Sept 29, 1979

~~Behavior of $\hat{f}$ at ∞?~~

Victor's problem: Let K be strictly-convex compact body in $\mathbb{R}^n$ with smooth boundary containing 0 in its interior. Define

$$N(t) = \text{number of lattice points in } tK$$
$$= \sum_{x \in \mathbb{Z}^n} \chi_K\left(\tfrac{1}{t}x\right)$$

The goal is to understand the asymptotic behavior of $N(\lambda)$ as $t \to \infty$. Need Poisson summation formula:

$$\sum_{\lambda \in \mathbb{Z}^n} f(x+\lambda) = \sum_{\mu \in \mathbb{Z}^n} a_\mu e^{2\pi i \mu x} \qquad a_\mu = \int f(x) e^{-2\pi i \mu x}\,dx$$
$$= \hat{f}(2\pi\mu)$$

$$\sum_{\lambda \in \mathbb{Z}^n} f(\lambda) = \sum_{\mu \in \mathbb{Z}^n} \hat{f}(2\pi\mu) \qquad \int f\left(\tfrac{x}{t}\right) e^{-i\xi x}\,dx$$
$$= \int f(x) e^{-i\xi t x}\, t^n dx$$

$$\sum_{\lambda \in \mathbb{Z}^n} f\left(\tfrac{\lambda}{t}\right) = t^n \sum_{\mu \in \mathbb{Z}^n} \hat{f}(2\pi t \mu)$$

or

$$\sum_{\lambda \in \mathbb{Z}^n} f\left(\tfrac{\lambda}{t}\right)\frac{1}{t^n} = \sum_{\mu \in \mathbb{Z}^n} \hat{f}(2\pi t \mu)$$

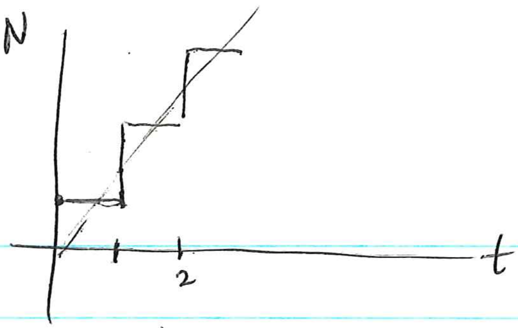
this has to be understood as distributions

Now one tries $f = \chi_K$. On the left one gets a cubical approx to the volume

$$\sum_{\lambda \in \mathbb{Z}^n} \chi_K\left(\tfrac{\lambda}{t}\right) \longrightarrow \operatorname{Vol} K = \int \chi_K\,dx = \hat{\chi}_K(0)$$

Example: $\mathbb{R}$ $\quad K = [-1, 1]$. Then we get

$$N(t) = \sum_{\lambda \in \mathbb{Z}} \chi_K\left(\tfrac{\lambda}{t}\right) = 2[t] + 1 = 2t + \text{sawtooth fn.}$$



$$\hat{\chi}_K(2\pi\mu t) = \int_{-1}^{1} e^{-2\pi i \mu t x}\, dx = \frac{e^{-2\pi i \mu t} - e^{+2\pi i \mu t}}{-2\pi i \mu t} = \frac{\sin(2\pi\mu t)}{\pi \mu t}$$

so

$$N(t) = 2[t] + 1 = t\left(2 + 2\sum_{\mu=1}^{\infty} \frac{\sin(2\pi\mu t)}{\pi\mu t}\right)$$

<u>September 30, 1979:</u>

There is a problem with using the Poisson summation formula for $f = \chi_K$ because $\hat{f}$ doesn't decay fast enough. The stationary phase lemma gives the asymptotic behavior $\hat{f}(t\xi)$ as $t \to {}^{+}\infty$, $\xi \neq 0$.

$$\hat{f}(t\xi) = \int e^{-it\xi\cdot x}\chi_K\, dx$$

If χ_K were smooth and radidly decreasing, then $\hat{f}$ would be also in $\mathcal{S}$, hence it should be possible to [redacted] describe the asymptotics of $\hat{f}$ in terms of the boundary.

K is compact, so we can cover it with small open patches and use a partition of unity $\sum_i \rho_i = 1$, so as to [redacted] worry about $f_i =$ [redacted] $\chi_K \rho_i$. Then we can choose coordinates so as to linearize the boundary at the expense of non-linearizing $x \mapsto \xi\cdot x$.

The first thing to understand is an integral of the

form
$$\int e^{-it\varphi(x)}\rho(x)\,dx$$

where $\rho\in C_0^\infty(\mathbb{R}^n)$ and φ is a real-valued function with $d\varphi\neq 0$ on $\operatorname{Supp}\rho$. By partition of 1 arguments we can make φ one of the coordinate functions. So look at
$$\int e^{-itx_1}\rho(x)\,dx = \int e^{-itx_1}dx_1\int d^{n-1}x'\,\rho(x_1,x')$$

This reduces you to a 1-dimensional situation. So the point seems to involve doing the integral along the level surfaces of φ first. This somehow allows the generalization to wavefront sets for a general distribution.

In order to do the integration by parts we use
$$d\left(\frac{e^{-it\varphi}}{-it}\right) = e^{-it\varphi}\,d\varphi$$
$$d\left(\frac{e^{-it\varphi}}{-it}\,\frac{\rho\,d^nx}{d\varphi}\right) = e^{-it\varphi}\rho\,d^nx - \frac{e^{-it\varphi}}{+it}\,d\left(\frac{\rho\,d^nx}{d\varphi}\right)$$

Here $\frac{\rho\,d^nx}{d\varphi}$ denotes any $n-1$ form with the property that $d\varphi\wedge\frac{\rho\,d^nx}{d\varphi} = \rho\,d^nx$. A simple way to construct this division is to choose a vector field X with $X\varphi\neq 0$ and then take
$$\frac{1}{X\varphi}\,i(X)\rho\,d^nx$$

In effect $0 = i(X)(d\varphi\wedge\rho\,d^nx) = (X\varphi)\rho\,d^nx - d\varphi\; i(X)(\rho\,d^nx)$

If $X=\frac{\partial}{\partial x_1}$, then $d\,\frac{1}{X\varphi}\,i(X)\rho\,d^nx = d\left(\frac{1}{\frac{\partial\varphi}{\partial x_1}}\rho\,dx_2\cdots dx_n\right) = \frac{\partial}{\partial x_1}\left(\frac{\rho}{\frac{\partial\varphi}{\partial x_1}}\right)d^nx$

and so we have

$$\int e^{-it\varphi}\rho\, d^n x = \frac{1}{it}\int e^{-it\varphi}\,\frac{\partial}{\partial x_1}\left(\frac{\rho}{\frac{\partial\varphi}{\partial x_1}}\right) d^n x$$

provided $\frac{\partial\varphi}{\partial x_1} \neq 0$ on Supp ρ. This sort of thing will establish uniformity of the estimates with variable φ, but it's probably not enough to establish the existence of wavefront sets.

So let's return to

$$\int e^{-it\varphi(x)}\rho\, d^n x$$

and let's suppose that φ has a non-degenerate critical point at $x=0$. Use the Morse lemma

$$\varphi(x) = (x')^2 - (x'')^2 \qquad x = (x'; x'')$$

and assume this coord. change exists on Supp ρ. Up to a Jacobian factor J we get

$$J\cdot\int e^{-it(x'^2 - x''^2)}\,\rho(x';x'')\, d^n x$$

$$= \frac{J}{t^{n/2}}\int e^{-i(x'^2 - x''^2)}\,\rho\left(\frac{x'}{\sqrt{t}}, \frac{x''}{\sqrt{t}}\right) d^n x$$

$$\sim \frac{J}{t^{n/2}}\,\rho(0)\,\pi^{n/2}\, e^{-i\frac{\pi}{4}\,\text{signature}}$$

J is the square root of the absolute value of the determinant of the Hessian of φ.

So we can summarize and say that if φ has only non-degenerate critical points on Supp ρ, then

$$\int e^{-it\varphi(x)} \rho \, d^n x = O(t^{-n/2}) \qquad \text{as } |t| \to \infty$$

and its a sum of contributions of the form

$$\rho(P) \, \frac{\pi^{n/2}}{t^{n/2}} \, \frac{e^{-i\frac{\pi}{4} \operatorname{sig}\left(\frac{1}{2}\varphi_{x_i x_j}\right)}}{\left|\det \frac{1}{2} \cdot \varphi_{x_i x_j}(P)\right|^{1/2}}$$

over all the critical points. Actually there is a whole asymptotic expansion in powers of t^{-1}. The powers are $t^{-n/2}$, $t^{-n/2-1}$, ... because

$$\int e^{-iQ(x)} x^{\alpha} \, dx = 0 \qquad \text{for } |\alpha| \text{ odd.}$$

Interesting problem: Take a G-orbit in $\mathfrak{g}$ and evaluate the above diffraction integral.

Return to $\displaystyle\int e^{-it\xi\cdot x} \chi_K \, d^n x = \int_K e^{-it\xi\cdot x} \, d^n x$ and integrate by "parts".

$$d\left(\frac{e^{-it\xi\cdot x}}{-it} \, \frac{d^n x}{\xi}\right) = e^{-it\xi\cdot x} d^n x$$

where $\frac{d^n x}{\xi}$ denotes any $n-1$ form with $\xi \cdot \frac{d^n x}{\xi} = d^n x$

Then

$$\int_K e^{-it\xi\cdot x} \, d^n x = \left(\frac{1}{-it}\right) \int_{\partial K} e^{-it\xi\cdot x} \, \frac{d^n x}{\xi} \, .$$

We can take $\displaystyle\frac{d^n x}{\xi} = \frac{1}{(X,\xi)} \, i(X) \, d^n x$ if $(X,\xi) \neq 0$ and X is a constant vector field. This combined with the above theory for what happens [redacted] on ∂K gives

$$\hat{f}(t\xi) = \int_K e^{-it\xi\cdot x}\, d^n x = O\left(t^{-\frac{n-1}{2}-1}\right) \quad -\left(\frac{n+1}{2}\right) \qquad \text{as} \quad |t| \to \infty$$

[redacted] and this is a sum of [redacted] contributions for each of the points on ∂K where the tangent plane coincides with $\xi\cdot x = \text{const}$. Assuming K is strictly convex there should be exactly 2 critical points.

Check: Take unit ball in $\mathbb{R}^3$, $\xi = (0,0,1)$

$$\hat{f}(t\xi) = \int_K e^{-itz}\, dV = 2\pi \int_0^1 r^2 dr \underbrace{\int_0^\pi e^{-itr\cos\theta} \sin\theta\, d\theta}_{\left[\frac{e^{-itr\cos\theta}}{itr}\right]_0^\pi = \frac{e^{+itr} - e^{-itr}}{itr}}$$

$$= \frac{2\pi}{it} \int_0^1 \left(e^{itr} - e^{-itr}\right) r\, dr$$

$$= \frac{2\pi}{it} \left[\frac{e^{itr}}{it} r - \frac{e^{itr}}{(it)^2} - \frac{e^{-itr}}{-it} r + \frac{e^{-itr}}{(-it)^2} \right]_0^1$$

$$= \frac{2\pi}{(it)^2}\left(e^{it} + e^{-it}\right) + \frac{2\pi}{(it)^3}\left(-e^{itr} + e^{-itr}\right)$$

$$= O\left(\frac{1}{t^2}\right) \qquad \text{checks } \frac{n+1}{2} = 2.$$

When we try to form $\sum_\mu' \hat{f}(2\pi t_\mu)$ we see that it converges conditionally [redacted] at best.

Eigenvalue interpretation. Consider a constant coefficient operator $Q(D)$ on a torus $\mathbb{R}^n/(2\pi\mathbb{Z})^n$. Then the exponentials $e^{-i\xi\cdot x}$ with $\xi\in\mathbb{Z}^n$ form a basis of eigenvectors and the eigenvalue for $e^{-i\xi x}$ is $Q(\xi)$. Thus

$$\begin{aligned}\text{no. of eigenvalues} \le t &= \text{no. of } Q(\xi)\le t\\ &= \text{no. of } \underbrace{\tfrac{1}{t}Q(\xi)}_{Q(\frac{\xi}{t})}\le 1 \qquad \text{if } Q \text{ has order } 1.\\ &= \text{no. of } \tfrac{\xi}{t} \text{ in } K\end{aligned}$$

where $K=\{\xi \mid Q(\xi)\le 1\}$. Therefore starting with K containing 0 in its interior, we can define Q to be homogeneous fn. of degree 1 with $K=\{\xi\in\mathbb{R}^n \mid Q(\xi)\le 1\}$.

The standard method for doing eigenvalue distribution is to look at the distribution on the t-line given by

$$\operatorname{tr} e^{-itQ(D)} = \sum_{\xi\in\mathbb{Z}^n} e^{-itQ(\xi)}$$

and then apply Tauberian theorems. The idea is that the above "Function" is the Fourier transform of the δ-measure

$$\sum_{\xi\in\mathbb{Z}^n}\delta(\omega-Q(\xi))$$

and hence behavior of the latter as $\omega\to\infty$ is related to the singularities of $\operatorname{tr}(e^{-itQ(D)})$.

Lets try to understand this trace as a Feynman path integral at least formally. We subdivide the interval $[0,t]$ into steps of size ε, and compute the relevant matrix element:

$$\langle x' | e^{-i\varepsilon Q(D)} | x \rangle$$

Recall
$$\left(e^{-i\varepsilon Q(D)} f\right)(x) = e^{-i\varepsilon Q(D)} \int \frac{d\xi}{2\pi} e^{-i\xi\cdot x} \hat{f}(\xi)$$

$$= \int \frac{d\xi}{2\pi} e^{-i\varepsilon Q(\xi) + i\xi\cdot x} \int e^{-ix'\xi} f(x')dx'$$

$$= \int dx' \int \frac{d\xi}{2\pi} e^{-i\varepsilon Q(\xi) + i\xi\cdot(x-x')} f(x')$$

Thus
$$\langle x | e^{-i\varepsilon Q(D)} | x' \rangle = \int \frac{d\xi}{2\pi} e^{-i\varepsilon Q(\xi) + i\xi(x-x')}$$
valid on $\mathbb{R}^n$

Now when we form the path integral by combining matrix multiplication and passing to the limit at $\varepsilon \to 0$ we get

$$\operatorname{tr} e^{-itQ(D)} = \int \left[\frac{dx\,d\xi}{2\pi}\right] e^{+i\int(\xi\,dx - Q(\xi)dt)}$$

where the integral is taken over closed paths $[0,t] \longrightarrow \mathbb{R}^{2n}$, $t' \longmapsto (x(t'), \xi(t'))$. Proceed formally and look at the stationary values of the exponential.
Wait: The closed paths take place in $(\mathbb{R}/2\pi\mathbb{Z})^n \times \mathbb{R}^n$??

$$\eta = \xi\,dx - Q(\xi)\,dt \quad \text{is like} \quad \eta = p\,dq - H\,dt$$

so we know the stationary curves are given by

$$\dot{x} = \frac{\partial Q}{\partial \xi} \qquad \dot{\xi} = -\frac{\partial Q}{\partial x} = 0$$

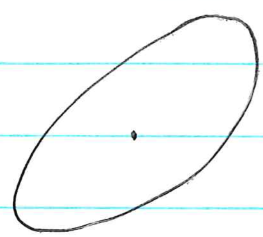
Thus $\dot{x}$ is constant, ξ is constant, so $x(t)$ is a 1-parameter subgroup in the torus. Note that because Q is homog. of degree 1, $\frac{\partial Q}{\partial \xi}$ is homogeneous of degree 0.

October 1, 1979

First let's consider critical [redacted] paths for the action integral $\int \xi dx - Q dt$. These are solutions of Hamilton's equations:

$$\dot{x} = \frac{\partial Q}{\partial \xi} \qquad \dot{\xi} = -\frac{\partial Q}{\partial x} = 0$$

hence are $x = \frac{\partial Q}{\partial \xi}(\xi_0)t + x_0$, $\xi = \xi_0$. We are assuming that $Q(\xi)$ is homogeneous of degree 1, and that $Q(\xi) \leq 1$ is a strictly convex compact region K with 0 in its interior and smooth boundary.



When we try to compute bicharacteristic curves on the torus which are closed and have period T, such a curve is given by x_0, ξ_0 such that

$$\frac{\partial Q}{\partial \xi}(\xi_0)\, T \quad \in \text{ lattice } (2\pi \mathbb{Z})^n$$

Notice that x_0 is an arbitrary point on the torus and that ξ_0 [redacted] can be multiplied by a strictly positive scalar. Hence the [redacted] periodic bicharacteristics can be described by points ξ on $Q(\xi) = 1$ whose tangent plane is rational. There are countably many of these and each one comes [redacted] with a height which [redacted] is the smallest T such that $\frac{\partial Q}{\partial \xi}(\xi)T \in (2\pi\mathbb{Z})^n$.

What is generalized Poisson summation formula in this situation? It should express $\mathrm{tr}(e^{-itQ(D)})$ in terms of a sum over rational points on ∂K, and this should hold modulo smooth functions. One has the usual Poisson summation formula:

$$\mathrm{tr}\left(e^{-itQ(D)}\right) = \sum_{\xi \in \mathbb{Z}^n} e^{-itQ(\xi)} = \sum_{\xi \in \mathbb{Z}^n} e^{-iQ(t\xi)}$$

$$= \sum_{x \in (2\pi \mathbb{Z})^n} \widehat{e^{-iQ}}\left(\frac{x}{t}\right) \frac{1}{t^n} .$$

Hence we should try to understand

$$(*) \qquad \int e^{ix\cdot\xi - itQ(\xi)} \frac{d\xi}{2\pi}$$

as a distribution in (x,t)-space. This is essentially the kernel of the operator $e^{-itQ(D)}$ on $\mathbb{R}^n$, so I should know where the singularities are.

First we show $(*)$ is a well-defined distribution on $\mathbb{R}^{n+1}$. Take $\varphi(x)$, $\psi(t)$ in C_0^∞ and integrate

$$\int dx\,dt\ \varphi(x)\psi(t) \int e^{ix\xi - itQ(\xi)} \frac{d\xi}{(2\pi)^n}$$

$$\overset{\text{defn}}{=} \int \frac{d\xi}{(2\pi)^n} \int dx\,dt\ \varphi(x)\psi(t)\, e^{i(x\xi - tQ(\xi))}$$

$$= \int \frac{d\xi}{(2\pi)^n}\ \hat{\varphi}(-\xi)\, \hat{\psi}(Q(\xi))$$

hence there is no problem with convergence. What's more we should be able to evaluate (*) by first integrating over $Q(\xi) = \omega$ and then integrating over $\omega \geq 0$, or by integrating radially first and then over the $(n-1)$-sphere.

Let $\xi = ru$ where $r = |\xi|$. Then

$$\int e^{i(x\xi - tQ(\xi))} \frac{d\xi}{(2\pi)^n} = (2\pi)^{-n} \int_{|u|=1} d\Omega_u \int_0^\infty r^{n-1} dr\, e^{ir(x\cdot u - tQ(u))}$$

Now on the unit sphere $Q(u)$ has a minimum value [redacted] >0, hence for $t \gg |x|$ one has

$$x\cdot u - tQ(u) < 0 \qquad \forall u \quad \text{[redacted]}$$

Since [redacted]

$$\int_0^\infty r^{n-1} dr\, e^{-ir\rho} = \int_0^\infty e^{-(i\rho)r} r^{n\text{[redacted]}} \frac{dr}{r} = \frac{\Gamma(n\text{[redacted]})}{(i\rho)^{n\text{[redacted]}}}$$

we get

$$\int_0^\infty r^{n-1} dr\, e^{ir(x\cdot u - tQ(u))} = \frac{\Gamma(n\text{[redacted]})}{[i(tQ(u) - x\cdot u)]^{n\text{[redacted]}}}$$

and so

$$(2\pi)^{-n} \int e^{i(x\cdot\xi - tQ(\xi))} d\xi = (2\pi)^{-n} \frac{\Gamma(n\text{[redacted]})}{i^{n\text{[redacted]}}} \int_{|u|=1} d\Omega_u \frac{1}{(tQ(u) - x\cdot u)^{n\text{[redacted]}}}$$

This nice and C^∞ in x, t. For example if $Q(\xi) = |\xi|$, then $tQ(u) - x\cdot u = t - |x|\cos\theta$ where θ is the angle between x and u. Hence the above formula is nice for $t > |x|$, but gives problems when $t < |x|$; we know that the singularities live only on $t = |x|$.

So probably we want to do the integration in the other order,

$$\int_0^\infty dr \int_{Q(\xi)=r} \frac{d^n\xi}{dQ}\, e^{i(x\xi - tQ(\xi))}$$

$$= \int_0^\infty dr\, e^{-irt} \int_{Q(\xi)=r} \frac{d^n\xi}{dQ}\, e^{i(x\cdot\xi)}$$

$$= \int_0^\infty dr\, e^{-irt} r^{n-1} \int_{Q(\xi)=1} \frac{d^n\xi}{dQ}\, e^{ir(x\cdot\xi)}$$

What are the critical points of $\xi \longmapsto x\cdot\xi$ on $Q(\xi)=1$?

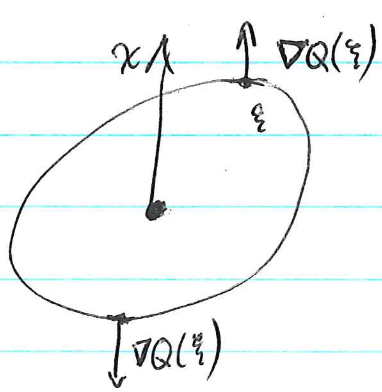
Use Lagrange multipliers:

$$F(\xi,\lambda) = x\cdot\xi - \lambda(Q(\xi)-1)$$

$$\nabla_\xi F = x - \lambda \nabla Q(\xi) = 0$$

$$\partial_\lambda F = Q(\xi) - 1 = 0$$

Recall $\nabla Q(\xi)$ is homogeneous of degree 0, hence once $\nabla Q(\xi)$ is proportional to x one can adjust $Q(\xi) = 1$. Thus the critical points are those ξ with $\nabla Q(\xi)$ proportional to x: There are 2 critical points.



Note: If $x = \lambda \nabla Q(\xi)$, then

$$x\cdot\xi = \lambda\, \nabla Q(\xi)\cdot\xi = \lambda Q(\xi) = \lambda$$

if $Q(\xi) = 1$.

Now argue that [redacted] we only have to worry about the $r \to +\infty$ behavior of $\int_{Q(\xi)=1} \frac{d^n\xi}{dQ}\, e^{ir(x\cdot\xi)}$, that one can partition off non-critical pieces. This is essentially working with a partition of 1 on the S^{n-1} sphere of rays.

A critical point ξ_c contributes a Gaussian [redacted] integral of the form

$$e^{ir(x\cdot\xi_c)} \cdot \text{constant} \cdot r^{-\frac{(n-1)}{2}} \quad + \text{ more neg. powers of } r$$

So we worry about

$$\int_0^\infty dr\, e^{-irt + ir\, x\cdot\xi_c}\, r^{\frac{n-1}{2}} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\left[i\left(t - x\cdot\xi_c\right)\right]^{\frac{n+1}{2}}}$$

It would appear therefore that the distribution

$$(2\pi)^{-n}\int d\xi\; e^{i\left(x\cdot\xi - tQ(\xi)\right)}$$

is smooth at those points (x,t) such that

$$(+) \qquad t - x\cdot\xi_c(x) \neq 0$$

where $\xi_c(x)$ denotes the point on $Q(\xi) = 1$ where $\nabla Q(\xi)$ the normal vector points in the same direction as x. (Assume $t > 0$, so that the other critical value doesn't have to be considered[redacted]. Also leave aside $x = 0$ for the moment.) Actually $x = 0$ should cause no trouble because for $t > 0$, nearby x won't provide [redacted] solutions of $(+)$.

[redacted] Therefore what emerges is that $\int \frac{d\xi}{(2\pi)^n} e^{i\xi x - itQ(\xi)}$ has singularities only at x, t satisfying

$$t = x\cdot\xi_c(x)$$

If one looks at $x\cdot\xi - tQ(\xi)$, things are OKAY if this is $\neq 0$, otherwise things are OKAY if along $Q(\xi) = 1$ we are at a point ξ where $x \cdot \xi$ has a non-critical point.

It seems to be easier to describe these in terms of the bicharacteristics:

$$\dot{x} = \frac{\partial Q}{\partial \xi} \qquad \dot{\xi} = 0.$$

We are interested in bicharacteristics beginning at $x = 0$ when $t = 0$, hence

$$x = t \frac{\partial Q}{\partial \xi}(\xi)$$

which means $\nabla Q(\xi)$ points in the direction of x and

$$\underline{x \cdot \xi = t\, Q(\xi)}$$

October 5, 1979

van der Corput situation

$$Q(\xi) = 1 \text{ on } \partial K, \quad Q \text{ homogeneous of degree } 1.$$

Assume ∂K smooth strictly convex. Then by stationary phase arguments one has

$$\hat{\chi}_K [redacted] (tu) = O(t^{-1} t^{-(\frac{n-1}{2})}) = O(t^{-(\frac{n+1}{2})}) \qquad \begin{array}{l} t \to +\infty \\ |u| = 1. \end{array}$$

to get onto ∂K — Gaussian int. factor

One wants to compute

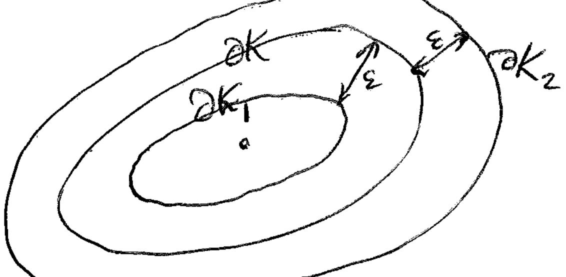
$$N(\lambda) = \sum_{x \in \mathbb{Z}^n} \chi_K\left(\frac{x}{\lambda}\right)$$

[redacted] using Poisson summation, but $\hat{\chi}_K$ doesn't decay fast enough. The idea is to replace χ_K by

$$\chi_K * g_\varepsilon \qquad\qquad g_\varepsilon(x) = \frac{1}{\varepsilon^n} g\left(\frac{x}{\varepsilon}\right)$$

where $g \geq 0$, has $\int g \, d^n x = 1$, [redacted] and $g \in C_0^\infty$. [redacted] suppose the support of g is inside $|z| < 1$. Then [redacted]

$$\chi_{K_1} \leq [redacted] \chi_K * g_\varepsilon \leq \chi_{K_2}$$



Also χ_K satisfies the above inequalities so that

$$\left| \sum_{x \in \mathbb{Z}^n} (\chi_K * g_\varepsilon)\left(\frac{x}{\lambda}\right) - \chi_K\left(\frac{x}{\lambda}\right) \right| \leq \sum_{x \in \mathbb{Z}^n} \chi_{K_2}\left(\frac{x}{\lambda}\right) - \chi_{K_1}\left(\frac{x}{\lambda}\right)$$

This last thing can be estimated by $\lambda^n \mathrm{vol}(K_2 - K_1)$ within an error of [redacted] size λ^{n-1}. Now we extend to let $\lambda \to \infty$, $\varepsilon \to 0$ in such a way that ?? This is not going to work because the [redacted] error λ^{n-1} is too big.

So turn to the second half:

$$\sum_{x \in \mathbb{Z}^n} (\chi_K * g_\varepsilon)\left(\frac{x}{\lambda}\right) = \lambda^n \sum_{x \in 2\pi\mathbb{Z}^n} \hat{\chi}_K(\lambda x)\, \hat{g}(\varepsilon \lambda x)$$

$$= \lambda^n \mathrm{vol}(K) + \lambda^n \sum_{x \in 2\pi\mathbb{Z}^n}{}' \hat{\chi}_K(\lambda x)\, \hat{g}(\varepsilon \lambda x)$$

Now we have the uniform estimate

$$\hat{\chi}_K(\lambda x) = O\left(|\lambda x|^{-\left(\frac{n+1}{2}\right)}\right) \qquad |x| \geq 2\pi$$

So the last term can be estimated by the integral

$$\lambda^n \int |\lambda x|^{-\left(\frac{n+1}{2}\right)} \hat{g}(\varepsilon \lambda x)\, d^n x = \lambda^{\frac{n-1}{2}} \int |x|^{-\left(\frac{n+1}{2}\right)} \hat{g}(\varepsilon \lambda x)\, d^n x$$

$$= \lambda^{\frac{n-1}{2}} (\varepsilon \lambda)^{\frac{n+1}{2}} (\varepsilon \lambda)^{-n} \int \underset{\substack{\uparrow \\ \text{integrable singularity}}}{|x|^{-\left(\frac{n+1}{2}\right)}} \underbrace{\hat{g}(x)}_{\text{rapidly decreasing}}\, d^n x$$

$$= [redacted]\; O\left(\varepsilon^{-\left(\frac{n-1}{2}\right)}\right).$$

The idea I had was that the error introduced in going from $\sum \chi_K\left(\frac{x}{\lambda}\right)$ to $\sum \chi_K * g_\varepsilon \left(\frac{x}{\lambda}\right)$ could be estimated by $\lambda^n \mathrm{vol}(K_2 - K_1)$ or $O(\lambda^n \varepsilon)$. Now one puts $\varepsilon = \lambda^{-d}$ ~~so that~~ and adjusts d so that the two errors

$$O(\lambda^n \varepsilon) + O\left(\varepsilon^{-\left(\frac{n-1}{2}\right)}\right)$$

are of the same size as $\lambda \to \infty$. Thus we want

$$n - d = +\left(\frac{n+1}{2}\right)d \qquad n = \frac{n+1}{2}d \quad \text{or}$$

hence error is $\overset{d = \frac{2n}{n+1}}{O(\lambda^{n-d})} = O\left(\lambda^{n-2+\frac{2}{n+1}}\right)$. So the end result is the Van der Corput result:

$$\boxed{\sum_{x \in \mathbb{Z}^n} \chi_K\left(\frac{x}{\lambda}\right) = \lambda^n \mathrm{vol}(K) + O\left(\lambda^{n-2+\frac{2}{n+1}}\right)}$$

What I did wrong on page 303 is to try to estimate the errors in replacing χ_K by $\chi_K * g_\varepsilon$ in terms of χ_K instead of $\chi_{K_j} * g_\varepsilon$ which one has better control of. Thus the good estimates are

$$\begin{aligned} \chi_{K_1} * g_\varepsilon \;\; &\le \chi_K \le \chi_{K_2} * g_\varepsilon \\ &\le \chi_K * g_\varepsilon \le \end{aligned}$$

so that

$$\left|\sum \chi_K\left(\frac{x}{\lambda}\right) - \chi_K * g_\varepsilon\left(\frac{x}{\lambda}\right)\right| \le \sum \left(\chi_{K_2} * g_\varepsilon - \chi_{K_1} * g_\varepsilon\right)\left(\frac{x}{\lambda}\right)$$

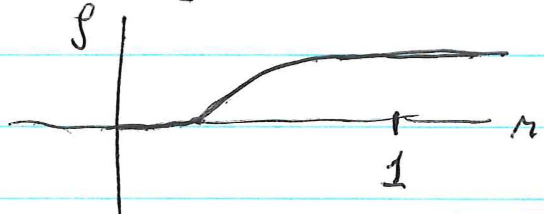
$$= \underbrace{\lambda^n \mathrm{vol}(K_2 - K_1)}_{O(\lambda^n \varepsilon)} + O\left(\varepsilon^{-\left(\frac{n-1}{2}\right)}\right)$$

The next question is whether one can write the error term more explicitly using the points on ∂K which have rational tangent plane. The Hormander mechanism is to replace $N(\lambda) = \sum_{\mathbb{Z}^n} \chi_K\left(\frac{x}{\lambda}\right)$ by $\mathrm{tr}\left(e^{-itQ(D)}\right)$ and to get info on the former by some sort of Tauberian arguments.

Here $Q(\xi)$ is homogeneous of degree 1 with value 1 on ∂K and it has a singularity at $\xi = 0$, which makes its Fourier transform

$$\check{Q}(x) = \int \frac{d\xi}{2\pi} e^{-ix\xi} Q(\xi)$$

non-rapidly decreasing at ∞. So we can choose a smooth function ρ:



and put $P(\xi) = \rho(|\xi|)\, Q(\xi)$. Then $\check{P}(x)$ should decay at ∞:

$$x^\alpha \check{P}(x) = \int \frac{d\xi}{2\pi} D_\xi^\alpha e^{-ix\xi} P(\xi) = \int \frac{d\xi}{2\pi} e^{-ix\xi} D_\xi^\alpha P(\xi).$$

When a homogeneous function is differentiated its degree goes down. So it's clear that $\check{P}(x)$ has a singularity at $x = 0$ and otherwise it is smooth and rapidly decreasing as $|x| \longrightarrow \infty$.

Since $P(\xi) = Q(\xi)$ for $\xi \in \mathbb{Z}^n$ we have

$$\operatorname{tr} e^{-itP(D)} = \operatorname{tr} e^{-itQ(D)}$$

Note that

$$\operatorname{tr} e^{-itP(D)} = \sum_{\xi \in \mathbb{Z}^n} e^{-itP(\xi)}$$

$$= \int e^{-it\lambda} \sum_{\xi \in \mathbb{Z}^n} \delta(\lambda - P(\xi))\, d\omega$$

$$N(\lambda) = \operatorname{card} \{ \xi \in \mathbb{Z}^n \mid P(\xi) \leq \lambda \}$$

jumps by 1 as λ passes through one of the values $P(\xi)$. Hence

$$\frac{d}{d\lambda} N(\lambda) = \sum_{\xi \in \mathbb{Z}^n} \delta(\lambda - P(\xi))$$

so we are taking the Fourier transform of $\frac{d}{d\lambda}N(\lambda)$:

$$\operatorname{tr} e^{-itP(D)} = \int e^{-it\lambda} \frac{d}{d\lambda} N(\lambda)\, d\lambda$$

hence the singularities of the distribution $\operatorname{tr} e^{-itP(D)}$ are related to the growth of $\frac{d}{d\lambda}N(\lambda)$ as $\lambda \to \infty$. To express these relations one needs Tauberian thms.

Let's work out the singularities of $\operatorname{tr}(e^{-itP(D)})$ in the situation under consideration where P is associated to the strictly convex gadget K. The operator $e^{-itP(D)}$ on the torus $\mathbb{R}^n/2\pi\mathbb{Z}^n$ has the kernel

$$K^T(x-x';t) = \sum_{\xi\in\mathbb{Z}^n} e^{i\xi(x-x')}\, e^{-itP(\xi)}$$

which should be the result of making

$$K(x-x';t) = \int \frac{d\xi}{(2\pi)^n} e^{i\xi(x-x')}\, e^{-itP(\xi)}$$

periodic.

$$\sum_{\nu\in 2\pi\mathbb{Z}^n} K(x+\nu;t) = \int \frac{d\xi}{(2\pi)^n} \sum_{\nu} e^{i\xi(x+\nu)}\, e^{-itP(\xi)}$$

$$\parallel$$

But $\displaystyle\sum_{\nu\in 2\pi\mathbb{Z}^n} e^{i\xi\nu} = (2\pi)^n \sum_{\mu\in\mathbb{Z}^n} \delta(\xi-\mu)$ so $\displaystyle\sum_{\xi\in\mathbb{Z}^n} e^{i\xi x - itP(\xi)}$.

Thus

$$K^T(x,t) = \sum_{\nu\in 2\pi\mathbb{Z}^n} K(x+\nu,t)$$

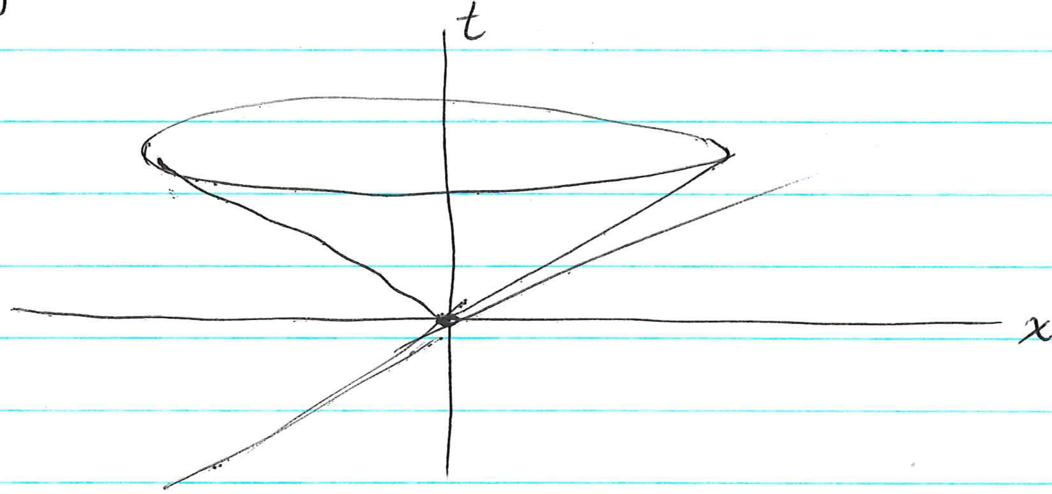
as distributions.

We now have to review what we know about

the singularities of $K(x,t) = \int \frac{d\xi}{(2\pi)^n} e^{i(\xi x - P(\xi)t)}$.
This [redacted] will be smooth near x,t where the phase factor has no stationary point. Stationary means

$$\nabla(\xi \cdot x - P(\xi)t) = x - \nabla P(\xi)t = 0$$

(Here we assume $|\xi| \geq 1$ so that $P(\xi) = Q(\xi)$ is homogeneous of degree 1). So to get the singularities of $K(x,t)$ we draw the lines $x = \nabla P(\xi)t$ as ξ ranges over $|\xi| = 1$.



Hence one has a nice cone of singularities. [redacted]

Now the picture for the periodic case $K^T(x,t)$ should be obtained by [redacted] summing [redacted] over the lattice $2\pi\mathbb{Z}^n$ all translates of the above cone. This leads to the following problems:

1) First understand the nature of the singularities of $K(x,t)$. You know they are located on the conical hypersurface [redacted] pictured above, but you really want a local description, i.e. some sort of recognizable distribution along the hypersurface.

2) Check that $K(x,t)$ decays fast enough in the x direction so that one can sum its translates

over the lattice $2\pi\mathbb{Z}^n$ without introducing new singularities.

Look again at

$$K(x,t) = \int \frac{d\xi}{(2\pi)^n} e^{i(x\cdot\xi - tP(\xi))}$$

[redacted] with [redacted] x, t near x_0, t_0 and $t_0 > 0$. Critical points [redacted] of the phase are given by

$$[redacted] \quad x - t\nabla P(\xi) = 0$$

and we are assuming that $x_0 = t_0 \nabla P(\xi_0)$ with $|\xi_0| = 1$. [redacted] We know for any x, t there is a unique ξ, $|\xi| = 1$ with

$$\nabla P(\xi) = \frac{x}{t} \,.$$

[redacted]

Now the real problem is to handle the fact that the stationary points form a stationary ray, and that the actual singularity results from the infinite integration along the ray. On page 300 we handled this by separating ξ into ru with $P(u) = 1$, but maybe it is possible to proceed directly. The point is that if we restrict $x\xi - tP(\xi)$ to $P(\xi) = 1$ or to $|\xi| = 1$, then the critical points for this integral [redacted] are more than we want - we want only these critical points such that $x\xi - tP(\xi) = 0$.

Let us change to

$$\tilde{K}(x,t) = \int \frac{d\xi}{(2\pi)^n} e^{i(x\xi - tQ(\xi))}$$

which differs from $K(x,t)$ by a smooth function. Then use homogeneity:

$$\tilde{K}(x,t) = \int \frac{d\xi}{(2\pi)^n}\, e^{i\left(\frac{x}{t}\cdot t\xi - Q(t\xi)\right)}$$

$$= \frac{1}{t^n}\int \frac{d\xi}{(2\pi)^n}\, e^{i\left(\frac{x}{t}\cdot\xi - Q(\xi)\right)} = \frac{1}{t^n}\tilde{K}\left(\frac{x}{t},1\right)$$

Let's treat this by a second order approximation at the critical point. Let ξ_0 satisfy $\nabla Q(\xi_0) = \frac{x}{t}$. Then

$$\frac{x}{t}\cdot\xi - Q(\xi) = \nabla Q(\xi_0)\cdot\xi - Q(\xi)$$

$$= \nabla Q(\xi_0)\cdot(\xi - \xi_0) + Q(\xi_0) - Q(\xi)$$

$$= -\frac{1}{2}\frac{\partial^2 Q}{\partial\xi_\alpha\partial\xi_\beta}(\xi_0)(\xi_\alpha - \xi_{0\alpha})(\xi_\beta - \xi_{0\beta}) + \cdots$$

but this doesn't seem to help, because the last expression is not homogeneous. So let us go back to our radial calculation. Suppose $t=1$.

$$(2\pi)^n\tilde{K}(x,1) = \int d\xi\, e^{i(x\cdot\xi - Q(\xi))}$$

$$= \int_0^\infty dr \int_{Q(\xi)=r} \frac{d^n\xi}{dQ}\, e^{i(x\cdot\xi - r)}$$

$$= \int_0^\infty r^{n-1}dr \int_{Q(\xi)=1}\frac{d^n\xi}{dQ}\, e^{ir(x\cdot\xi - 1)}$$

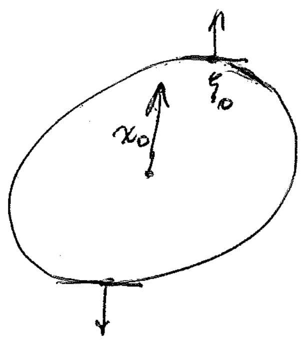
The program is to understand the singularities of

$$K(x,t) = (2\pi)^{-n} \int d\xi \, e^{i(x\cdot\xi - tQ(\xi))}$$

which lie on the conical hypersurface swept out by the lines $x = t\,\nabla Q(\xi)$ for various ξ. For the purpose of the trace we need understand only the singularities of $K(x_0, t)$ where x_0 is fixed. In this case the phase has one critical point with $t>0$, namely, where

$$x_0 = t_0 \nabla Q(\xi_0)$$

Moreover the function $x_0\cdot\xi$ on $Q(\xi)=1$ has two critical points where x_0 is proportional to $\nabla Q(\xi)$.



So we do the integral over $Q(\xi)=r$ first:

$$(2\pi)^n K(x_0,t) = \int_0^\infty dr\, e^{-itr} \int_{Q(\xi)=r} \frac{d^n\xi}{dQ} e^{i(x_0\cdot\xi)}$$

$$= \int_0^\infty dr\, e^{-itr} r^{n-1} \int_{Q(\xi)=1} \frac{d^n\xi}{dQ} e^{ir(x_0\cdot\xi)}$$

Now apply stationary phase to the latter integral. It gives as $r\to\infty$

$$\int_{Q(\xi)=1} \frac{d^n\xi}{dQ} e^{ir(x_0\cdot\xi)} = e^{ir(x_0\cdot\xi_0)} \left(\sqrt{2\pi}\right)^{n-1} e^{\frac{i\pi}{4}\operatorname{sg} H} \left|\det H\right|^{-1/2}$$

$$\times r^{-\left(\frac{n-1}{2}\right)} \cdot \left(\text{factor giving } \frac{d^n\xi}{dQ}(\xi_0) \text{ rel. to Euclidean volume}\right) \quad +$$

where H is the Hessian of $x_0\cdot\xi$ on $Q(\xi)=1$ at ξ_0

There's another term do the the other critical point and an error $O(r^{-(\frac{n-1}{2})-1})$. The other critical point [redacted] won't contribute a singularity when one does the r integration since $t > 0$. But the error term probably means we are only getting the leading part of the singulatiy.

$$\int_0^\infty dr\, e^{-itr}\, r^{n-1}\, e^{ir\overbrace{(x_0\cdot\xi_0)}^{t_0}}\ \text{[redacted]}\ (\sqrt{2\pi})^{n-1}\ \overbrace{e^{-i\frac{\pi}{4}\operatorname{sg} H}}^{e^{-i\frac{\pi}{4}(n-1)}}\ r^{-(\frac{n-1}{2})}$$

$$= \int_0^\infty \frac{dr}{r}\, e^{-i(t-t_0)r}\, r^{\frac{n+1}{2}}\ \text{[redacted]}\ e^{-\frac{i\pi}{4}(n-1)}(\sqrt{2\pi})^{n-1}$$

$$= \frac{\Gamma(\frac{n+1}{2})}{(i(t-t_0))^{\frac{n+1}{2}}}\, e^{-\frac{i\pi}{4}(n-1)}(\sqrt{2\pi})^{n-1} = \frac{\Gamma(\frac{n+1}{2})\, 2\pi^{\frac{n-1}{2}}}{(t-t_0)^{\frac{n+1}{2}}}(-i)^n$$

I need an example: Take $Q(\xi) = |\xi|$ in 3 dim.

$$\int d\xi\, e^{i(\text{[redacted]}\, x_0\cdot\xi - t|\xi|)}$$

will converge nicely for t in [redacted] LHP

[redacted] Let θ be the angle between ξ and x_0.

$$x_0\cdot\xi = t_0 r\cos\theta \qquad t_0 = |x_0|$$

$$\int d\xi\, e^{i(x_0\cdot\xi - t|\xi|)} = 2\pi\int_0^\infty r^2 dr\, e^{-itr}\int_0^\pi \frac{e^{it_0 r\cos\theta}}{-it_0 r}\,(-it_0 r\sin\theta)\, d\theta$$

$$= 2\pi\int_0^\infty r^2 dr\, e^{-itr}\left[\frac{e^{it_0 r\cos\theta}}{-it_0 r}\right]_0^\pi = \frac{2\pi}{it_0}\int_0^\infty r^2\frac{dr}{r}\left[e^{-i(t-t_0)r} - e^{-i(t+t_0)r}\right]$$

$$(e^{it_0 r} - e^{-it_0 r})/it_0 r$$

$$= \frac{2\pi}{i t_0}\left(\frac{\Gamma(2)}{(i(t-t_0))^2} - \frac{\Gamma(2)}{(i(t+t_0)^2)}\right) = \frac{2\pi i}{t_0}\left(\frac{1}{(t-t_0)^2} - \frac{1}{(t+t_0)^2}\right)$$

where $t_0 = |x_0|$. The above integrations are rigorous for $t \in$ LHP, and should hold as distributions as t becomes real.

Note

$$\frac{1}{|x_0|}\left(\frac{1}{(t-|x_0|)^2} - \frac{1}{(t+|x_0|)^2}\right) = \frac{1}{\cancel{|x_0|}} \frac{4t\cancel{|x_0|}}{(t^2-x_0^2)^2}$$

hence it should be true that

$$\frac{4t}{(t^2-x^2)^2} = -2\frac{\partial}{\partial t}\left(\frac{1}{t^2-x^2}\right) \quad \text{and hence} \quad \frac{1}{t^2-x^2}$$

satisfy the wave equation $(\partial_t^2 - \partial_x^2)u = 0$ in 3 dims.

(OKAY because we know that $\Delta\left(\frac{1}{r^2}\right) = 0$ in 4 dims.

Recall in n dims.

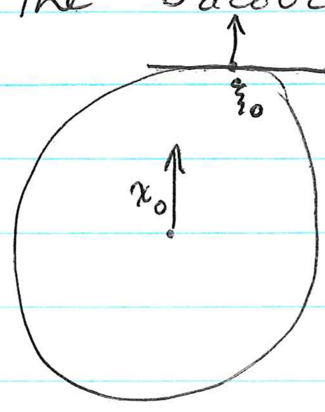
$$\Delta = \frac{1}{r^{n-1}}\frac{\partial}{\partial r} r^{n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\text{ spherical Laplacian}$$

hence the radial harmonic functions are given by

$$0 = r^{n-1}\Delta r^s = \frac{\partial}{\partial r} r^{n-1} s r^{s-1} = s(n+s-2) \qquad \text{or} \qquad s = 0,\ 2-n$$

so these are $1, \frac{1}{r^{n-2}}$.)

What remains is to figure out the Jacobian type factors.



Somewhat simpler approach:

Since we are concerned only about ξ near the ray $\mathbb{R}_{>0}\xi_0$, we can first integrate over the planes $x_0 \cdot \xi =$ [redacted] r and then [redacted] over [redacted] r.

Let's choose orthog. coords in ξ space so that positive ξ_n-axis points along x_0, hence $\xi_n = \xi \cdot \frac{x_0}{|x_0|}$. Then

$$\int d\xi \, e^{i(x_0 \cdot \xi - tQ(\xi))} = \int d\xi_n \, e^{i|x_0|\xi_n} \int d\xi_1 \cdots d\xi_{n-1} \, e^{-it\xi_n Q\left(\frac{\xi_1}{\xi_n}, \cdots, \frac{\xi_n}{\xi_n}\right)}$$

$$= \int dr \, e^{i|x_0|r} \, r^{n-1} \int d^{n-1}\xi' \, e^{-itr Q(\xi', 1)} \qquad \begin{cases} r = \xi_n \\ \xi_i = \xi_i' r & i < n \end{cases}$$

Remember that $x_0 \cdot \xi - tQ(\xi)$ has critical point at ξ_0, t_0 where

$$x_0 = t_0 \nabla Q(\xi_0)$$

and that $x_0 \cdot \xi_0 = t_0 Q(\xi_0)$. Let's choose ξ_0 so that $x_0 \cdot \xi_0 = |x_0|$, hence $\xi_0 = (\xi_0', 1)$ and let's expand $Q(\xi', 1)$ about its critical point ξ_0':

$$Q(\xi', 1) = Q(\xi_0', 1) + \tfrac{1}{2} H_{\alpha\beta} \text{ [redacted] } (\xi' - \xi_0')^\alpha (\xi' - \xi_0')^\beta + \cdots$$

where H is positive-definite. Then the above integral gets estimated by

$$\int_0^\infty dr \, r^{n-1} e^{ir(|x_0| - tQ(\xi_0))} \frac{1}{(tr)^{\frac{n-1}{2}}} |\det H|^{-1/2} (2\pi)^{\frac{n-1}{2}} \, e^{-i\frac{\pi}{4}(n-1)}$$

$$|x_0| = x_0 \cdot \xi_0 = t_0 \nabla Q(\xi_0) \cdot \xi_0$$
$$= t_0 Q(\xi_0) \qquad \therefore \; |x_0| - tQ(\xi_0) = (t_0 - t) Q(\xi_0)$$

$$\int_0^\infty \frac{dr}{r} \, r^{\frac{n+1}{2}} \, e^{-i(t-t_0)Q(\xi_0) r} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\left[i(t-t_0)Q(\xi_0)\right]^{\frac{n+1}{2}}}$$

So we get

$$\int d\xi\, e^{i(x_0\cdot\xi - tQ(\xi))} \approx \frac{\Gamma\left(\frac{n+1}{2}\right)}{\left[i(t-t_0)Q(\xi_0)\right]^{\frac{n+1}{2}}} \frac{1}{t^{\frac{n-1}{2}}} |\det H|^{-1/2} (2\pi)^{\frac{n-1}{2}} \times e^{-i\frac{\pi}{2}\left(\frac{n-1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right) \quad (2\pi)^{\frac{n-1}{2}}\, i^{-n}}{(t-t_0)^{\frac{n+1}{2}}\, Q(\xi_0)^{\frac{n+1}{2}}\, t^{\frac{n-1}{2}}\, |\det H|^{1/2}}$$

The only thing strange about this is the $t^{\frac{n-1}{2}}$, however note that the error is one more power of $t-t_0$ in the denominator, so that modulo the error this factor can be replaced by $t_0^{\frac{n-1}{2}}$.

It's more or less clear that the method used so far isn't going to give the full singularity on the t-line without much more work. In the Duistermaat-Guillemin situation the singularities are of the form $\frac{1}{t-t_0}$, but one assumes some kind of non-degeneracy hypothesis.

The program is to understand the singularities of

$$K(x,t) = (2\pi)^{-n}\int d\xi \; e^{-i(\xi\cdot x - P(\xi)\cdot t)}$$

Note that $K(x,t)$ is the solution of

$$i\frac{\partial\psi}{\partial t} = P(D)\psi$$

such that $K(x,0) = \delta(x)$.

Possibility: The singularity structure should propagate by means of ODE's. Recall what we did for

$$\partial_t^2\psi = (\partial_x^2 - V(x))\psi .$$

One can solve $(-\partial_x^2 + V)u = k^2 u$ formally by ~~asymptotic series~~ a series

$$u(x,k) = e^{ikx}\left(1 + \frac{a_1(x)}{k} + \frac{a_2(x)}{k^2} + \cdots\right)$$

namely

$$a_1' = \frac{1}{2i}V$$

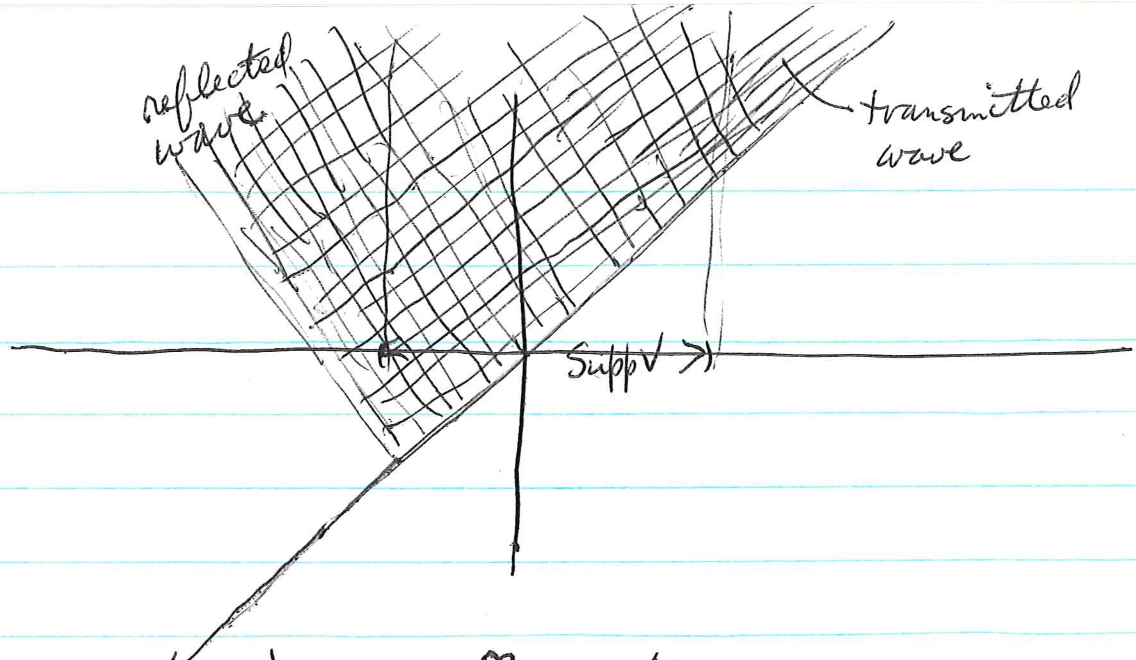
$$a_2' = \frac{1}{4}V' + \frac{1}{2i}Va_1$$

The constants can be determined by requiring $u(x,k) = e^{-ikx}$ for $x \ll 0$ (assuming V has compact support).

Next suppose $\psi(x,t)$ and $u(x,k)$ are related by

$$\psi(x,t) = \int \frac{dk}{2\pi}\, u(x,k)e^{-ikt}$$

so that ψ is the solution of the wave equation which is $\delta(x-t)$ for $x \ll 0$. Then ψ has support:



and so $\psi(x,t)$ is C^∞ with [redacted] a jump along $t=x$, and this is reflected in the fact that

$$u(x,k) = \int_{t\geq x} dt\, e^{+ikt}\,\psi(x,t)$$

has the asymptotic expansion described above.

Let's try the same thing for

$$K(x,t) = (2\pi)^{-n}\int d\xi\, e^{i(\xi\cdot x - P(\xi)\cdot t)}$$

$$u(x,k) = \int dt\, e^{ikt}\, K(x,t)$$

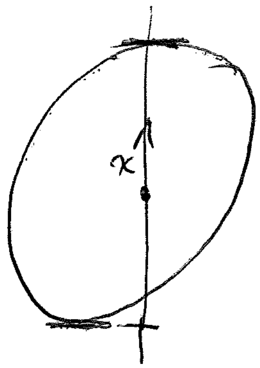
$$= (2\pi)^{-n}\int d\xi\, e^{i\xi\cdot x}\,\delta(P(\xi)-k)\, 2\pi$$

$$= (2\pi)^{-n+1}\int_{P(\xi)=k} \frac{d^n\xi}{dP}\, e^{i\xi\cdot x}$$

$P=Q$ once $k \gg 0$.

$$= (2\pi)^{-(n-1)}\, k^{n-1}\int_{Q(\xi)=1}\frac{d^n\xi}{dQ}\, e^{ik\xi\cdot x}$$

Now this has a nice asymptotic expansion [redacted] given by stationary phase. If $|x|=1$, then $\xi\cdot x$ is the projection of ξ onto the line of x, so there are two critical points which means as we already know that there are two singularities in t for $K(x,t)$.



$$K(x,t) = (2\pi)^{-n}\int d\xi \; e^{-i(\xi x - P(\xi)t)}$$

$$u(x,k) = \int K(x,t)\, e^{ikt}\, dt = (2\pi)^{-n}\int d\xi \; e^{i\xi x} \; 2\pi\, \delta(P(\xi)-k)$$

$$= (2\pi)^{-n+1} \int_{P(\xi)=k} \frac{d^n\xi}{dP}\, e^{-i\xi x} \underset{|k|\gg 0}{\overset{\uparrow}{=}} (2\pi)^{-n+1} k^{n-1} \int_{Q(\xi)=1} \frac{d^n\xi}{dQ}\, e^{ik\xi\cdot x}$$

We know from stationary phase that this last integral has an asymptotic expansion coming from the two critical points of $\xi\cdot x$ on $P(\xi)=1$. Let's concentrate on the critical point with $\xi\cdot x > 0$, and let's put $t(x) = \xi_c(x)\cdot x$. Thus $t = t(x)$ describes the cone $x = \nabla Q(\xi)t$ for different ξ. Then the upper half of the asymptotic expansion for $u(x,k)$ is of the form

$$e^{ikt(x)}\, k^{\frac{n-1}{2}}\left(a_0(x) + \frac{a_1(x)}{k} + \cdots \right)$$

What we should do now is to try to directly construct a formal solution of

$$P(D)u = ku$$

of the form

$$e^{ikt(x)}\left(a_0(x) + \frac{a_1(x)}{k} + \frac{a_2(x)}{k^2} + \cdots\cdots\right)$$

This obviously requires understanding the asymptotic exp. of

$$e^{-ikt(x)} P(D)\, e^{ikt(x)} = P(D + k\nabla t)$$

as $k\to\infty$, which is part of standard ΨDO theory.

Let's compute the first few terms

$D = \frac{1}{i}\frac{\partial}{\partial x}$

$$D^n(e^{ik\varphi} v) = D^n(e^{ik\varphi})v + nD^{n-1}(e^{ik\varphi})Dv + \cdot$$

$$D^n(e^{ik\varphi}) = D^{n-1}(e^{ik\varphi}\, kD\varphi) = D^{n-2}\left(e^{ik\varphi}\left(k^2(D\varphi)^2 + kD^2\varphi\right)\right)$$

$$= D^{n-3}\left(e^{ik\varphi}\left(k^3(D\varphi)^3 + k^2 D\varphi\, D^2\varphi + k^2 D\varphi\, D^2\varphi + O(k)\right)\right)$$

$$= D^{n-3}\left(e^{ik\varphi}\left(k^3(D\varphi)^3 + 3k^2 D\varphi\, D^2\varphi + O(k)\right)\right)$$

$$= D^{n-4}\left(e^{ik\varphi}\left(k^4(D\varphi)^4 + 6k^3(D\varphi)^2 D^2\varphi + O(k^2)\right)\right)$$

$$= e^{ik\varphi}\left(k^n(D\varphi)^n + \binom{n}{2}k^{n-1}(D\varphi)^{n-2}D^2\varphi + O(k^{n-2})\right).$$

Thus
$$e^{-ik\varphi}D^n(e^{ik\varphi}) = k^n(D\varphi)^n + nk^{n-1}(D\varphi)^{n-2}D^2\varphi + O(k^{n-2})$$

so
$$e^{-ik\varphi}D^n(e^{ik\varphi}v) = k^n(D\varphi)^n v + k^{n-1}\left[\left(n(D\varphi)^{n-1}Dv\right) + \binom{n}{2}(D\varphi)^{n-2}D^2\varphi\, v\right] + O(k^{n-2}).$$

Better method: $\quad e^{-ik\varphi}P(D)e^{ik\varphi} = P(D + k\nabla\varphi)$.

You want to evaluate this at x where $\nabla\varphi(x) = \eta$

so write $\quad \nabla\varphi = \eta + \rho \quad$ where $\rho(x) = 0$. Then

$$P(D + k\nabla\varphi) = P(k\eta) + \frac{\partial P}{\partial\xi}(k\eta)(D + k\rho) + \frac{1}{2}\frac{\partial^2 P}{\partial\xi^2}(k\eta)\underbrace{(D + k\rho)^2}_{D^2 + 2k\rho D + k^2\rho^2 + k(D\rho)} + \ldots$$

since ρ vanishes at η this gives

$$P(D + k\nabla\varphi) = P(k\nabla\varphi) + \frac{\partial P}{\partial\xi}(k\nabla\varphi)D + \frac{1}{2}\frac{\partial^2 P}{\partial\xi^2}(k\nabla\varphi)kD^2\varphi +$$

and if Q is homogeneous of degree 1 we get

$$Q(D + k\nabla\varphi) = kQ(\nabla\varphi) + \left[\frac{\partial Q}{\partial\xi}(\nabla\varphi)D + \frac{1}{2}\frac{\partial^2 Q}{\partial\xi^2}(\nabla\varphi)D^2\varphi\right] + O\left(\frac{1}{k}\right)$$

So to solve $P(D+k\nabla t)\, v = kv$ via an asymptotic expansion amounts to:

$$(*)\qquad \left\{ k\,Q(\nabla t) + \left[\frac{\partial Q}{\partial \xi}(\nabla t)\, D + \frac{1}{2}\frac{\partial^2 Q}{\partial \xi^2}(\nabla t)(-\nabla^2 t)\right] + \frac{1}{k}\cdots \right\} v = kv$$

Recall $t = t(x)$ describes the singularity cone of lines $x = t\nabla Q(\xi)$, hence $\nabla t(x)$ points in the direction of x, and since $t(x)$ is homogeneous of degree 1, one has

$$\nabla t(x)\cdot x = t(x)$$

Thus $\nabla t(x) = \xi(x)$ = point on $Q(\xi) = 1$ where normal vector has same direction as x. Thus

$$Q(\nabla t) \equiv 1.$$

Also $\frac{\partial Q}{\partial \xi}(\nabla t) = \nabla Q(\xi(x)) = \frac{x}{t(x)}$. I can sort of see how to use the above $*$ to grind out a ~~asymptotic~~ formal series solution $a_0(x) + \frac{a_1(x)}{k} + \cdots$ starting from an $a_0(x)$ satisfying

$$\left[\frac{\partial Q}{\partial \xi}(\nabla t)\, D + \frac{1}{2}\frac{\partial^2 Q}{\partial \xi^2}(\nabla t)\,(-\nabla^2 t)\right] a_0 = 0$$

or

$$\left[x\cdot D + t(x)\,\frac{1}{2}\frac{\partial^2 Q}{\partial \xi^2}(\nabla t)\,(-\nabla^2 t)\right] a_0 = 0$$

You integrate this along rays thru O. It seems possible to determine how a_0 should look as one approaches O along a ray, so one maybe gets the whole singularity structure of $K(x,t)$ in this way.

Dual convex body: Recall that if K is a convex body in ξ-space which is symmetric $(-K = K)$, then K defines a norm in ξ-space. In fact [redacted] $\|\xi\| =$ the homogeneous function $Q(\xi)$ of degree 1 with [redacted] value 1 on ∂K. To [redacted] see Q is a norm suppose given $\xi, \eta \neq 0$. Then $\frac{\xi}{|\xi|}, \frac{\eta}{|\eta|} \in K$ so

$$\frac{\xi+\eta}{|\xi|+|\eta|} = \frac{|\xi|}{|\xi|+|\eta|} \cdot \frac{\xi}{|\xi|} + \frac{|\eta|}{|\xi|+|\eta|} \frac{\eta}{|\eta|} \in K, \text{ hence } |\xi+\eta| \leq |\xi|+|\eta|.$$

Then from Banach space theory we know that the dual space, x-space, to ξ-space carries a dual norm:

$$|x| = \sup \frac{\xi \cdot x}{|\xi|}$$

But all this works even if K isn't symmetric. In fact define as we have done already

$$t(x) = \sup \frac{\xi \cdot x}{Q(\xi)}$$

Then clearly $t(\lambda x) = \lambda t(x)$ for $\lambda > 0$ and also $t(x+y) \leq t(x) + t(y)$, so that

$$\{x \mid t(x) \leq 1\}$$

is a dual convex body to K in x-space.

October 9, 1979

$$\sum_{\xi\in\mathbb{Z}^n} \delta(\lambda - Q(\xi)) = \sum_{x\in 2\pi\mathbb{Z}^n} \int e^{i\xi\cdot x}\,\delta(\lambda - Q(\xi))\,d\xi$$

$$= \sum_{x\in 2\pi\mathbb{Z}^n} \int_{Q(\xi)=\lambda} \frac{d^n\xi}{dQ}\, e^{i\xi x}$$

$$= \sum_{x\in 2\pi\mathbb{Z}^n} \lambda^{n-1} \int_{Q(\xi)=1} \frac{d^n\xi}{dQ}\, e^{i\lambda\xi x}$$

This has the "leading" term $\lambda^{n-1}\int_{Q(\xi)=1}\frac{d^n\xi}{dQ}$ [redacted]. The idea is to write the rest as a sum over the rational rays in x-space. [redacted] The point is that the critical point on $Q(\xi)=1$ belonging to the x in the same ray is the same.

The real question is as follows: We have an asymptotic expansion for

$$\int_{Q(\xi)=1} \frac{d^n\xi}{dQ}\, e^{i\lambda\xi\cdot x} \sim e^{i\lambda t(x)}\,\lambda^{-\left(\frac{n-1}{2}\right)} \text{ etc.}$$

and you would like to try to sum this over the lattice. Is there some sort of simple relation between the counting problems for K and the dual convex body?

October 10, 1979:

Consider $-\Delta + m^2 + V$ on the line where $V \in C_0^\infty(\mathbb{R})$ is small. The operator

$$(-\Delta + m^2 + V)^{-s} = \frac{1}{2\pi i} \oint \lambda^{-s} \frac{1}{\lambda + \Delta - m^2 - V} \, d\lambda$$

doesn't have a trace because its' spectrum is continuous, however upon subtracting $(-\Delta + m^2)^{-s}$ it perhaps does:

$$\mathrm{tr}\left[(-\Delta+m^2+V)^{-s} - (-\Delta+m^2)^{-s}\right] = \frac{1}{2\pi i}\oint \lambda^{-s}\, \overset{\mathrm{tr}}{\wedge}\left(\frac{1}{\lambda+\Delta-m^2-V} - \frac{1}{\lambda+\Delta-m^2}\right) d\lambda$$

$$= \frac{1}{2\pi i}\oint (\lambda+m^2)^{-s}\, \mathrm{tr}\left(\frac{1}{\lambda+\Delta-V} - \frac{1}{\lambda+\Delta}\right) d\lambda$$

$$= \frac{-1}{2\pi i}\int_{-\infty}^{\infty} (k^2+m^2)^{-s}\, \mathrm{tr}\left(G_k - G_k^0\right) 2k\,dk$$

Here k is approached from the UHP so

$$G_k^0 = \frac{e^{ik|x-x'|}}{2ik}$$

Recall the Born series:

$$G = G^0 + G^0 V G^0 + G^0 V G^0 V G^0 + \cdots$$

$$\mathrm{tr}(G - G^0) = \mathrm{tr}(G^0 V G^0 + \cdots\cdots)$$

$$\mathrm{tr}(G^0 V G^0) = \frac{1}{(2ik)^2}\int e^{ik|x_1-x_2|} V(x_2) e^{ik|x_2-x_1|}\, dx_1 dx_2$$

$$= \frac{1}{(2ik)^2}\int V(x)dx \int e^{2ik|x|} dx$$

$$= \frac{1}{(2ik)^2}\left(\int V\right) 2 \frac{1}{-2ik} = \frac{1}{4ik^3}\int V$$

$$\operatorname{tr}(G^\circ V G^\circ V G^\circ) = \frac{1}{(2ik)^3}\int e^{ik|x_1-x_2|}V(x_2)e^{-ik|x_2-x_3|}V(x_3)e^{ik|x_3-x_1|}\,dx_1\,dx_2\,dx_3$$

$$= \frac{1}{(2ik)^3}\,2\int_{x_2<x_3} V(x_2)V(x_3)\,dx\,dx_2 \underbrace{\int dx_1\; e^{ik\underbrace{[|x_1-x_2|+|x_2-x_3|+|x_3-x_1|]}_{2\text{ diam of }\{x_1,x_2,x_3\}}}}$$

$$\int_{x_1<x_2} dx_1\, e^{2ik(x_3-x_1)} + \int_{x_2<x_1<x_3} dx_1 e^{2ik(x_3-x_2)} + \int_{x_1>x_3} dx_1\, e^{2ik(x_1-x_2)}$$

$$e^{2ik(x_3-x_2)}\left\{\int_{x_1<x_2} dx_1 e^{2ik(x_2-x_1)} + x_3-x_2 + \int_{x_1>x_3} dx_1\, e^{2ik(x_1-x_3)}\right.$$

$$\int_{-\infty}^{0} du\, e^{-2iku} + x_3-x_2 + \int_0^\infty du\, e^{2iku}$$

$$-\frac{1}{2ik} + x_3-x_2 + -\frac{1}{2ik}$$

So

$$\operatorname{tr}(G^\circ V G^\circ V G^\circ) = \frac{1}{(2ik)^3}\int dx_2\,dx_3\, V(x_2)V(x_3)\left[-\frac{1}{ik}+|x_3-x_2|\right]e^{2ik|x_2-x_3|}$$

Another approach

$$\operatorname{tr}\left(\frac{1}{k^2+\Delta-V} - \frac{1}{k^2+\Delta}\right) = \frac{1}{2k}\frac{d}{dk}\operatorname{tr}\log\left(\frac{k^2+\Delta-V}{k^2+\Delta}\right)$$

$$= \frac{1}{2k}\frac{d}{dk}\log\det(1-G_k^\circ V)$$

$$\operatorname{tr}\left(\log\left(1-G^\circ V\right)\right) = -\operatorname{tr}(G^\circ V) - \frac{1}{2}\operatorname{tr}\left((G^\circ V)^2\right)$$

$$\operatorname{tr}(G^\circ V) = \int \frac{1}{2ik}V = \frac{1}{2ik}\int V$$

$$\operatorname{tr}\left((G^\circ V)^2\right) = \frac{1}{(2ik)^2}\int e^{ik|x_1-x_2|}V(x_2)e^{-ik|x_2-x_1|}V(x_1)\,dx_1\,dx_2$$

$$= \frac{1}{(2ik)^2}\int V(x_1)V(x_2)\,e^{2ik|x_1-x_2|}\,dx_1\,dx_2$$

$$-\frac{1}{2k}\frac{d}{dk}\left(\frac{1}{2ik}\int V\right) = -\frac{1}{2k\cdot 2i}\left(-\frac{1}{k^2}\right)\int V = \frac{1}{4ik^3}\int V$$

so the two approaches lead to the same formula which is

$$\operatorname{tr}\left(\frac{1}{k^2+\Delta-V} - \frac{1}{k^2+\Delta}\right) = \frac{1}{4ik^3}\int V$$

$$+ \frac{1}{(2ik)^3}\int dx_1\,dx_2\, V(x_1)V(x_2)\left[-\frac{1}{ik} + |x_1-x_2|\right]e^{2ik|x_1-x_2|}$$

$$+ \;O(V^3)$$

Next lets do the ~~same~~ same calculations but in momentum space:

$$\langle x|G^0|x'\rangle = \frac{e^{ik|x-x'|}}{2ik} = \int\frac{d\xi}{2\pi}\,\frac{e^{i\xi(x-x')}}{k^2-\xi^2} \qquad \operatorname{Im} k>0$$

$$\langle \xi|G^0|\xi'\rangle = \int dx\,dx'\langle\xi|x\rangle\langle x|G^0|x'\rangle\langle x'|\xi'\rangle \qquad \langle x|\xi\rangle = e^{i\xi x}$$

$$= \int dx\,dx'\; e^{-i\xi x + i\xi' x'}\,\frac{e^{ik|x-x'|}}{2ik}$$

$$= \int dx\,dx'\; e^{-i\xi(x+x')+i\xi' x'}\,\frac{e^{ik|x|}}{2ik}$$

$$= 2\pi\,\delta(\xi-\xi')\,\frac{1}{k^2-\xi^2}$$

$$\langle\xi|V|\xi'\rangle = \int dx\, V(x)\, e^{-i\xi x + i\xi' x} = \hat V(\xi-\xi')$$

$$\operatorname{tr}(G^0 V) = \int\frac{d\xi}{2\pi}\,\frac{1}{k^2-\xi^2}\,\hat V(0) = \frac{1}{2ik}\,\hat V(0) = \frac{1}{2ik}\int V$$

$$\operatorname{tr}(G^0VG^0V) = \int \frac{1}{k^2-\xi_1^2}\hat V(\xi_1-\xi_2)\frac{1}{k^2-\xi_2^2}\hat V(\xi_2-\xi_1)\,\frac{d\xi_1}{2\pi}\,\frac{d\xi_2}{2\pi}$$

$$= \int \frac{d\xi_1\, d\xi_2}{(2\pi)^2} \underbrace{\hat{V}(\xi_1)\hat{V}(-\xi_1)}_{|\hat{V}(\xi_1)|^2 \text{ when } V \text{ is real}} \frac{1}{k^2-(\xi_1+\xi_2)^2} \frac{1}{k^2-\xi_1^2}$$

In order to evaluate the ζ function on needs to evaluate things like

$$\frac{-1}{2\pi i} \int_{-\infty}^{\infty} (k^2+m^2)^{-s} \frac{2k\,dk}{4ik^3}$$

These are Eulerian integrals I think. In any case for $s = -\frac{1}{2}$, which gives the ground energy shift (see p 215), the integral diverges logarithmically, which shows $\int V(x)dx = 0$ is necessary (+ probably sufficient) for a <u>finite energy shift.</u>