fermion algebra: This is the way the physicist handles systems of identical particles satisfying the Pauli exclusion principle such as electrons. One is given a system of N electrons and one writes a Hamiltonian of the form

$$(*) \quad H_N = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + U(x_i) \right) + \frac{1}{2} \sum_{i \neq j}' V(x_i, x_j)$$

which is interpreted as follows. $\frac{p_i^2}{2m}$ is the kinetic energy of the i-particle. One thinks of these electrons as being around a nucleus and $U(x)$ is the potential energy from the nucleus. Thus the first term is the energy of the electrons without their mutual interaction. The second term represents the interaction energy; one assumes $V(x,x') = V(x',x)$ since the two terms $\frac{1}{2}V(x_i,x_j) + \frac{1}{2}V(x_j,x_i)$ are symmetric. Typical examples are Coulomb interaction

$$V(x,x') = \frac{e^2}{|x-x'|} \qquad e^2 > 0 \text{ representing repulsion}$$

$$U(x) = \frac{-Ce}{|x|} \qquad -Ce < 0 \text{ representing attraction.}$$

How is the Hamiltonian interpreted as an operator? A single particle is described by a wave function $\psi_\alpha(x)$ with several components representing the spin degrees of freedom. N distinguishable particles would be described by a vector in $\mathcal{H}^{\otimes N}$ where $\mathcal{H}$ is the one-particle

space, that is by a wave function

$$\psi = \psi(x_1,\alpha_1; x_2,\alpha_2; \cdots; x_N,\alpha_N)$$

One puts in the Pauli exclusion principle by requiring anti-symmetry, i.e. that

$$\psi \in (\mathcal{H}^{\otimes N})_{anti}.$$

The Hamiltonian H_N is an operator on this anti-symmetric space, since $\sigma H_N \sigma^{-1} = H_N$ for all $\sigma \in \Sigma_N$.

Point: The above description is not well-suited to calculation (Mattuck refers to it as the Neanderthal description), so one translates it into 2nd quantization language. Before doing this, let's record the orthonormal basis for $(\mathcal{H}^{\otimes N})_{anti}$ furnished by Slater determinants.

Slater determinants: Let φ_i ($i \in I$) be an orthonormal basis for $\mathcal{H}$. Then an orth. basis for $\mathcal{H}^{\otimes N}$ is given by the vectors $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_N}$ for $i_1, \ldots, i_N \in I$. An orth. basis for anti-symm. subspace is furnished by the vectors

$$\Phi_{i_1 \cdots i_N} = \frac{1}{\sqrt{N!}} \sum_{\sigma \in \Sigma_N} (-1)^\sigma \varphi_{i_{\sigma 1}} \otimes \cdots \otimes \varphi_{i_{\sigma N}}$$

for each N-fold subset $i_1 < \cdots < i_N$ of the index set I. As wave functions:

$$(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_N})(x_1\alpha_1, \ldots, x_N\alpha_N) = \varphi_{i_1}(x_1\alpha_1) \cdots \varphi_{i_N}(x_N\alpha_N)$$

$$(\Phi_{i_1 \cdots i_N})(x_1\alpha_1, \ldots, x_N\alpha_N) = \frac{1}{\sqrt{N!}} \det\left[\varphi_{i_j}(x_k\alpha_k)\right]_{j,k=1}^{N}$$

2nd quantization: (This name is misleading – better to call it Fock space: Instead of $(\mathcal{H}^{\otimes N})_{anti}$ with N fixed one forms the exterior algebra

$$\bigwedge \mathcal{H} = \bigoplus_{N \geq 0} \bigwedge^N \mathcal{H}$$

Recall that if φ_i, $i \in I$, is an orth basis for $\mathcal{H}$, then $\bigwedge^N \mathcal{H}$ has the orthonormal basis

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_N} \qquad i_1 < \cdots < i_N$$

so we can identify $\bigwedge^N \mathcal{H}$ and $(\mathcal{H}^{\otimes N})_{anti}$ via $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_N} \mapsto \Phi_{i_1 \cdots i_N}$. On the exterior algebra we have creation and annihilation operators

$$a_i^* = e(\varphi_i) \qquad \text{ext. mult.}$$

$$a_i = i(\varphi_i^*) \qquad \text{int. mult. by } \varphi_i^* \text{ where } \varphi_i^* = \langle \varphi_i |$$

in terms of which we can write the total Hamiltonian $H = \bigoplus H_N$. This requires some calculation but the formulas are as follows:

First of all take a 1-particle operator, i.e. an operator on $\mathcal{H}$, say for example $\frac{p^2}{2} + U(x)$, and extend it to N particles by adding:

$$H_0 = \sum_i \frac{p_i^2}{2} + U(x_i)$$

Then the corresponding operator in Fock space notation is obtained by writing the one particle operator as a matrix relative to the basis φ_i:

$$\sum_{k,l} |\varphi_k\rangle \langle \varphi_k | \frac{p^2}{2} + U | \varphi_k \rangle \langle \varphi_l |$$

and then forming

$$\sum_{k,l} a_k^* \langle \varphi_k | \tfrac{p^2}{2} + u | \varphi_l \rangle a_l .$$

Thus the one particle operator is extended to a derivation on the exterior algebra.

In this case one often takes φ_k to be the orthonormal basis of eigenfunctions for $\frac{p^2}{2} + u(x)$ so that

$$\left(\tfrac{p^2}{2} + u\right)\varphi_k = \varepsilon_k \varphi_k$$

and then the Fock space operator is just

$$\sum_k \varepsilon_k a_k^* a_k .$$

Next consider a 2-particle operator, e.g.

$$\tfrac{1}{2} \sum_{i \neq j} V(x_i, x_j) .$$

You write this as a matrix on the 2 particle space:

$$V_{klmn} = \langle \varphi_k \wedge \varphi_l | V | \varphi_m \wedge \varphi_n \rangle$$

so that

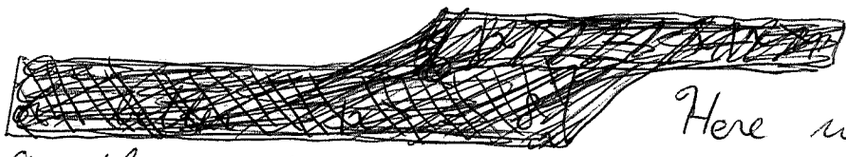
$$V = \sum_{klmn} | \varphi_k \wedge \varphi_l \rangle V_{klmn} \langle \varphi_m \wedge \varphi_n | \qquad \text{on } \Lambda^2$$

$$\boxed{V = \sum_{k<l,\, m<n} a_k^* a_l^* V_{klmn} a_n a_m}$$

It's now necessary to look at some examples.

First let's note that when V is a function of the

coordinates, the natural basis for $\mathcal{H}$ to use is the δ-function states, defined by



$$\delta_x(x') = \delta(x'-x).$$

Here we ignore spins. Another notation is $\delta_x = |x\rangle$. One has

$$\varphi_i(x) = \langle x|\varphi_i$$

$$\varphi_i = \int |x\rangle dx \langle x|\varphi_i$$

Let's write the destruction operators relative to this basis.

$$\langle x|f\rangle = \sum_i \langle x|\varphi_i\rangle\langle\varphi_i|f\rangle$$

so the destruction operator i (= derivation of degree -1 on the exterior alg) belonging to $f \longmapsto \langle x|f\rangle$ is

$$\psi(x) = \sum_k \varphi_k(x)\, a_k$$

The representation for a 2-particle operator is

$$V = \frac{1}{4}\int dx_1 dx_2 dx_3 dx_4\ \psi(x_1)^*\psi(x_2)^*\langle \delta_{x_1}\wedge\delta_{x_2}|\, V\,|\delta_{x_3}\wedge\delta_{x_4}\rangle\psi(x_4)\psi(x_3)$$

because we don't require $x_1<x_2$, $x_3<x_4$

Let's compute the matrix coefficient when V multiplies by $V(x,x')$ on the 2-particle space. Recall that

$$(\delta_{x_3}\wedge\delta_{x_4})(x,x') = \frac{1}{\sqrt{2}}\begin{vmatrix}\delta_{x_3}(x) & \delta_{x_3}(x')\\ \delta_{x_4}(x) & \delta_{x_4}(x')\end{vmatrix}$$

hence

$$\langle \delta_{x_1}\wedge\delta_{x_2}|\, V\,|\delta_{x_3}\wedge\delta_{x_4}\rangle = \frac{1}{2}\int dx dx' \begin{vmatrix}\delta_{x_1}(x) & \delta_{x_1}(x')\\ \delta_{x_2}(x) & \delta_{x_2}(x')\end{vmatrix} V(x,x') \begin{vmatrix}\delta_{x_3}(x) & \delta_{x_3}(x')\\ \delta_{x_4}(x) & \delta_{x_4}(x')\end{vmatrix}$$

$$= \frac{1}{2} V(x_1,x_2)\begin{bmatrix}\delta(x_1-x_3)\,\delta(x_2-x_4)\\ -\delta(x_1-x_4)\,\delta(x_2-x_3)\end{bmatrix} - \frac{1}{2}V(x_2,x_1)\begin{bmatrix}\delta(x_2-x_3)\,\delta(x_1-x_4)\\ -\delta(x_2-x_4)\,\delta(x_1-x_3)\end{bmatrix}$$

$$= V(x_1,x_2)\left[\delta(x_1-x_3)\,\delta(x_2-x_4) - \delta(x_2-x_3)\,\delta(x_1-x_4)\right]$$

so we get

$$V = \frac{1}{2}\int dx_1 dx_2\, \psi(x_1)^*\psi(x_2)^* V(x_1,x_2)\psi(x_2)\psi(x_1)$$

What to remember: $\psi(x)$ is the destruction operator on the exterior algebra with $\psi(x)f = \langle x|f\rangle = f(x)$ for $f \in \mathcal{H}$. Hence if ψ_k is an orthonormal basis, since

$$\langle x|f\rangle = \sum \langle x|\psi_k\rangle\langle\psi_k|f\rangle$$

we have $\psi(x) = \sum \psi_k(x)\, a_k$, $\quad a_k = -i(\langle\psi_k|)$

Degenerate electron gas: This is a model for a metal in which the lattice atoms are replaced by a uniform positive background so as to make the whole system neutral. Think of N electrons in a cubical box of volume V; the electrons interact with each other and the background; ignore spin. The Hamiltonian is

$$H = H_{el} + H_b + H_{el\text{-}b}$$

$$H_{el} = \sum_{i=1}^{N}\frac{p_i^2}{2m} + \frac{1}{2}e^2\sum_{i\neq j=1}^{N}{}' \frac{e^{-\mu|x_i-x_j|}}{|x_i-x_j|}.$$

$$H_b = \frac{1}{2}e^2\int d^3x\, d^3x' \frac{n(x)\,n(x')\,e^{-\mu|x-x'|}}{|x-x'|}$$

where the background has charge density $n(x)\,e\,d^3x$

and

$$H_{el\text{-}b} = -e^2 \sum_{i=1}^{N} \int d^3x \, \frac{n(x)\, e^{-\mu|x-x_i|}}{|x-x_i|}$$

Here μ is a positive convergence factor which is eventually allowed to go to 0, but first one takes the limit as $N, V \to \infty$ with $N/V = n$ constant. Somehow this allows one to take advantage of translation invariance and obtain with $n(x) = n$

$$H_b = \tfrac{1}{2} e^2 \int d^3x\, d^3x' \, \frac{n^2 e^{-\mu|x-x'|}}{|x-x'|}$$

$$= \tfrac{1}{2} e^2 n^2 V \int d^3x \, \frac{e^{-\mu|x|}}{|x|} = \tfrac{1}{2} e^2 n^2 V \, \frac{4\pi}{\mu^2}$$

$$= \tfrac{1}{2} e^2 \frac{N^2}{V} \frac{4\pi}{\mu^2}$$

Here I have used the fact that $\dfrac{e^{-\mu|x|}}{4\pi|x|}$ is the Green's function for $-\Delta + \mu^2$, so that

$$\frac{e^{-\mu|x|}}{4\pi|x|} = \int \frac{d^3k}{(2\pi)^3} \, \frac{e^{ikx}}{k^2+\mu^2}$$

$$\frac{1}{k^2+\mu^2} = \int \frac{e^{-\mu|x|}\, e^{-ikx}}{4\pi|x|} \, d^3x$$

Similarly $H_{el\text{-}b} = -e^2 N n \dfrac{4\pi}{\mu^2} = -e^2 \dfrac{N^2}{V} \dfrac{4\pi}{\mu^2}$

so that

$$H = -\tfrac{1}{2} e^2 \frac{N^2}{V} \frac{4\pi}{\mu^2} + H_{el}$$

Hence the effect of the background is to renormalize.

Next we write H_{el} in 2nd quantization. [redacted] since [redacted] the electrons are

moving freely in the box except for Coulomb interactions we take for orthonormal basis the states of definite momentum with periodic boundary conditions:

$$\psi_k(x) = \frac{1}{\sqrt{V}} e^{ik\cdot x}$$

where k runs over the lattice $\frac{2\pi}{L}\mathbb{Z}^3$, the box being cubical of side L. Then

$$\psi(x) = \frac{1}{\sqrt{V}} \sum e^{ikx} a_k$$

so the kinetic energy is

$$\int dx\ \psi(x)^* \left(\frac{-\Delta}{2m}\right) \psi(x) = \int \frac{dx}{V} \sum_{k,k'} e^{-ikx} a_k^* \left(\frac{-\Delta}{2m}\right) e^{ik'x} a_{k'}$$

$$= \sum_k \frac{k^2}{2m} a_k^* a_k$$

The potential energy is

$$\frac{1}{2} e^2 \int dx_1 dx_2\ \psi(x_1)^* \psi(x_2)^* \frac{e^{-\mu|x_1-x_2|}}{|x_1-x_2|} \psi(x_2)\psi(x_1)$$

$$= \frac{1}{2} e^2 \frac{1}{V^2} \sum_{k_1,k_2,k_3,k_4} a_{k_1}^* a_{k_2}^* \int dx_1 dx_2\ e^{-ik_1x_1 - ik_2x_2} e^{ik_4x_2 + ik_3x_1} \cdot \frac{e^{-\mu|x_1-x_2|}}{|x_1-x_2|} a_{k_4} a_{k_3}$$

Now

$$\int dx_1 dx_2\ e^{\underbrace{i(k_3-k_1)x_1 + i(k_4-k_2)x_2}_{x_1+x_2}} \underbrace{\frac{e^{-\mu|x_1-x_2|}}{|x_1-x_2|}}_{x_1}$$

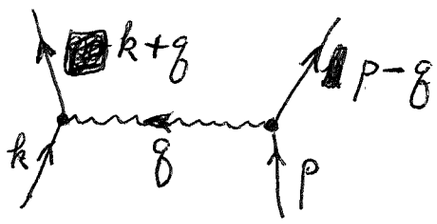
Put $x_1 \mapsto x_1 + x_2$

$$= V\,\delta(k_3 - k_1 + k_4 - k_2)\, \frac{4\pi}{(k_3-k_1)^2 + \mu^2}$$

so the potential term becomes

$$\frac{1}{2}\frac{e^2}{V}\sum_{k_1,k_2,k_3,k_4}{}' \delta(k_3+k_4-k_1-k_2)\frac{4\pi}{(k_3-k_1)^2+\mu^2} a^*_{k_1} a^*_{k_2} a_{k_4} a_{k_3}$$

Change variables: $k = k_3$, $p = k_4$ denote the incoming momenta, $q = k_1 - k_3$ is the momentum transfer. The picture is



and the potential energy term is

$$\frac{1}{2}\frac{e^2}{V}\sum_{k,p,q}{}' \frac{4\pi}{q^2+\mu^2} a^*_{k+q} a^*_{p-q} a_p a_k$$

so

$$H_{el} = \sum_k{}' \frac{k^2}{2m} a^*_k a_k + \frac{e^2}{2V}\sum_{k,p,q}{}' \frac{4\pi}{q^2+\mu^2} a^*_{k+q} a^*_{p-q} a_p a_k$$

Suppose we were to try to operate with this Hamiltonian and no background. Then as we let $\mu \to 0$ we get in trouble when the momentum transfer q is zero. The part with $q=0$ is

$$\frac{e^2}{2V}\sum_{k,p}{}' \frac{4\pi}{\mu^2} a^*_k a^*_p a_p a_k$$

$$= \frac{e^2}{2V}\sum_k{}' \frac{4\pi}{\mu^2} a^*_k \underbrace{\hat{N} a_k}_{a_k(N-1)} = \frac{e^2}{2V}\frac{4\pi}{\mu^2}\left(\hat{N}^2 - \hat{N}\right)$$

where $\hat{N} = \sum a_p^* a_p$ is the number of particles operator. Since we work in dimension N, this means $\hat{N} = N$, and so this $q=0$ is a constant

$$\frac{e^2}{2V}\frac{4\pi}{\mu^2}(N^2-N) = \frac{1}{2}\frac{e^2}{V}\frac{N^2}{\mu^2}4\pi - \frac{e^2}{2}\frac{N}{V}\frac{4\pi}{\mu^2}$$

The first term is cancelled by the background; the second term represents an energy per particle of

$$-\frac{e^2}{2}4\pi\frac{1}{V\mu^2}$$

which goes to zero if we let $V\to\infty$ first (whatever this means). In any case infinite additive constants to the Hamiltonian don't have much significance, so the good Hamiltonian is

$$H = \sum_k \frac{k^2}{2m} + \frac{e^2}{2V}\sum_{k,p,q}{}' \frac{4\pi}{q^2} a_{k+q}^* a_{p-q}^* a_p a_k$$

$q\neq 0$

where we ultimately take the limit $\frac{N}{V}\to n$, $V\to\infty$. (N tells you what piece of Fock space you are in.)

[redacted] One is after the ground state energy. First we compute this [redacted] for the free gas:

$$H = \sum_k \frac{k^2}{2m}$$

One fills up all the states in order of increasing energy according to the exclusion principle. Recall that k runs over the lattice $\frac{2\pi}{L}\mathbb{Z}$ where $L^3 = V$. So we are interested in all the lattice points inside a sphere $|k| < k_F$, where k_F is the so-

called Fermi momentum and

$$\epsilon_F = \frac{k_F^2}{2m}$$

is the Fermi energy. Each value of k comes with 2 spin degrees, so the total number of states N with $|k| < k_F$ can be found by a volume calculation

$$\frac{N}{2} \cdot \left(\frac{2\pi}{L}\right)^3 \simeq \frac{4}{3}\pi k_F^3$$

↑ no of lattice points; ↑ volume per lattice point

$$\therefore \quad k_F = \left(3\pi^2 \frac{N}{V}\right)^{1/3}$$

The ground state energy is

$$E = \sum_{|k| k_F} 2 \frac{k^2}{2m}$$

↑ spin

$$E \cdot \left(\frac{2\pi}{L}\right)^3 = \sum_{|k|<k_F} 2\frac{k^2}{2m}\left(\frac{2\pi}{L}\right)^3 \simeq \int_{|k|<k_F} 2\frac{k^2}{2m}\, d^3k$$

$$= \frac{1}{m} 4\pi \int_0^{k_F} r^2 r^2 dr = \frac{4\pi}{m}\frac{k_F^5}{5}$$

$$E = \frac{V}{2\pi^2 m 5} k_F^5 = \frac{V}{5\pi^2} \underbrace{\frac{k_F^2}{2m}}_{\text{Fermi energy } \epsilon_F} k_F^3$$

($k_F^3 = 3\pi^2 \frac{N}{V}$)

$$= N \frac{3}{5}\left(\frac{k_F^2}{2m}\right)$$

$$\therefore \quad \boxed{\frac{E}{N} = \frac{3}{5}\epsilon_F \qquad \text{energy per particle is } 3/5 \text{ the Fermi energy}}$$

This formula is the same if spin is ignored, however one then has $k_F^3 = 6\pi^2 \frac{N}{V}$

Degenerate electron gas (continued). The Hamiltonian is

$$H = \sum \frac{k^2}{2m} a_k^* a_k + \frac{e^2}{2V} \sum_{\substack{k,p,q \\ q \neq 0}} \frac{4\pi}{q^2} a_{k+q}^* a_{p-q}^* a_p a_k$$

where I have left out the spin variables (I think the two spins run independently, so the real ground energy gets doubled.). The unperturbed ground state is

$$\Phi_0 = \bigwedge_{|k|<k_F} \psi_k$$

and we found the unperturbed ground energy to be

$$E^{(0)} = N \cdot \frac{3}{5} \left(\frac{k_F^2}{2m} \right)$$

Now we want the first order correction. Since $|\Phi_0| = 1$ this is

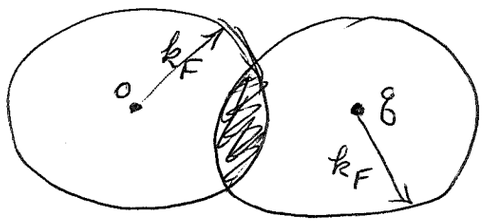
$$E^{(1)} = \langle \Phi_0 | H' | \Phi_0 \rangle$$

$$= \frac{e^2}{2V} \sum_{\substack{k,p,q \\ q \neq 0}} \frac{4\pi}{q^2} \underbrace{\langle \Phi_0 | a_{k+q}^* a_{p-q}^* a_p a_k | \Phi_0 \rangle}_{\text{this is } \pm 1 \text{ or } 0.}$$

We must have $|k|, |p| < k_F$ for $a_p a_k |\Phi_0\rangle \neq 0$. If the $\langle\ \rangle$ term is $\neq 0$, the creation operators must cancel out the destruction operators, and since $q \neq 0$ we have $p - q \neq p$, hence we must have $p - q = k$ or $p = k + q$. In this case the sign is -1. So

$$E^{(1)} = -\frac{e^2}{2V} \sum_{\substack{k,q \\ q \neq 0}} \frac{4\pi}{q^2} \theta(k_F - |k|)\, \theta(k_F - |k+q|)$$

Now when we pass to the limit as $V \to \infty$ the sum overk gets replaced by an integral over a region



Let's now understand Feynman diagrams in the many body problem. Ignore spin, so that $\psi(x)$ (destruction operator) has one component. $\psi(x) = i(\langle x|$.

$$H_0 = \int d^3x \; \psi(x)^* \left(-\frac{\nabla^2}{2m} + U(x) \right) \psi(x)$$

$$H_1 = \frac{1}{2} \int d^3x \, d^3x' \; \psi(x)^* \psi(x')^* \; V(x-x') \; \psi(x') \psi(x)$$

We are after the Green's function

$$iG(xt, x't') = \langle F_0 | T[\psi(xt) \psi(x't')] | F_0 \rangle / \langle F_0 | F_0 \rangle$$

where F_0 is the (Heisenberg, i.e. honest) ground state and $\psi(xt)$ is the Heisenberg operator

$$\psi(xt) = e^{iHt} \psi(x) e^{-iHt}$$

Also the T operator has the fermion sign, (which means there is some ambiguity when $t = t'$?)

There is a certain ambiguity with the two particle operator which should be clarified. One is usually given an interaction potential $V(x,x')$ which is symmetric. Define its matrix elements relative

to an orth. basis φ_k for $\mathcal{H}$ by

$$V_{klmn} = \langle \varphi_k \otimes \varphi_l | V | \varphi_m \otimes \varphi_n \rangle$$

symmetry $V(x,x') = V(x',x)$ implies that

$$V_{klmn} = V_{eknm}.$$

One has

$$V|\varphi_m \otimes \varphi_n\rangle = \sum_{kl} |\varphi_k \otimes \varphi_l\rangle V_{klmn}$$

$$V|\varphi_n \otimes \varphi_m\rangle = \sum_{kl} |\varphi_l \otimes \varphi_k\rangle V_{klmn}$$

so

$$V|\varphi_m \wedge \varphi_n\rangle = \sum_{k,l}{}' |\varphi_k \wedge \varphi_l\rangle V_{klmn}$$

and hence

$$\langle \varphi_k \wedge \varphi_l | V | \varphi_m \wedge \varphi_n \rangle = V_{klmn} - V_{eknm}$$

hence

$$\hat{V} = \frac{1}{4} \sum_{klmn} a_k^* a_l^* \langle \varphi_k \wedge \varphi_l | V | \varphi_m \wedge \varphi_n \rangle a_n a_m$$

$$= \frac{1}{2} \sum_{klmn}{}' V_{klmn} a_k^* a_e^* a_n a_m$$

which is the formula found in the books (e.g. Fetter-Walecka) Hence V_{klmn} is not antisymmetric in k,l and m,n although its antisymmetrization is the only thing that matters.

Key example: V given by $V(x-x')$ and the one-particle eigenfunctions are plane waves

$$\varphi_k(x) = \frac{1}{\sqrt{V}} e^{ikx}$$

Then

$$V_{\substack{k_1 k_2 k_3 k_4 \\ 1\ 2\ 3\ 4}} = \frac{1}{V^2}\int e^{-ik_1 x - ik_2 x'} V(x-x') e^{-ik_3 x + ik_4 x'} dx dx'$$

$$= \frac{1}{V^2}\int V(x) e^{i(k_3-k_1)(x+x') + i(k_4-k_2)x'} dx dx'$$

$$= \frac{1}{V}\delta(k_1+k_2-k_3-k_4)\ V_{k_1-k_3}$$

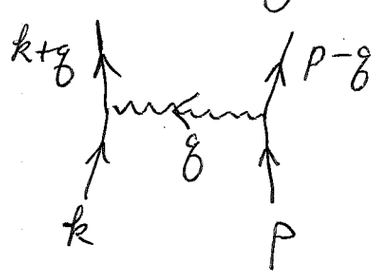
where $V_q = \int V(x) e^{-iqx} dx$ satisfies

$$V_{-q} = V_q$$

because $V(x) = V(-x)$. So the usual way to write the interaction operator is

$$\frac{1}{2V}\sum_{k,p,q} V_q\, a^*_{k+q}\, a^*_{p-q}\, a_p\, a_k$$

and to denote it by a diagram



many particle system of fermions

$$H_0 = \int dx\ \psi(x)^*\left(-\frac{\Delta}{2m} + U(x)\right)\psi(x)$$

$$H_1 = \frac{1}{2}\int dx\,dx'\ \psi(x)^*\psi(x')^*\ V(x,x')\,\psi(x')\,\psi(x)$$

The problem is to compute the Green's function

$$iG(xt, x't') = \langle \Psi_0 |\ \psi_H(xt)\ \psi_H(x't')^*\ | \Psi_0 \rangle$$

using diagram methods. One needs the formula

$$iG(xt, x't') = \frac{\langle 0 |\ T[S\,\psi(xt)\,\psi(x't')^*]\ | 0\rangle}{\langle 0 | S | 0\rangle}$$

where $\psi(xt)$ is an interaction picture operator defined by

$$\psi(xt) = e^{iH_0 t}\,\psi(x)\,e^{-iH_0 t}$$

Suppose $\varphi_k(x)$ is a complete orthonormal system of eigenvectors for H_0 with $H_0\varphi_k = \varepsilon_k\varphi_k$. Then

$$\psi(x) = \sum \varphi_k(x)\,a_k \qquad a_k = i(\langle \varphi_k |)$$

$$H_0 = \sum_k \varepsilon_k a_k^* a_k$$

and

$$\psi(xt) = \sum \varphi_k(x)\,e^{-i\varepsilon_k t}\,a_k$$

Let's compute the free Green's function

$$iG^0(xt, x't') = \langle 0 |\ T[\psi(xt)\,\psi(x't')^*]\ | 0\rangle$$

$$= \sum_k \varphi_k(x)\,\varphi_k(x')^*\,e^{-i\varepsilon_k(t-t')} \begin{cases} \langle 0 | a_k a_k^* | 0\rangle & t > t' \\ -\langle 0 | a_k^* a_k | 0\rangle & t < t' \end{cases}$$

(Since $\langle 0 | a_k a_{k'}^* | 0\rangle = 0 \quad k \neq k'$)

But $\langle 0 | a_k a_k^* | 0 \rangle = \begin{cases} 0 & k < k_F \\ 1 & k > k_F \end{cases}$

so

$$iG^0(xt, x't') = \sum_k \varphi_k(x)\varphi_k(x')^* e^{-i\varepsilon_k(t-t')}\left[\theta(t-t')\theta(k-k_F) - \theta(t'-t)\theta(k_F-k)\right]$$

Next use Fourier transform.

$$\int e^{-i\varepsilon t}\,\theta(t)\, e^{i\omega t}\,dt = \int_0^\infty e^{(-i\varepsilon+i\omega)t}\,dt$$

$$= \frac{-1}{-i\varepsilon+i\omega} = \frac{+i}{\omega-\varepsilon}$$

analytic for ω in the UHP so one writes

$$-ie^{-i\varepsilon t}\theta(t) = \int \frac{d\omega}{2\pi}\,\frac{e^{-i\omega t}}{\omega-\varepsilon+i\eta}$$

Hence

$$G^0(xt, x't') = \sum_k \varphi_k(x)\varphi_k(x')^* \int \frac{d\omega}{2\pi} e^{-i\omega t}\left[\frac{\theta(k-k_F)}{\omega-\varepsilon_k+i\eta} + \frac{\theta(k_F-k)}{\omega-\varepsilon_k-i\eta}\right]$$

so

$$G^0(x, x'; \omega) = \sum_k \varphi_k(x)\varphi_k(x')^*\left[\frac{\theta(k-k_F)}{\omega-\varepsilon_k+i\eta} + \frac{\theta(k_F-k)}{\omega-\varepsilon_k-i\eta}\right]$$

Perhaps a way to remember this is to think of the energy adjusted so that $\varepsilon_k > 0$ for $k > k_F$ and $\varepsilon_k < 0$ for $k < k_F$ whence G^0 as a function of $t-t'$ has positive frequencies for $t-t' > 0$ and negative frequencies for $t-t' < 0$.

Lehmann Representation: Let Φ_n be a complete orth. basis of energy eigenfunctions for Λ $\mathcal{H}$. Then

$$iG(xt,x't') = \langle \Psi_0 | T[\psi(xt)\psi(x't')^*] | \Psi_0 \rangle$$

$$= \theta(t-t') \sum_n \langle \Psi_0 | e^{iHt}\psi(x)e^{-iHt} | \Psi_n \rangle \langle \Psi_n | e^{iHt'}\psi(x')^* e^{-iHt'} | \Psi_0 \rangle$$

$$- \theta(t'-t) \langle \Psi_0 | e^{iHt'}\psi(x')^* e^{-iHt'} e^{iHt}\psi(x) e^{-iHt} | \Psi_0 \rangle$$

$$= \sum_n \theta(t-t') e^{-i(E_n - E)(t-t')} \langle \Psi_0 | \psi(x) | \Psi_n \rangle \langle \Psi_n | \psi(x')^* | \Psi_0 \rangle$$

$$- \sum_n \theta(t'-t) e^{-i(E - E_n)(t-t')} \langle \Psi_0 | \psi(x')^* | \Psi_n \rangle \langle \Psi_n | \psi(x) | \Psi_0 \rangle$$

In the first term E_n can be restricted to $E_n(N+1)$ i.e. the energies on Λ^{N+1}; similarly the second term uses $E_n(N-1)$. So the Fourier transform is

$$G(xt,x't') = \int \frac{d\omega}{2\pi} e^{-i\omega t} G(x,x',\omega)$$

$$G(x,x',\omega) = \sum \frac{\langle \Psi_0^N | \psi(x) | \Psi_n^{N+1} \rangle \langle \Psi_n^{N+1} | \psi(x')^* | \Psi_0^N \rangle}{\omega - (E_n^{N+1} - E_0^N) + i\eta}$$

$$+ \sum \frac{\langle \Psi_0^N | \psi(x')^* | \Psi_n^{N-1} \rangle \langle \Psi_n^{N-1} | \psi(x) | \Psi_0^N \rangle}{\omega + (E_n^{N-1} - E_0^N) - i\eta}$$

Now it is argued that in the thermodynamic limit $N \to \infty$, $V \to \infty$, $\frac{N}{V}$ constant, one has

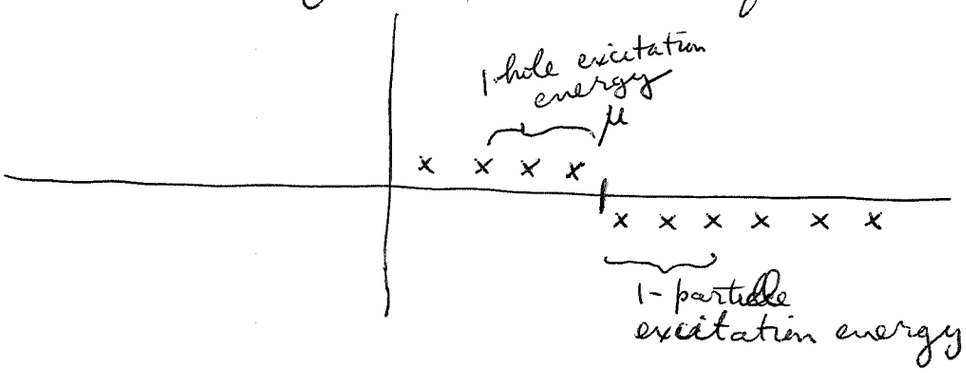
$$E_0^{N+1} - E_0^N = \mu + O\left(\frac{1}{N}\right)$$

where μ = chemical potential (extra energy needed to get another particle into the system); here μ is a constant. One writes the LHP poles in the form

$$\omega = (E_n^{N+1} - E_0^N) - i\eta$$

$$\omega = \underbrace{\left(E_n^{N+1} - E_0^{N+1}\right)}_{\text{possible excitation energy for } N+1 \text{ particle system}} + \underbrace{\left(E_0^{N+1} - E_0^{N}\right)}_{\mu} - i\eta$$

So the poles of $G(\omega)$ look as follows



In the limit we get

$$G(x,x';\omega) = \int_0^\infty d\omega' \left[\frac{A(x,x';\omega')}{\omega - \mu - \omega' + i\eta} + \frac{B(x,x';\omega)}{\omega - \mu + \omega' - i\eta} \right]$$

and the analytic structure has been completely changed.

Many body problem:

$$H_0 = \int dx\, \psi(x)^* \left(-\frac{\Delta^2}{2m} + U(x)\right) \psi(x)$$

$$H' = \frac{1}{2}\int dx\, dx'\, \psi(x)^* \psi(x')^* V(x,x') \psi(x') \psi(x)$$

where $V(x,x') = V(x',x)$. These operate on $\wedge \mathcal{H}$ where $\mathcal{H}$ = Hilbert space on which $-\frac{\Delta^2}{2m} + U$ works, hence $\mathcal{H} = L^2(\mathbb{R}^3)$. $\psi(x)$ = interior multiplication by $\langle x|$.

Let ϕ_k be an orth basis of eigenvectors for $-\frac{\Delta^2}{2m} + U$ with eigenvalues ω_k. Example: $U = 0$ and we work in a box, then $\phi_k = \frac{1}{\sqrt{V}} e^{ikx}$, $k \in \frac{2\pi}{L}\mathbb{Z}$, and $\omega_k = \frac{k^2}{2m}$. Then

$$\psi(x) = \sum_k \phi_k(x)\, a_k \qquad a_k = i(\phi_k^*)$$

and we can express H_0, H' in terms of the a_k, a_k^*

$$H_0 = \sum_k \omega_k a_k^* a_k$$

$$H' = \frac{1}{2} \sum_{klmn} V_{klmn}\, a_k^* a_l^* a_n a_m$$

$$V_{klmn} = \int dx\, dx' \underbrace{V(x,x')}\, \phi_k(x)^* \phi_l(x')^* \phi_m(x) \phi_n(x')$$

The reason one likes this representation is that

$$[iH_0, a_k] = -i\omega_k a_k$$

$$a_k(t) \overset{\text{defn}}{=} e^{iH_0 t} a_k e^{-iH_0 t} = e^{-i\omega_k t} a_k$$

and so $$\psi(xt) \overset{\text{defn}}{=} e^{iH_0 t} \psi(x) e^{-iH_0 t} = \sum_k \phi_k(x) e^{-i\omega_k t} a_k$$

Our goal is to compute the Green's function

$$iG(xt,x't') = \langle \underset{\uparrow}{\Psi_0} \mid T[\psi_H(xt)\psi_H(x't')^*] \mid \Psi_0 \rangle$$

interacting ground state

$\psi_H(x) = e^{iHt}\psi(x)e^{-iHt}$.

We use the formula

$$iG(xt,x't') = \frac{\langle 0 \mid T[S\psi(xt)\psi(x't')^*] \mid 0\rangle}{\langle 0 \mid S \mid 0\rangle}$$

Concentrate on the numerator and denote it $i\tilde{G}$. One has

$$i\tilde{G} = \langle 0 \mid T[\psi(xt)\psi(x't')^*] \mid 0\rangle$$

$$+ \frac{(-i)}{1!}\int dt_1 \langle 0 \mid T[H'(t_1)\psi(xt)\psi(x't')^*] \mid 0\rangle + \dots$$

$$+ \frac{(-i)^n}{n!}\int dt_1 \dots dt_n \langle 0 \mid T[H'(t_1)\dots H'(t_n)\psi(xt)\psi(x't')^*] \mid 0\rangle + \dots$$

The first term is $iG_0(xt,x't')$.

$$\langle 0 \mid T[H'(t_1)\psi(xt)\psi(xt')] \mid 0\rangle$$

$$= \int dx_1 dx_2 \overbrace{\tfrac{1}{2}V(x_1,x_2)}\langle 0 \mid T[\psi(x_1t_1)^*\psi(x_2t_1)^*\psi(x_2t_1)\psi(x_1t_1)\psi(xt)\psi(x't')^*] \mid 0\rangle$$

This has to be integrated over all t_1. A nice idea is to add another variable t_2 and to put

$$V(x_1t_1, x_2t_2) = V(x_1,x_2)\delta(t_1-t_2)$$

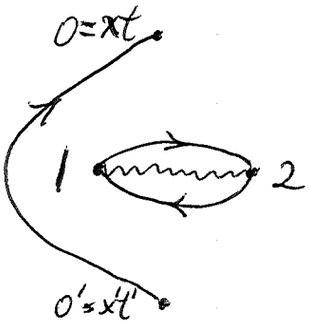
Then the first order term becomes

$$\frac{(-i)}{1!}\int dx_1t_1\, dx_2t_2 \; \tfrac{1}{2} V(x_1t_1,x_2t_2)$$

$$\cdot\langle 0 \mid T[\psi(x_1t_1)^*\psi(x_2t_2)^*\psi(x_2t_2)\psi(x_1t_1)\psi(xt)\psi(x't')^*] \mid 0\rangle$$

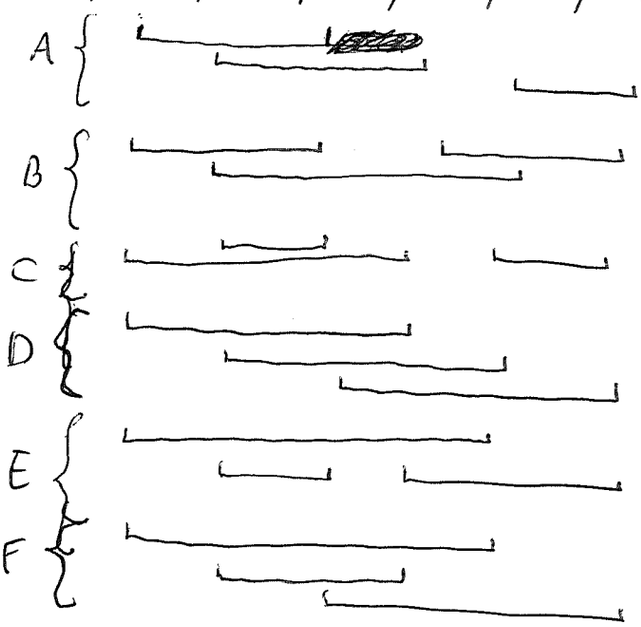
except we have to be careful how to interpret things at equal times.

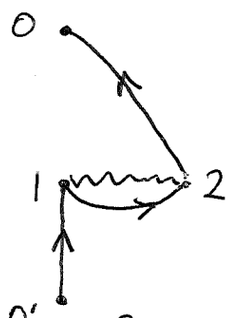
Now one has to use Wick's thm in order to evaluate the vacuum expectation value. This breaks the thing down into a sum of terms over possible pairwise contractions of a ψ^* against a ψ. The possibilities are labelled by diagrams.



A

$\psi(1)^*\psi(2)^*\psi(2)\psi(1)\psi(0)\psi(0')^*$





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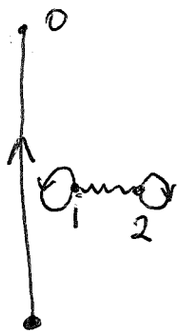

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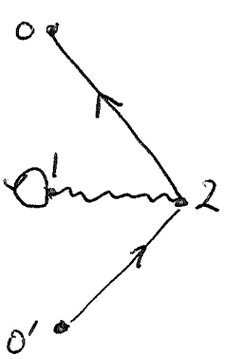

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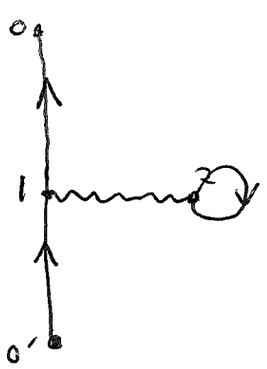

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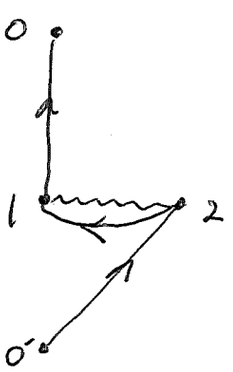


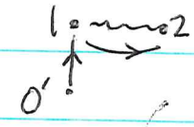
F

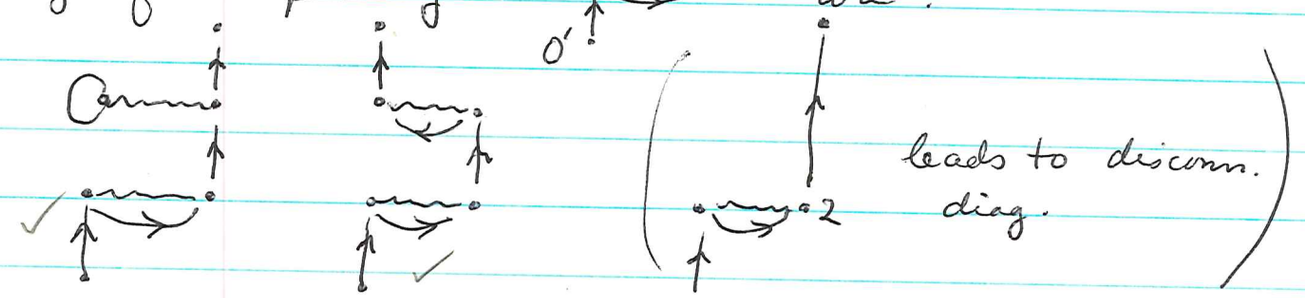
Construct all 2nd order diagrams. This involves the contractions belonging to

$$V(1,2)\,V(3,4)\,\langle 0|T[\psi(1)^*\psi(2)^*\psi(2)\psi(1)\,\psi(3)^*\psi(4)^*\psi(4)\psi(3)\,\psi(0)\psi(0')^\dagger]|0\rangle$$

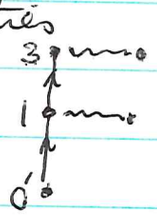
For each space time point 1, 2, 3, 4 you have a ψ^* from which an arrow leaves and a ψ for which an arrow comes in. Total number of ways of contracting seems to be $5! = 120$. Taking advantage of the symmetries $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ and $(1,2) \leftrightarrow (3,4)$ should cut the number of diagrams to something manageable.

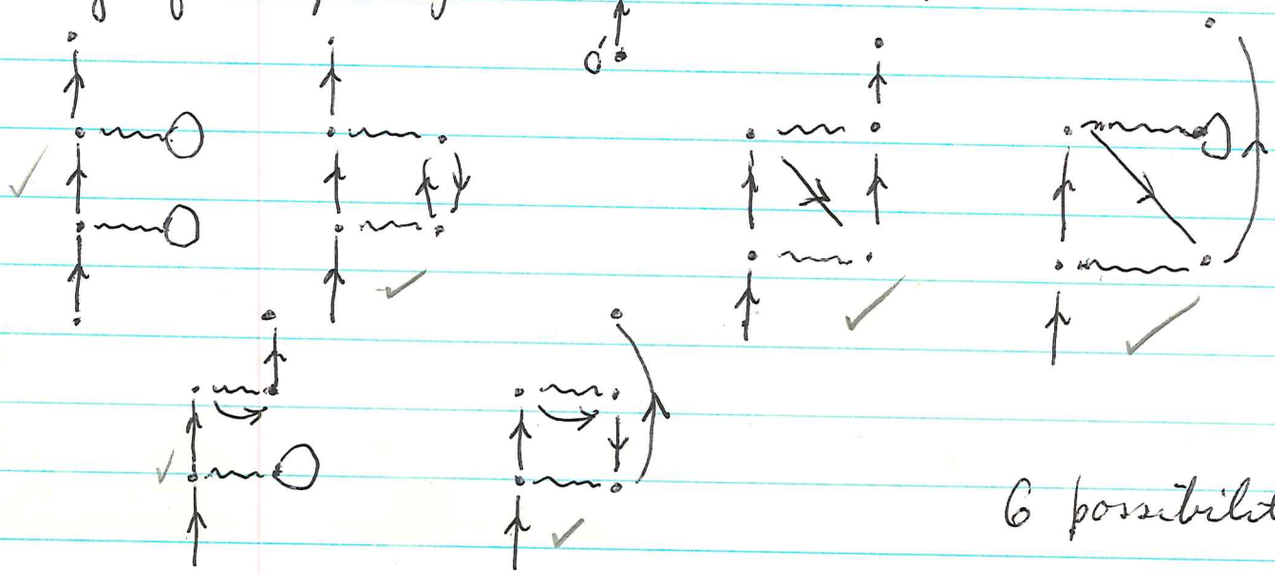
Consider only connected diagrams. Then $0'$ can be assumed joined to 1 which can then be joined to either 2, 3, or 0. So

Ways of completing  are:

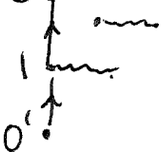


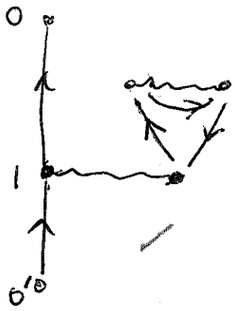
2 possibilities

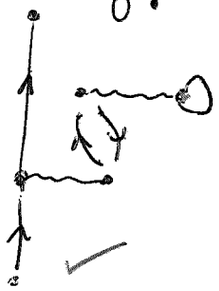
Ways of completing  are:

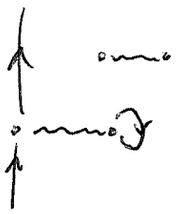


6 possibilities

Ways of completing  are





( leads to discon.)

2 possibilities.

Hence there are 10 2nd order connected Feynman diagrams. (Fetter-Walecka p. 100).

We already saw [redacted] when the 2-particle interaction is written

$$V = \frac{1}{2} \sum V_{klmn}\, a_k^* a_l^* a_n a_m$$

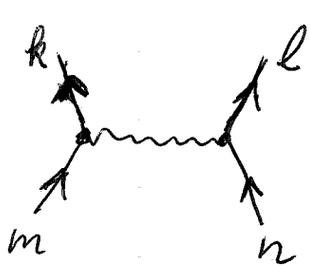
with $V_{klmn} = \langle \varphi_k \otimes \varphi_l | V | \varphi_m \otimes \varphi_n \rangle$, that this V_{klmn} is not the same as

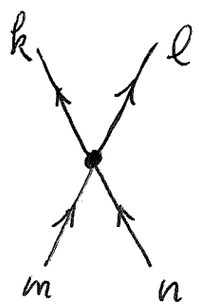
$$\tilde{V}_{klmn} = \langle \varphi_k \wedge \varphi_l | V | \varphi_m \wedge \varphi_n \rangle$$

which is anti-symmetric in m,n and k,l. In fact

$$\tilde{V}_{klmn} = \frac{1}{2}\left(V_{klmn} - V_{klnm}\right)$$

What this means is that our diagrams are different for the V and $\tilde{V}$ formulas:





This maybe is the difference between Goldstone + Hugenholtz diagrams. The two first order diagrams

collapse into a single diagram  .

Problem: I really don't understand what one is trying to compute when one subjects a system such as the simple harmonic oscillator to an external perturbation. There are two cases of interest:

1) time-dependent. Here the perturbation works for a finite time interval, say $[0,T]$, and one is interested in the transition amplitudes between unperturbed energy eigenstates. Thus one computes the S matrix $e^{iH_0T}U(T,0)$.

2) time-independent. Here one wants to understand the way the unperturbed energy eigenstates move into the perturbed ones. In particular, one wants the ground state energy shift for discrete energy levels, and some sort of S-matrix $\tilde{U}(\infty,-\infty)$ in the scattering situation.

Looking at $e^{itH_0}e^{-itH}$ reminds me of buildings. Recall what I did for $G=GL_n(\mathbb{C})$. One considers 1-parameter subgps e^{tX} and introduces an equivalence relation $e^{tX}\sim e^{tY}$ if $e^{-tX}e^{tY}$ remains in a compact subset of G as $t\to+\infty$. This isn't quite correct because one wants all unipotent 1-parameter groups equivalent. So replace the requirement by requiring $e^{-tX}e^{tY}$ to have at most polynomial growth as $t\to+\infty$. Recall that the matrix elements of e^{tX} are exponential polys.

The key point of the equivalence relation is that the equivalence class of X is obtained as follows. One arranges the eigenvalues of X in order of increasing real part and constructs a unique flag in $\mathbb{C}^n$, say $0 < V_1 < \cdots < V_k = \mathbb{C}^n$, such that it is invariant under X and such that on V_i/V_{i-1} the eigenvalues have the same real part which decreases with i. It follows there is a unique hermitian matrix H equivalent to X, namely you split this flag orthogonally and take the real parts of the corresponding eigenvalues of X.

Can one compute $e^{-A} e^{A+\varepsilon B}$ as a power series in ε? We can treat this by Dyson's method.

$$\tilde{U}(t) = e^{-tA} e^{t(A+\varepsilon B)}$$

$$\frac{d}{dt}\tilde{u}(t) = e^{-tA}(-A + A + \varepsilon B)\, e^{t(A+\varepsilon B)}$$

$$= e^{-tA} \varepsilon B e^{tA}\, \tilde{u}(t)$$

$$= e^{-t\,\mathrm{ad}A}(\varepsilon B)\quad \tilde{u}(t)$$

so

$$\tilde{u}(t) = I + \int_0^t dt_1\ e^{-t_1\,\mathrm{ad}A}(\varepsilon B)\cdot \tilde{u}(t_1)$$

$$= I + \int_0^t dt_1\, e^{-t_1\,\mathrm{ad}(A)}(\varepsilon B) + \int_0^t dt_1\, e^{-t_1\,\mathrm{ad}A}(\varepsilon B)\cdot\int_0^{t_1} dt_2\, e^{-t_2\,\mathrm{ad}A}(\varepsilon B) + \cdots$$

First order term is $\displaystyle\int_0^t dt_1\, e^{-t_1\,\mathrm{ad}(A)}(\varepsilon B) = \frac{1 - e^{-t\,\mathrm{ad}A}}{\mathrm{ad}A}(\varepsilon B)$

Let's suppose A is diagonal: $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \end{pmatrix}$.

Then

$$(\operatorname{ad} A)(E_{ij}) = [A, E_{ij}] = (\lambda_i - \lambda_j) E_{ij}$$

and so $\operatorname{ad} A$ is an isomorphism on the set of matrices with 0 diagonal entries. Let's suppose B is of this type, and put $\alpha = \operatorname{ad} A$, $\frac{1}{\alpha}(B)$ = the unique matrix with 0 diag. entries such that $\alpha(\frac{1}{\alpha}B) = B$.

$$\int_0^t dt_1\, e^{-t_1\alpha}(B) = \int_0^t dt_1\, e^{-t_1\alpha}\alpha\left(\frac{1}{\alpha}B\right)$$

$$= (1 - e^{-t\alpha})\left(\frac{1}{\alpha}B\right)$$

Next look at 2nd order

$$\int_0^t dt_1\, e^{-t_1\alpha}(B) \cdot \int_0^{t_1} dt_2\, e^{-t_2\alpha}(B)$$

$$= \int_0^t dt_1\, e^{-t_1\alpha}(B) \cdot (1 - e^{-t_1\alpha})\left(\frac{1}{\alpha}B\right)$$

Since $e^{-t_1\alpha}(X) = e^{-t_1 A} X e^{t_1 A}$ is an automorphism we see the above equals

$$= \int_0^t dt_1 \left[e^{-t_1\alpha}(B) \cdot \tfrac{1}{\alpha}(B) - e^{-t_1\alpha}\left(B \cdot \tfrac{1}{\alpha}(B)\right)\right]$$

$$= (1 - e^{-t\alpha})\left(\tfrac{1}{\alpha}(B)\right) \cdot \tfrac{1}{\alpha}(B) - \left(\frac{1 - e^{-t\alpha}}{\alpha}\right)\left(B \cdot \tfrac{1}{\alpha}(B)\right)$$

Now there's a problem in that $B \cdot \frac{1}{\alpha}(B)$ might have diagonal entries. Hence we ought to see what can be done without assuming B has zero diagonal entries.

Now the series

$$\frac{1-e^{-t\alpha}}{\alpha} = \sum_{n\geq 1} (-1)^{n-1} \frac{\alpha^{n-1}}{n!} = 1 + O(\alpha)$$

[redacted] leaves the diagonal part of B [redacted] alone

So put B^{Δ} = diagonal part of B.

Then we have

$$\frac{1-e^{-t\alpha}}{\alpha}(B) = [\text{redacted}]B^{\Delta} + (1-e^{-t\alpha})\frac{1}{\alpha}(B-B^{\Delta})$$

$$= B^{\Delta} + \frac{1}{\alpha}(B-B^{\Delta}) - e^{-t\alpha}\frac{1}{\alpha}(B-B^{\Delta})$$

so the 2nd order term is

$$\int_0^t dt_1\, e^{-t_1\alpha}(B)\cdot\left[B^{\Delta} + \frac{1}{\alpha}(B-B^{\Delta}) - e^{-t_1\alpha}\frac{1}{\alpha}(B-B^{\Delta})\right]$$

$$= \frac{1-e^{-t\alpha}}{\alpha}(B)\cdot\left[B^{\Delta} + \frac{1}{\alpha}(B-B^{\Delta})\right] - \frac{1-e^{-t\alpha}}{\alpha}\left(B\cdot\frac{1}{\alpha}(B-B^{\Delta})\right)$$

This seems to be too complicated to be useful.

Recall the expansion

$$e^{t(A+B)} = e^{tA} + \int_0^t dt_1\, e^{(t-t_1)A} B e^{t_1 A} + \int_0^t dt_1 \int_0^{t_1} dt_2\; e^{(t-t_1)A} B e^{(t_1-t_2)A} B e^{t_2 A} + \cdots \tag{1}$$

The integrals on the right are convolutions, so another way of viewing the above is to take Laplace transforms

$$\int_0^\infty e^{-st} e^{t(A+B)}\, dt = \frac{1}{s-A-B} \tag{2}$$

$$= \frac{1}{s-A} + \frac{1}{s-A} B \frac{1}{s-A} + \cdots\cdots\cdots$$

In particular the second order term in (1) is given by the inversion formula

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{s-A} B \frac{1}{s-A} B \frac{1}{s-A}\; e^{st}\, ds$$

If we take a complete set of eigenvectors $|n\rangle$ for A, say $A|n\rangle = \lambda_n |n\rangle$, then we can compute the matrix elements of $e^{t(A+B)}$ relative to this basis. First order term is

$$\langle n' | \frac{1}{2\pi i} \int e^{st} ds \frac{1}{s-A} B \frac{1}{s-A} | n \rangle$$

$$= \langle n'|B|n\rangle \frac{1}{2\pi i} \int \frac{e^{st}\, ds}{(s-\lambda_{n'})(s-\lambda_n)} = \langle n'|B|n\rangle \underbrace{\frac{e^{\lambda_{n'} t} - e^{\lambda_n t}}{\lambda_{n'} - \lambda_n}}$$

to be interpreted as t if $\lambda_{n'} = \lambda_n$.

Let us consider now the quantum-mechanical situation: $H = H_0 + V$. We are interested in V as a perturbation of H_0 and how it gives rise to transitions between the H_0-eigenstates. If you let the perturbation act for a while, say T, then you get a transition amplitude from $|n\rangle$ to $|n'\rangle$ whose abs. value squared is the probability of transition. Hence I want to understand the behavior of

$$\langle n'|e^{iH_0t}e^{-iHt}|n\rangle = e^{i\lambda_n t}\langle n'|e^{-iHt}|n\rangle$$

as $t \to +\infty$. But if $\{\psi_m\}$ is an orthonormal basis of H eigenvectors with $H\psi_m = \lambda'_m\psi_n$, then we know

$$\langle n'|e^{-iHt}|n\rangle = \sum e^{-i\lambda'_m t}\langle n'|\psi_m\rangle\langle\psi_m|n\rangle$$

which is oscillatory as $t \to +\infty$.

Let's try to understand the "adiabatic" formalism as presented in Schweber's book. The idea here is that one replaces the potential V by the potential $e^{\varepsilon t}V$ which vanishes as $t \to -\infty$. Then one looks at what happens to the eigenfunctions φ_a of H_0 in going from $t = -\infty$ to 0, and finally one lets $\varepsilon \to 0^+$. We have Dyson's formula

$$\tilde{U}_\varepsilon(t,-\infty) = I + (-i)\int_{-\infty}^{t} dt_1\, \tilde{V}_\varepsilon(t_1) + (-i)^2\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\, \tilde{V}_\varepsilon(t_1)\tilde{V}_\varepsilon(t_2)$$

$$+ \cdots\cdots$$

where $\tilde{V}_\varepsilon(t) = e^{iH_0 t} e^{\varepsilon t} V_\varepsilon e^{-iH_0 t}$

Put $\psi_\varepsilon(t) = \tilde{U}_\varepsilon(t,-\infty)\varphi_a$ so that one has

$$\begin{cases} \frac{d}{dt}\psi_\varepsilon(t) = -i\tilde{V}_\varepsilon(t)\psi_\varepsilon(t) \\ \psi_\varepsilon(-\infty) = \varphi_a \end{cases}$$

or

$$\psi_\varepsilon(t) = \varphi_a - i\int_{-\infty}^{t} e^{\varepsilon t_1 + iH_0 t_1} V e^{-iH_0 t_1}\psi_\varepsilon(t_1)\,dt_1$$

Let's solve this by Picard iteration.

$$\psi_\varepsilon^{(0)}(t) = \varphi_a$$

$$\psi_\varepsilon^{(1)}(t) = \varphi_a - i\int_{-\infty}^{t} e^{(\varepsilon+iH_0)t_1} V \underbrace{e^{-iH_0 t_1}\varphi_a}_{e^{-iE_a t_1}\varphi_a} dt_1$$

$$e^{(\varepsilon+iH_0-iE_a)t_1} V\varphi_a$$

$$= \varphi_a - i\left[\frac{e^{(\varepsilon+iH_0-iE_a)t_1}}{\varepsilon+iH_0-iE_a}\right]_{-\infty}^{t} V\varphi_a$$

$$= \varphi_a + \frac{e^{(iH_0-iE_a)t + \varepsilon t}}{E_a - H_0 + i\varepsilon} V\varphi_a$$

$$\psi_\varepsilon^{(2)}(t) = \psi_\varepsilon^{(1)}(t) + (-i)\int_{-\infty}^{t} e^{(\varepsilon+iH_0)t_1} V e^{-iH_0 t_1} \frac{e^{(iH_0-iE_a+\varepsilon)t_1}}{E_a-H_0+i\varepsilon} V\varphi_a$$

$$= \psi_\varepsilon^{(1)}(t) + \frac{e^{(iH_0-iE_a+2\varepsilon)t}}{E_a-H_0+2i\varepsilon} V \frac{1}{E_a-H_0+i\varepsilon} V\varphi_a$$

So it seems that we obtain the formula

$$\psi_\varepsilon(0) = \varphi_a + \frac{1}{E_a - H_0 + i\varepsilon} V\varphi_a + \frac{1}{E_a - H_0 + 2i\varepsilon} V \frac{1}{E_a - H_0 + i\varepsilon} V\varphi_a$$

$$+ \frac{1}{E_a - H_0 + 3i\varepsilon} V \frac{1}{E_a - H_0 + 2i\varepsilon} V \frac{1}{E_a - H_0 + i\varepsilon} V\varphi_a + \cdots\cdots$$

I can check this result as follows. Put

$$\Psi(t) = e^{-iE_at}\varphi_a + e^{\varepsilon t - iE_at} \frac{1}{E_a - H_0 + i\varepsilon} V\varphi_a$$

$$+ e^{2\varepsilon t - iE_at} \frac{1}{E_a - H_0 + 2i\varepsilon} V \frac{1}{E_a - H_0 + i\varepsilon} V\varphi_a + \cdots$$

Then

$$H_0\Psi(t) = E_a e^{-iE_at}\varphi_a + e^{\varepsilon t - iE_at}\left\{-1 + \frac{E_a + i\varepsilon}{E_a - H_0 + i\varepsilon}\right\} V\varphi_a$$

$$+ e^{2\varepsilon t - iE_at}\left\{-1 + \frac{E_a + 2i\varepsilon}{E_a - H_0 + 2i\varepsilon}\right\} V \frac{1}{E_a - H_0 + i\varepsilon} V\varphi_a + \cdots$$

$$i\frac{d}{dt}\Psi(t) = E_a e^{-iE_at}\varphi_a + e^{\varepsilon t - iE_at}\left\{i\varepsilon + E_a\right\} \frac{1}{E_a - H_0 + i\varepsilon} V\varphi_a$$

$$+ e^{2\varepsilon t - iE_at}\left\{2i\varepsilon + E_a\right\} \frac{1}{E_a - H_0 + 2i\varepsilon} V \frac{1}{E_a - H_0 + i\varepsilon} V\varphi_a + \cdots$$

$$\therefore \left(-H_0 + i\frac{d}{dt}\right)\Psi(t) = e^{\varepsilon t} V e^{-iE_at}\varphi_a + e^{\varepsilon t} V e^{\varepsilon t - iE_at} \cdot \frac{1}{E_a - H_0 + i\varepsilon} V\varphi_a + \cdots$$

$$= e^{\varepsilon t} V \Psi(t)$$

Thus $\Psi(t)$ satisfies the Schroedinger equation

$$i\frac{d}{dt}\Psi(t) = \left(H_0 + e^{\varepsilon t} V\right)\Psi(t)$$

so $\psi_\varepsilon(t) = e^{iH_0t}\Psi(t)$ satisfies $i\frac{d}{dt}\psi_\varepsilon(t) = \tilde{V}(t)\psi_\varepsilon(t)$

Let's consider

(*)
$$\begin{cases} \frac{d\psi}{dt} = -i(H_0 + e^{\varepsilon t}V)\psi \\ \psi(t) \sim e^{-iE_a t}\varphi_a \quad \text{as } t \to -\infty \end{cases}$$

Do this by iterating. If ψ is a sum of exponentials then

$$\left(\frac{d}{dt} + iH_0\right)\psi = -ie^{\varepsilon t}V\psi$$

has a solution obtained by undetermined coeffs. Start with

$$\psi^{(0)} = e^{-iE_a t}\varphi_a$$

$$\left(\frac{d}{dt} + iH_0\right)\psi^{(1)} = -ie^{(\varepsilon - iE_a)t}V\varphi_a$$

This has the solution

$$\psi^{(1)} = e^{-iE_a t}\varphi_a + (-i)\frac{e^{(\varepsilon - iE_a)t}}{\varepsilon + iH_0 - iE_a}V\varphi_a .$$

↑ handles $t \to -\infty$

Iterating again gives

$$\left(i\frac{d}{dt} - H_0\right)\psi^{(2)} = e^{\varepsilon t}V\psi^{(1)}$$

$$= e^{(\varepsilon - iE_a)t}V\varphi_a + e^{(2\varepsilon - iE_a)t}V\frac{1}{i\varepsilon + E_a - H_0}V\varphi_a$$

so

$$\psi^{(2)} = e^{-iE_a t}\varphi_a + \frac{e^{(\varepsilon - iE_a)t}}{i\varepsilon + E_a - H_0}V\varphi_a + \frac{e^{(2\varepsilon - iE_a)t}}{2i\varepsilon + E_a - H_0}V\frac{1}{i\varepsilon + E_a - H_0}V\varphi_a$$

etc.

Question: When might this series converge?

Suppose H_0 and V commute, so that we can solve this like a first order DE:

$$\psi(t) = e^{-i\int_{t'}^{t}(H_0 + e^{\varepsilon t_1}V)dt_1}\psi(t')$$

$$= e^{-iE_a t}\, e^{-i\frac{e^{\varepsilon t}}{\varepsilon}V}\varphi_a$$

$$= e^{-iE_a t}\left(1 + \frac{e^{\varepsilon t}}{i\varepsilon}V + \frac{e^{2\varepsilon t}}{2i\varepsilon}V\frac{1}{i\varepsilon}V + \cdots\right)\varphi_a$$

So in this case the series converges very nicely. Also we see that the limit as $\varepsilon \to 0+$ does not exist.

Let us denote by $\psi_\varepsilon(t)$ the solution of (*) so that

$$\psi_\varepsilon(t) = e^{-iE_a t}\varphi_a + e^{(\varepsilon - iE_a)t}\frac{1}{i\varepsilon + E_a - H_0}V\varphi_a + \cdots$$

Since

$$\left(i\frac{d}{dt} - H_0\right)\psi_\varepsilon = e^{\varepsilon t}V\psi_\varepsilon$$

it follows (at least when H_0, V are finite-dim'l matrices) that the line spanned by ψ_ε has as limit, as point, as $\varepsilon \to 0$ a solution of

$$\left(i\frac{d}{dt} - H_0\right)\psi = V\psi.$$

Hence we expect quite generally to be able to normalize ψ_ε so that it converges as $\varepsilon \to 0$. The Gell-Mann Low normalization is to divide by $\langle\varphi_a|\psi_\varepsilon(0)\rangle$:

$$\tilde{\psi}_\varepsilon(0) = \frac{\psi_\varepsilon(0)}{\langle\varphi_a|\psi_\varepsilon(0)\rangle} = \frac{U_\varepsilon(0,-\infty)\varphi_a}{\langle\varphi_a|U_\varepsilon(0,-\infty)|\varphi_a\rangle}$$

This obviously works provided the limiting eigenfunction is not orthogonal to φ_a.

Gell-Mann - Low formula for the ground energy shift. This time let's introduce λ into V so that one has

$$\left(i\frac{d}{dt} - H_0\right)\psi_\varepsilon(t) = \lambda e^{\varepsilon t} V \psi_\varepsilon(t)$$

with

$$\psi_\varepsilon(t) = e^{-iE_a t} + e^{(\varepsilon - iE_a)t}\frac{1}{i\varepsilon + E_a - H_0}\lambda V \varphi_a + \cdots$$

$$+ e^{(2\varepsilon - iE_a)t}\frac{1}{2i\varepsilon + E_a - H_0}\lambda V \frac{1}{i\varepsilon + E_a - H_0}\lambda V \varphi_a + \cdots$$

One sees that

$$\frac{\partial}{\partial t} e^{iE_a t}\psi_\varepsilon(t) = \varepsilon\lambda \frac{\partial}{\partial \lambda} e^{iE_a t}\psi_\varepsilon(t)$$

$$\frac{\partial \psi_\varepsilon}{\partial t}(t) + iE_a\psi_\varepsilon(t) = \varepsilon\lambda\frac{\partial}{\partial\lambda}\psi_\varepsilon(t)$$

$$\underbrace{i\frac{\partial \psi_\varepsilon}{\partial t}(t)}_{(H_0 + \lambda e^{\varepsilon t}V)\psi_\varepsilon(t)} - E_a\psi_\varepsilon(t) = i\varepsilon\lambda\frac{\partial}{\partial\lambda}\psi_\varepsilon(t)$$

Hence setting $t = 0$ we get

$$\text{(+)} \qquad i\varepsilon\lambda\frac{\partial}{\partial\lambda}\psi_\varepsilon(0) = (H - E_a)\psi_\varepsilon(0) \qquad H = H_0 + V$$

Now you assume that as $\varepsilon \to 0$, $\psi_\varepsilon(0)$ converges to an eigenvector for H except for a normalization factor. Specifically assume

$$\frac{\psi_\varepsilon(0)}{\langle \varphi_a | \psi_\varepsilon(0)\rangle} \to \psi_a \qquad H\psi_a = E_a'\psi_a$$

Then

$$\lim_{\varepsilon\to 0+} \frac{i\varepsilon\lambda\frac{\partial}{\partial\lambda}\left(\psi_\varepsilon(0)\right)}{\langle\varphi_a|\psi_\varepsilon(0)\rangle} = \left(E_a'-E_a\right)\frac{\psi_\varepsilon(0)}{\langle\varphi_a|\psi_\varepsilon(0)\rangle}$$

taking the dot product with φ_a

or

$$E_a'-E_a = \lim_{\varepsilon\to 0}\frac{i\varepsilon\lambda\frac{\partial}{\partial\lambda}\langle\varphi_a|\psi_\varepsilon(0)\rangle}{\langle\varphi_a|\psi_\varepsilon(0)\rangle}$$

But you should recall that

$$\psi_\varepsilon(0) = \tilde{U}_\varepsilon(0,-\infty)\varphi_a$$

which gives us the formula

$$E_a'-E_a = \lim_{\varepsilon\to 0} i\varepsilon\lambda\frac{\partial}{\partial\lambda}\log\langle\varphi_a|\tilde{U}_\varepsilon(0,-\infty)|\varphi_a\rangle$$

Assuming that the eigenstate ψ_a is discrete like φ_a (and if this doesn't follow) that

$$\psi_{-\varepsilon}(0) = U_\varepsilon(0,\infty)\varphi_a$$

when normalized by dividing by $\langle\varphi_a|\psi_{-\varepsilon}(0)\rangle$ converges to ψ. Then taking the dot product of ⊕ with $\psi_{-\varepsilon}(0)$ gives

$$E_a'-E_a = \lim_{\varepsilon\to 0}\frac{i\varepsilon\lambda\langle\psi_{-\varepsilon}(0)|\frac{\partial}{\partial\lambda}\psi_\varepsilon(0)\rangle}{\langle\psi_{-\varepsilon}(0)|\psi_\varepsilon(0)\rangle}$$

The $\frac{\partial}{\partial\lambda}$ should be symmetrical so that

$$E_a'-E_a = \lim_{\varepsilon\to 0}\frac{1}{2}i\varepsilon\lambda\frac{\partial}{\partial\lambda}\log\langle\varphi_a|S_\varepsilon|\varphi_a\rangle$$

where $S_\varepsilon = \tilde{U}_\varepsilon(\infty,0)\cdot\tilde{U}_\varepsilon(0,-\infty)$

Review adiabatic business. Let

$$H(t) = H_0 + e^{-\varepsilon|t|}V$$

with $\varepsilon > 0$. Because of the exponential damping one expects that the operators

$$\tilde{U}(0, \pm\infty) = \lim_{t\to\pm\infty} U(0,t)e^{-iH_0 t}$$

exist quite generally. Put $\psi_a^{\pm}(t) = \tilde{U}(t, \mp\infty)\varphi_a$ where φ_a is an eigenvector for H_0: $H_0\varphi_a = E_a\varphi_a$. Then for $t \le 0$, $\psi^+ = \psi_a^+$ satisfies

$$\left(i\frac{\partial}{\partial t} - H_0\right)\psi_a^+ = e^{+\varepsilon t}V\psi_a^+$$

$$\psi_a^+ \sim e^{-iE_a t}\varphi_a.$$

and so we can grind out the following series for ψ_a^+ by iteration

$$\psi_a^+ = e^{-iE_a t}\varphi_a + e^{(\varepsilon - iE_a)t}\frac{1}{i\varepsilon + E_a - H_0}V\varphi_a$$

$$+ e^{(2\varepsilon - iE_a)t}\frac{1}{2i\varepsilon + E_a - H_0}V\frac{1}{i\varepsilon + E_a - H_0}V\varphi_a + \cdots\cdots.$$

Now replace V by λV. Then operating with $-i\varepsilon\lambda\frac{\partial}{\partial\lambda}$ on $\psi_a^+(0)$ multiplies the nth term by $ni\varepsilon$, so that

$$\left(i\varepsilon\lambda\frac{\partial}{\partial\lambda} + E_a - H_0\right)\psi_a^+(0) = \lambda V\psi_a^+(0)$$

or

$$i\varepsilon\lambda\frac{\partial}{\partial\lambda}\psi_a^+(0) = (H - E_a)\psi_a^+(0)$$

In a similar way we have

$$\psi_a^-(0) = \varphi_a + \frac{1}{-i\varepsilon + E_a - H_0} V\varphi_a + \cdots$$

$$-i\varepsilon\lambda\frac{\partial}{\partial\lambda}\psi_a^-(0) = (H-E_a)\psi_a^-(0)$$

Hence

$$\tfrac{1}{2}i\varepsilon\lambda\frac{\partial}{\partial\lambda}\langle\psi_a^-(0)|\psi_a^+(0)\rangle$$

$$= \tfrac{1}{2}\langle\psi_a^-(0)|(H-E_a)\psi_a^+(0)\rangle + \tfrac{1}{2}\langle(H-E_a)\psi_a^-(0)|\psi_a^+(0)\rangle$$

$$= \langle\psi_a^-(0)|(H-E_a)|\psi_a^+(0)\rangle$$

$$= \langle\varphi_a|\tilde{U}(\infty,0)(H-E_a)|\tilde{U}(0,-\infty)|\varphi_a\rangle$$

Hence dividing by $\langle\psi_a^-(0)|\psi_a^+(0)\rangle = \langle\varphi_a|S|\varphi_a\rangle$ we get

$$\boxed{\tfrac{1}{2}i\varepsilon\lambda\frac{\partial}{\partial\lambda}\log\langle\varphi_a|S|\varphi_a\rangle = \frac{\langle\varphi_a|\tilde{U}(\infty,0)(H-E_a)|\tilde{U}(0,\infty)|\varphi_a\rangle}{\langle\varphi_a|S|\varphi_a\rangle}}$$

(Actually it should be possible to prove this formula directly by using Schwinger style methods: Assume $E_a = 0$ and write $\varphi_a = |0\rangle$ to get a more familiar notation. Then

$$\delta\log\langle 0|S|0\rangle = \lim_{T\to\infty}\delta\log\langle 0|U(T,-T)|0\rangle$$

$$= \lim_{T\to\infty}\int_{-T}^{T}(-i)\frac{\langle 0|U(T,t)\,\delta H(t)\,U(t,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}\,dt$$

$$i\langle 0|U(0,t)\,\delta H(t)\,U(t,0)|0\rangle\,dt$$

In the present case $\delta H(t) = \delta\lambda\, e^{-\varepsilon|t|} V$. Also

$$\frac{d}{dt}\langle 0|U(T,t)\,H(t)\,U(t,-T)|0\rangle$$

$$= \langle 0|U(T,t)\Big(\underbrace{[iH(t),H(t)]}_{0} + \underbrace{\frac{\partial H}{\partial t}(t)}_{-\varepsilon\,\mathrm{sign}(t)\,\lambda e^{-\varepsilon|t|}V}\Big)\,U(t,-T)|0\rangle$$

So

$$\int_0^T (-i)\langle 0|U(T,t)\,\underbrace{\delta H(t)}_{\delta\lambda\, e^{-\varepsilon|t|}V}\,U(t,-T)|0\rangle\,dt$$

$$= \int_0^T (-i)\frac{\delta\lambda}{\lambda}\left(\frac{-1}{\varepsilon}\right)\frac{d}{dt}\langle 0|U(T,t)\,H(t)\,U(t,-T)|0\rangle\,dt$$

$$= -\frac{i\,\delta\lambda}{\lambda\varepsilon}\Big[\langle 0|U(T,0)\,H\,U(0,-T)|0\rangle - \blacksquare\,o(T)\Big]$$

$$\therefore \tfrac{1}{2} i\varepsilon\lambda\frac{\partial}{\partial\lambda}\log\langle 0|S|0\rangle = \lim_{T\to\infty}\frac{\langle 0|U(T,0)\,H\,U(0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

which was to be proved.)

If we assume $\tilde{U}(0,\pm\infty)\varphi_a$ as $\varepsilon\to 0+$ converges up to ■ normalization constants to the same eigenvector φ_a for H, then we get the Gell-Mann-Low formula

$$\blacksquare\quad \Delta E = \lim_{\varepsilon\to 0+}\tfrac{1}{2} i\varepsilon\lambda\frac{\partial}{\partial\lambda}\log\langle\varphi_a|S|\varphi_a\rangle$$

Our next project will be to understand this formula in concrete cases involving the harmonic

oscillator.

So lets consider the s.h.o. with a constant forcing term as perturbation

$$H = \frac{1}{2}p^2 + \frac{1}{2}(\omega q)^2 - cq \qquad c \in \mathbb{R}$$

$$= \frac{1}{2}p^2 + \frac{1}{2}\omega^2\left(q - \frac{c}{\omega^2}\right)^2 - \frac{c^2}{2\omega^2}$$

Here the ground state energy shift is

$$\Delta E = -\frac{c^2}{2\omega^2}$$

Recall that for a source perturbation $-J(t)q$ we have

$$\log \langle 0|S|0\rangle = \frac{i}{2}\int J(t)\, G_0(t-t')\, J(t')\, dt\, dt'$$

where

$$G_0(t) = \frac{e^{-i\omega|t|}}{-2i\omega} = \int \frac{dk}{2\pi} \frac{e^{-ikt}}{-k^2+\omega^2}$$

Feynman contour where $\mathrm{Im}\,\omega < 0$

So want to evaluate this when $J(t) = \lambda e^{-\varepsilon|t|}$ c.

$$\varepsilon \int e^{-\varepsilon|t|} \frac{e^{-i\omega|t-t'|}}{-2i\omega} e^{-\varepsilon|t'|}\, dt\, dt'$$

$$= \varepsilon \int dt\, dt' \int \frac{dk}{2\pi}\, e^{-\varepsilon|t|} e^{-\varepsilon|t'|} \frac{e^{-ikt+ikt'}}{-k^2+\omega^2}$$

$$= \varepsilon \int \frac{dk}{2\pi} \frac{1}{-k^2+\omega^2} \left(\int dt\; e^{-\varepsilon|t| - ikt} \right)^2 = \varepsilon \int \frac{dk}{2\pi} \left(\frac{2\varepsilon}{\varepsilon^2+k^2}\right)^2 \frac{1}{-k^2+\omega^2}$$

$$= \int 4 \frac{dk}{2\pi} \left(\frac{1}{1+k^2}\right)^2 \frac{1}{\omega^2 - k^2\varepsilon^2} \xrightarrow[\varepsilon \to 0+]{\text{as}} \frac{4}{\omega^2\, 2\pi} \int dk\, \frac{1}{(1+k^2)^2}$$

$$\int_{\text{⌒}} dk \frac{1}{(k-i)^2(k+i)^2} = 2\pi i \text{ Res at } i = 2\pi i \frac{d}{dk}(k+i)^{-2}\Big|_{k=i}$$

$$= 2\pi i\ (-2)(2i)^{-3} = 2\pi i \frac{-2}{-8i} = \frac{\pi}{2}$$

So

$$\frac{1}{2} i\varepsilon \lambda \frac{\partial}{\partial \lambda} \log \langle 0|S|0\rangle = \frac{1}{2} i\varepsilon \lambda \frac{\partial}{\partial \lambda} \frac{i}{2}(\lambda c)^2 \int e^{-\varepsilon|t|} G_0 e^{-\varepsilon|t'|}$$

$$\xrightarrow[\varepsilon \to 0^+]{\text{as}} \frac{1}{2} i\varepsilon \not{2} \frac{i}{2} \lambda^2 c^2 \frac{\not{\pi}}{\omega^2 \not{2} \not{\pi}} \cdot \frac{\not{\pi}}{\not{k}} = -\frac{\lambda^2 c^2}{2\omega^2}$$

which checks.

When one uses perturbation theory, one would like the eigenstates for H_0 to evolve into the eigenstates for $H = H_0 + V$ as the perturbation is turned on, which means we deal with $H_\lambda = H_0 + \lambda V$ and let λ go from 0 to 1. Consider for example the forced oscillator

$$H_0 = \omega a^* a \qquad V = c a + \bar{c} a^*$$

$$H = \omega a^* a + c a + \bar{c} a^*$$

$$= \omega \left(a^* + \frac{c}{\omega}\right)\left(a + \frac{\bar{c}}{\omega}\right) - \frac{|c|^2}{\omega}$$

This is again an oscillator. The ground state is

$$\psi_0 = e^{-\frac{\bar{c}}{\omega} a^*} |0\rangle / \text{normalization const.}$$

Except for the energy shift $-\frac{|c|^2}{\omega}$ the operators H, H_0 are unitarily equivalent:

$$e^{-\left(\frac{c}{\omega} a + \frac{\bar{c}}{\omega} a^*\right)} a\, e^{\left(\frac{c}{\omega} a + \frac{\bar{c}}{\omega} a^*\right)} = e^{-\mathrm{ad}\left(\frac{c}{\omega} a + \frac{\bar{c}}{\omega} a^*\right)} a$$

$$= a - \left[\frac{c}{\omega} a + \frac{\bar{c}}{\omega} a^*, a\right] + \frac{1}{2!}[\,[\quad]] + \cdots$$

$$= a + \frac{\bar{c}}{\omega}$$

similarly $e^{-\left(\frac{c}{\omega} a + \frac{\bar{c}}{\omega} a^*\right)} a^*\, e^{\left(\frac{c}{\omega} a + \frac{\bar{c}}{\omega} a^*\right)} = a^* + \frac{c}{\omega}$

and so $e^{-\left(\frac{c}{\omega} a + \frac{\bar{c}}{\omega} a^*\right)} H_0\, e^{\left(\frac{c}{\omega} a + \frac{\bar{c}}{\omega} a^*\right)} = H + \frac{|c|^2}{\omega}$.

In general one doesn't expect a unitary equivalence between H and $H_0 + \text{const}$, because the energy shifts for the different states of H_0 will differ. However let's concentrate on the correspondence between eigenvectors.

The adiabatic way to establish the correspondence is to put in a damping factor

$$H(t) = H_0 + e^{\varepsilon t} V$$

and then see how the free eigenfunction $e^{-iE_a t}\varphi_a$ evolves from $t=-\infty$ to 0:

$$\psi(t) = \lim_{t'\to-\infty} \tilde{U}_\varepsilon(t,t')\, e^{-iE_a t'} \varphi_a \qquad \psi(0) = \tilde{U}_\varepsilon(0,-\infty)\varphi_a$$

Now one normalizes this to have inner prod. 1 with φ_a and takes the limit as $\varepsilon \to 0+$.

$$\psi_a = \lim_{\varepsilon\to 0+} \frac{\tilde{U}_\varepsilon(0,-\infty)\varphi_a}{\langle \varphi_a | \tilde{U}_\varepsilon(0,-\infty)|\varphi_a\rangle}$$

(Perhaps a simpler method is to watch how the state ψ_a evolves as λ goes from 0 to 1).

Example: Suppose we take a simple harmonic oscillator and perturb the spring constant:

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2 + \tfrac{1}{2}\varepsilon q^2$$

The eigenvalues of H are $(n+\frac{1}{2})\sqrt{\omega^2+\varepsilon}$, $n\in\mathbb{N}$, so it is no longer possible to connect H_0, H by a unitary operator + energy shift.

Let us next consider the case of a KG field, where ω^2 is replaced by the operator $-\frac{d^2}{dx^2}+m^2$ on the line and where $\frac{1}{2}\varepsilon q^2$ becomes

$$\frac{1}{2}\int \varepsilon(x)\phi(x)^2\,dx$$

where $\varepsilon(x) \in C_0^\infty(R)$ is small. Thus my Hamiltonian is

$$H = \frac{1}{2}\int\left[\pi(x)^2 + \phi(x)\left(-\frac{d^2}{dx^2}+m^2\right)\phi(x) + \varepsilon(x)\phi(x)^2\right]dx$$

and the field equation is

$$\frac{\partial^2\phi}{\partial t^2} = -\left(-\frac{\partial^2}{\partial x^2}+m^2+\varepsilon(x)\right)\phi$$

I've seen before that the ground state energy shift is

$$\Delta E = \frac{1}{2}\operatorname{tr}\left\{\left(-\frac{\partial^2}{\partial x^2}+m^2+\varepsilon\right)^{1/2} - \left(-\frac{\partial^2}{\partial x^2}+m^2\right)^{1/2}\right\}$$

and that there is some way of computing this using scattering data. This should be finite for ε small of compact support, since we are working on the line. (?)

Example: Work in a box $[0, L]$ with periodic boundary conditions, and assume ε is constant. The eigenvalues of $-\frac{\partial^2}{\partial x^2}+m^2$ are

$$\omega_k^2 = k^2+m^2 \qquad \text{where} \quad k \in \frac{2\pi}{L}\mathbb{Z}$$

Hence $$\Delta E = \frac{1}{2}\sum_{k\in\frac{2\pi}{L}\mathbb{Z}}\left(\sqrt{k^2+m^2+\varepsilon} - \sqrt{k^2+m^2}\right)$$

$$\sim \frac{\varepsilon}{2\sqrt{k^2+m^2}} \qquad \therefore \Delta E = \infty.$$