November 4, 1979

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Program: To understand the quantum mechanics of $H = \frac{p^2}{2} + V(q)$ where V looks like  say $V = \frac{1}{2}\omega^2 x^2 + g x^4$. The first problem is to compute the ground state energy. This problem is accessible by several methods I have encountered - better I have seen several ways of treating this problems.

1) The most elementary is to write

$$H = \underbrace{\frac{p^2}{2} + \frac{1}{2}\omega^2 q^2}_{H_0} + \underbrace{g q^4}_{H'}$$

and to use power series

$$\psi = \varphi_0 + g\psi_1 + g^2\psi_2 + \cdots$$
$$E = \tfrac{1}{2}\omega + g E_{(1)} + \cdots$$

(so called Rayleigh-Schrödinger series)

and grind out the coefficients requiring that $\psi_1, \psi_2, \ldots$ be orthogonal to φ_0. For example one gets

$$E^{(1)} = \langle \varphi_0 | H' | \varphi_0 \rangle \qquad \text{if} \quad \langle \varphi_0 | \varphi_0 \rangle = 1$$

2.) Kato's method: The projection operator on the ground state is given by

$$P_0 = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda - H_0}$$

where the contour circles just the ground eigenvalue. We're assuming the ground state evolves nicely under the perturbation, so that at least for small g the analogous formula for P holds, but with the same contour. Then if we use the series for the resolvent:

$$\frac{1}{\lambda - H} = \frac{1}{\lambda - H_0} + \frac{1}{\lambda - H_0} H' \frac{1}{\lambda - H_0} + \cdots$$

we get

$$P = P_0 + \frac{1}{2\pi i} \oint \frac{1}{\lambda - H_0} H' \frac{1}{\lambda - H_0} \, d\lambda + \cdots\cdots .$$

A similar sort of series can be found for the ground energy $\quad E = tr\,(HP).$

3) Gell-Mann-Low thm. gives the new ground state by adiabatic perturbation: Let $\psi_\varepsilon(t)$ = soln of

$$i\,\partial_t \psi_\varepsilon(t) = (H_0 + e^{\varepsilon t} V)\, \psi_\varepsilon(t)$$

which is asymptotic to $e^{-iH_0 t}\varphi_0$ as $t \to -\infty$. Then

$$\psi = \lim_{\varepsilon \to 0} \frac{\psi_\varepsilon(0)}{\langle \varphi_0 | \psi_\varepsilon(0) \rangle}$$

is the ground state for H. There's another formula for E, and using it one can derive Goldstone's thm. asserting that E is given by a sum of connected Feynman diagrams. (Fetter-Walecka book).

4) If we put $H_\lambda = H_0 + \lambda H'$ and denote by Ψ_λ the ground state of H_λ normalized so that $\|\Psi_\lambda\| = 1$, then one has for $E_\lambda = \langle \Psi_\lambda | H_\lambda | \Psi_\lambda \rangle$, the formula

$$\frac{dE_\lambda}{d\lambda} = \langle \Psi_\lambda | H' | \Psi_\lambda \rangle$$

by first order perturbation theory. Hence integrating

$$E = \int_0^1 \langle \Psi_\lambda | H' | \Psi_\lambda \rangle \, d\lambda$$

5) Functional integral approach. Although not essential for what follows, let's start off with the thermal partition function:

$$Z(\beta) = \operatorname{tr}(e^{-\beta H})$$

which can be expressed as a path integral

$$Z(\beta) = \int_{x(0)=x(\beta)} dx(t)\, e^{-\int [\frac{1}{2}\dot{x}^2 + V(x)]dt}$$

Now split $V(x)$ into $\frac{1}{2}\omega^2 x^2 + W(x)$ and we get

$$Z(\beta) = \int dx(t)\, e^{-\int(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2)dx}\, e^{-\int W(x)dt}$$

Introduce

$$Z_0(\beta, J) = \int dx(t)\, e^{-\int_0^\beta [\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2]dt + i\int_0^\beta J(t)x(t)\,dt}$$

Using the formula

$$f\left(\frac{1}{i}\frac{d}{dJ}\right) \int e^{iJx}\, d\mu(x) = \int f(x)\, e^{iJx}\, d\mu(x)$$

we see that

$$Z(\beta, J) = e^{-\int W(\frac{1}{i}\frac{\partial}{\partial J(t)})dt}\, Z_0(\beta, J)$$

Now $Z_0(\beta, J)$ is the Fourier transform of a Gaussian measure so we have

$$Z_0(\beta, J) = e^{-\frac{1}{2}\iint J(t)\, G(t,t')\, J(t')\, dt\,dt'}\, Z_0(\beta)$$

where G is the inverse of $-\partial_t^2 + \omega^2$ on periodic functions on $[0,\beta]$. Also we know $Z_0(\beta) = \operatorname{tr}(e^{-\beta H_0})$

$$Z_0(\beta) = \sum e^{-\beta(n+\frac{1}{2})\omega} = \frac{1}{2\sinh(\frac{\beta\omega}{2})}$$

but maybe this isn't very important.

(Is the following relevant:

$$(+) \qquad Z_0(\beta) = \int\limits_{x(0)=x(\beta)} \mathcal{D}x(t)\, e^{-\int\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2\right)dt} = \left(\frac{\det(-\partial_t^2+\omega^2)}{\det(-\partial_t^2)}\right)^{-1/2}$$

NO

where the ratio of determinants is to be evaluated on periodic functions of period β. One has the basis $e^{in\frac{2\pi}{\beta}x}$ of eigenfunctions [redacted] so

$$\frac{\det(-\partial_t^2+\omega^2)}{\det(-\partial_t^2)} = \prod_{n\in\mathbb{Z}} \frac{\left(n\frac{2\pi}{\beta}\right)^2+\omega^2}{\left(n\frac{2\pi}{\beta}\right)^2} = \frac{\left(0\frac{2\pi}{\beta}\right)^2+\omega^2}{\left(0\frac{2\pi}{\beta}\right)^2}\left(\frac{\sinh\left(\frac{\beta\omega}{2}\right)}{\frac{\beta\omega}{2}}\right)^2$$

but there is some trouble at $n=0$. ? [redacted] No missing from (+) is $\int \mathcal{D}x(t)\, e^{-\int\frac{1}{2}\dot{x}^2} = tr\left(e^{-\beta\frac{p^2}{2}}\right)$ which is an infinity cancelling the $n=0$.)

Returning to the derivation at hand we have

$$\frac{Z(\beta)}{Z_0(\beta)} = e^{-\int W\left(\frac{1}{i}\frac{\partial}{\partial J(t)}\right)dt} \; e^{-\frac{1}{2}\int JGJ} \Bigg|_{J=0}$$

where in this formula $W(x) = gx^4$ is the interaction and where G is the periodic Green's fn. for $-\partial_t^2+\omega^2$ of period β. Now the right side has a diagram interpretation, and the logarithm of the right side has a connected diagram interpretation. [redacted] Now recall that

$$Z(\beta) = e^{-\beta F}$$

where F is the "free energy", hence the shift in free energy has a connected diagram interpretation. Finally letting $\beta \longrightarrow \infty$

$$\frac{Z(\beta)}{Z_0(\beta)} \approx \frac{e^{-\beta E}}{e^{-\beta E_0}}$$

so that we get a connected diagram interpretation of the ground state energy shift. This is Goldstone's thm.

November 5, 1979

This time instead of the partition fn. $\mathrm{tr}(e^{-\beta H})$ let us work with the amplitude

$$\langle x| e^{-TH} |x'\rangle = \int_{\substack{x(-\frac{T}{2})=x' \\ x(\frac{T}{2})=x}} \mathcal{D}x(t)\, e^{-\int(\frac{1}{2}\dot{x}^2 + V(x))dt}$$

Again we expand

$$V(x) = \frac{1}{2}\omega^2 x^2 + W(x)$$

and then we have

$$\langle x| e^{-TH} |x'\rangle = e^{-\int W(\frac{1}{i}\frac{\partial}{\partial J(t)})dt} \int \mathcal{D}x(t)\, e^{-\int \frac{1}{2}(\dot{x}^2+\omega^2x^2)dt + i\int Jx\,dt}$$

set $J=0$

To simplify suppose $x=x'=0$, so that the latter integral is a Gaussian which we can evaluate as

$$e^{-\frac{1}{2}\int JGJ} \underbrace{\langle x=0| e^{-TH_0} |x'=0\rangle}_{\sqrt{\frac{\omega}{2\pi\sinh(\omega T)}}}$$

where G is the Green's function for $-\partial_t^2 + \omega^2$ on $[-\frac{T}{2}, \frac{T}{2}]$ with 0 endpoint conditions.

Now the idea is to let $T \to \infty$ in which case

$$G \sim \frac{e^{-\omega|t-t'|}}{+2\omega}$$

and

$$\sqrt{\frac{\omega}{2\pi\sinh(\omega T)}} \approx \sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega}{2}T}$$

To see what's happening we should take an example say $\quad W = \frac{1}{2} g x^2$. Then we get something like

$$e^{-\int W\left(\frac{1}{i}\frac{\partial}{\partial J(t)}\right) dt} \; e^{-\frac{1}{2}\int JGJ}\Big|_{J=0} = \sum_{\text{diagrams}}$$

Return to computing $\mathrm{tr}(e^{-\beta H})$, only let's work out the diagrams in energy coordinates, rather than time. So the idea here is to describe periodic functions $x(t)$ of period β as Fourier series

$$x(t) = \sum a_k e^{ikt} \qquad k \in \frac{2\pi}{\beta}\mathbb{Z}.$$

$$\overline{a_k} = a_{-k}.$$

$$\int_0^\beta \left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2\right) dt = \frac{1}{2}\int_0^\beta x(-\partial_t^2 + \omega^2)\, x \, dt$$

$$= \beta \frac{1}{2} \sum_k (k^2 + \omega^2) |a_k|^2$$

$$\int_0^\beta x^4 \, dt = \beta \sum_{k_1, \ldots, k_4} a_{k_1} a_{k_2} a_{k_3} a_{k_4} \, \delta(k_1 + \cdots + k_4)$$

so our path integral is

$$\int \underbrace{[da_0 \textstyle\prod da_k \, da_{-k}]}_{[da]} \; e^{-\beta\left(\frac{1}{2}\sum (k^2+\omega^2)|a_k|^2 + g\sum a_{k_1} a_{k_2} a_{k_3} a_{k_4} \, \delta(k_1 + \cdots + k_4)\right)}$$

As before we reduce this to a Gaussian integral

$$\int [da] \; e^{-\beta\left(\frac{1}{2}\sum_k (k^2+\omega^2)|a_k|^2\right) + i \sum_k J_k a_k}$$

being acted on by $\quad e^{-\beta g \sum \left(\frac{1}{i}\frac{\partial}{\partial J_{k_1}}\right) \cdots \left(\frac{1}{i}\frac{\partial}{\partial J_{k_4}}\right) \delta(k_1 + \cdots + k_4)}$

November 7, 1979

The problem is to understand the connected generating functional. This arises as follows. Suppose we have a Hamiltonian $H = \frac{p^2}{2} + V(q)$ and we want to compute the amplitude

$$\langle x | e^{-TH} | x' \rangle = \int_{\substack{x(0)=x' \\ x(T)=x}} \mathcal{D}x(t) \, e^{-\int_0^T (\frac{1}{2}\dot{x}^2 + V(x))dt}$$

We do a stationary phase approximation on the path integral. To simplify suppose that $x = x' = 0$ and that V has an absolute minimum at $x=0$, say

$$V(x) = \tfrac{1}{2}\omega^2 x^2 + \underbrace{\tilde{V}(x)}_{O(x^3)}$$

Then

$$\langle 0 | e^{-TH} | 0 \rangle = \int_{x(0)=x(T)=0} \mathcal{D}x(t) \, e^{-\int (\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2) dt} \, e^{-\int \tilde{V}(x) dt} .$$

The last exponential factor is expanded out in a series; what one gets is a series of integrals of the form

$$\int x^\alpha \, d\mu \qquad d\mu \text{ is a Gaussian measure}$$

which can be evaluated by Wick's thm. This leads to a big series for $\langle 0 | e^{-TH} | 0 \rangle$ whose terms are described by Feynman diagrams. Moreover

$$\log \frac{\langle 0 | e^{-TH} | 0 \rangle}{\langle 0 | e^{-TH_0} | 0 \rangle} = e^W$$

where W is a sum of connected diagrams.

Let's consider a simplified situation in which the space of paths is replaced by R, and where the path integral is

$$\int e^{-f(x)}\, dx$$

where $f(x)$ has an absolute minimum at $x=0$, and say

$$f(x) = \frac{1}{2}\omega^2 x^2 + \frac{g_3}{3!}x^3 + \frac{g_4}{4!}x^4 + \cdots\cdots$$

Then

$$\int e^{-f(x)}\, dx = \int e^{-\frac{1}{2}\omega^2 x^2} dx \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{g_3}{3!}x^3 + \frac{g_4}{4!}x^4 + \cdots\right)^n$$

Let's take a typical term when the exponential series is expanded out, for example the term involving $g_3 g_5$ which is

$$\frac{(-1)^2}{2!}\, 2\frac{g_3 g_5}{3!\,5!} \int e^{-\frac{1}{2}\omega^2 x^2} x^3 x^5\, dx$$

According to Wick's theorem the integral is the sum over all possible pairwise contractions. We can represent these ways of contracting by drawing edges

The total number of contractions possibilities is $\frac{8!}{2^4 4!} = 1\cdot3\cdot5\cdot7$. Somehow one has to count the symmetries of the graph. The goal is collect together the different contraction possibilities leading to the same graph. In the above example we can permute the lines issuing from the vertices. This gives us a group of order $3!\,5!$ acting on the contraction possibilities. This $3!\,5!$ cancels the denominator, so the net effect is that each graph is counted with a factor of $1/$the order of its symmetry group.

So the general rule is as follows. Suppose we want the coefficient of $g_3^a g_4^b \cdots g_n^c$. Then we consider all graphs with a-vertices of mult. 3, b-vertices of mult. 4, etc. We let $\Sigma_a \times \Sigma_b \times \cdots$ act as well as Σ_3 on each 3-fold vertex, Σ_4 on each 4-fold vertex, etc. and identify equivalent graphs. Finally we have for each inequivalent graph a term

$$\frac{(-1)^n}{|\mathrm{Aut}(\text{graph})|}\Big(\underbrace{\int e^{-\frac{1}{2}\omega^2 x^2} x^2\,dx}_{\sqrt{\frac{\pi}{2}}\,\frac{1}{\omega^3}}\Big)^n \qquad \text{not quite correct.}$$

Correction:
$$\int e^{-\frac{1}{2}\omega^2 x^2} x^{2m}\,dx = \left(\frac{1}{i}\frac{d}{dJ}\right)^{2m}\int e^{-\frac{1}{2}\omega^2x^2 + iJx}\,dx\Big|_{J=0}$$

$$= (-1)^m\left(\frac{d}{dJ}\right)^{2m}\left(e^{-\frac{1}{2}\frac{J^2}{\omega^2}}\,\frac{\sqrt{2\pi}}{\omega}\right)\Big|_{J=0}$$

$$= (-1)^m\left(\frac{d}{dJ}\right)^{2m}\frac{\sqrt{2\pi}}{\omega}\sum \frac{\left(-\frac{1}{2}\right)^m}{m!}\frac{J^{2m}}{\omega^{2m}}\Big|_{J=0} = \frac{\sqrt{2\pi}}{\omega^{2m+1}}\,\frac{(2m)!}{2^m\, m!}$$

$$= \frac{\sqrt{2\pi}}{\omega}\,\frac{1}{\omega^{2m}}\,\underbrace{(2m-1)(2m-3)\cdots 3\cdot 1}_{\text{number of possible contractions in } x^{2m}.}$$

Hence the contribution of a given graph is

$$\prod_{\text{vertices}} g_i\,(-1)^n\,\frac{1}{|\mathrm{Aut}(\text{graph})|}\;\frac{1}{\omega^{3m}}\;\frac{\sqrt{2\pi}}{\omega}$$

$\frac{1}{\omega^{3m}}$: product over the edges

$\frac{\sqrt{2\pi}}{\omega}$: normalization factor for the Gaussian integral which is 1 if $\int d\mu = 1$.

n = number of vertices

m = number of edges

November 9, 1979

We are trying to understand the diagram expansion for an integral of the form

$$\int d\mu \; e^{+\sum \frac{g_\alpha}{\alpha!} x^\alpha}$$

where $d\mu$ is a Gaussian measure on the space with coordinates x_i. Let's first work out the contractions

$$\int e^{-\frac{1}{2}(x,Ax)} e^{iJx} dx = (2\pi)^{n/2} (\det A)^{-1/2} e^{-\frac{1}{2}(J,A^{-1}J)}$$

$$\int \underbrace{e^{-\frac{1}{2}(x,Ax)} \frac{dx}{(2\pi)^{n/2}} (\det A)^{1/2}}_{d\mu} e^{Jx} = e^{\frac{1}{2}(J,A^{-1}J)}$$

Hence $$\int d\mu \; x_i x_j = \frac{\partial^2}{\partial J_i \partial J_j} e^{\frac{1}{2}(J,A^{-1}J)} \Big|_{J=0}$$

$$= (A^{-1})_{ij}$$

Wick's thm says that if $I = (i_1, \ldots, i_n)$ and $x_I = x_{i_1} \cdots x_{i_n}$ then

$\int d\mu \; x_I$ = sum over all possible pairwise contractions of I of the products of the contracted factors. The number of these is $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

Yesterday I found is advantageous to write

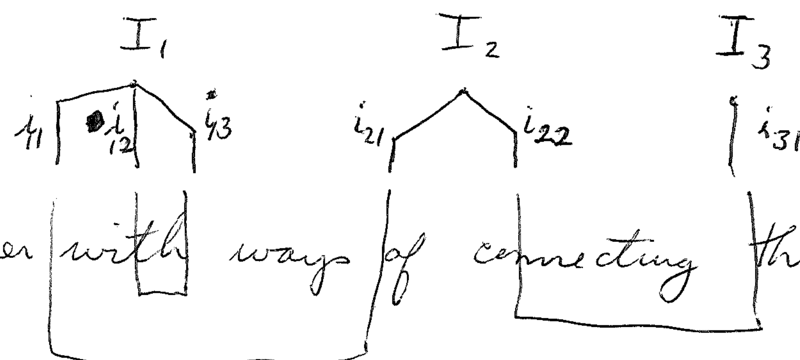
$$\sum_\alpha \frac{g_\alpha}{\alpha!} x^\alpha = \sum_I \frac{g_I}{|I|!} x_I$$

where I runs over all finite sequences $i_1, \ldots, i_n$ in the variables. [redacted] Then the exponential to be integrated is

$$e^{\sum \frac{g_I}{|I|!} x_I} = \sum_n \frac{1}{n!} \left(\sum \frac{g_I}{|I|!} x_I \right)^n$$

$$= \sum_n \frac{1}{n!} \sum_{I_1, I_2, \cdots I_n} \frac{g_{I_1}}{|I_1|!} \cdots \frac{g_{I_n}}{|I_n|!} x_{I_1} x_{I_2} \cdots x_{I_n}$$

This means that the integral $\int d\mu\, e^{\sum \frac{g_\alpha}{\alpha!} x^\alpha}$ is a sum of terms each described by a finite sequence $(I_1, \cdots, I_n)$ of finite sequences of variables together with all ways of pairwise contracting all the variables. Such a term can be drawn as a diagram



together with ways of connecting this up by wires

Such a diagram receives a weight of $\frac{1}{n!} \frac{1}{|I_1|!} \cdots \frac{1}{|I_n|!}$.
But now to take the sum over all such labelled diagrams is very inefficient, because in computing the g-factors and the contraction factors one doesn't care about the ordering of $I_1, I_2, \cdots, I_n$, nor the ordering of the variables inside each I_j. Consequently we group together equivalent diagrams. Two graphs are equivalent when there is a one-one correspondence between vertices and between edges which is compatible with the variables attached to the ends of each edge. The problem is to count the number of labelled graphs in each equivalence class.

, . . Start again. A point you're missing is that one draws a graph first then adds all terms resulting from labelling the edges issuing from a vertex. This amounts to selecting n, the number of vertices, and $p_1, p_2, \ldots, p_n$, the multiplicity of each vertex, and then one of the $(2n-1)!!$ ways of contracting. Once this data is given you then sum over all possible ways of putting a variable at the end of each edge. So now each of these sums is weighted by $\frac{1}{n!}\frac{1}{p_1!}\cdots\frac{1}{p_n!}$.

The problem is to decide when two sums are the same. For example

obviously contribute the same. The thing to do is look at isomorphic graphs and to figure out how many times a graph type contributes, and use this to sum once over each graph type with a modified symmetry factor. So we want to look at "contraction diagrams" leading to the same graph as above. One has an obvious group of symmetries: one can permute the lines issuing from a vertex, and one can permute vertices of the same multiplicity. So arrange vertices in order of increasing multiplicities $p_1 \le p_2 \le \cdots \le p_n$. This means we write

$$e^{\sum_I \frac{g_I}{|I|!} x_I} = e^{\sum_p \frac{1}{p!} \sum_{|I|=p} g_I x_I}$$

$$= \sum_{k_1, k_2, \ldots} \frac{1}{k_1!}\frac{1}{k_2!}\cdots \left(\frac{1}{1!}\sum_{|I|=1} g_I x_I\right)^{k_1} \left(\frac{1}{2!}\sum_{|I|=2} g_I x_I\right)^{k_2} \cdots$$

Once we agree that graphs have the multiplicity of their vertices fixed in order, then we have the wreath product $\Sigma_{k_p} \wr \Sigma_p$ acting on the vertices of multiplicity p. In the expansion

$$e^{\sum_I \frac{g_I}{|I|!} x_I} = \sum_{k_1, k_2, \cdots} \frac{1}{k_1!(1!)^{k_1}} \frac{1}{k_2!(2!)^{k_2}} \cdots \Big(\sum_{|I|=1} g_I x_I\Big)^{k_1} \Big(\sum_{|I|=2} g_I x_I\Big)^{k_2} \cdots$$

Each way of contracting is weighted by dividing by the order of the symmetry group

$$(\Sigma_{k_1} \wr \Sigma_1) \times (\Sigma_{k_2} \wr \Sigma_2) \times \cdots$$

associated to vertex multiplicities. So it follows we can sum over inequivalent graphs provided we divide by the order of the symmetry group of the graph.

Example: Take 1 variable and compute

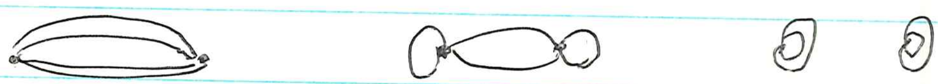
$$\int d\mu \, e^{+g\frac{x^4}{4!}} \qquad \text{where} \quad d\mu = e^{-\frac{1}{2}\omega^2 x^2} \frac{\omega\, dx}{\sqrt{2\pi}}$$

$$= \int d\mu \sum \frac{g^n}{n!} \left(\frac{x^4}{4!}\right)^n = \sum_n \frac{g^n}{n!} (4n-1)!! \left(\frac{1}{\omega^2}\right)^{2n} \frac{1}{(4!)^n}$$

first order graphs: There's only one  which has $2 \cdot 2 \cdot 2 = 2^3$ symmetries.

$$\frac{g}{8}\left(\frac{1}{\omega^2}\right)^2 \qquad\qquad \frac{g}{1!}\, 1 \cdot 3 \left(\frac{1}{\omega^2}\right)^2 \frac{1}{4!} = \frac{g}{2 \cdot 4} \frac{1}{\omega^4} \checkmark$$

2nd order graphs



$$2 \cdot 4! = 2^4 3 \qquad\qquad 2^4 \qquad\qquad 2 \cdot (2^3)^2 = 2^7$$

$$g^2 \frac{1}{\omega^8}\left(\frac{1}{2^4 3} + \frac{1}{2^4} + \frac{1}{2^7}\right) = \frac{g^2}{\omega^8}\frac{1}{2^7 3}\left(\underbrace{\frac{2^3 + 2^3 3 + 3}{8 + 24 + 3}}_{35}\right) = \frac{g^2}{\omega^8}\frac{5\cdot 7}{2^7 3}$$

other formulas $\qquad \dfrac{g^2}{2!}\, 1\cdot 3\cdot 5\cdot 7 \,\dfrac{1}{\omega^8}\,\dfrac{1}{\underbrace{(4!)^2}_{2^2\cdot 3^2\cdot 2^4}} = \dfrac{g^2}{\omega^8}\,\dfrac{5\cdot 7}{2^7 3}$

so it checks.

Important: these graphs as well as their symmetry factors are completely independent of the number of variables. The variables enter when you label the [redacted] ends of the edges and compute the contraction factors.

Next consider the example of finding the ground state for

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 + g\frac{q^4}{4!}.$$

We want to compute the behavior of

$$tr\left(e^{-TH}\right) = \int \mathcal{D}x(t)\, e^{-\frac{1}{2}\int(\dot{x}^2 + \omega^2 x^2)dt \; - \int g\frac{x^4}{4!}dt}$$

as $T \to \infty$ where the path integral is taken over periodic paths $x(t)$ of period T. Expand in Fourier series

$$x(t) = \sum a_k e^{-ikt} \qquad k \in \frac{2\pi}{T}\mathbb{Z}$$

The fact x is real-valued signifies that $\bar{a}_k = a_{-k}$. Now one has

$$\int\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2\right)dt = \sum_k \frac{1}{2}(k^2+\omega^2)|a_k|^2 \cdot T$$

$$\int \frac{x(t)^4}{4!}dt = \sum_{k_1,k_2,k_3,k_4} \frac{1}{4!} a_{k_1}a_{k_2}a_{k_3}a_{k_4}\;\delta(k_1+\dots+k_4)\,T$$

When we expand $e^{-g\int\frac{x^4}{4!}}$ in powers of g we get a sum of terms involving integrals

$$\int d\mu \quad a_{k_1}\cdots\cdots a_{k_n}$$

where $d\mu$ is an appropriate measure, essentially

$$e^{-\frac{1}{2}\sum_k' (k^2+\omega^2)|a_k|^2} \quad \frac{da_0}{\sqrt{2\pi}} \prod_{k=1}^{\infty} \frac{da_k\, d\bar{a}_k}{2\pi} .$$

Let's use the generating function

$$\int e^{-\frac{1}{2}\sum_k' (k^2+\omega^2)|a_k|^2 + \sum J_k a_k} \quad \frac{da_0}{\sqrt{2\pi}} \prod_{k=1}^{\infty} \frac{da_k\, d\bar{a}_k}{2\pi}$$

where the J_k are independent variables. I need

$$\int e^{-\alpha^2|z|^2 + Jz + \tilde{J}\bar{z}} \quad \alpha^2\frac{dz\, d\bar{z}}{2\pi}$$

$$= \int e^{-\alpha^2(x^2+y^2) + (J+\tilde{J})x + i(J-\tilde{J})y} \quad \alpha^2\frac{2dx\, dy}{2\pi}$$

$$= \quad e^{\frac{1}{2}\frac{1}{2\alpha^2}\underbrace{\{(J+\tilde{J})^2 - (J-\tilde{J})^2\}}_{2J\tilde{J}\ +2J\tilde{J}}}$$

$$= e^{\frac{1}{\alpha^2}J\tilde{J}}$$

Hence the generating function is

$$N\, e^{\frac{1}{2}\frac{1}{\omega^2}J_0^2 + \frac{1}{2}\sum\frac{1}{k^2+\omega^2}J_k J_{-k}}$$

Review: I want to compute the first few terms of the perturbation expansion of

$$\int \mathcal{D}x \; e^{-\int\left(\frac{1}{2}\dot{x}^2+\frac{1}{2}\omega^2x^2\right)dt \; - \int g\frac{x^4}{4!}dt}$$

where the integral is taken over periodic paths of period T. We describe such paths by Fourier series

$$x(t) = \sum a_k e^{ikt} \qquad k \in \frac{2\pi}{T}\mathbb{Z}$$

where x real means $\overline{a_k} = a_{-k}$. We use this formula to change variables in the path integral. One has

$$\int\left(\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}\omega^2x^2\right)dt = \frac{1}{2}\sum_k (k^2+\omega^2)|a_k|^2\, T$$

hence up to a normalization factor N

$$\mathcal{D}x \; e^{-\int \frac{1}{2}\dot{x}^2+\frac{1}{2}\omega^2x^2} \longmapsto N e^{-\frac{T}{2}\sum (k^2+\omega^2)|a_k|^2} da_0 \prod_{k>0} da_k d\bar{a}_k .$$

We need

$$\int e^{-\alpha^2|z|^2 + Jz + \tilde{J}\bar{z}}\, dz\, d\bar{z} = \int e^{-\frac{1}{2}2\alpha^2(x^2+y^2) + (J+\tilde{J})x + i(J-\tilde{J})y}\, 2\,dx\,dy$$

$$= \frac{(\sqrt{2\pi})^2}{(\sqrt{2\alpha^2})^2}\, e^{\frac{1}{2}\frac{1}{2\alpha^2}\left((J+\tilde{J})^2-(J-\tilde{J})^2\right)} = \frac{\pi}{\alpha^2}\, e^{\frac{1}{\alpha^2}J\tilde{J}}$$

It follows that

$$\int da_0 \prod_{k>0} da_k\, d\bar{a}_k \; e^{-\frac{T}{2}\sum (k^2+\omega^2)|a_k|^2 + \sum J_k a_k}$$

$$= N \cdot e^{\frac{1}{2}\sum \frac{1}{(k^2+\omega^2)T} J_k J_{-k}}$$

Compute ground energy for the anharmonic oscillator

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 + \frac{g}{4!} q^4$$

through the second order in g. Recall for $H = H_0 + V$ with $H_0 \varphi_n = E_n \varphi_n$, φ_n an orthonormal basis, then the ground energy for H is

$$E = \underbrace{\lambda_0}_{E_0 = \frac{1}{2}\omega} + \lambda_1 + \lambda_2 + \cdots$$

$$\lambda_1 = \langle \varphi_0 | V | \varphi_0 \rangle$$

$$\lambda_2 = \sum_{n \neq 0} \frac{|\langle \varphi_n | V | \varphi_0 \rangle|^2}{E_0 - E_n} \qquad E_n = \left(n + \tfrac{1}{2}\right)\omega$$

In the case $V = \frac{g}{4!} q^4$ we have

$$\lambda_1 = \frac{g}{4!} \langle 0 | q^4 | 0 \rangle = \frac{g}{4!}\, 3 \left(\frac{1}{2\omega}\right)^2 = \frac{g}{8} \frac{1}{(2\omega)^2}$$

since $\langle 0 | q^2 | 0 \rangle = \langle 0 | \left(\frac{a + a^*}{\sqrt{2\omega}}\right)^2 | 0 \rangle = \frac{1}{2\omega} \langle 0 | a a^* + \cancel{a^* a} | 0 \rangle = \frac{1}{2\omega}$

Now for the harmonic oscillator $\varphi_n = \frac{1}{\sqrt{n!}} (a^*)^n |0\rangle$, hence

$$\langle \varphi_0 | V | \varphi_n \rangle = \frac{1}{\sqrt{n!}} \frac{g}{4!} \frac{1}{(2\omega)^2} \langle 0 | (a + a^*)^4 a^{*n} | 0 \rangle$$

When $(a + a^*)^4$ is put into normal product form one gets terms $a^4 + (4) a^3 a^* + (6) a^2 a^{*2} + 4 a a^{*3} + a^{*4}$

$$+ (\ ) a^2 + (\ ) a a^* + (\ ) a^{*2} + (\ ),$$

hence one only gets contributions for $n = 4$ from the a^4 and $n = 2$ from the $4 a^3 a^*$ and $(\ ) a^2$ terms. So

$$\langle \varphi_0 | V | \varphi_4 \rangle = \frac{1}{\sqrt{4!}} \frac{g}{4!} \frac{1}{(2\omega)^2} 4! = \frac{g}{\sqrt{4!}\,(2\omega)^2}$$

$$\langle \varphi_0 | V | \varphi_2 \rangle = \frac{1}{\sqrt{2}} \frac{g}{4!} \frac{1}{(2\omega)^2} \underbrace{\langle 0 | \left\{ \begin{matrix} a^3 a^* & a^{*2} \\ + a^2 a^* a & a^{*2} \\ + a\, a^* a^2 & a^{*2} \\ + a^* a^3 & a^{*2} \end{matrix} \right. | 0 \rangle}_{(3! + 2\cdot 2 + 2) = 12}$$

$$= \frac{1}{\sqrt{2}} \frac{g}{2} \frac{1}{(2\omega)^2}$$

Hence

$$\lambda_2 = \left(\frac{g}{\sqrt{4!}\,(2\omega)^2} \right)^2 \frac{1}{-4\omega} + \left(\frac{g}{\sqrt{2}\, 2\, (2\omega)^2} \right)^2 \frac{1}{-2\omega}$$

$$= -\frac{g^2}{(2\omega)^5} \left(\frac{1}{\underbrace{4!\,2}_{8\cdot 6}} + \frac{1}{8} \right) = -\frac{g^2}{2^8 \omega^5} \frac{7}{6}$$

Now let's attack the same problem by Feynman diagrams. We start with the path integral expression for the partition fn.

$$\mathrm{tr}\left(e^{-\frac{T}{\hbar} H} \right) = \int \mathcal{D}x \; e^{-\frac{1}{\hbar}\int \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2}\omega^2 x^2 \right) dt - \frac{g}{4!\hbar} \int x^4 dt}$$

where the paths are periodic of period T. Then one uses Fourier series to change variables in the path integral.

$$x(t) = \frac{1}{T} \sum a_k e^{ikt} \qquad k \in \frac{2\pi}{T}\mathbb{Z}$$

We use this so that as $T \to \infty$ one has

$$x(t) = \int a_k e^{ikt} \frac{dk}{2\pi} .$$

So

$$\int \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 \right) dt = \frac{1}{2} \sum \left(\frac{k^2 + \omega^2}{T} \right) |a_k|^2 \longmapsto \frac{1}{2} \int (k^2+\omega^2) |a_k|^2 \frac{dk}{2\pi}$$

$$\int x^4 dt = \frac{1}{T^3} \sum a_{k_1} a_{k_2} a_{k_3} a_{k_4} \, \delta(k_1 + \cdots + k_4) \longmapsto \int \frac{dk_1 \cdots dk_4}{(2\pi)^4} a_{k_1} \cdots a_{k_4} \, \delta(k_1 + \cdots + k_4)$$

Our partition function becomes

$$N \int (da_0 \textstyle\prod da_k \, da_{-k})\, e^{-\frac{1}{2}\sum \frac{k^2+\omega^2}{hT}|a_k|^2} \; e^{-\frac{g}{h\,4!}\frac{1}{T}\overset{③}{\sum} a_{k_1}\cdots a_{k_4}\,\delta(k_1+\cdots+k_4)}$$

By yesterday's analysis we know that the non-zero contractions occur between the variables a_k, a_{-k} and lead to the "propagator" factor $\dfrac{hT}{k^2+\omega^2}$.

First order diagrams: There's only one: 

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and its contribution to the partition function is

$$-\frac{g}{h\,8T^3}\sum_{k_1,k_2}\frac{hT}{k_1^2+\omega^2}\,\frac{hT}{k_2^2+\omega^2}$$

$$= -T\underbrace{\left(\frac{gh}{8}\sum\left(\frac{1}{k_1^2+\omega^2}\,\frac{1}{T}\right)\left(\frac{1}{k_2^2+\omega^2}\,\frac{1}{T}\right)\right)}$$

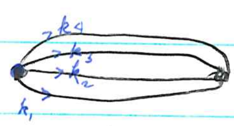
[redacted] $\xrightarrow[T\to\infty]{\text{as}} \dfrac{gh}{8}\Bigg(\underbrace{\int\frac{dk}{(k^2+\omega^2)\,2\pi}}\Bigg)^2$

$$\left[\frac{1}{2\pi}\,\frac{1}{\omega}\arctan\frac{k}{\omega}\right]_{-\infty}^{+\infty} = \frac{1}{2\omega}$$

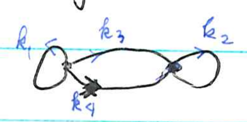
So the first order correction to the ground energy is

$$\lambda_1 = \frac{gh}{8}\left(\frac{1}{2\omega}\right)^2$$

There are 2 second order diagrams



2·4!



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The former gives the contribution

$$\frac{1}{2\cdot 4!}\left(-\frac{g}{h\,[redacted]\,T^3}\right)^2 \sum_{k_1,\ldots,k_4}\left(\prod_{i=1}^{4}\frac{hT}{k_i^2+\omega^2}\right)\delta(k_1+\cdots+k_4)$$

$$= T\frac{1}{2\cdot 4!}\, g^2 h^2 \sum_{k_1,\dots,k_4} \prod \frac{1}{k_i^2+\omega^2}\, \delta(k_1+\cdots+k_4)\, \frac{1}{T^3}$$

$$\sim T\left(\frac{1}{2\cdot 4!}\, g^2 h^2 \int \frac{2\pi\,\delta(k_1+\cdots+k_4)}{(k_1^2+\omega^2)\cdots(k_4^2+\omega^2)}\, \frac{dk_1\cdots dk_4}{(2\pi)^4}\right)$$

The latter gives the contribution to the partition fn

$$\frac{1}{16}\left(-\frac{g}{hT^3}\right)^2 \sum \frac{hT}{k_1^2+\omega^2}\, \frac{\delta(k_3+k_4)\,(hT)^2}{(k_3^2+\omega^2)(k_4^2+\omega^2)}\, \frac{hT}{k_2^2+\omega^2}$$

$$= T\left(\frac{1}{16}\, g^2 h^2 \sum \frac{1}{(k_1^2+\omega^2)(k_2^2+\omega^2)(k_3^2+\omega^2)^2}\, \frac{1}{T^3}\right)$$

$$\sim T\, \frac{1}{16}\, g^2 h^2 \int \frac{1}{(k_1^2+\omega^2)(k_2^2+\omega^2)(k_3^2+\omega^2)^2}\, \frac{dk_1\, dk_2\, dk_3}{(2\pi)^3}$$

But
$$\int_{-\infty}^{\infty} \frac{dk}{(k^2+\omega^2)^2\, 2\pi} = \int_{-\infty}^{\infty} \frac{d\,\omega\tan\theta}{\omega^4\sec^4\theta\;\, 2\pi} = \frac{1}{2\pi\omega^3}\int_{-\pi/2}^{\pi/2} \cos^2\theta = \frac{1}{2\pi\omega^3}\,\frac{\pi}{2} = \frac{1}{4\omega^3}$$

So the contribution is $\sim T\, \frac{1}{16}\, g^2h^2\, \frac{1}{4\omega^3}\cdot\frac{1}{(2\omega)^2} = T\left(\frac{g^2h^2}{8\,(2\omega)^5}\right)$

It remains to evaluate

$$\int \frac{2\pi\,\delta(k_1+\cdots+k_4)}{\prod(k_i^2+\omega^2)} \prod \frac{dk_i}{2\pi}$$

$$\int \frac{dk_3/2\pi}{(k_3^2+\omega^2)\left((\underbrace{k_1+k_2}_{-b}+k_3)^2+\omega^2\right)} = \int \frac{dx}{2\pi}\, \frac{1}{(x^2+\omega^2)((x-b)^2+\omega^2)}$$

$$= 2\pi i\, \frac{1}{2\pi}\left(\mathrm{Res}_{i\omega} + \mathrm{Res}_{b+i\omega}\right)$$

$$= i\left(\frac{1}{2i\omega\,(\underbrace{(i\omega-b)^2}_{\cancel{-\omega^2}-2i\omega b+b^2}+\omega^2)} + \frac{1}{((b+i\omega)^2+\omega^2)(2i\omega)}\right)$$

$$= \frac{1}{2\omega}\left(\frac{1}{b^2-2i\omega b} + \frac{1}{b^2+2i\omega b}\right) = \frac{1}{2\omega b}\left(\frac{1}{b-2i\omega} + \frac{1}{b+2i\omega}\right)$$

$$= \frac{1}{2\omega b}\,\frac{2b}{b^2+4\omega^2} = \frac{1}{\omega}\,\frac{1}{b^2+4\omega^2} = \frac{1}{\omega\left((k_1+k_2)^2+4\omega^2\right)}.$$

$$\xcancel{\int \frac{dk_2/2\pi}{k_2^2+\omega^2}\,\frac{1}{\omega\big((\underbrace{k_1}_{-b}+\underbrace{k_2}_{x})^2+4\omega^2\big)} = \frac{1}{\omega}\int\frac{dx}{2\pi}\,\frac{1}{(x^2+\omega^2)\left((x-b)^2+(2\omega)^2\right)} = \frac{1}{\omega}\,\frac{1}{2\omega}\,\frac{1}{k_1^2+16\omega^2}}$$

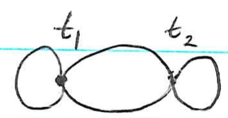
It seems that the [redacted] integrations are [redacted] easier if done over time. So go back to

$$\int \mathcal{D}x\; e^{-\frac{1}{\hbar}\int\left(\frac{1}{2}\dot{x}^2+\frac{1}{2}\omega^2x^2\right)dt \;-\frac{g}{\hbar 4!}\int x^4\,dt}$$

and do the diagrams directly thinking of there being one variable $x(t)$ for each t. Recall that once the diagram is given we then add up over all ways of assigning variables to the ends of the edges. So in 2nd order we have

$2\cdot 4!$

$$\frac{1}{2\cdot 4!}\left(\frac{-g}{\hbar}\right)^2 \iint \left(\hbar G(t_1,t_2)\right)^4 dt_1\,dt_2$$



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$$\frac{1}{16}\left(\frac{-g}{\hbar}\right)^2 \iint \hbar G(t_1,t_1)\left(\hbar G(t_1,t_2)\right)^2 \hbar G(t_2,t_2)\cdot dt_1\,dt_2$$

Notice that with periodic boundary conditions [redacted] on $[0,T]$

$$G(t_1,t_2) = \sum_k \frac{e^{ikt_1}e^{-ikt_2}}{k^2+\omega^2}\,\frac{1}{T} \longrightarrow \int \frac{e^{ik(t_1-t_2)}}{k^2+\omega^2}\,\frac{dk}{2\pi} = \frac{e^{-\omega|t_1-t_2|}}{+2\omega}$$

is a function of t_1-t_2 and hence

$$\iint G(t_1,t_2)^4\,dt_1\,dt_2 = T\int G(t,0)^4\,dt$$

$$\sim T\frac{1}{(2\omega)^4}\int e^{-4\omega|t|}\,dt$$

$$= T\frac{1}{(2\omega)^4}\,\frac{2}{4\omega} = T\frac{1}{(2\omega)^5}$$

So the first diagram contributes to the partition fn

$$\sim T\left(\frac{1}{2\cdot 4!}\,g^2\hbar^2\,\frac{1}{(2\omega)^5}\right)$$

and the second contributes

$$\sim T\left(\frac{1}{16}\,g^2\hbar^2\,\frac{1}{(2\omega)^4}\,\frac{2}{2\omega}\right) = T\left(\frac{1}{8}\,g^2\hbar^2\,\frac{1}{(2\omega)^5}\right).$$

Hence the second order correction to the ground energy is

$$\lambda_2 = -\frac{g^2\hbar^2}{(2\omega)^5}\left(\frac{1}{2\cdot 4!}+\frac{1}{8}\right)$$

which agrees with the elementary calculation.

Some basic general facts. This time consider a general interaction, which means effectively that one has vertices of different multiplicities. Let's use Fourier coefficient description and check that the power of T comes out correctly. Let's suppose we have a diagram with vertices of multiplicities $p_1 \le p_2 \le \cdots \le p_v$ where v is the number of vertices. To a vertex of mult. p belongs the power

$$\frac{1}{T^{p-1}}\,\frac{g_p}{\hbar p!}\sum a_{k_1}\cdots a_{k_p}\ \delta\text{ factor}$$

and each [redacted] edge furnishes us with $\frac{hT}{k^2+\omega^2}$
so we have an expression

$$(*) \qquad \frac{1}{h^v}\,\frac{1}{T^{\Sigma(p_i-1)}}\,(hT)^e \sum_{\substack{\text{indep}\\ \text{momenta}}}$$

The number of independent momenta is found as follows: The set of all momenta is the [redacted] group $\sum \frac{2\pi}{T}\mathbb{Z}$ of oriented 1-chains on the graph. The momenta adding up to zero at each vertex is the group of 1-cycles which [redacted] is H_1(graph), which has rank = no. of loops.

$$\boxed{\text{number of loops in the graph} \quad = \quad \text{number of independent momenta}}$$

Call this [redacted] number L. Note by [redacted] Euler

[redacted] $$v - e = c - L$$

where c is the number of components of the graph. Each momentum integration requires a $\frac{1}{T}$ factor, so [redacted] that we can take the $T \to \infty$ limit. Hence $*$ becomes

$$h^{e-v}\,\frac{1}{T^{-v}\,T^{2e}}\,T^e\,T^L \underbrace{\sum \qquad \frac{1}{T^L}}_{\to \int_{\text{ind. momenta}}}$$

where we use $\sum p_i = 2e$. Hence we get

$$h^{L-c}\,T^c \int_{\text{ind. momenta}}$$

in the partition fn. This shows that the connected graphs are $\sim T \cdot \text{const}$ and that if we restrict to connected graphs, then expanding in powers of $\hbar$ is the same as the loop expansion.

So far I have a sort of understanding of the ground energy. The next obstacle seems to be to understand the meaning of the Green's functions. The two-point Green's function is defined as follows. Suppose we fix an inverse temperature β. Then the Green's function is, up to a factor of i,

$$\langle T[e^{iHt} q e^{-iHt} e^{iHt'} q e^{-iHt'}] \rangle,$$

where $\langle \ \rangle$ denotes thermal average $\langle A \rangle = tr(e^{-\beta H}A)/tr(e^{-\beta H})$.

November 14, 1979

Review Victor's lectures on Tauberian theorems:

Thm: Suppose a_n bounded below and $\sum a_n x^n$ converges for $|x|<1$, (i.e. $\overline{\lim} |a_n|^{1/n} \leq 1$). Then

$$\lim_{x\to 1^-} (1-x)\sum a_n x^n = A \implies \frac{1}{N}\sum_{n\leq N} a_n \longrightarrow A$$

Proof: Can suppose $a_n \geq 0$ by adding a constant. Putting x^k in form x gives

$$\underbrace{\left(1-x^k\right)}_{1-x}(1-x)\sum a_n x^{nk} \longrightarrow A$$

or

$$(1-x)\sum a_n x^n \left(x^n\right)^{k-1} \longrightarrow \frac{A}{k} = A\int_0^1 x^{k-1}dx$$

hence for any polynomial $P(x)$ we have

$$\circledast \qquad (1-x)\sum a_n x^n P(x^n) \longrightarrow A\int_0^1 P(x)\,dx$$

But now it is easily seen that for any piecewise continuous f on $0\leq x\leq 1$ we can find polys P_1, P_2 with $P_1 \leq f \leq P_2$ on $[0,1]$ and $\int (P_2-P_1)\,dx < \varepsilon$. It follows because $a_n \geq 0$ that these inequalities persist for the left side of $\circledast$ with P replaced by P_1, f, P_2, hence

$$(1-x)\sum a_n x^n f(x^n) \longrightarrow A\int_0^1 f\,dx$$

Now take $f(x) = \begin{cases} 0 & 0\leq x<\frac{1}{e} \\ 1/x & \frac{1}{e}\leq x\leq 1\end{cases}$ $\qquad \int f(x)\,dx = 1$

and

$$(1-x)\sum a_n x^n f(x^n) = (1-x)\sum_{n\leq N} a_n \qquad \text{for } x^n \geq \tfrac{1}{e} \qquad \begin{matrix} n\log x \geq -1 \\ n \leq \frac{1}{\log(\frac{1}{x})}\end{matrix}$$

where $N = \frac{1}{\log(\frac{1}{x})}$, or $x = e^{-1/N}$. Thus we have

$$\left(1-e^{-1/N}\right)\sum_{n\leq N} a_n \longrightarrow A \quad \text{so} \quad \frac{1}{N}\sum_{n\leq N} a_n \longrightarrow A.$$

QED.

Reformulation: Suppose a_n is the sequence of partial sums of a series $\sum b_n$ so that

$$a_n = \sum_{k \le n}' b_k \qquad\qquad x^k/_{1-x}$$

Then

$$\sum a_n x^n = \sum_n x^n \Big(\sum_{k \le n} b_k\Big) = \sum_k b_k \sum_{n \ge k}' x^n = \frac{\sum' b_k x^k}{1-x}$$

so the preceding thm. implies

$$\sum' b_n = A \quad \text{in the sense of Abel} \implies \sum' b_n = A \quad \text{in } (C, 1) \text{ sense}$$

assuming that the partial sums $a_n = \sum_{k \le n}' b_k$ are bounded.

The above thm. can be quasi-generalized to

$$\lim_{s \downarrow 0} s \int_0^\infty e^{-st}\, d\mu(t) = A \implies \lim_{s \searrow 0} s \int_0^{1/s} d\mu(t) = A$$

for any measure on $[0, \infty]$ such that $\int e^{-st}\, d\mu$ converges for $s > 0$.

Wiener Tauberian thm. is a consequence of

<u>Thm:</u> $f \in L^1(\mathbb{R})$, $\hat{f}(\xi) = \int e^{ix\xi} f(x)\,dx \neq 0$ all $\xi \in \mathbb{R}$
$\Rightarrow$ closed ideal in L^1 generated by f is all of L^1.

<u>Proof</u> in the case where f decays fast enough at ∞ (after Kac): What we have to show is that elements of the form $g * f$ with $g \in L^1$ are dense in L^1. As

$$\widehat{g*f} = \hat{g}\hat{f}$$

what we want to do ~~is~~ take an element h of L^1 and divide $\hat{h}$ by $\hat{f}$. ~~Now we know $\widehat{C_0^\infty}$ is dense in L' hence the Schwartz space S is dense in L^1, hence $\hat{S} = S$ is dense in~~ This is possible if $\hat{h} \in C_0^\infty$ and $\hat{f} \in C^\infty$ for then $\hat{h}/\hat{f} \in C_0^\infty \subset \widehat{L^1}$. Note $\hat{f} \in C^\infty$ follows if $x^n f \in L^1$ for all n. The only problem is why $\widehat{C_0^\infty}$ is dense in L^1. But C_0^∞ is dense in the Schwartz space S in the Schwartz topology, so because $S \simeq \hat{S}$ is a topological isomorphism, $\widehat{C_0^\infty}$ is dense in S. But C_0^∞ is dense in L^1, and $C_0^\infty \subset S \subset L^1$ so S is dense in L^1. $\therefore$ $\widehat{C_0^\infty}$ is dense in L^1.

Wiener Tauberian thm. Suppose $f \in L^1$ and $\hat{f}(\xi) \neq 0$ $\forall \xi$. Then for any $f' \in L^1$ and $h \in L^\infty$ one has

$$\lim_{x\to\infty} (f*h)(x) = A\int f \implies \lim_{x\to\infty} (f'*h)(x) = A\int f'$$

<u>Proof.</u> Clear if $f' = g*f$ with $g \in L^1$ because

$$(f'*h)(x) = [g*(f*h)](x) = \int g(y) \underbrace{(f*h)(x-y)}_{\to A\int f} \to A \underbrace{\int g \cdot \int f}_{\int f'}$$

where we have used $f * h$ is bounded, continuous and the dominated convergence thm. Reason:

$$\left|(f*h)(x)\right| = \left|\int_{T} f(x-y)\, h(y)\, dy\right| \leq \|f\|_1 \cdot \|h\|_\infty \qquad \text{etc.}$$

Similarly if $f', f'' \in L^1$ are close then $f' * h, f'' * h$ are sup norm close, etc., etc.,

Let's briefly review something about the ζ function and prime numbers. The starting point is Euler's proof of infinitely many primes using

$$\zeta(s) = \sum{}' \frac{1}{n^s} = \prod (1-p^{-s})^{-1} \; ;$$

let $s \searrow 1$, and use that $\zeta(s) \longrightarrow +\infty$. Then

$$\log \zeta(s) = \sum_p{}' -\log(1-p^{-s}) = \sum_p \sum_{k \geq 1} \frac{1}{k}(p^{-s})^k$$

$$= \sum p^{-s} + \sum_{k \geq 2}{}' \frac{1}{k}\Big(\sum_p{}' p^{-ks}\Big)$$

and the rearrangement of summation is possible [redacted] for $\operatorname{Re} s > 1$ because the series converges absolutely.

Now estimates show that the second term for $\log \zeta(s)$ is analytic for $\operatorname{Re} s > \frac{1}{2}$. In effect put $f(s) = \sum p^{-s}$; so that

$$\log \zeta(s) = f(s) + \tfrac{1}{2} f(2s) + \tfrac{1}{3} f(3s) + \cdots\cdots$$

We have for s real

$$f(s) \leq [redacted] \sum_{n \geq 2} n^{-s} \leq 2^{-s} + \underbrace{\int_2^\infty x^{-s}\, dx}$$

$$\therefore\; f(s) \leq 2^{-s}\left[1 + \frac{2}{s-1}\right] = 2^{-s}\left(\frac{s+1}{s-1}\right) \qquad \left.\frac{x^{-s+1}}{-s+1}\right]_2^\infty = \frac{2^{-s+1}}{s-1}$$

Hence

$$\sum_k \frac{1}{k} f(ks) \leq \sum_k \frac{1}{k} 2^{-ks} \underbrace{\left(\frac{ks+1}{ks-1}\right)}_{\to 1 \text{ as } k \to \infty}$$

This shows that $\left|\sum_{k \geq 2} \frac{1}{k} f(ks)\right| \leq C$ uniformly for $\mathrm{Re}\, s \geq \frac{1}{2} + \varepsilon$. So therefore $\log \zeta(s)$ and $f(s)$ have the same singularities for $\mathrm{Re}\, s > \frac{1}{2}$.

Now

$$f(s) = \int_0^\infty e^{-st} d\mu(t) \qquad d\mu(t) = \sum_p \delta(t - \log p)\, dt$$

$$\mu(t) = \sum_{\log p \leq t} 1 = \pi(e^t)$$

The prime number theorem asserts

$$\pi(x) \sim x/\log x \qquad \text{or} \qquad \mu(t) \sim e^t/t$$

and somehow this results from the singularity $\zeta(s) \sim \frac{1}{s-1}$ as $s \to 1$. Laplace inversion gives formally

$$\frac{d\mu}{dt} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} f(s)\, ds$$

but there are problems with the fact $f(s)$ doesn't decay vertically. So instead one can do the following

$$f(s) = \int_0^\infty e^{-st} d\mu(t) = \left[e^{-st}\mu(t)\right]_0^\infty + s\int_0^\infty e^{-st} \mu(t)\, dt$$

whence

$$\mu(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} \frac{f(s)}{s}\, ds$$

and this should probably hold except where $\mu(t)$ jumps, since $\mu(t)$ is piecewise continuous. It probably is necessary to go to ∞ symmetrically, à la Eisenstein.

Now we know $\zeta(s) \sim \frac{1}{s-1}$ as $s \to 1$, so we really ought to understand

$$-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\log(s-1)\,\frac{ds}{s}$$

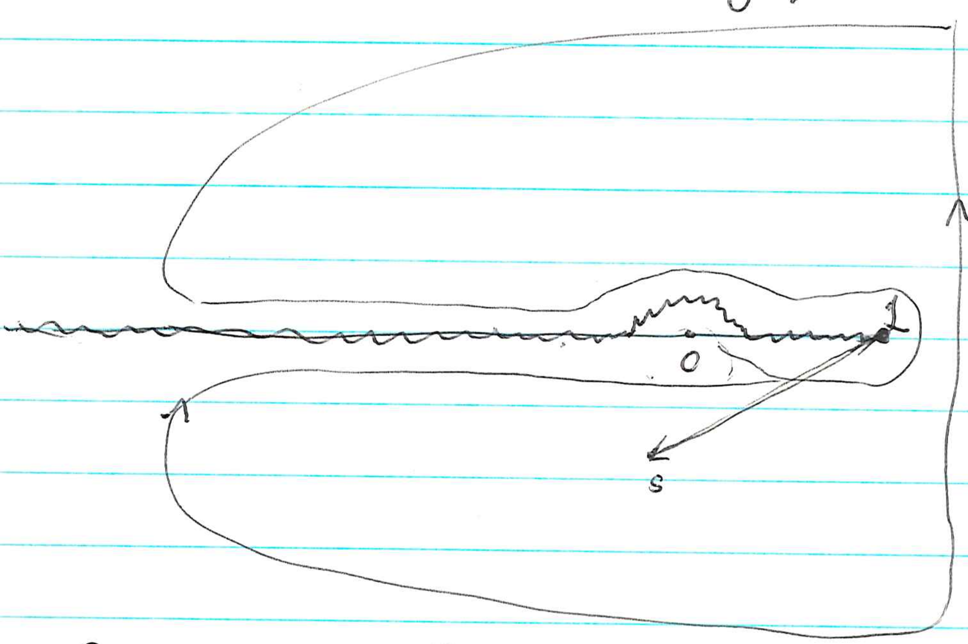
First note that this integral converges, for if we put $s=1+iy$ then

$$e^{t+iyt}\underbrace{\log(1+iy)}_{\left(+\pi/2+\log y\right)}\frac{i\,dy}{1+iy}$$

$$\frac{i}{1+iy} = \frac{1}{y}\left(\frac{1}{1-i/y}\right) = \frac{1}{y}+O\left(\frac{1}{y^2}\right)$$

so the non-absolutely convergent part is $e^{iyt}\,\frac{\log y}{y}$. Since $\frac{\log y}{y}\searrow 0$ this is like an alternating series.

The next thing is to deform the contour to the branch cut for $\log(1-s)$, which we will take to be $-\infty<s\le 1$ except we push it away from $s=0$.



As we push the contour thru 0 we get the residue at 0 which is

$$-e^{0t}\log(0-1) = +\pi i$$

Define log$(s-1)$ to have $\mathrm{Im}=-\pi$ on the bottom of the cut.

It then increases to $2\pi i+\log(s-1)$ on the top of the cut.

So we get

$$+\pi i + \frac{1}{2\pi i}\int_{-\infty}^{1} e^{st}\,\frac{\log(s-1)}{s}\,ds + \frac{1}{2\pi i}\int_{-\infty}^{1} e^{st}\,\frac{\log(s-1)+2\pi i}{s}\,ds$$

$$= +\pi i + \int_{-\infty}^{1} e^{st}\,\frac{ds}{s} = +P\int_{-\infty}^{1}\frac{e^{st}}{s}\,ds$$

where in the integral the contour passes above the pole at $s=0$. For $t>0$ we can change variables to get

$$\frac{1}{2\pi i}\int e^{st}\,\frac{-\log(s-1)}{s}\,ds = +P\int_{-\infty}^{t}\frac{e^{s}}{s}\,ds$$

Up to a constant this is

$$\int_{1}^{t}\frac{e^{s}}{s}\,ds$$

Summary:

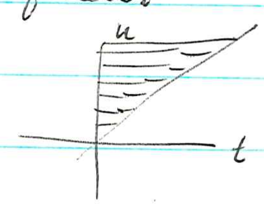
$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\left(\frac{-\log(s-1)}{s}\right)ds = P\int_{-\infty}^{t} e^{x}\,\frac{dx}{x}$$

The latter is the so-called exponential integral function. It has a logarithmic singularity at $t=0$, and hence its Laplace transform is defined. Actually things might be simpler. Put

$$E(t) = \int_{t}^{\infty} e^{-u}\,\frac{du}{u}$$

. Then $E(t)$ has a log. singularity at $t=0$, so its Laplace transform is defined.

$$\int_{0}^{\infty} e^{-st}E(t)\,dt = \int_{0}^{\infty}dt\int_{t}^{\infty} e^{-st}e^{-u}\,\frac{du}{u}$$



$$= \int_0^\infty \frac{du}{u} e^{-u} \int_0^u e^{-st}\,dt = \int_0^\infty \frac{du}{u} e^{-u} \left[\frac{e^{-st}}{-s}\right]_0^u$$

$$= \int_0^\infty \frac{du}{u} e^{-u}\, \frac{1-e^{-su}}{s} = \frac{1}{s}\int_0^\infty du\; \frac{e^{-u} - e^{-(s+1)u}}{u}$$

[redacted] $= \frac{1}{s}\log(s+1).$ Here I have used

$$\int_0^\infty \frac{e^{-au} - e^{-bu}}{u}\,du = \int_a^b ds \int_0^\infty e^{-su}\,du = \int_a^b \frac{1}{s}\,ds = \log\left(\frac{b}{a}\right).$$

More generally for $a > 0$

$$\frac{1}{2\pi i}\int e^{st}\,\frac{\log(s+a)}{s}\,ds = \log(a) - \int_{-\infty}^{-a} \frac{e^{st}}{s}\,ds$$

$$= \log(a) + \int_a^\infty \frac{e^{-tx}}{x}\,dx$$

$$= \log(a) + \int_{at}^\infty \frac{e^{-x}}{x}\,dx = \log(a) + E(at)$$

but this results from $a=1$ by scaling. Suppose $a \searrow 0$:

$$\frac{1}{2\pi i}\int e^{st}\,\frac{\log s}{s}\,ds = \left[\log a + \int_1^\infty \frac{e^{-x}}{x}\,dx + \int_{at}^1 \frac{e^{-x}-1}{x}\,dx + \underbrace{\int_{at}^\infty \frac{1}{x}\,dx}_{-\log a - \log t}\right]_{a\to 0}$$

$$= \underbrace{\int_1^\infty \frac{e^{-x}}{x}\,dx + \int_0^1 \frac{e^{-x}-1}{x}\,dx}_{\text{constant}} - \log t$$

As a check $\displaystyle\int_0^\infty e^{-st}\,t^\alpha\,\frac{dt}{t} = \frac{\Gamma(\alpha)}{s^\alpha}$

$$\int_0^\infty e^{-st}\,t^\alpha \log t\,\frac{dt}{t} = \frac{\Gamma'(\alpha)}{s^\alpha} + \frac{\Gamma(\alpha)}{s^\alpha}(-\log s)$$

$$\int_0^\infty e^{-st}\log t\,dt = \frac{\Gamma'(1)}{s} - \frac{\log s}{s}$$

which shows
$$\frac{\log s}{s} = \mathcal{L}\left\{ \Gamma'(1) - \frac{\log t}{\text{[redacted]}} \right\}$$

Finally we want to estimate
$$F(t) = P\int_{-\infty}^{t} \frac{e^{x}}{x}\,dx \;=\; P\int_{-\infty}^{1} e^{xt}\,\frac{dx}{x}$$
as $t \to +\infty$. We have
$$F(t) = c + \int_1^t \frac{e^x}{x}\,dx \le c + \int_1^t e^x\,dx \le c - 1 + e^t$$
Also $F \in \uparrow$ for $t > 0$ and
[redacted]
$$F(t) - F(t/2) = \int_{t/2}^{t} \frac{e^x}{x}\,dx = \int_{1/2}^{1} e^{tx}\,\frac{dx}{x}$$

Laplace's method gives the asymptotic behavior of the latter. Actually we should use this method directly on
$$F(t) = \int_{-\infty}^{1} e^{xt}\,\frac{dx}{x} \qquad \frac{1}{x} \text{ interpreted as a distribution}$$
$$= e^t\int_0^\infty e^{-yt}\,\frac{1}{1-y}\,dy \sim e^t\left\{\frac{1}{t} + \frac{\Gamma(2)}{t^2} + \cdots\cdots\right\}$$
The upshot is that
$$\frac{1}{2\pi i}\int e^{st}\left(\frac{-\log(s-1)}{s}\right)ds = P\int_{-\infty}^{1} e^{xt}\,\frac{dx}{x} \sim \frac{e^t}{t}$$

which in principle yields the leading term for $\pi(e^t)$ and the prime number thm.

It remains to understand the Tauberian business that deduces the PNT from $\zeta(1+iy) \neq 0$

How Tauberian results are used in spectral theory. Victor took $H = -\Delta + V$ on $\mathbb{R}^3$ with V growing sufficiently fast and by using a parametrix for $(u+H)^{-1}$ he was able to show

$$\sum_n \frac{1}{(u+\lambda_n)^2} \sim \int \frac{1}{(u+\xi^2+V(x))^2} \frac{dx\,d\xi}{(2\pi)^3} \qquad \text{as } u \to \infty$$

$$\| \qquad\qquad\qquad \|$$

$$\int \frac{1}{(u+\lambda)^2} dN(\lambda) \qquad \int \frac{1}{(u+\lambda)^2} dV(\lambda)$$

where $N(\lambda)$ = number of $\lambda_n \leq \lambda$ $\qquad V(\lambda) = \frac{1}{(2\pi)^3}$ symplectic volume of $H \leq \lambda$.

He will use a Tauberian result to deduce

$$N(\lambda) \sim V(\lambda).$$

A slightly different version considers the heat operator which gives the partition function

$$\operatorname{tr}\left(e^{-\beta H}\right) = \int e^{-\beta\lambda}\, dN(\lambda)$$

$$= \int_{\text{paths of period } \beta} \mathcal{D}x(t)\; e^{-\int_0^\beta \left(\frac{\dot{x}^2}{2} + V(x)\right) dt}$$

and compares it with classical partition fn.

$$\int e^{-\beta H} \frac{dx\,d\xi}{(2\pi)^3} = \int e^{-\beta\lambda}\, dV(\lambda).$$

as $\beta \searrow 0$. This case has been studied by Atiyah [redacted] and others [redacted] using the fact that ΨDO theory gives an asymptotic expansion for $\langle x | e^{-\beta H} | x' \rangle$ as $\beta \searrow 0$.