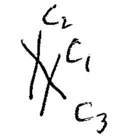
Mumford's theorem concerns what happens when you resolve the singularities of an isolated singular point on a surface. The inverse image of the singular point is a positive divisor $D = \sum' n_i C_i$, C_i irred curve, n_i integer > 0. One then has for the intersection pairing:



$$C_i \cdot C_j \geq 0 \qquad i \neq j$$

$$C_i \cdot D = \sum' n_j C_i \cdot C_j = 0 \qquad \forall i$$

singular pt.

and hence $C_i^2 = C_i \cdot C_i \leq 0$. Mumford's theorem says that the intersection matrix $C_i \cdot C_j$ is negative semi-definite. (Possibly also if one knows the graph, with vertices C_i and edges (i,j) for those C_i and C_j which intersect non-trivially, is connected, then on $\mathbb{Z}C_1 + \cdots + \mathbb{Z}C_n / \mathbb{Z}D$ the intersection matrix is negative-definite).

Suppose we change $C_i \cdot C_j$ to $-C_i \cdot C_j$ and then rescale so that we get a real symmetric matrix, i.e. a bilinear form $x \cdot y$ on $\mathbb{R}^n$ satisfying

$$e_i \cdot e_j \leq 0 \qquad i \neq j$$

$$e_i \left(\sum' e_j\right) = 0$$

Thus Mumford's thm. is a special case of:

Prop: Let $A = (a_{ij})$ be a real-symmetric matrix whose off diagonal entries are ≤ 0 and such that $\sum_{j=1}^{n} a_{ij} \geq 0 \quad \forall i$ [redacted] Then $A \geq 0$.

Proof: Using induction on n, we know $X^t A X \geq 0$

if X is a vector with at least one zero entry. Next note that we can reduce the diagonal elements of A to obtain a matrix A' with $A' \leq A$ and the row sums of A are 0. So we can suppose the row sums of A are zero, i.e. $e_1 + \cdots + e_n \in \mathrm{Ker}(A)$. Next given X we can add a multiple of $e_1 + \cdots + e_n$ to it so as to render one of its entries 0, whence we win by induction.

Next suppose we have a matrix a_{ij} with off-diagonal entries ≤ 0 and row sums $= 0$.

Thanksgiving trip.

Before the Thanksgiving trip we were looking at Mumford's theorems concerning real symmetric matrices $A = (a_{ij})$ with $a_{ij} \leq 0$ for $i \neq j$. [redacted] This says that if the row[redacted] sums $\sum_{j=1}^{n} a_{ij}$ are ≥ 0, then $A \geq 0$. <u>Proof:</u> Induction on n permits us to assume known that $X^t A X \geq 0$ when at least one coordinate of X is zero. Then by decreasing diagonal entries of A we can suppose the row sums of A are 0, i.e $A(e_1 + \cdots + e_n) = 0$. Then given X we can subtract a multiple of $e_1 + \cdots + e_n$ to obtain X' with $(X')^t A X' = X^t A X$, and so that one entry of X' is zero.

Next suppose A with off-diag. entries ≤ 0 and row sums $= 0$. Given $X = \sum x_i e_i$ not a multiple of $e_1 + \cdots + e_n$ we can add multiple of $e_1 + \cdots + e_n$ to it so as to obtain a new vector X with the same $X^t A X$ but with all $x_i \neq 0$ and some x_i positive and some x_i negative. So $X = X^+ + X^-$ where $X^+ = \sum_{x_i > 0} x_i e_i$, $X^- = \sum_{x_i < 0} x_i e_i$. Then

$$X^t A X = \underset{\geq 0}{(X^+)^t A X^+} + \underset{\geq 0}{(X^-)^t A X^-} + 2\underbrace{(X^+)^t A X^-}_{\sum_{x_j < 0,\, x_i > 0} x_i a_{ij} x_j}$$

The last term is ≥ 0 and in fact > 0 provided we know that whenever we divide up $\{1, .., n\}$ into 2 disjoint sets, there is an i in the first and a j in the second such that [redacted] a_{ij}. In other words [redacted] if we form a graph with vertices $\{1, .., n\}$ and edges for each (i, j) such that $a_{ij} \neq 0$, we want this graph to be connected. Thus we conclude

Prop: Let A be a real symmetric matrix with off-diag. entries ≤ 0 and row sums $= 0$. If the graph defined by the non-zero off diagonal entries is connected, then $e_1 + \cdots + e_n$ spans $\operatorname{Ker} A$, so that we know $X^t A X > 0$ for X not a multiple of $e_1 + \cdots + e_n$.

Cor: If off-diag. entries are ≤ 0 and give a connected graph and if at least one row sum is > 0, then $A > 0$.

Proof of Cor: Let A' be obtained from A by adjusting the row sums to zero. Then $X^t A X = X^t A' X + X^t D X$ where D is a diagonal matrix ≥ 0 with at least one strictly positive entry. Then $X^t A X = 0 \Rightarrow X^t A' X = 0$ so $X = c(e_1 + \cdots + e_n)$ and then $X^t D X \Rightarrow c = 0$.

Next we want to understand how the natural symmetry group, namely the diagonal matrices $(\mathbb{R}_{>0})^n$, affects things. It leaves off-diagonal entries ≤ 0 but changes the vector $e_1 + \cdots + e_n$ into $\lambda_1 e_1 + \cdots + \lambda_n e_n$ where $\lambda_i > 0$. Note that Λ acts on A by sending A to $\Lambda A \Lambda$, not conjugation.

So let us start with the $a_{ij} \leq 0$ given for $i \neq j$ giving a connected graph, and let $\tilde{A}$ be the matrix with these off-diagonal entries and 0's on the diagonal. Then a typical A will be in the form $A = D + \tilde{A}$ with D diagonal. Suppose we are after the least eigenvalue for A. The argument in the above proposition, where one splits X into X^+ and X^- shows that on going from $X^+ + X^-$ to $X^+ - X^-$, the A value decreases, although the ℓ^2 norm stays the same. Hence a minimum-eigenvalue eigenvector has all $x_i \geq 0$ (or ≤ 0). If some $x_i = 0$, then by increasing $X = \sum x_j e_j$ to $X + \varepsilon e_i$, then the ℓ^2 norm increases as ε^2 but the A-value decreases $\sim \varepsilon$ (assuming the vertex i is connected to a j with $x_j > 0$, which we can do by looking at these i first.) Thus we see that an eigenvector with minimum eigen-value has all $x_i > 0$ (or < 0), and moreover it is unique up to scalar multiplication.

Notice that once $\tilde{A}$ is given, one can choose a vector $X = (x_i)$ with all $x_i > 0$ and then define D so that $A = D + \tilde{A}$ kills X. It follows that X is the unique minimum-eigenvalue eigenvector for A.

Given $A = D + \tilde{A}$, let λ be the minimum eigenvalue, so that $X^t A X \geq \lambda X^t X$ for all X. It follows that $A - \lambda I$ has its kernel generated by a vector with strictly positive coefficients.

Once $\tilde{A}$ is given we get a 1-1 correspondence between D's such that $A = D + \tilde{A}$ has minimum eigenvalue 0 and lines in $\mathbb{R}^n$ spanned by vectors with strictly positive coefficients. This set of lines is an open simplex because each line contain a unique vector of the form (λ_i) with $\lambda_i > 0$, $\sum \lambda_i = 1$.

Question: One sees that the minimum eigenvector for $A = D + \tilde{A}$ has all components of the same sign. Do the higher eigenvectors have increasing numbers of components of opposite sign?

We should relate the Mumford business to Frobenius' theory of matrices with positive entries. Call such a matrix P. The key point is that the power series matrix

$$\frac{1}{1-zP} = 1 + zP + z^2P^2 + \cdots\cdots$$

note P^2, P^3 have pos. entries

is a matrix of power series with positive coefficients, so by a basic fact of complex variables, it has a singularity at $z=R$ where R is the radius of convergence. In this case the singularities are of the form $1/\lambda_i$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues. So $R = 1/|\lambda_1|$ where $|\lambda_1| \geq |\lambda_i|$ for $i = 2, \ldots, n$. Since the singularity occurs at $z=R$ we have $1/R = \lambda_i$ for some i say $i=1$. Thus we see P has a positive real eigenvalue λ_1 with $|\lambda_i| \leq \lambda_1$ for all the other eigenvalues.

Suppose first that P is semi-simple. We can assume $\lambda_1 = 1$. Then from the Jordan form it is clear that

$$\lim_{n\to\infty} P^n = E$$

is the projection on the $\lambda = 1$ eigenspace. Wait – this happens only if the other eigenvalues have modulus < 1. Instead take the average

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k = E$$

and this kills the other eigenvalues.

$$\frac{1}{n}\sum_{k=0}^{n-1} P^k = \frac{1}{n}\,\frac{P^n - 1}{P - 1} \qquad \text{where } P \neq 1.$$

It follows that if we apply E to any of the basis element e_i we get a vector with positive entries, hence the 1-eigenspace is spanned by vectors with entries ≥ 0.

Next let us consider the general case where P isn't semi-simple. By Jordan $P = S + N$, where S is semi-simple, N is nilpotent, and $[S,N] = 0$. We know all eigenvalues of P and hence S satisfy $|\lambda| \leq 1$ and hence we can form the series

$$\frac{1}{1-zP} = \sum z^n P^n$$

which converges for $|z| < 1$. The idea is to multiply this by $(1-z)^{k+1}$ and then let $z \nearrow 1$. What we get is a matrix L with positive coefficients, provided this limit exists. Also L clearly kills all the generalized eigenspaces of P belonging to the eigenvalues $\neq 1$. To see what happens for the generalized eigenspace belonging to $\lambda = 1$, we can suppose $P = I + N$, then

$$\frac{1}{1-zP} = \frac{1}{1-z-zN} = \frac{1}{1-z}\,\frac{1}{1-\frac{zN}{1-z}}$$

$$= \frac{1}{1-z}\left(1 + \frac{z}{1-z}N + \cdots + \left(\frac{z}{1-z}\right)^k N^k\right)$$

where k is such that $N^k \neq 0$, but $N^{k+1} = 0$. Then

$$\lim_{z \to 1} (1-z)^{k+1} \frac{1}{1-zP} = N^k$$

has its image contained in the space of eigenvectors for P with eigenvalue 1.

Consequently, one sees that a matrix with positive entries has an eigenvector with positive entries for its maximum eigenvalue, and there is a standard procedure for constructing it

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November 26, 1979

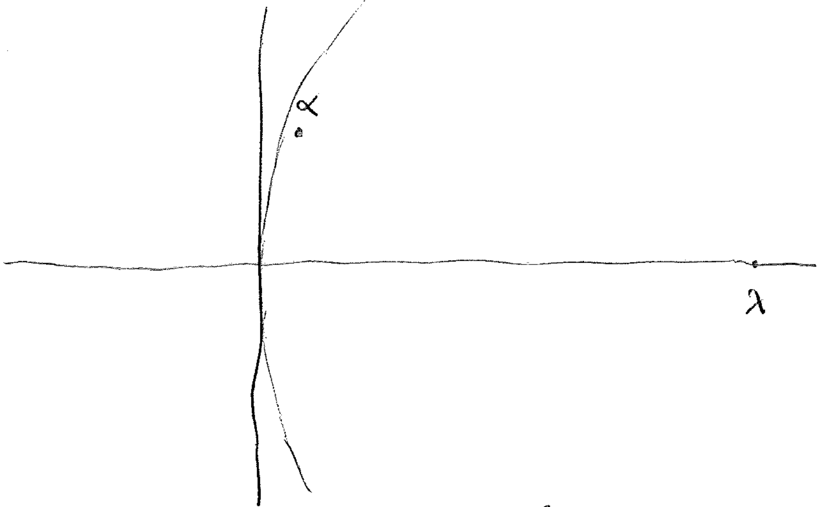
Return to Mumford situation of a matrix A, real symmetric, with negative off-diagonal entries giving rise to a connected graph. Let $A = D + \tilde{A}$, where D is the diagonal part of A. If we choose $\lambda \geq$ all entries of D, then $\lambda - A = (\lambda - D) - \tilde{A}$ is a matrix with positive entries, so it has by Frobenius a maximum-eigenvalue eigenvector with strictly positive coefficients. So A has a minimum-eigenvalue eigenvector with strictly positive coefficients. But the Frobenius theory, or the part of it that I know (see above), doesn't give the fact that this minimum eigenvalue is of multiplicity one.

Hence the Frobenius business is more general in that it deals with non-symmetric matrices, and less precise.

Finally note that if A has negative off-diagonal entries and has its spectrum in $Re(\lambda) > 0$, then for $\lambda \gg 0$

$$\frac{1}{A} = \frac{1}{\lambda - (\lambda - A)} = \frac{1}{\lambda} \cdot \frac{1}{1 - \frac{\lambda - A}{\lambda}} = \frac{1}{\lambda} + \frac{\lambda - A}{\lambda^2} + \frac{(\lambda - A)^2}{\lambda^3} + \cdots$$

is a matrix with positive entries. This series converges for $\left| \frac{\lambda - \alpha}{\lambda} \right| < 1$ for each eigenvalue α of A which is the case for λ large and $Re(\alpha) > 0$.



Also if $t > 0$ then

$$e^{-tA} = \lim_{n \to \infty} \left(1 - \frac{tA}{n}\right)^n$$

will be a matrix with positive entries. Since

$$\frac{1}{A} = \int_0^\infty e^{-tA}\, dt$$

this gives another proof that $\frac{1}{A}$ has positive [redacted] entries.

Note that e^{-tA} has positive entries when the off-diagonal entries of A are [redacted] negative, and then we have that

$$\frac{1}{u+A} = \int_0^\infty e^{-tA} e^{-tu}\, dt$$

has positive entries for u + spectrum of A in the right half-planes.

At this stage I understand the Mumford result pretty well. It would be desirable to work in, if possible, the Hodge Index thm. - Grothendieck proof. Let's go over the proof of this result.

Let F be a projective non-singular surface over an algebraically closed [redacted] field, say $\mathbb{C}$. On $\mathrm{Pic}(X)$ we have the intersection pairing $(L_1, L_2) \longmapsto c_1(L_1)\cdot c_1(L_2)$. Actually this pairing is defined on $\mathrm{Im}\{c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})\}$.

The Hodge index thm. says this pairing has signature [redacted] $(+,-,-,-,\cdots)$, like Minkowski's metric. If we assume the pairing has this signature, we know the positive cone $L\cdot L>0$ has [redacted] two components, and the forward time component can be selected out by the condition [redacted] $\mathcal{O}(1)\cdot L>0$ where $\mathcal{O}(1)$ is a fixed very ample line bundle. If we start out with an $\mathcal{O}(1)$ and construct an orthogonal basis with signature $(+,+,-,\cdots)$. ?

More Tauberian business from Victor's course

Let f be increasing on $[0,\infty)$, and $f\geq 0$, and let $\sigma \geq 1$. Claim

$$\int_0^T f(t)\,dt \sim AT^\sigma \implies f(T) \sim A\sigma T^{\sigma-1} \qquad (\text{here } T\to\infty).$$

~~Proof~~ The hypothesis implies that $\forall \varepsilon>0$ one has

$$(A-\varepsilon)T^\sigma \leq \int_0^T f(t)\,dt \leq (A+\varepsilon)T^\sigma$$

for suff. large T. Let $\delta>0$ be fixed. Then

$$\delta T f(T) \underset{\substack{\uparrow \\ f \text{ increasing}}}{\leq} \int_T^{T(1+\delta)} f(t)\,dt \leq (A+\varepsilon)T^\sigma(1+\delta)^\sigma - (A-\varepsilon)T^\sigma$$

so

$$\limsup_{T\to\infty} \frac{f(T)}{T^{\sigma-1}} \leq \frac{(A+\varepsilon)(1+\delta)^\sigma - (A-\varepsilon)}{\delta}$$

for any $\varepsilon>0$. Now let $\varepsilon\to 0$ to get $\limsup \frac{f(T)}{T^{\sigma-1}} \leq A\,\frac{(1+\delta)^\sigma-1}{\delta}$, and then let $\delta \to 0$ to get

$$\limsup \frac{f(T)}{T^{\sigma-1}} \leq A\sigma$$

Similarly

$$\delta T\, f(T) \geq \int_{T(1-\delta)}^{T} f(t)\,dt \geq (A-\varepsilon)T^{\sigma} - (A+\varepsilon)T^{\sigma}(1-\delta)^{\sigma}$$

yields

$$\liminf \frac{f(T)}{T^{\sigma-1}} \geq A\sigma \qquad \text{Q.E.D.}$$

The converse $f(T) \sim A\sigma T^{\sigma-1} \Rightarrow \int_0^T f \sim AT^{\sigma}$ is more or less clear because for any $\varepsilon > 0$ we have

$$(A-\varepsilon)\sigma T^{\sigma-1} \leq f(T) \leq (A+\varepsilon)\sigma T^{\sigma-1} \qquad T \text{ large}$$

$$(A-\varepsilon)(T^{\sigma} - T_0^{\sigma}) \leq \int_{T_0}^{T} f \leq (A+\varepsilon)(T^{\sigma} - T_0^{\sigma})$$

$$(A-\varepsilon)\left(1 - \frac{T_0^{\sigma}}{T^{\sigma}}\right) + \frac{1}{T^{\sigma}}\int_0^{T_0} f \leq \frac{1}{T^{\sigma}}\int_0^{T} f \leq (A+\varepsilon)\left(1 - \frac{T_0^{\sigma}}{T^{\sigma}}\right) + \frac{1}{T^{\sigma}}\int_0^{T_0} f \qquad \text{etc.}$$

This even works for $\sigma > 0$.

Karamata Tauberian thm. f increasing on $[0,\infty)$, and ≥ 0, $\sigma \geq 1$. Then

$$\int_0^{\infty} e^{-st} f(t)\,dt \sim \frac{A}{s^{\sigma}} \text{ as } s \searrow 0 \quad \Longrightarrow \quad f(t) \sim A\frac{t^{\sigma-1}}{\Gamma(\sigma)} \quad \text{as } t \to \infty$$

One deduces this from Wiener Tauberian thm. $s = e^{-x}$ $t = e^{y}$

$$\int_{-\infty}^{\infty} e^{-e^{-(x-y)}} f(e^{y})\, e^{y}\,dy \sim A e^{\sigma x}$$

$$\int_{-\infty}^{\infty} \underbrace{e^{-\sigma(x-y)}\, e^{-e^{-(x-y)}}}_{K(x-y)} \underbrace{e^{-\sigma y} f(e^{y}) e^{y}}_{g(y)}\,dy \sim \frac{A}{\Gamma(\sigma)}\cdot\Gamma(\sigma)$$

$K \in L^1$ for $\sigma > 0$

Now $\hat{K}(\xi) = \int_{-\infty}^{\infty} e^{-\sigma x} e^{-e^{-x}} e^{i\xi x}\, dx$ $\qquad t = e^{-x} \qquad dx = -\frac{dt}{t}$

$$= \int_0^\infty e^{-t} t^{\sigma - i\xi} \frac{dt}{t} = \Gamma(\sigma - i\xi)$$ non-vanishing in ξ.

Assuming we can show $g \in L^\infty$, the Wiener Tauberian thm says

$$\int_{-\infty}^{\infty} K'(x-y)\, g(y) \sim \frac{A}{\Gamma(\sigma)} \int_{-\infty} K'$$

for any $K' \in L^1$. Take $K'(x) = e^{-\sigma x} \underbrace{H(x)}_{\text{Heaviside}}$ $= \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$

Then

$$\int K' = \int_0^\infty e^{-\sigma x}\, dx = \frac{1}{\sigma}$$

$$\int_{-\infty}^{x} e^{-\sigma(x-y)} e^{-\sigma y} f(e^y) e^y\, dy = e^{-\sigma x} \int_0^{e^x} f(t)\, dt \sim \frac{A}{\Gamma(\sigma)\sigma}$$

so $\displaystyle\int_0^T f(t)\, dt \sim \frac{A T^\sigma}{\Gamma(\sigma)\sigma}$ and so by first result

$$f(t) \sim \frac{A t^{\sigma-1}}{\Gamma(\sigma)}$$

Thus it remains to show that g is bounded, i.e. that $f(t)/t^{\sigma-1}$ is bounded.

Now

$$A s^{-\sigma} \sim \int_0^\infty e^{-st} f(t)\, dt \geq \int_0^{1/s} e^{-st} f(t)\, dt \geq e^{-1} \int_0^{1/s} f(t)\, dt$$

So $\int_0^T f(t)\, dt = O(T^\sigma)$. Then

$$T f(T) \leq \int_T^{2T} f(t)\, dt = O(T^\sigma) \implies f(t) \in O(t^{\sigma-1})$$

which is what we need.

October 28, 1979

Amit's account of Ising partition function. The energy of an assignment $s = \{s_i\}$ of spins is

$$E(s) = -\sum_{ij}{}' J_{ij} s_i s_j - \sum_i h_i s_i$$

and the partition function is

$$Z = \sum_s{}' e^{-\beta E(s)} = \sum_s e^{\sum K_{ij} s_i s_j + \sum H_i s_i}$$

where $K_{ij} = \beta J_{ij}$ and $H_i = \beta h_i$. Then the average magnetization of the i-th site is

$$\langle s_i \rangle = \frac{\partial}{\partial H_i} \log Z$$

In the case where we have translation invariance, this is independent of i, and is called the magnetization, as one has

$$M = \frac{1}{N}\frac{\partial}{\partial H} \log Z \qquad \text{here all } H_i = H$$

The susceptibility is

$$\frac{\partial M}{\partial H} = \frac{1}{N}\frac{\partial^2}{\partial H^2} \log Z$$

$$= \frac{1}{N}\left(\frac{1}{Z}\frac{\partial^2}{\partial H^2} \text{[redacted]}\, Z - \frac{1}{Z^2}\left(\frac{\partial Z}{\partial H}\right)^2\right)$$

$$= \frac{1}{N}\sum_{ij}\left(\langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle\right)$$

$$= \frac{1}{N}\sum_{ij}\left\langle (s_i - \langle s_i\rangle)(s_j - \langle s_j\rangle)\right\rangle$$

$$= \sum_j \underbrace{\left\langle (s_0 - \langle s_0\rangle)(s_j - \langle s_j\rangle)\right\rangle}_{\text{correlation between 0th and jth spin}}$$

Amit interprets this formula as relating the singularities of the susceptibility to the long-range behavior of the spin correlation.

Note that

$$Z(\beta) = \int e^{-\beta E} \, d\mu(E)$$

then

$$-\frac{\partial}{\partial\beta}\log Z = \frac{-1}{Z}\frac{\partial Z}{\partial\beta} = \frac{1}{Z}\int E e^{-\beta E}\, d\mu(E) = \langle E \rangle$$

and

$$\frac{\partial^2}{\partial\beta^2}\log Z = \frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2} - \left(\frac{1}{Z}\frac{\partial Z}{\partial\beta}\right)^2 = \langle E^2\rangle - \langle E\rangle^2$$

$$= \langle (E - \langle E\rangle)^2 \rangle$$

Now the specific heat is $C = \dfrac{\partial\langle E\rangle}{\partial T} = \dfrac{\partial\langle E\rangle}{\partial\beta} \Big/ \underbrace{\dfrac{\partial T}{\partial\beta}}_{-1/\beta^2}$

or ,

$$C = \beta^2\frac{\partial^2}{\partial\beta^2}\log Z$$

hence one sees that the specific heat is given by the deviation of E around its average value.

Somehow the key problem with these partition functions involves passing to the limit as the number of sites or the volume goes to ∞.

Let's recall what happens for the Bose-~~Einstein~~ gas. We consider independent Bose particles in a cubical box of volume $V = L^3$. The 1-particle Hilbert space has basis $\varphi_k = \frac{1}{\sqrt{V}} e^{ikx}$, $k \in \left(\frac{2\pi}{L}\mathbb{Z}\right)^3$. and φ_k has energy $k^2/2$. The Hilbert space for the gas is the symmetric algebra on the 1-particle space and it is given the Hamiltonian H which is the derivation extending the 1-particle Hamiltonian. To form the grand partition function one needs also the number of particle operators N and then

$$Z_{grand} = tr\ e^{-\beta(H-\mu N)}$$

where μ is a chemical potential whose purpose is to make the density what it should be. Clearly

$$Z_{grand} = \prod_k \sum_{n\geq 0} e^{-\beta(\frac{k^2}{2}-\mu)n} = \prod_k \frac{1}{1-e^{-\beta\frac{k^2}{2}+\beta\mu}}$$

$$\log Z_{gr} = \sum_{k\in\left(\frac{2\pi}{L}\mathbb{Z}\right)^3} -\log\left(1-e^{\beta\mu-\beta\frac{k^2}{2}}\right)$$

First note that we have to have $\mu < 0$ for Z_{gr} to make sense. Next note that if we let $V = L^3 \to \infty$ and divide by V, the sum goes into an integral

Put $\alpha = e^{\beta\mu}$

$$\lim_{V\to\infty} \frac{1}{V}\log Z_{gr} = \int \frac{d^3k}{(2\pi)^3}\left(-\log\left(1-e^{\beta\mu-\beta\frac{k^2}{2}}\right)\right)$$

$$= \int \frac{d^3k}{(2\pi)^3} \sum_{n\geq 1} \frac{1}{n}\alpha^n e^{-n\beta k^2/2}$$

~~Recall here that k^2 stands for $|\vec{k}|^2$, so we can do this integral radially where [illegible]~~

Recall that $k^2 = |\vec{k}|^2$, so we can do the integral in spherical coords getting

$$\int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} \sum \frac{1}{n}\alpha^n e^{-n\beta k^2/2}$$

$$= \frac{2}{(2\pi)^2} \sum_{n\geq 1} \frac{1}{n}\alpha^n \int_0^\infty e^{-\left(\frac{n\beta}{2}\right)k^2} \frac{k^{3/2}dk}{k\ 2}$$

$$= \frac{1}{(2\pi)^2} \sum_{n\geq 1} \frac{\alpha^n}{n} \frac{\overbrace{\Gamma(3/2)}^{\frac{1}{2}\pi^{1/2}}}{\left(\frac{n\beta}{2}\right)^{3/2}}$$

$$\frac{(2\pi)^{1/2}}{n^{3/2}\beta^{3/2}}$$

So the partition function is

$$\lim_{V\to\infty} \frac{1}{V} \log Z_{gr} = (2\pi\beta)^{-3/2} \sum_{n\geq 1} \frac{\alpha^n}{n^{5/2}} \qquad \alpha = e^{\beta\mu}$$

The density is given by

$$\rho = \lim_{V\to\infty} \frac{\langle N \rangle}{V} = \frac{1}{\beta}\frac{\partial}{\partial\mu} \lim \frac{1}{V} \log Z$$

$$= \frac{1}{\beta} (2\pi\beta)^{-3/2} \sum_{n\geq 1} \frac{e^{n\beta\mu}\, n\beta}{n^{5/2}}$$

$$= (2\pi\beta)^{-3/2} \sum_{n\geq 1} \frac{\alpha^n}{n^{3/2}}$$

The question is now can we adjust μ so as to get a desired density. Now $\alpha = e^{\beta\mu} < 1$, so we want to choose α so that

$$\sum_{n\geq 1} \frac{\alpha^n}{n^{3/2}} = \rho (2\pi\beta)^{3/2}$$

But the maximum the left side can be is $\zeta(3/2)$ and hence there is a problem if the density ρ is too high or the temperature $T = 1/\beta$ is too low. This is the phenomenon of Bose-Einstein condensation.

November 30, 1979

Ising model partition function is of the form

$$\sum_s{}' \; e^{-\frac{1}{2} s\cdot Ks \; + \; Hs} \;.$$

Let's consider the simpler problem where the sum over spins $s_i = \pm 1$ is replaced by an integral

$$\int e^{-\frac{1}{2} s\cdot Ks + Hs} \, ds$$

Provided the matrix K is > 0, this is a Gaussian integral which we can evaluate. We get

$$(2\pi)^{N/2} (\det K)^{-1/2} \, e^{\frac{1}{2} H\cdot K^{-1} H}$$

The first example to consider is a linear chain with nearest neighbor interaction which gives

$$K_{n,n+1} = -\beta$$

and all the other off-diagonal entries are zero. We are going to have to adjust the diagonal elements so as to get $K > 0$, so we ought to begin by determining the spectrum of the Jacobi matrix

$$\begin{matrix} 0 & 1 & \\ 1 & 0 & 1 \\ & 1 & \ddots \end{matrix}$$

The eigenvalue equations are

$$u_{n+1} + u_{n-1} = \lambda u_n$$

and solutions are $u_n = e^{ikn}$ or better $u_n = \rho^n$

where $\zeta + \zeta^{-1} = \lambda$. Thus the spectrum is the interval $[-2, 2]$.

(Similarly in 2 dimensions the eigenvalue equation is

$$u_{m,n+1} + u_{m,n-1} + u_{m+1,n} + u_{m-1,n} = \lambda u_{mn}$$

and you get spectrum $\lambda = 2(\cos k_1 + \cos k_2)$ from plane waves $u_{mn} = e^{i(k_1 m + k_2 n)}$. Thus the spectrum is $[-4, 4]$. From the general theory of matrices with off-diag entries ≥ 0 I know that by subtracting 4 from main diagonal yields a matrix ≤ 0 with the eigenvector consisting of all 1's. The general theory doesn't immediately imply that putting 4 on the diagonal gives a matrix ≥ 0, however in this lattice case, by conjugating $u_{mn} \mapsto (-1)^{m+n} u_{mn}$ one flips the sign of the matrix. This seems to be related to anti-ferromagnetism being modelled with two lattices.)

So in order to get $K > 0$ we are going to have to add a multiple of the identity to our basic off-diagonal matrix $K_{n,n+1} = -\beta$. Denote by $\tilde{K}$ the off-diagonal matrix so that $K = \lambda I + \tilde{K}$.

Let's scale things differently. It will be convenient to work with the J-matrix Δ defined by

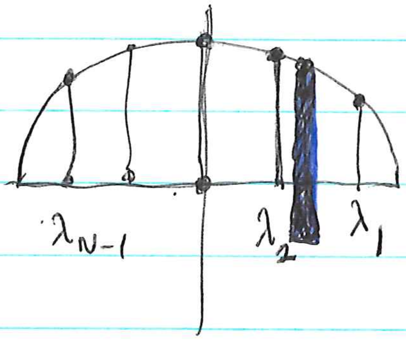
$$(\Delta u)_n = \tfrac{1}{2}(u_{n+1} + u_{n-1})$$

Let's concentrate on this matrix restricted to the interval $[0, N]$ with Dirichlet condition. This gives the $(N-1)\times(N-1)$ J-matrix Δ_N. The eigenfunction vanishing at zero is

$$\varphi_n = \sin(n\theta)$$

and the eigenvalue is $\cos\theta$. This vanishes at N when $\sin(N\theta) = 0$ or $\theta = \frac{j\pi}{N}$ $j = 1, 2, \ldots, N-1$. Thus the eigenvalues are

$$\lambda_j = \cos\left(\frac{j\pi}{N}\right) \qquad 1 \le j < N$$



and so

$$\det(\lambda - \Delta_N) = \prod_{j=1}^{N-1} \left(\lambda - \cos\frac{j\pi}{N}\right)$$

The partition function is

$$Z_N = \int e^{-\frac{1}{2}\beta\, s\cdot(\lambda-\Delta_N)s \,+\, Hs}\, ds$$

$$= \left(\frac{2\pi}{\beta}\right)^{(N-1)/2} \left(\det(\lambda-\Delta_N)\right)^{-1/2} e^{\frac{1}{2}\frac{1}{\beta} H\cdot(\lambda-\Delta_N)^{-1} H}$$

and the goal is to understand this as $N \to \infty$. Actually one should think of Δ_N as being on the interval $[-N/2, N/2]$, say N is even, so that as $N \to \infty$ we get Δ.

Notice that $\lambda = 1$ might be interesting even though we really want $\lambda > 1$ in order that $(\lambda - \Delta)^{-1}$ exists. We have

$$\left((\lambda-\Delta_N)^{-1}\right)_{nn'} = \sum_{1\le j<N} \frac{\sin(n\theta_j)\sin(n'\theta_j)}{\lambda - \cos\theta_j}\, \frac{1}{c_j^2}$$

where $$c_j^2 = \sum_{n=1}^{N-1} (\sin n\theta_j)^2 = \sum_{n=0}^{N-1} \frac{1-\cos\left(2n\frac{j\pi}{N}\right)}{2} = \frac{1}{2}N$$

but this seems too complicated.

The point perhaps is that you want to be able to divide $\log Z_N$ by $N-1$ and get a limit as $N \to \infty$. I think that $(\lambda - \Delta_N)^{-1} \to (\lambda - \Delta)^{-1}$ so that this ~~partition~~ piece of the partition function disappears at least for $\lambda > 1$. Hence look at the determinant factor:

$$\frac{1}{N-1} \log \det(\lambda - \Delta_N) = \frac{1}{N-1} \sum_{j=1}^{N-1} \log\left(\lambda - \cos \frac{j\pi}{N}\right) \longrightarrow \int_0^1 \log(\lambda - \cos \pi x)\, dx$$

Hence

$$\frac{1}{N} \log Z_N \longrightarrow \frac{1}{2} \log\left(\frac{2\pi}{\beta}\right) - \frac{1}{2} \int_0^1 \log(\lambda - \cos \pi x)\, dx$$

and this limit makes sense even for $\lambda = 1$. Unfortunately H doesn't appear.

Let's see what happens for $\lambda = 1$. The eigenvectors with eigenvalue $= 1$ are linear functions $u_n = An + B$, so

$$\frac{1}{1 - \Delta_N}(n, n') = \frac{\varphi(n_<)\,\psi(n_>)}{W}$$

where $\varphi(n) = n + N/2$ $\quad \psi(n) = n - N/2$ and W is adjusted so that $1 - \Delta_N$ applied to the above gives $\delta(n-n')$.

$$\frac{\varphi(n')\psi(n')}{W} - \frac{1}{2}\left(\frac{\varphi(n'-1)\psi(n') + \varphi(n')\psi(n'+1)}{W}\right) = 1$$

$$= \frac{(\varphi(n') - \varphi(n'-1))\,\psi(n')}{2W} - \frac{\varphi(n')(\psi(n'+1) - \psi(n'))}{2W}$$

$$= \frac{\psi(n') - \varphi(n')}{2W} = -N/2W = 1 \qquad \therefore W = -N/2$$

and so

$$\langle n | (1-\Delta_N)^{-1} | n' \rangle = -\frac{2}{N}\left(n_< + N/2\right)\left(n_> - N/2\right)$$

Hence $\quad \frac{1}{N}\langle n | (1-\Delta_N)^{-1} | n' \rangle \longrightarrow \frac{1}{2}$ so we see that if

$$Z_N = \int e^{-\frac{1}{2}\beta\, s\cdot(1-\Delta_N)s + Hs}\, ds$$

$$= \left(\frac{2\pi}{\beta}\right)^{(N-1)/2} \det(1-\Delta_N)^{-1/2}\, e^{\frac{1}{2}\frac{1}{\beta}H\cdot(1-\Delta_N)^{-1}H}$$

then $\lim_{N\to\infty}\frac{1}{N}\log Z_N$ exists and it is

$$+\frac{1}{2}\log\left(\frac{2\pi}{\beta}\right) - \frac{1}{2}\int_0^1 \log(1-\cos\pi x)\,dx + \frac{1}{2\beta}\,\frac{1}{2}\left(\sum H_n\right)^2$$

It seems this isn't too interesting because ultimately we want the external field H_n to be constant in n.

December 1, 1979

Let us compute the partition function $tr(e^{-\beta H})$ for the 1-diml harmonic oscillator $H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$. The simplest method is to use the known spectrum of the operator: the eigenvalues of H are $(n+\frac{1}{2})\omega$, $n \geq 0$ an integer, so

$$tr\, e^{-\beta H} = \sum_{n\geq 0} e^{-\beta(n+\frac{1}{2})\omega} = \frac{e^{-\frac{1}{2}\beta\omega}}{1-e^{-\beta\omega}} = \frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)}$$

The other method will be to compute explicitly the path integral for the partition function

$$tr\left(e^{-\beta H}\right) = \int e^{-\int_0^\beta \left(\frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2\right)dt}\, \mathcal{D}q$$

where q runs over paths of period β and $\mathcal{D}q$ is appropriately normalized. Recall the path integral arises because we use the "Trotter product formula"

$$e^{-\beta H} = e^{-\beta\left(\frac{p^2}{2}+\frac{\omega^2 q^2}{2}\right)} = \lim_{N\to\infty}\left(e^{-\frac{1}{N}\beta\frac{p^2}{2}}\, e^{-\frac{1}{N}\beta\frac{\omega^2 q^2}{2}}\right)^N$$

together with the explicit representation of the kernel for $e^{-a\frac{p^2}{2}}$

$$\langle q| e^{-a\frac{p^2}{2}} |q'\rangle = \int \frac{dp}{2\pi} \underbrace{\langle q|p\rangle}_{e^{ipq}} e^{-a\frac{p^2}{2}} \langle p|q'\rangle$$

$$= \int \frac{dp}{2\pi}\, e^{-a\frac{p^2}{2} + ip(q-q')} = \frac{1}{\sqrt{2\pi a}}\, e^{-\frac{1}{2}\frac{(q-q')^2}{a}}$$

Specifically

$$\langle q_N | e^{-\beta H} | q_0\rangle = \left(\int \prod_1^{N-1} \frac{dq_j}{\sqrt{2\pi a}}\, e^{-\frac{1}{2}\sum_1^N \frac{(q_j-q_{j-1})^2}{a} - \frac{1}{2}a\omega^2 \sum_1^N q_{j-1}^2}\right) \frac{1}{\sqrt{2\pi a}}$$

total of N segments

where $a = \frac{\beta}{N}$. Let's check this works for $q_0 = q_N = 0$

and $w=0$. On the right we have a Gaussian integral over $q_1,\dots,q_{N-1}$ space. Notice that the a can be replaced by 1 via $q_j \mapsto \sqrt{a}\, q_j$. Recall

$$\int \prod \frac{dq_j}{\sqrt{2\pi}}\, e^{-\frac{1}{2} q\cdot Aq} = (\det A)^{-1/2}$$

The quadratic form in this case is

$$q_1^2 + (q_2-q_1)^2 + \cdots + (q_{N-1}-q_{N-2})^2 + q_{N-1}^2$$

and it belongs to the matrix

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \cdot & & \\ & & & \cdot & -1 \\ & & & -1 & 2 \end{pmatrix}$$

which is $2(1-\Delta_N)$.

We know the eigenvalues of Δ_N are as follows. The eigenfunctions are $\sin kn$ where $k = \frac{\pi}{N}, \frac{2\pi}{N}, \dots, \frac{N-1}{N}\pi$ and the eigenvalue is $\cos k$. So

$$\det 2(1-\Delta_N) = \prod_{j=1}^{N-1} 2\left(1-\cos\frac{j\pi}{N}\right)$$

Calculation for $N=2,3,4$ shows this product $=N$, and a general proof is easily found by induction on N for the determinant. So we get

$$\langle q=0|\, e^{-\beta H}\,|q=0\rangle = \frac{1}{\sqrt{2\pi a}}\,\frac{1}{\sqrt{N}} = \frac{1}{\sqrt{2\pi\beta}}$$

since $Na=\beta$.

Notice that the above computation of $\det(1-\Delta_N) = \frac{N}{2^{N-1}}$ is not obviously consistent with the computation on p. 445

$$\frac{1}{N}\log\det(\lambda-\Delta_N) = \frac{1}{N}\sum_1^{N-1}\log\left(\lambda-\cos\frac{j\pi}{n}\right) \longrightarrow \int_0^1 \log(\lambda-\cos\pi x)\,dx$$

although maybe it is.

Actually it seems OKAY. If $\lambda \geq 1$ let $\lambda = \frac{1}{2}(a+a^{-1})$

with $a \leq 1$. Then

$$\lambda - \cos\theta = \frac{a+a^{-1}}{2} - \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad z = e^{i\theta}$$

$$= \frac{1}{2}\left(a+a^{-1}-z-z^{-1}\right) = \frac{1}{2}z^{-1}\left(-z^2+(a+a^{-1})z-1\right)$$

$$= -\frac{1}{2}z^{-1}(z-a)(z-a^{-1}) = \frac{1}{2}(1-az^{-1})(1-az)\frac{1}{a}$$

Thus

[redacted] $$\lambda - \cos\theta = \frac{1}{2a}\left|1-ae^{i\theta}\right|^2$$

so

$$\frac{1}{2\pi}\int_0^{2\pi}\log(\lambda-\cos\theta)\,d\theta = \log\left(\frac{1}{2a}\right) + \underbrace{\frac{1}{\pi}\int_0^{2\pi}\log\left|1-ae^{i\theta}\right|d\theta}_{\frac{1}{\pi}\operatorname{Re}\underbrace{\int_0^{2\pi}\log(1-ae^{i\theta})\,d\theta}_{2\pi\log 1 = 0}}$$

$$\boxed{\frac{1}{2\pi}\int_0^{2\pi}\log(\lambda-\cos\theta)\,d\theta = \log\left(\frac{1}{2a}\right) \qquad a = \lambda - \sqrt{\lambda^2-1}}$$

Hence

$$\frac{1}{N}\log\det(\lambda-\Delta_N) \rightarrow \frac{1}{\pi}\int_0^{\pi}\log(\lambda-\cos\theta)\,d\theta = \log\left(\frac{1}{2a}\right)$$

But we can do much better than this, because we know that

$$\det(\lambda-\Delta_N) = C\sin(N\theta)$$

where $\cos\theta = \lambda$ and C is adjusted so that the right-side is a monic polynomial of degree $N-1$ in λ.

$$\sin(N\theta) = \frac{(e^{i\theta})^N - (e^{-i\theta})^N}{2i}$$

$$\frac{\sin(N\theta)}{\sin(\theta)} = \frac{(e^{i\theta})^N - (e^{-i\theta})^N}{e^{i\theta}-e^{-i\theta}} = (e^{i\theta})^{N-1} + \cdots + (e^{-i\theta})^{N-1}$$

$$= (2\lambda)^{N-1} + \text{lower terms}$$

Thus

$$\boxed{\det(\lambda - \Delta_N) = \frac{1}{2^{N-1}} \frac{(a^{-1})^N - a^N}{a^{-1} - a} \qquad \begin{aligned} a^{-1} &= \lambda + \sqrt{\lambda^2 - 1} \\ a &= \lambda - \sqrt{\lambda^2-1} \end{aligned}}$$

from which it follows that

$$\frac{1}{N} \log \det(\lambda - \Delta_N) \longrightarrow \log\left(\frac{1}{2a}\right)$$

Another proof:

$$\det(\lambda - \Delta_N) = \prod_1^{N-1} \lambda - \cos\frac{j\pi}{N} = \prod_1^{N-1} \frac{1}{2a} \left|1 - e^{\frac{ij\pi}{N}}\right|^2$$

$$= \frac{1}{(2a)^{N-1}} \prod_{\substack{-N<j<N \\ j \neq 0, N}} \left(1 - e^{\frac{ij\pi}{N}}\right) = \frac{1}{(2a)^{N-1}} \frac{1 - a^{2N}}{1 - a^2}$$

standard cyclotomic factorization of $1 - X^{2N}$.

$$= \frac{1}{2^{N-1}} \frac{a^{-N} - a^N}{a^{-1} - a}$$

Let us return to computing $\mathrm{tr}(e^{-\beta H})$ by the finite path integral. To calculate

$$(*) \qquad \int \prod_{j=0}^{N-1} \frac{dq_j}{\sqrt{2\pi a}} \; e^{-\frac{1}{2} \sum_1^N \frac{(q_j - q_{j-1})^2}{a} - \frac{1}{2} a\omega^2 \sum_1^N q_{j-1}^2}$$

where $q_0 = q_N$. The quadratic form is given by the matrix

$$\left(\frac{2}{a} + a\omega^2\right) I - \frac{2}{a} \Delta_{N,\text{periodic}}$$

and $\Delta_{N,\text{periodic}}$ has eigenvectors e^{ikn} where $k \in \frac{2\pi}{N}\mathbb{Z}$ for periodicity and $0 \leq k < 2\pi$ to avoid duplication. These the eigenvalues are $\cos j\frac{2\pi}{N}$ $\quad j=0,\dots,N-1$.

Hence our Gaussian integral is

$$\frac{1}{\sqrt{a}^{\,N}}\left[\frac{1}{\prod_{j=0}^{N-1}\left(\left(\frac{2}{a}+a\omega^2\right)-\frac{2}{a}\cos j\frac{2\pi}{N}\right)}\right]^{1/2} \qquad \text{here } a=\frac{\beta}{N}$$

We need

$$\prod_{j=1}^{N-1} 2\left(\lambda-\cos j\frac{2\pi}{N}\right)=\prod_{j=1}^{N-1}\frac{1}{\alpha}\left|1-\alpha e^{ij\frac{2\pi}{N}}\right|^2 \qquad \text{here } \alpha=\lambda-\sqrt{\lambda^2-1}$$

$$=\frac{1}{\alpha^{N-1}}\left(\frac{1-\alpha^N}{1-\alpha}\right)^2$$

(Idea: You somehow are using finite approximations to the Laplacean on S^1. What are the corresponding $\int$ functions like?)

First finish the calculation

$$(*)=\left[\prod_{j=0}^{N-1}(2+a^2\omega^2)-2\cos j\frac{2\pi}{N}\right]^{-1/2}$$

$$=\left[a^2\omega^2\,\frac{1}{\alpha^{N-1}}\cdot\left(\frac{1-\alpha^N}{1-\alpha}\right)^2\right]^{-1/2}$$

$$=\left[a\omega\,\frac{1}{\alpha^{(N-1)/2}}\,\frac{1-\alpha^N}{1-\alpha}\right]^{-1}=\left[a\omega\,\frac{\alpha^{-(N/2)}-\alpha^{N/2}}{\alpha^{-1/2}-\alpha^{1/2}}\right]^{-1}$$

where $\lambda=1+\frac{a^2\omega^2}{2}=1+\frac{1}{2}\omega^2\frac{\beta^2}{N^2}$. Now

$$\sqrt{\lambda^2-1}=\sqrt{\frac{\omega^2\beta^2}{N^2}+\left(\frac{\omega^2\beta^2}{2N^2}\right)^2}\sim\frac{\omega\beta}{N}$$

$$\alpha\sim 1-\frac{\omega\beta}{N} \qquad 1-\alpha\sim\frac{\omega\beta}{N}$$

So as $N\to\infty$

$$(*)\longrightarrow\left[\beta\omega\,\frac{e^{\frac{\beta\omega}{2}}-e^{-\frac{\beta\omega}{2}}}{\beta\omega}\right]^{-1}=\frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)}.$$

Return to field theory. To fix the ideas consider a field theory with 0 space dimensions, that is, an ordinary 1-dimensional quantum-mechanical problem described by a Hamiltonian

$$H = \frac{p^2}{2} + V(q)$$

Here the partition function is

$$Z(T) = tr\,(e^{-TH}) = \int_{q \text{ periodic of period } T} e^{-\int_0^T [\frac{1}{2}\dot{q}^2 + V(q)]\,dt}\,\mathcal{D}q$$

and the analogue of the infinite volume limit is

$$[redacted]\ \lim_{T\to\infty} -\frac{1}{T}\log Z(T) = \text{ground energy } E_0$$

As this problem stands there are no extra parameters to vary as one does in statistical mechanics.

By analogy with the Ising model, let's add a source term to H ~~to~~ get

$$H_J = \frac{p^2}{2} + V(q) - Jq$$

Then E_0 becomes a function of J. [redacted] J is analogous to [redacted] an external magnetic field, and

$$\frac{dE_0}{dJ}, \quad \frac{d^2E_0}{dJ^2}$$

is the analogue of the magnetization M, resp. susceptibility χ.

<u>Example</u>: $H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$

$$H_J = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} - Jq = \frac{p^2}{2} + \frac{1}{2}\omega^2\left(q^2 - 2\frac{J}{\omega^2}q + \frac{J^2}{\omega^4}\right) - \frac{J^2}{2\omega^2}$$

$$= \frac{p^2}{2} + \frac{1}{2}\omega^2\left(q - \frac{J}{\omega^2}\right)^2 - \frac{J^2}{2\omega^2}$$

so H_J is just an oscillator centered at $\frac{J}{\omega^2}$ with a constant energy shift. The ground energy is

$$E_0(J) = \frac{1}{2}\omega - \frac{J^2}{2\omega^2}$$

so
$$M = \frac{dE_0}{dJ} = -\frac{J}{\omega^2} \qquad \chi = -\frac{1}{\omega^2} .$$

As in the Ising model having a source with $J = J(t)$ is used to get hold of the correlation functions of the field at different times, i.e. the Green's functions.

Finally one studies a simple example like the anharmonic oscillator

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 + \frac{g}{4!} q^4 \qquad g > 0$$

In some sense this is like the Ising model because the q^4 tends to kill off large q, which is similar to requiring $q = \pm 1$.

December 3, 1979

Lesson of yesterday: View

$$tr\left(e^{-TH}\right) = \int \mathcal{D}q \; e^{-\int_{-T/2}^{T/2}\left(\frac{\dot{q}^2}{2}+V(q)\right)dt}$$

q has period T

as the analogue of a Ising partition function, and the $T \to \infty$ limit as the analogue of the infinite volume limit. In other words we replace the finite sum over spin configurations by a path integral which should be more computable. [redacted] The goal will be to understand the effect of different boundary conditions as $T \to \infty$.

The first thing to do is understand the situation where a source term is put in:

$$H = \frac{p^2}{2} + V(q) - J(t)\,q$$

analogous to the external field in the Ising case. First take $V(q) = \frac{1}{2}\omega^2 q^2$. Then you [redacted] can explicitly evaluate the path integral. The boundary conditions of interest are where $q(-T/2)$, $q(T/2)$ are fixed, and also the periodic cases.

[redacted] Let's proceed with a general H as above, and let's use paths with $q(-T/2) = q_{in}$ $q(T/2) = q_{out}$. Then

$$\langle q_{out} \,|\, U(T/2,-T/2)^J \,|\, q_{in} \rangle = \int \mathcal{D}q \; e^{-\int_{-T/2}^{T/2}\left(\frac{\dot{q}^2}{2}+V(q)\right)dt + \int Jq\,dt}$$

$$= \sum_{n\geq 0} \frac{1}{n!} \int_{-T/2}^{T/2} dt_1 \cdots dt_n \; J(t_1)\cdots J(t_n)\; G_T^{(n)}(t_1,\ldots,t_n)$$

where $G_T^{(n)}(t_1,\ldots,t_n) = \langle q_{out} \,|\, T\left[U(T/2,-T/2)\, q(t_1)\cdots q(t_n)\right] |\, q_{in} \rangle$ or better for $t_1 \geq \cdots \geq t_n$

$$G_T^{(n)}(t_1, \ldots, t_n) = \langle q_{out} | U(T/2, t_1)\, q\, U(t_1, t_2) \cdots\cdots q\, U(t_n, -T/2) | q_{in} \rangle$$

$$= \langle q_{out} | \underbrace{U(T/2)}_{e^{-T/2 H}}\; q(t_1) \cdots q(t_n)\; \underbrace{U(0, -T/2)}_{e^{-T/2 H} |m'\rangle\langle m'| q_{in}\rangle} | q_{in} \rangle$$

$$= \sum_{m, n} e^{-\frac{T}{2} E_m} \langle q_{out} | m \rangle \langle m | q(t_1) \cdots q(t_n) | m' \rangle e^{-\frac{T}{2} E_{m'}} \langle m' | q_{in} \rangle$$

$$\sim e^{-T(E_0)} \langle q_{out} | 0 \rangle \langle 0 | q(t_1) \cdots q(t_n) | 0 \rangle \langle 0 | q_{in} \rangle$$

Curious: $\langle 0 | T[q(t_1) \cdots q(t_n)] | 0 \rangle$ makes use of operators, but the path integral interpretation is that of an expectation value. We are using the operator interpretation to understand the path integral as $T \to \infty$. This has something to do with transfer matrices.

So for J of compact support at least, it is clear what the asymptotic behavior of the path integral is.

Look at the Ising model again. Then

$$\langle s_i \rangle = \frac{\partial}{\partial H_i} \log Z_N$$

converges as $N \to \infty$ although $\log Z_N$ doesn't. One has to divide by N to get $\frac{1}{N} \log Z_N$ to converge. In particular

$$\lim_{N \to \infty} \frac{1}{N} \log Z_N$$

should be independent of each H_i separately, however if we put all $H_i = H$, then

$$\frac{\partial}{\partial H} \log Z_N = \sum_i \langle s_i \rangle$$

so $\quad \frac{\partial}{\partial H}\left(\frac{1}{N}\log Z_N\right) \longrightarrow$ average magnetization $\langle s \rangle$.

This is curious: Z_N is a function of the H_i for the i in the region N (Better notation: Z_Ω) yet

$$\lim \frac{1}{|\Omega|}\log Z_\Omega$$

is independent of finitely many H_i. The question is whether it is a function of $\lim \frac{1}{|\Omega|}\sum_{i\in\Omega} H_i$.

Let's work this out for $H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2}$. Use periodic boundary conditions of period T.

$$Z_T(J) = \int e^{-\int\left(\frac{1}{2}\dot q^2 + \frac{1}{2}\omega^2 q^2\right)dt + \int Jq}\,\mathcal{D}q = N e^{\frac{1}{2}\int J \underset{T}{G} J}$$

where N is the path integral for $J=0$

$$N = \operatorname{tr}\left(e^{-TH}\right) = \frac{1}{2\sinh\frac{\omega T}{2}}$$

Now

$$\underset{T}{G}(t,t') = \sum_{k\in\frac{2\pi}{T}\mathbb{Z}} \frac{e^{ikt}\,e^{-ikt'}}{k^2+\omega^2}\,\frac{1}{T}$$

so

$$\int J \underset{T}{G} J = \sum_{k\in\frac{2\pi}{T}\mathbb{Z}} \frac{\hat J(k)\,\hat J(-k)}{k^2+\omega^2}\,\frac{1}{T} \qquad \hat J(k) = \int_{-T/2}^{T/2} J e^{ikt}\,dt$$

If J is constant then $\hat J(k) = JT\delta(k)$ and so

$$\int J \underset{T}{G} J = \frac{J^2}{\omega^2}\,T$$

which leads to a new ground energy of $\frac{\omega}{2} - \frac{J^2}{2\omega^2}$. It checks!

$$-\log Z_T(J) = \log\left(2\sinh\frac{\omega T}{2}\right) - \sum_{k\in\frac{2\pi}{T}\mathbb{Z}} \frac{|\hat{J}(k)|^2}{k^2+\omega^2}\frac{1}{T}$$

It's known that

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} J(t)\, e^{-ikt}\, dt$$

exists for an almost periodic function $J(t)$, and hence it is clear that the limit

$$\lim_{T\to\infty} \frac{1}{T}\log Z_T(J)$$

exists for J almost periodic. Does this mean that $P^2/2 + \frac{1}{2}\omega^2 q^2 - J(t)q$ has a meaningful ground energy for J almost periodic?

Generalize to several dimensions. Form

$$\int \mathcal{D}\phi\; e^{-\int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\omega^2\phi^2\right)dx \;+\; \int_\Omega J\phi\, dx}.$$

where Ω is a box in n-space, and $\mathcal{D}\phi$ has to be defined properly. This is a Gaussian integral so this integral is given by

$$N_\Omega\, e^{\frac{1}{2}\int_\Omega J G_\Omega J\, dx} \qquad N_\Omega = \text{normal. constant}$$

where G_Ω is the appropriate Green's fn. Just as in the case of 1-dimension, one has to specify the boundary behavior of ϕ. The simplest thing is to use periodic conditions. The eigenfunctions are $\frac{1}{\sqrt{V}}e^{ikx}$ where $k\in\left(\frac{2\pi}{L}\mathbb{Z}\right)^d$, $V = L^d$ and then

$$G(x,x') = \sum \frac{e^{ik(x-x')}}{k^2+\omega^2}\frac{1}{V}$$

Is there a meaninful way to define N_Ω?

The quadratic form has the eigenvalues $k^2+\omega^2$ where $k \in \left(\frac{2\pi}{L}\mathbb{Z}\right)^d$, ~~except that because ϕ is real, one should~~ so the normalization constant is some way of making sense out of

$$\prod (k^2+\omega^2)^{-1/2}$$

which is independent of ω. In dimension 1 you write this as

$$\omega^{-1} \prod_{k>0} (k^2+\omega^2)^{-1} = \Big[\omega \underbrace{\prod_{k>0}\left(1+\frac{\omega^2}{k^2}\right)}_{\prod_{n\geq 1}\left(1+\left(\frac{\omega L}{2\pi n}\right)^2\right) = \left(\sinh\frac{\omega L}{2}\right)\Big/\frac{\omega L}{2}} \cdot \prod_{k>0} k^2\Big]^{-1}$$

December 5, 1979

The problem now is to understand a free field theory. In this case [redacted] a configuration is given by a function $\phi(x)$ on R^d and the energy function is

$$E(\phi) = \int \left(\tfrac{1}{2}|\nabla\phi|^2 + \tfrac{1}{2}\mu^2\phi^2\right) dx$$

The idea is to do this integral over a box Ω of volume [redacted] $V = L^d$ for ϕ defined on Ω with periodic boundary conditions. Then you wish to average [redacted] to get the partition fn:

$$Z_\Omega(J) = \int \mathcal{D}\phi \; e^{-\int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\mu^2\phi^2\right) dx + \int_\Omega J\phi\, dx}$$

Already in 2 dimensions we have a problem with $\mathcal{D}\phi$. To see, let's use the Fourier transform to diagonal the energy function:

$$\phi(k) = \int_\Omega \phi(x)\, e^{-ikx}\, dx \xrightarrow{\text{as } \Omega\to\infty} \int \phi(x) e^{-ikx} dx$$

$$\phi(x) = \frac{1}{V}\sum_{k\in\left(\frac{2\pi}{L}\mathbb{Z}\right)^d} \phi(k) e^{ikx} \longrightarrow \int \phi(k) e^{ikx} \frac{dk}{(2\pi)^d}$$

Then

$$\int_\Omega \left(\tfrac{1}{2}|\nabla\phi|^2 + \tfrac{1}{2}\mu^2\phi^2\right) dx = \frac{1}{V}\sum_k \left(k^2+\mu^2\right)|\phi(k)|^2$$

$$\int_\Omega J\phi\, dx = \frac{1}{V}\sum_k J(-k)\phi(k)$$

Do the Gaussian integral to get $\quad Z_\Omega(J) = \text{Const}\; e^{\frac{1}{2}\int J G_\Omega J}$

where
$$G_\Omega(x,x') = \sum{}' \frac{e^{ikx}e^{-ikx'}}{V(k^2+\omega^2)}$$

so
$$\int J G_\Omega J = \frac{1}{V}\sum_k \frac{|J(k)|^2}{k^2+\omega^2} \qquad J \text{ real}.$$

Now the normalization constant in the Gaussian integral is something like 1/SQRT of the determinant of the energy form. Notice that the natural orthonormal basis for periodic functions is $e^{ikx}/\sqrt{V}$, so that in orthonormal coords

$$E(\phi) = \sum_k{}' (k^2+\mu^2)\left|\frac{\phi(k)}{\sqrt{V}}\right|^2$$

and hence we should have

$$(*) \qquad \det E = \prod_{k\in\left(\frac{2\pi}{L}\mathbb{Z}\right)^d} (k^2+\mu^2)$$

Notice that this is a finite volume problem that has to be settled before we worry about taking the $\Omega\to\infty$ limit. In one dimension we know from Feynman's formula that there is a sensible way to define $\mathcal{D}\phi$, or rather

$$\mathcal{D}\phi\; e^{-\int \frac{1}{2}\dot\phi^2\,dt}.$$

It gives Wiener measure on paths. Another way to see that there is reasonable way to do things in dimension 1, is to notice that the expression $(*)$ for $\det E$ can be regularized in a fashion independent of μ by dividing by $\prod_k k^2$. One gets

$$\prod_{k} \left(1+\frac{\mu^2}{k^2}\right)$$

which converges in dim. $\underline{1}$, but diverges in higher dims.

Compare with

$$\frac{1}{V}\sum{}' \frac{1}{k^2} \sim \int \frac{1}{k^2}\frac{dk}{(2\pi)^d} = \frac{\mathrm{vol}(S^{d-1})}{(2\pi)^d}\int^{\infty} \frac{1}{r^2} r^{d-1} dr.$$

so you logarithmic divergence for $d=2$ and worse for $d>2$.

This is what they call ultra-violet divergence because it has to do with large k. One consequence is that there is an infinity depending on the choice of μ.

December 6, 1979

Victor's lectures on Szego's thm. (Widom's proof.)

Szego thm: Let H be the Hardy space in $L^2(\frac{d\theta}{2\pi}, S^1)$ and $\pi : L^2(S^1) \longrightarrow H$ the orthogonal projection. If f is a continuous function on S^1 the corresponding Toeplitz operator T_f is defined to be $T_f = \pi M_f \pi$, where M_f = multiplication by f.

$$\langle z^{n'} | T_f | z^n \rangle = \int f z^{n-n'} \frac{d\theta}{2\pi}$$

so the matrix of T_f is

$$\begin{pmatrix} c_0 & c_{+1} & c_2 & \\ c_{-1} & c_0 & c_{+1} & \ddots \\ \ddots & c_{-1} & \ddots & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \qquad c_n = \int f z^{-n} \frac{d\theta}{2\pi}$$

Let π_n = orthogonal projection onto $\{1, z, \dots, z^{n-1}\}$. Then Szego's thm says

$$\frac{1}{n} \sum_{\lambda \in \operatorname{spec} \pi_n T_g \pi_n} \delta(x-\lambda) \xrightarrow{\text{weak}} g_* \left(\frac{d\theta}{2\pi}\right) \qquad g \in C^0(S^1).$$

In other words for f continuous on $g(S^1)$ one has

$$\frac{1}{n} \operatorname{trace} f(\pi_n (T_g) \pi_n) \longrightarrow \int f(g(z)) \frac{d\theta}{2\pi}$$

By Weierstrass it suffices to prove this for $f = x^k$, $k \in \mathbb{N}$. Notice that

$$\frac{1}{n} \operatorname{trace} \pi_n (T_g) \pi_n = \frac{1}{n} \cdot n c_0 = c_0 = \int g \frac{d\theta}{2\pi}$$

so the formula is trivially true for $f(x) = x$. Hence the main point is to see when

$$\frac{1}{n}\,\mathrm{tr}\left[\pi_n A^k \pi_n - (\pi_n A \pi_n)^k\right] \to 0.$$

The useful corollary of Szegö's thm. is as follows:

<u>Cor</u>: Suppose $g > 0$ everywhere. Then

$$\frac{1}{n}\log\det\left(\pi_n T_g \pi_n\right) \longrightarrow \int \log(g)\,\frac{d\theta}{2\pi}$$

This follows by taking the continuous function on $\mathrm{Im}\,g$ to be $\log x$.

Victor's generalization of the Szego thm. is to take a positive elliptic pseudo-differential operator P on a compact manifold X and let π_λ be the projection onto where $P \le \lambda$. Then

$$\frac{1}{\dim \pi_\lambda} \sum_{\mu \in \mathrm{Spec}(\pi_\lambda M_g \pi_\lambda)} \delta(x-\mu) \longrightarrow g_* \qquad ?$$