.

K comp. conn. Lie group

$\nu : K \to \operatorname{Aut} V \qquad \dim V < \infty$

Let D = set of equivalence classes of irreducible representations of K.

Ex: $K = T$, $\dim T = \ell$, $T = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{\ell}$ where $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. $Z = \mathbb{Z} \times \cdots \times \mathbb{Z}$

Then $\lambda \in Z \xrightarrow[\text{isom}]{} \hat{T} \ni \eta^\lambda : T \to \mathbb{C}^* = \operatorname{Aut} V'$

If $\lambda = (n_1, \ldots, n_\ell)$ and $a \in T = (z_1, \ldots, z_\ell)$, then $\eta^\lambda(a) = \prod z_i^{n_i}$. Here $D = Z$ and any function on T can be expanded $f = \sum_{\lambda \in Z} a_\lambda \eta^\lambda$ trig poly.

If ν rep of K, then ν is determined by $\chi(a) = \operatorname{tr} \nu(a)$ $a \in K$. $\chi(a)$ is a class fn $\Rightarrow$ it is known if $\chi \mid T$ known where T is a maximal torus of K.

Def: $\Delta(\nu) = \{ \lambda \in Z \mid \chi_\nu = \sum a_\lambda \eta^\lambda$ on T and $a_\lambda \neq 0 \}$

$\Delta(\nu) \subseteq Z$ is the set of weights of ν.

Def: $W = N(T)/T$ finite gp acts on T and $\therefore Z$.

Since χ class fn. $\sigma(\Delta(\nu)) = \Delta(\nu)$ for any $\sigma \in W$.

Fundamental Theorem of Cartan Weyl theory: For any $\lambda \in Z$, $\exists$ unique ν^λ $\ni$

(1) $\lambda \in \Delta(\nu^\lambda)$

(2) $\Delta(\nu^\lambda)$ = convex hull of $\{\sigma\lambda\}_{\sigma \in W}$

(3) Any irred. rep $\simeq \nu^\lambda$ for some $\lambda \in Z$. and $\nu^\lambda \simeq \nu^\mu \Leftrightarrow \lambda = \sigma\mu$ for some $\sigma \in W$

The Weyl algebra (after Kostant)

Let V be a real fin-dim vector space and let $\langle , \rangle$ be a nondegenerate skew symmetric bilinear form on V. Actually for most of the below the non degeneracy is not important. The Stone von Neumann theorem guarantees the uniqueness of a ~~representation of a~~ linear map A of V into self adjoint operators on a Hilbert space such that

$$(1) \qquad [A(x), A(y)] = \frac{1}{i}\langle x, y\rangle \qquad \text{for } x, y \in V$$

or more precisely such that the Weyl relations hold:

$$(2) \qquad e^{iA(x)} \cdot e^{iA(y)} = e^{iA(x+y)} \cdot e^{i\langle x, y\rangle}$$

If V is thought of as a locally compact abelian group (2) says that $x \longmapsto e^{iA(x)}$ is a projective representation of V. Consequently it seems natural to extend B to $L^1(V)$ by setting

$$B(f) = \int f(x) e^{iA(x)}\, dx \qquad \text{for } f \in L^1(V)$$

whence

$$\| B(f) \| \le \| f \|_1$$

and

$$B(f) \cdot B(g) = B(f * g)$$

where

$$(3) \qquad (f*g)(x) = \int f(z)\, g(x-z)\, e^{i\langle z, x\rangle}\, dz$$

As in usual way one demonstrates that $L^1(V)$ with multiplication defined by (3) is a ~~Banach~~ C^* algebra which we call the analytic Weyl algebra of V. Stone von Neumann theorem says that the analytic Weyl algebra has a unique irreducible module.

Now let $W(\langle\ \rangle)$ be the tensor algebra of V modulo the relations $x \otimes y - y \otimes x = \frac{1}{i}\langle x, y\rangle$. Define a map $\varphi: S(V) \longrightarrow W(V)$ so that $\varphi(P(x)) = P_\varphi(x)$ all $x \in V$. Then (1) gives a representation of $W(V)$ and hence a map of $S(V)$ functions on V^* into endos on Hilbert space which we call $A(P)$. This can be extended to more functions on V^*.

The relation between A and B is simple. We have that

$$A(e^{ix}) = e^{iA(x)}$$

If f is a function on V^* and

$$f = \int_{y\in V} \hat{f}(y) e^{iy}\, dy$$

is a representation of it as a sum of exponentials then

$$A(f) = \int \hat{f}(y)\, e^{iA(y)}\, dy = B(\hat{f}).$$

Rmk. If G a Lie gp + $\mathfrak{g}$ is Lie algebra, Note analogy between

$L'(V)$ ~~with~~ $*$	$W(V)$	$S(V)$ = functions on V^*
$L'(G)$ ——	$U(\mathfrak{g})$	$S(\mathfrak{g}^*)$ = —— $\mathfrak{g}^*$

Poisson bracket defined on $S(V)$; $\langle x,y\rangle = \{x,y\}$ for $x,y\in V$. One shows that

(4) $$x*f = xf + \frac{1}{2i}\{x,f\} \qquad \text{if } x\in V \quad f\in S(V).$$

hence $$x*f - f*x = \frac{1}{i}\{x,f\}$$

(5) Also one can show that

$$f*g - g*f = \frac{1}{i}\{f,g\} \qquad g,f\in S(V)$$

Provided that degree f or degree $g \le 2$.

False if both of degree 3.

Def: A s.a. (unbdd). $\lambda \in \operatorname{Spec} A$ is a limit point of $\operatorname{Spec} A$ if either λ is in the continuous spectrum of A, or λ is a limit point of the point spectrum, or λ is in the point spectrum of A and the multiplicity of A is infinite

equiv. cond. $\forall a,b$ with $a<\lambda<b$ rank $E_b - E_a = \infty$.

Weyl's Criterion: μ limit point of $\operatorname{Spec} A \Leftrightarrow \exists f_n \in \mathcal{D}_A \ni \|f_n\| = 1$ $f_n \to 0$ weakly and $(A-\mu)f_n \to 0$ strongly.

A comp. cont. $\Leftrightarrow$ 0 only limit point.

A transf $A: L^2(a,b) \to L^2(a,b)$ of the form $Af(x) = \int K(x,y) f(y)\,dy$ is of Carleman type if $\int |K(x,y)|^2 dy < \infty$ a.e. x.

von Neumann: ~~Any Carleman s.a. transf has 0 as limit point of its spectrum and~~ Any s.a. transf of $L^2(a,b)$ which comes from a Carleman kernel has 0 for a limit point of its spectrum. Conversely any such s.a. transf of $L^2(a,b)$ is unitarily equiv. to a Carleman s.a. transf.

On Clifford algs.

~~Think of~~

It is very natural to replace the exterior algebra by the Clifford algebra when dealing with Riemannian manifold. Bott, I believe, claimed that the Hodge formulas appeared much nicer. In particular if G is a **compact** Lie group one should study the Clifford algebra $C(\mathfrak{g}^*)$ instead of $\Lambda\mathfrak{g}^*$ in order to get the good cohomology **pictures**.

Bert claims that **if** $\mathfrak{g}$ semi-simple ~~and~~ then as $\mathfrak{g}$ mods $C(\mathfrak{g}) \simeq M \otimes J$ where **M is an** irreducible $\mathfrak{g}$ module and J = **invariants**. Moreover $J \simeq C(P)$ where P = space of primitives in $H^*(\mathfrak{g})$. In other words $\exists$ analogy between $U(\mathfrak{g}) - S(\mathfrak{g})$, $C(\mathfrak{g}) - \Lambda(\mathfrak{g})$. Moreover there is a transgression from certain elements of $U(\mathfrak{g})$ to elements of $C(\mathfrak{g})$.

In calculating a harmonic form for fundamental class of G/T need to evaluate a determinant $\frac{\partial p_i}{\partial x_j}$ p_i are invariants x_j basis for cartan. Also by Chevalley $H^*(G/T)$ is regular repn of ~~W~~ W so if ~~$\mathfrak{g}$~~ G simple h occurs in $H^*(G/T)$ l times — where?

<u>Bert on Clifford algebras</u>

The problem is to calculate the cohomology of K/T. General thm. of Cartan: ~~If~~ G/H is a homog. space, G compact, then

$$H^*(G/H) = \mathrm{Tor}_*^{I(g')}(k, I(h'))$$

$I =$ invariants in S

$I(h') = H^*(B_H)$?

~~The problem is therefore to show that~~ The idea is ~~that~~ ~~$H^*(G/H)$~~ to write $g = h + p$ orth direct sum so that $H^*(G/H) \simeq$ cohomology of $\Lambda(p)^h$. But p carries a non-degenerate form so we can form $C(p) = \Lambda(p)$ with a different multiplication. Now $\Lambda^2(p)$ is a Lie subalg of $C(p)$ ~~[illegible]~~ namely the Lie alg of the orthogonal group. Thus action of h on p gives a map $h \to \Lambda^2(p)$ of Lie algebras hence an algebra map

$$\rho : U(h) \longrightarrow C(p)$$

Moreover this is such that ~~[illegible]~~ if $u \in C(p)$ $x \in h$ then $x * u$ (group action) $= xu - ux$ (Clifford mult.). Thus $C(p)^h =$ commutant of $\mathrm{Im}\,\rho$. Now $C(p)$ is the matrix algebra of the spin representation. Thus ~~cohomology classes~~ we have interpreted the ~~[illegible]~~ cochains $\Lambda(p)^h$ as operators on the ~~spinor~~ representation of h we get by spinning p.

Now in the case of K/T one finds that $Z\Lambda(p)^h/B$ consists of ~~[illegible]~~ the orbit of ρ under W. ~~There is the~~

~~problem of associating~~ Thus we get operators associated to the various elements of W and to calculate ~~[illegible]~~ Shubert multiplication one has ~~[illegible]~~ only to determine how operator multiplication compares with exterior multiplication.

A special case is the adjoint action. Then

$$\rho : U(\mathfrak{g}) \longrightarrow C(\mathfrak{g})$$

is a kind of transgression and carries elements of the center into primitives?

A theorem: Given an orthogonal rep of k on p it comes from a ~~reductive situation of $k+p$ with g/k~~ ~~compact~~ compact homogeneous space $\Longleftrightarrow$ under $\rho : U(k) \to C(p)$ the Casimir op of k goes into a perfect square in $\Lambda^4 p$ (Squares of elts in $\Lambda^3 p$ ~~[illegible]~~ lie in $\Lambda^4 p$). It comes from a symmetric space $\Longleftrightarrow$ $\rho X = 0$.

A theorem of Weil: Suppose ~~[illegible]~~ $a \in \Lambda^2 V \subset C(V)$. To calculate $\exp a$ $\in \mathrm{Spin}\, V$ one proceeds as follows. Write

$$\exp a = \frac{u-1}{u+1} \qquad \text{Cayley transf}$$

Then $\quad \exp_{\text{exterior}} u = \exp_{\text{Cliff.}} a \quad$?

Relation between repns. of general linear + symmetric gp.

Fact: An irred. repn V of $\mathfrak{g}$ is a representation of the adjoint group iff the 0-weight occurs.

For $sl(n)$, $\dim V = n$ one has that $V^{\otimes k}$ is a rep. of adjoint group $\Leftrightarrow k = r \cdot n$. For $k = n$ have that $V^{\otimes n} = V_1 \otimes U_1 + \cdots + V_{p(n)} \otimes U_{p(n)}$ where U_j are the full set of irreducibles of the symmetric grp. Theorem: $V_1, \ldots, V_{p(n)}$ are those irred. reps. of ~~$sl(n)$~~ the adjoint group for which twice a root is not a weight, and moreover V_i^T is the complete set of irreducible reps. of Weyl group. $\exists$ nice relation between the reps V_i^T and U_i. Moreover $V^{\otimes rn}$ consists of reps. of the adjoint group for which $r+1$ times a root is not a weight.

More on Clifford algebras

$$\mathrm{Spin}(\mathfrak{g}) = 2^{[\frac{\ell}{2}]} \cdot V\rho$$

so
$$\Lambda \mathfrak{g} = \mathrm{Spin}\,\mathfrak{g} \otimes \mathrm{Spin}\,\mathfrak{g} = 2^{\ell}(V\rho \otimes V\rho)$$

Choice of ρ similar to that of an ordering

Bert has a Clifford algebra approach which ~~gives~~ ~~eigenvalues~~ interprets elements of exterior algebra as operators. But operators have eigenvalues. These should be roots of unity.

It is usual to discuss orientation in terms of the ~~[illegible]~~ 1 dimensional repn $\Sigma_n \to Z_2$ given by sign. Look for orientation with values in any repn. of symmetric group.

$$u: H_c^*(X) \otimes H^*(Y) \longrightarrow H_c^*(X \times Y)$$

Proof: $H_c^i(X) \otimes H$

$\searrow$

Bott–Morse theory shows that a manifold can be put together ~~using open set all~~ by attaching an open cell U to M with $U \cap M$ a vector bundle over a sphere.

Kostants method for calculating $H^*(K/T)$.
By Chevalley-Eilenberg one knows that

$$H^*(K/T) = H^*\{[\Lambda(\mathfrak{g}/\mathfrak{h})']^{h}, d\}.$$

In virtue of the ~~[illegible]~~ Killing form on $\mathfrak{g}$ this may be rewritten

$$H^* \ [\Lambda(\mathfrak{g}/\mathfrak{h})]^{h}$$

Bert proposes to use the Clifford algebra of $\mathfrak{g}/\mathfrak{h}$ instead of the exterior alg. ~~Thus~~

Let V ~~be~~ be a vector space over $\mathbb{C}$ with ~~a~~ a non-degenerate symm. quad. form Q and ~~let~~ $C(V)$ be the Clifford algebra of V. ~~Thus~~
By definition this means that there is a canonical map $\iota : V \to C(V)$ with $(\iota v)^2 = -Q(v)\cdot 1$ with the obvious universal property. Writing V as an ODS $V = \mathbb{C}e_1 + \cdots + \mathbb{C}e_r$ ~~&~~ $Q(\sum \alpha_i e_i) = \sum \alpha_i^2$ one has

$$C(V) = \bigotimes_{\mathbb{Z}_2 gr.} C(\mathbb{C})$$

$\mathbb{Z}_2$-graded tensor product. ~~that~~ $\therefore$ $C(V)$ has gen. e_i relations

$$\begin{cases} e_i e_j = -e_j e_i & i \neq j \\ e_i^2 = -1. \end{cases}$$

and a basis $e_{\iota_1} \cdots e_{\iota_q}$ $\quad \iota_1 < \cdots < \iota_q$.

$\therefore$ $\dim C(V) = 2^{\dim V}$. Now $C(\mathbb{C}) = \underline{\mathbb{C} + \mathbb{C}e}$ $\quad e^2 = -1$

$$C(V) = C^+(V) \oplus C^-(V).$$

observe that elements of the form $\sum a_{ij} e_i e_j$ form a Lie subalg.

$$e_i e_j,\ e_k e_l$$

In fact one should be able to identify in a canonical way the Lie algebra $\sigma(V)$ with degree 2 elts of $C^+(V)$ such that they operate on $V \subset C^-(V)$ by bracket.

set $\quad W = \{ x \in C^+(V) \mid [x, V] \subset V \}.$

Then

$$\begin{tikzcd} W \arrow[r, "\tau"] & Hom(V,V) \\ \Lambda^2 V \arrow[u, "\rho", "s"'] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

where $\quad \tau(w)(v) = [w, v]$

$$\rho(v_1 \wedge v_2) = [v_1, v_2]$$

$$[[v_1, v_2], v] \in V \quad ?$$

$$\sigma(v_1 \wedge v_2)(x) = v_1(v_2, x) - v_2(v_1, x)$$

so is

$$\tau\rho(v_1 \wedge v_2)(x) = [\rho(v_1 \wedge v_2), x] = [[v_1, v_2], x]$$
$$\| ?$$
$$\sigma(v_1 \wedge v_2)(x) = v_1(v_2, x) - v_2(v_1, x).$$

something is wrong.

$$v_1 v_2 + v_2 v_1 = -Q(v_1+v_2) + Q(v_1) + Q(v_2)$$
$$= -2(v_1, v_2).$$

$$(v_2, x) = -\frac{1}{2}(v_2 x + x v_2)$$
$$(v_1, x) = -\frac{1}{2}(v_1 x + x v_1)$$

$$\begin{matrix} v_1(v_2, x) \\ -v_2(v_1, x) \end{matrix} = -\frac{1}{2}\left\{ v_1(v_2 x + x v_2) - v_2(v_1 x + x v_2)\right\}.$$

$$= -\frac{1}{2}\, [v_1, v_2]\, x + \{v_1 x v_2 - v_2 x v_1\} \quad ?$$

---

we have a basis

$$e_{i_1} \wedge \cdots \wedge e_{i_r} \qquad i_1 < \cdots < i_r$$

from which we can calc when

$$\left[\sum_{a_{i\ldots}} a_{i\ldots} e_{i\ldots}, V\right] \subset V.$$

idea being that for a term.

$$[\underline{e_{i_1} \cdots e_{i_r}}, v] \qquad v$$

even

has the leading term $\underline{Q\, e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge v}$

$$[e_i, e_j] = e_ie_j - e_je_i = \underline{2e_ie_j} \qquad i<j$$
$$= -2e_je_i \qquad i>j.$$
$$= 0 \qquad i=j.$$

$$[e_{i_1} \cdots e_{i_r}, e_j] = \quad ?$$

Suppose $x \in C^+(V)$ and
$[x, V] = 0$
ie x belongs to center of $C^+(V)$.
$\Longrightarrow$ x is a scalar ?

The real thing to do is to interpret $C(V)$ as operators on an exterior alg. of dim half that of V.

If dim V even, then we can choose an isomorphism

$$V = W + W'$$

with $Q(v) = \langle w, \lambda \rangle$ if $v = w + \lambda$.

Bert's idea is that $C(\mathfrak{g}) = \Lambda(\mathfrak{g}')$ as G modules. Thus to calculate $\Lambda(\mathfrak{g}')^G$ it suffices to calculate

$$C(\mathfrak{g})^G.$$

But there is a map ~~an action~~

$$\begin{array}{c} \Lambda^2\mathfrak{g} \longrightarrow C(\mathfrak{g}) \\ \uparrow \\ \mathfrak{g} \end{array}$$

where we think of $\Lambda^2\mathfrak{g}$ as $\mathfrak{o}(\mathfrak{g})$ and moreover the $\mathfrak{g}$-action on $C(\mathfrak{g})$ is thus ~~conjugation~~ bracketing with the image of canon map $\rho : \mathcal{U}(\mathfrak{g}) \longrightarrow C(\mathfrak{g})$

however $C(\mathfrak{g})$ is a matrix alg. ~~whose~~ and all irred. reps thus $\operatorname{Im}\rho$ is a semi-simple subalg. Bert claims that

$$\boxed{C(\mathfrak{g}) \simeq J \otimes H}$$

not always true

says that ~~and~~ multiplicities are same

$$C(\mathfrak{g}) \simeq \operatorname{End}\{\operatorname{Spin}(\mathfrak{g})\}$$

<u>Claim:</u> $\operatorname{Spin}\mathfrak{g} = \underline{c} \cdot V\delta$.

$$\boxed{C(\mathfrak{g}) \simeq M_c \otimes \operatorname{Hom}(V\delta, V\delta).}$$

take the Cartan series $\mathfrak{gl}_n$ $n \geq 1$. what are the various ways

Proposition 1: $[n] \longrightarrow \mathfrak{gl}_{n+1}(V)$ is a ~~co~~ covariant functor

In other words if we consider Cartan's series and how they are constructed we can recover the category Δ inside! but <u>not</u> the ~~degen~~ faces? ~~The~~ ~~[illegible]~~

$$U(p) \times U(q) \longrightarrow U(p+q)$$

$$\Sigma_p \times \Sigma_q \longrightarrow \Sigma_{p+q}$$

orbits i.e. cells somehow parameterized by cosets in this case shuffles. - should be related to ~~[illegible]~~ how a $U(p+q)$ representation decomposes over the subgps $U(p) \times U(q)$.

Atiyah has shown how to define a map

$$\varinjlim R(\Sigma_p)' \longrightarrow Op(K).$$

Think over $\mathcal{L}$ and $\mathcal{C}$.

τ transgression

~~$\tau: \mathcal{C} \to \mathcal{L}$~~

$\tau: C \longrightarrow L$

Groth point of view:

covering spaces, Galois theory, simple algebras

Use BU somehow

Fundamental problem is that the relevant coalgebras are non-commutative by Steenrod.

~~Theorem: Let $\tau: \mathcal{C} \to \mathcal{L}$~~

~~2nd part: Take~~

Ideas: Write up the DG math part thinking about

1) ~~[illegible]~~ Clifford alg. instead of exterior alg.

Heisenberg alg. ——— symmetric alg.

symmetric spaces | Wall's paper.

algs. with involution |

2)

Recalculate Clifford algebra

V vector space $/\mathbb{C}$ with quadratic form Q.

$C(V) = T(V)/$ ideal gen by $v \otimes v - Q(v)$

standard basis $e_{i_1} \cdots e_{i_q}$ $\quad i_1 < \cdots < i_q$

$$e_i e_j = -e_j e_i \qquad i \neq j$$

$$e_i^2 = -1. \qquad \text{where } e_i \text{ orth for } V$$

Example: suppose W vector space. ~~Then get map~~

Consider $V = W \oplus W'$ with

$$Q(w, \lambda) = \langle w, \lambda \rangle$$

Then $Q((w_1, \lambda_1) + (w_2, \lambda_2)) - Q(w_1, \lambda_1) - Q(w_2, \lambda_2)$

$= \langle w_1 + w_2, \lambda_1 + \lambda_2 \rangle - \langle w_1, \lambda_1 \rangle - \langle w_2, \lambda_2 \rangle$

$= \langle w_2, \lambda_1 \rangle + \langle w_1, \lambda_2 \rangle$

Thus bilinear form is

$\cancel{\langle w_1, \lambda_1 \rangle \cdot \langle w_2, \lambda_2 \rangle}$

$$(w_1, \lambda_1) \cdot (w_2, \lambda_2) = \frac{1}{2}(\langle w_1, \lambda_2 \rangle + \langle w_2, \lambda_1 \rangle)$$

Now define a map

$$V \longrightarrow \mathrm{End}(\Lambda W)$$

by

$$w \oplus \lambda \longmapsto \underline{\iota}(w) + \underline{e}(\lambda)$$

But $(\iota(w) + e(\lambda))^2 = \iota(w)e(\lambda) + e(\lambda)\iota(w) = \langle w, \lambda \rangle = Q(w \oplus \lambda)$

Therefore ~~…~~ one obtains ~~an isom~~ an algebra map

$$\# \qquad \boxed{\begin{array}{ccc} C_{+Q}(W \oplus W') & \xrightarrow{\simeq} & \mathrm{End}(\Lambda W') \\ w \oplus \lambda & \longmapsto & \underline{\underline{i}}(w) + \underline{\underline{e}}(\lambda) \end{array}}$$

where $Q(w \oplus \lambda) = \langle w, \lambda \rangle$.

Proposition: $\#$ is an isomorphism.

Proof: Reduce by tensor products to case where $\dim W = 1$ and calculate.

So if we are given a V with a non-degenerate Q And V is even-dimensional, then we may choose a max. isotropic subspace W and get an isom

$$V = W \oplus W'.$$

In this case we get an isom. $C_{-Q}(V)$ with $\mathrm{End}(\Lambda W')$.
$\therefore$ $C_Q(V)$ for Q non-degenerate and symmetric

$(\mathrm{Pfaff}\ OSO^{-1}) = \det O \cdot \mathrm{Pfaff}\ S.$ very similar to $\sqrt{\det}$

Clifford algebra: generators $p_i, q_i \ni$

$$p_i q_i + q_i p_i = 1$$

$$p_i q_j + q_j p_i = 0 \qquad i \neq j.$$

here
$$\boxed{\begin{aligned} p_i &= \underline{i}(v_i) \\ q_i &= \underline{e}(v_i). \end{aligned}}$$

This is a definite representation of the Lie algebra of the orthogonal group. The question then becomes how ~~does the~~ do we ~~prove~~ find $\underline{\Lambda^2 V} \subset C(V)$?

Denote scalar ~~product~~ on V by $(\,,\,)$. Then ~~state~~ recall

$$\mathrm{Hom}(V,V)' \simeq \mathrm{Hom}(V,V)$$

self dual by $\operatorname{tr} AB$.

and can define A anti-symmetric if

$$A = -A^t.$$

ie $\quad (Av, w) + (v, Aw) = 0.$

Now define

$$\Lambda^2 V \longrightarrow \mathrm{Hom}(V,V)$$

by $\quad v_1 \wedge v_2 \longmapsto \varphi_{v_1, v_2}$

where $\quad \varphi_{v_1,v_2}(w) = (v_1, w)v_2 - (v_2, w)v_1$

perhaps same as $\quad w \mapsto \underline{i}(w)(v_1 \wedge v_2) = (w, v_1)v_2 - (w, v_2)v_1.$

Thus have $\Lambda^2 V \longrightarrow \mathrm{Hom}(V,V)$
$\searrow$ $\cup$ of antisymmetric transf.

Should also be able to **recover** $\Lambda^2 V$ within $C(V)$ as transformations preserving V. Can you define a derivation $\underline{\iota}(v)$ of $C(V)$?

$$\cancel{\underline{\iota}(v \otimes v - Q(v))}$$

$$\underline{\iota}(w)(v \otimes v - Q(v)) = (w,v)\,v - v\,(w,v) - 0 = 0.$$

Yes.

Prop: $T(V) \xrightarrow{\pi} C(V)$. Let $\underline{\iota}(v)$ be ~~inner deriv.~~ interior multiplication on $T(V)$. Then $\underline{\iota}(v)$ ~~acts on~~ induces a derivation on $C(V)$.

Proof: $\operatorname{Ker}\pi$ gen. by $\underbrace{w \otimes w - Q(w)}_{\gamma}$. have

$$\theta(\alpha \cdot \gamma \cdot \beta) = \theta\alpha \cdot \gamma + \beta \pm \alpha \cdot \theta\gamma \cdot \beta \pm \alpha \cdot \gamma \cdot \theta\beta$$

$\therefore$ to show

$$\underline{\iota}(v)\,\{w \otimes w - Q(w)\} \in \operatorname{Ker}\pi$$

$$(v,w)\,w - w\,(v,w) - 0 = 0$$

$\iota(v)\,\iota(v)$ derivation

$\iota(v)\,\iota(v)\,w = 0.$

$\therefore$ have ΛV acting on $C(V)$ thru ι.

$\cancel{\iota(v)}$

$$\iota(v)\,e(w) + e(w)\,\iota(v)$$

$i(v)\, e(w) + e(w)\, i(v) = (v, w)$ since ι is a derivation!!!

We should be able to realize $\underline{\Lambda^2 V}$ within ~~Hom(V,V)~~ $C(V)$ as a subalgebra!

Thus take $v_1 \wedge v_2 \longrightarrow v_1 v_2$ if $v_1 \perp v_2$.

$$v_1 \wedge v_2 \longmapsto v_1 \cdot \left(v_2 - \frac{(v_2, v_1)}{(v_1, v_1)} v_1\right) = v_1 \cdot v_2 - (v_2, v_1)$$

Take

$$\boxed{v_1 \wedge v_2 \longmapsto v_1 v_2 - (v_1, v_2)}$$

bilinear and vanishes if $v_1 = v_2$ so is correct.

Thus get $\Lambda^2 V \hookrightarrow C(V)$

claim a Lie subalgebra

Show that $1 + \Lambda^2 V \subset C^0(V)$ is the $\{w \mid [w, V] \subset V\}$.

(Is Pfaffian same as reduced norm in $C(V)$.)

$\cancel{[v_1 v_2 - (v_1, v_2), w] = [v_1, w] v_2 + v_1 [v_2, w]}$

$w \longmapsto \iota(w)(v_1 v_2 - (v_1, v_2))$ ✓

$$\iota(w)[v_1v_2, w_1w_2] = [\underline{\iota}(w)v_1v_2, w_1w_2] + [v_1v_2, \underline{\iota}(w)w_1w_2]$$

$$v_1v_2w_1w_2 - w_1w_2v_1v_2$$
$$\|$$
$$-v_1w_1v_2w_2 + (v_2, w_1)v_1w_2$$
$$\|$$
$$w_1v_1$$

$$[v_1v_2, w] = v_1v_2w - wv_1v_2$$
$$= v_1(\underline{v_2w + wv_2}) - (v_1w + wv_1)v_2$$
$$= v_1(v_2, w) - (v_1, w)v_2$$

Thus to the element

$$[v_1v_2, w] = v_1(v_2, w) - (v_1, w)v_2 = -\underline{\iota}(w)(v_1 \wedge v_2).$$

then $[v_1v_2, w_1w_2] = [v_1v_2, w_1]w_2 + w_1[v_1v_2, w_2].$

If we define
$$\begin{array}{ccc} \Lambda^2 V & \xrightarrow{\varphi} & \mathrm{Hom}(V, V) \\ \downarrow{\scriptstyle\alpha} & & \\ C(V) & & \end{array}$$

$$\alpha(v_1 \wedge v_2) = v_1v_2 - (v_1, v_2)$$

then $\varphi(v_1 \wedge v_2)(w) = [\alpha(v_1 \wedge v_2), w].$

Important point if we now set $\mathfrak{g} = \Lambda^2 V$, then we get map

$$\begin{array}{lll} U(\mathfrak{g}) \longrightarrow C^{\circ}(V) & & \text{onto} \\ \quad \searrow \mathrm{Hom}(V, V) & & \text{onto.} \end{array}$$