August 10, 1979

perturbing an oscillator by a source; translation on holom. fn. repn.
coherent states + metaplectic rep 160-65
perturbing oscillator by $\varepsilon(t)\frac{q^2}{2}$ and $e^{aP^2/2}e^{bx^2/2}$

According to Roman's book one can see renormalization already for KG field with external source. Hence it makes sense to get the formulas for the oscillator in good shape.

Consider $H = \omega a^*a + \tilde{J}a + Ja^*$ where $\tilde{J}, J$ are functions of time with support in $[0, T]$. We want to compute the S operator using the coherent states

$$e_\lambda = e^{\lambda z} = e^{\lambda a^*}|0\rangle .$$

Recall
$$\langle e_{\lambda'}|e_\lambda\rangle = e^{\bar{\lambda}'\lambda}$$

$$e^{-iH_0 t}e_\lambda = e_{e^{-i\omega t}\lambda} \qquad H_0 = \omega a^*a$$

$$\left[a^*a, \begin{Bmatrix} a \\ a^* \end{Bmatrix}\right] = \begin{Bmatrix} -a \\ a^* \end{Bmatrix}$$

Now

$$\langle e_{\lambda'}|S|e_\lambda\rangle = \langle e_{\lambda'}|\,U_0(T,0)^{-1}U(T,0)\,|e_\lambda\rangle$$
$$= \langle e_{\lambda''}|\,U(T,0)|e_\lambda\rangle$$

where $\lambda'' = e^{-i\omega T}\lambda'$ or $\lambda' = e^{i\omega T}\lambda''$. Let $\tilde{J}, J$ undergo infinitesimal displacements $\delta\tilde{J}, \delta J$ and recall

$$\delta U(T,0) = -i\int_0^T U(T,t)\left(\delta\tilde{J}a + \delta J a^*\right)U(t,0)\,dt$$

hence

$$\delta\log\langle e_{\lambda'}|S|e_\lambda\rangle = -i\int_0^T\left[\delta\tilde{J}(t)\langle a(t)\rangle + \delta J(t)\langle a^*(t)\rangle\right]dt$$

where $\langle a(t)\rangle = \langle e_{\lambda''}|\,U(T,t)\,a\,U(t,0)|e_\lambda\rangle \Big/ \langle e_{\lambda''}|U(T,0)|e_\lambda\rangle .$

Next

$$\frac{d}{dt}\langle a(t)\rangle = \langle [iH, a](t)\rangle$$

$$= i\langle [\omega a^* a + \tilde{J} a + J a^*, a](t)\rangle$$

$$= -i\omega\langle a(t)\rangle - iJ(t)$$

$$\frac{d}{dt}\langle a^*(t)\rangle = +i\omega\langle a^*(t)\rangle + i\tilde{J}(t)$$

$$\langle a(0)\rangle = \lambda \qquad \text{since} \quad a e_\lambda = \lambda e_\lambda$$

$$\langle a^*(T)\rangle = \overline{\lambda''}$$

Hence

$$\langle a(t)\rangle = -i\int_0^t e^{-i\omega(t-t')} J(t')\,dt' + \lambda e^{-i\omega t}$$

$$\langle a^*(t)\rangle = -i\int_t^T e^{+i\omega(t-t')}\tilde{J}(t')\,dt' + \underbrace{\overline{\lambda''}\, e^{i\omega t - i\omega T}}_{\overline{\lambda'}\, e^{i\omega t}}$$

So

$$\delta \log\langle e_{\lambda'}|S|e_\lambda\rangle = -i\lambda\int_0^T \delta\tilde{J}(t)\, e^{-i\omega t}\,dt$$

$$- i\overline{\lambda'}\int_0^T \delta J(t)\, e^{i\omega t}\,dt$$

$$-\int_0^T dt\int_0^t dt'\;\delta\tilde{J}(t)\, e^{-i\omega(t-t')} J(t') - \int_0^T dt\int_t^T dt'\;\delta J(t)\, e^{i\omega(t-t')}\tilde{J}(t')$$

In the last term we interchange t, t' and then interchange the order of integration to get

$$-\int_0^T dt'\int_{t'}^T dt = -\int_0^T dt\int_0^t dt'\;\tilde{J}(t)\, e^{-i\omega(t-t')}\,\delta J(t')$$

hence the last two terms can be combined into

$$- \iint_{t>t'} dt\,dt' \underbrace{\left[\delta\tilde{J}(t)\, J(t') + \tilde{J}(t)\,\delta J(t')\right]}_{\delta[\tilde{J}(t)\, J(t')]} e^{-i\omega(t-t')}$$

Now we can integrate out the δ ~~[illegible]~~ starting from $\tilde{J}, J = 0$, where $\langle e_{\lambda'} | S | e_\lambda \rangle = e^{\bar{\lambda}'\lambda}$

$$\boxed{\begin{aligned} \log \langle e_{\lambda'} | S | e_\lambda \rangle &= \bar{\lambda}'\lambda - i\lambda \int \tilde{J}(t) e^{-i\omega t}\, dt - i\bar{\lambda}' \int J(t)\, e^{i\omega t}\, dt \\ &\quad - \iint_{t>t'} dt\,dt' \; \tilde{J}(t)\, J(t')\, e^{-i\omega(t-t')} \end{aligned}}$$

Next we need the translation action on holomorphic representation:

$$\int |f(z)|^2 e^{-|z|^2}\, dV = \int |f(z-\alpha)|^2 e^{-z\bar{z} + \alpha\bar{z} + \bar{\alpha}z - \alpha\bar{\alpha}}\, dV$$

$$= \int \left| f(z-\alpha)\, e^{\bar{\alpha}z - \frac{1}{2}|\alpha|^2} \right|^2 e^{-|z|^2}\, dV$$

hence

$$\boxed{(T_\alpha f)(z) = e^{\bar{\alpha}z - \frac{1}{2}|\alpha|^2} f(z-\alpha)}$$

is unitary. Also

$$(T_\alpha e_\lambda)(z) = e^{-\frac{1}{2}|\alpha|^2 + \bar{\alpha}z}\, e^{\lambda(z-\alpha)}$$

$$= e^{-\frac{1}{2}|\alpha|^2 - \alpha\lambda}\, e^{(\lambda + \bar{\alpha})z}$$

$$\langle e_{\lambda'} | T_\alpha | e_\lambda \rangle = e^{-\frac{1}{2}|\alpha|^2 - \alpha\lambda + \bar{\lambda}'\lambda + \bar{\lambda}'\bar{\alpha}}$$

$$\boxed{\langle e_{\lambda'} | T_\alpha | e_\lambda \rangle = e^{\bar{\lambda}'\lambda - \alpha\lambda + \bar{\alpha}\bar{\lambda}' - \frac{1}{2}|\alpha|^2}}$$

Now suppose $\tilde{J} = \bar{J}$ so that S is unitary. Then

$$\log\langle e_{\lambda'}|S|e_\lambda\rangle = \bar{\lambda}'\lambda - \underbrace{\left(i\int \tilde{J} e^{-i\omega t}dt\right)}_{\alpha}\lambda + \underbrace{\left(-i\int J e^{i\omega t}dt\right)}_{\bar{\alpha}}\bar{\lambda}'$$

$$- \iint_{t>t'} dt\, dt' \; \tilde{J}(t)\,\bar{\tilde{J}}(t')\, e^{-i\omega(t-t')}$$

Note that

$$\mathrm{Re}\iint_{t>t'} dt\,dt'\; \tilde{J}(t)e^{-i\omega t}\;\overline{\tilde{J}(t')e^{-i\omega t'}}$$

$$= \frac{1}{2}\iint dt\,dt'\; \mathrm{Re}\left[\tilde{J}(t)e^{-i\omega t}\;\overline{\tilde{J}(t')e^{-i\omega t'}}\right] = \frac{1}{2}\alpha\bar{\alpha}$$

whence we see that S agrees with T_α

$$\alpha = i\int \tilde{J}(t)e^{-i\omega t}dt$$

up to a scalar of modulus 1. The scalar can be obtained by looking at $\langle 0|S|0\rangle$.

Digression on uncertainty principle and coherent states. To derive the uncertainty principle one uses

$$\frac{d}{dx}x - x\frac{d}{dx} = 1$$

whence

$$\|\psi\|^2 = \left(\frac{d}{dx}x\psi, \psi\right) - \left(x\frac{d}{dx}\psi, \psi\right)$$

$$= -\left(x\psi, \frac{d}{dx}\psi\right) - \left(\frac{d}{dx}\psi, x\psi\right) = -2\,\mathrm{Re}\left(x\psi, \frac{d\psi}{dx}\right)$$

so by Cauchy-Schwarz

(*) $$\|\psi\|^2 \leq 2\,\|x\psi\|\cdot\left\|\frac{d\psi}{dx}\right\|.$$

Now
$$\frac{\mathrm{Re}\left(x\psi, \frac{d\psi}{dx}\right)}{\|x\psi\|\cdot\left\|\frac{d\psi}{dx}\right\|} = \cos\theta$$

where θ is the angle between $x\psi$ and $\frac{d\psi}{dx}$, hence when the above (*) is an equality, we have $\cos\theta = -1$ or
$$\frac{d\psi}{dx} = -a x\psi \qquad \text{with} \quad a > 0$$
or
$$\psi = Ce^{-\frac{a}{2}x^2}$$

Now if ψ should be such that its average position ~~and momentum zero~~ is zero:
$$\langle\psi|x|\psi\rangle = \int x|\psi|^2 dx = 0$$

then $\|x\psi\|^2 = \int x^2|\psi|^2 dx$ is the $(\text{deviation})^2$ or $(\text{uncertainty})^2$ in the position measurement, so $\|x\psi\|$ is the uncertainty in position when $\langle\psi|x|\psi\rangle = 0$. Similarly $\left\|\frac{d\psi}{dx}\right\|$ is essentially the uncertainty in momentum when the average momentum is zero. Now by translation and multiplication by e^{ikx} one can always move a ψ so that $\langle x\rangle + \langle\frac{d}{dx}\rangle = 0$ in which case (*) is essentially the uncertainty principle. We see that the uncertainty inequality is an equality for states of the form

(+) $$\psi = C e^{-\frac{a}{2}x^2 + \alpha x}$$

where $a > 0$, $\alpha \in \mathbb{C}$, and C is a normalization constant.

What are the wave functions belonging to the states $e^{\lambda z}$? Modulo scalars one has that the operators e^{itp}, $e^{it'q}$ commute because they agree essentially with $e^{i(tp+t'q)}$; this follows from

$$e^A e^B = e^{A+B} e^{+\frac{1}{2}[A,B]}$$

when $[A,B]$ is a scalar. Hence $e^{\lambda z} = e^{\lambda a^*}|0\rangle$ will be the result of applying e^{itp} (translation) and $e^{it'q}$ (mult.) to $|0\rangle = e^{-\frac{1}{2}x^2}$. Hence the states $e^{\lambda z}$ coincide with states of the above type (+) with $a = 1$. $\blacksquare$

Next project is to understand quantizing the KG field:

$$\ddot{\phi} = -(-\Delta + m^2)\phi$$

which is a big harmonic oscillator. It is perhaps best to approach this from the phonon situation where instead of the field $\phi(x)$ one has a real function q_γ on a finite abelian group Γ. The Hamiltonian is

$$H = \frac{1}{2}\sum_\gamma p_\gamma^2 + \frac{1}{2}\sum_{\gamma,\gamma'} q_\gamma D(\gamma-\gamma') q_{\gamma'}$$

where the matrix $D(\gamma-\gamma')$ is positive-definite (and clearly translation invariant). The classical equations of motion are

$$\ddot{q}_\gamma = -\sum_{\gamma'} D(\gamma-\gamma')q_{\gamma'} .$$

Because of translation invariance the [redacted] modes of vibration are given by waves: (Note $\gamma \mapsto e^{ik\gamma}$ is a character of Γ so that k is determined in a Brillouin zone)

$$q_\gamma = e^{ik\gamma - i\omega t}$$

where $\quad -\omega^2 e^{ik\gamma} = -\sum_{\gamma'} D(\gamma-\gamma')e^{ik\gamma'}$ $\quad$ or

$$\omega^2 = \sum_\gamma D(\gamma)e^{-ik\gamma} = \hat{D}(k)$$

Here $\hat{D}(k) > 0$ because $D(\gamma-\gamma')$ is positive-definite (Bochner thm.). Also $\hat{D}(k) = \hat{D}(-k)$ because D is real. Let

$$\omega_k = \sqrt{\hat{D}(k)} \qquad \text{pos. square root}$$

whence $\quad \omega_k = \omega_{-k} > 0 .$

Now use the Fourier transform to diagonalize the Hamiltonian: Put

$$Q_k = \frac{1}{\sqrt{N}}\sum_\gamma q_\gamma e^{-ik\gamma} \qquad N = \operatorname{card}\Gamma$$

so that Q_k sees the k-th mode $\quad q_\gamma = e^{ik\gamma - i\omega_k t}$ and none of the others. Put

$$P_k = \frac{1}{\sqrt{N}}\sum_\gamma p_\gamma e^{ik\gamma}$$

so that we have the commutation relations

$$[Q_k, Q_{k'}] = [P_k, P_{k'}] = 0$$

$$[P_k, Q_{k'}] = \frac{1}{N}\sum_{\gamma,\gamma'} \underbrace{[p_\gamma, q_{\gamma'}]}_{\frac{1}{i}\delta_{\gamma\gamma'}} e^{ik\gamma}e^{-ik'\gamma'} = \frac{1}{i}\sum_\gamma \frac{1}{N}e^{i(k-k')\gamma}$$

or $$[P_k, Q_{k'}] = \frac{1}{i}\delta_{kk'}$$

Also $P_k^* = P_{-k}$, $Q_k^* = Q_{-k}$. Now rewrite the Hamiltonian using the inversion formulas

$$q_r = \frac{1}{\sqrt{N}}\sum_k Q_k e^{-ikr} \qquad p_r = \frac{1}{\sqrt{N}}\sum_k P_k e^{-ikr}$$

$$\sum_r p_r^2 = \sum_r \frac{1}{N}\sum_k P_k e^{-ikr}\sum_{k'} P_{k'} e^{-ik'r} = \sum_k P_k P_{-k} = \sum_k P_{-k} P_k$$

$$\sum_{r,r'} q_r D(r-r') q_{r'} = \sum_{\substack{r,r'\\k,k'}} \frac{1}{N} Q_{k'} e^{-ik'r} D(r-r') Q_k e^{-ikr'}$$

$r - r' + r'$

$$= \frac{1}{N}\sum_{k,k'}\sum_{r'} Q_{k'} \hat{D}_{k'}\, e^{-ik'r'} Q_k e^{-ikr'} = \sum_k Q_{-k}\hat{D}_k Q_k$$

So $$H = \frac{1}{2}\sum_k P_{-k} P_k + \frac{1}{2}\sum_k Q_{-k}\,\omega_k^2\, Q_k$$

Now you introduce creation + annih. ops. Recall annihilators are linear combinations of

$$i p_r + \sum_{r'} \omega_{r-r'} q_{r'}$$

hence we put

$$a_k = \frac{1}{\sqrt{2\omega_k}}\left(i P_{-k} + \omega_k Q_k\right)$$

$$a_k^* = \frac{1}{\sqrt{2\omega_k}}\left(-i P_k + \omega_k Q_{-k}\right)$$

so $$[a_k, a_{k'}^*] = \delta_{k,k'} \quad , \quad [a_k, a_{k'}] = [a_k^*, a_{k'}^*] = 0$$

and $$\omega_k a_k^* a_k = \frac{1}{2}\left(P_k P_{-k} + \omega_k^2 Q_{-k} Q_k \right.$$ [redacted] $$\left. - i\omega_k P_k Q_k + i\omega_k Q_{-k} P_{-k} \right)$$

so $$\sum_k \omega_k a_k^* a_k = \frac{1}{2} \sum{}' \left(P_{-k} P_k + \omega_k^2 Q_{-k} Q_k \right) - \frac{1}{2} \sum{}' \omega_k$$

and so $$H = \sum_k{}' \omega_k a_k^* a_k + \frac{1}{2} \sum{}' \omega_k \, .$$

August 11, 1979

Yesterday we went over the diagonalization of the Hamiltonian for scalar phonon oscillator on a finite abelian grp. Γ. The formulas were

$$H = \frac{1}{2}\sum_\gamma p_\gamma^2 + \frac{1}{2}\sum_{\gamma,\gamma'} q_\gamma D(\gamma-\gamma') q_{\gamma'}$$

$$q_\gamma = \frac{1}{\sqrt{N}}\sum_k Q_k e^{-ik\gamma} \qquad Q_k = \frac{1}{\sqrt{N}}\sum_\gamma q_\gamma e^{-ik\gamma}$$

(hence Q_k sees the basic wave mode $q_\gamma = e^{ik\gamma}$).

$$p_\gamma = \frac{1}{\sqrt{N}}\sum P_k e^{-ik\gamma} \qquad P_k = \frac{1}{\sqrt{N}}\sum_\gamma p_\gamma e^{-ik\gamma}$$

Then
$$H = \frac{1}{2}\sum_k \left(P_k^* P_k + \omega_k^2 Q_k^* Q_k\right). \qquad \omega_k^2 = \sum D(\gamma) e^{+i\gamma k}$$

Put
$$a_k = \frac{1}{\sqrt{2\omega_k}}\left(iP_{-k} + \omega_k Q_k\right)$$
$$a_k^* = \frac{1}{\sqrt{2\omega_k}}\left(-iP_k + \omega_k Q_{-k}\right)$$

$$[P_k, Q_{k'}] = \delta_{kk'}\frac{1}{i}$$
$$[P_k, P_{k'}] = [Q_k, Q_{k'}] = 0$$
$$[a_k, a_{k'}^*] = \delta_{kk'}$$
$$[a_k, a_{k'}] = [a_k^*, a_{k'}^*] = 0$$

then
$$H = \sum_k \omega_k \left(a_k^* a_k + \frac{1}{2}\right)$$

Notice also that we have
$$Q_k = \frac{1}{\sqrt{2\omega_k}}\left(a_k + a_{-k}^*\right)$$
$$q_\gamma = \frac{1}{\sqrt{N}}\sum_k \frac{1}{\sqrt{2\omega_k}}\left(a_k e^{ik\gamma} + a_k^* e^{-ik\gamma}\right)$$

Now we want to discuss quantizing the KG field which has the field equation
$$\ddot{\phi} = -(-\Delta + m^2)\phi$$

This is a harmonic oscillator with $D = -\Delta + m^2$, which can be diagonalized use the wave mode

$$\phi(x) = e^{-ikx}$$

which has frequency $\omega_k = \sqrt{k^2+m^2}$. We work in a box of volume V with periodic boundary conditions, and do the Fourier transform

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_k \phi_k e^{ikx} \qquad \phi_k = \frac{1}{\sqrt{V}} \int \phi(x) e^{-ikx}\, dx$$

$$\dot\phi(x) = \frac{1}{\sqrt{V}} \sum_k \pi_k e^{-ikx} \qquad \pi_k = \frac{1}{\sqrt{V}} \int \dot\phi(x) e^{ikx}\, dx$$

Then

$$H = \frac{1}{2} \sum_k \left(\pi_k^* \pi_k + \omega_k^2 \phi_k^* \phi_k \right)$$

Put

$$a_k = \frac{1}{\sqrt{2\omega_k}} \left(i\pi_{-k} + \omega_k \phi_k \right)$$

then

$$H = \sum_k \omega_k a_k^* a_k + \frac{1}{2} \sum \omega_k$$

and

$$\phi_k = \frac{1}{\sqrt{2\omega_k}} \left(a_k + a_{-k}^* \right)$$

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{ikx} + a_k^* e^{-ikx} \right)$$

Now the ground state energy for an oscillator in this case is

$$\frac{1}{2} \sum \omega_k$$

which is infinite, but one ignores this because "only energy differences are measurable."

Conclusion: The above formulas show how to quantize the Klein-Gordon field in a box of volume V. The important point is that one gets an oscillator Hamiltonian

$$H = \sum_k \omega_k a_k^* a_k$$

where k runs over the lattice of wave vectors belonging to our box and $\omega_k = \sqrt{k^2+m^2}$.

Now I want to perturb the KG field by a source, so the field equation then becomes

$$\ddot{\phi} = -(-\Delta + m^2)\phi + \rho$$

and it comes from the Hamiltonian

$$H = \int \left[\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\phi(-\Delta + m^2)\phi - \rho\phi\right] dx$$

$$-\int \rho(x)\phi(x)\,dx = -\int \rho(x) \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{ikx} + a_{+k}^* e^{-ikx}\right) dx$$

$$= \sum_k \tilde{J}_k a_k + J_k a_k^*$$

where

$$\tilde{J}_k = -\frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_k}} \int \rho(x) e^{ikx}\,dx$$

$$J_k = -\frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_k}} \int \rho(x) e^{-ikx}\,dx$$

so that

$$\bar{\tilde{J}}_k = J_k \qquad \text{Also} \quad \bar{\tilde{J}}_k = \tilde{J}_{-k}.$$

So what we have is an oscillator with a

constant forcing term.

Example: Go back to $H = \frac{1}{2}p^2 + \frac{1}{2}(\omega q)^2 - Jq$. where J is a real constant. The classical motion is

$$\ddot{q} = -\omega^2 q + J$$

so the equilibrium position is

$$q_0 = \frac{J}{\omega^2}$$

So [redacted] let's shift the origin; put

$$q = \tilde{q} + \frac{J}{\omega^2}$$

and H becomes

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\left(\tilde{q}^2 + 2\frac{J}{\omega^2}\tilde{q} + \frac{J^2}{\omega^4}\right) - J\left(\tilde{q} + \frac{J}{\omega^2}\right)$$

$$= \frac{1}{2}p^2 + \frac{1}{2}\omega^2\tilde{q}^2 - \frac{1}{2}\frac{J^2}{\omega^2}.$$

Hence there is a shift of the classical ground energy to $-\frac{1}{2}\frac{J^2}{\omega^2}$.

Next consider

$$H = \omega a^* a + \tilde{J} a + J a^*$$

and make the change [redacted] $a = \tilde{a} + v$

$H =$ [redacted] $\omega(\tilde{a}^* + v^*)(\tilde{a} + v) + \tilde{J}(\tilde{a} + v) + J(\tilde{a}^* + v^*)$

$$= \omega\tilde{a}^*\tilde{a} + (\omega v^* + \tilde{J})\tilde{a} + (\omega v + J)\tilde{a}^* + \omega v^* v + \tilde{J} v + J v^*$$

Then choose $v = -\frac{J}{\omega}$. $\omega\frac{\tilde{J}}{\omega}\frac{J}{\omega} -$ [redacted] $\tilde{J}\frac{J}{\omega}$ [redacted] $- J\frac{\tilde{J}}{\omega} = -\frac{J\tilde{J}}{\omega}$

So

$$H = \omega \tilde{a}^* \tilde{a} - \frac{\bar{J}\tilde{J}}{\omega}$$

and the [redacted] ground energy decreases to $-\frac{|\tilde{J}|^2}{\omega}$. The ground state for H, denote it Φ_0, satisfies

$$\tilde{a}\,\Phi_0 = (a - v)\Phi_0 = 0$$

and hence in terms of the ground state $|0\rangle$ for $H_0 = \omega a^* a$, we have

$$\Phi_0 = \sqrt{Z}\, e^{v a^*} |0\rangle$$

where $\sqrt{Z}$ is a normalization constant making $\langle \Phi_0 | \Phi_0 \rangle = 1$

$$\left(\frac{1}{\sqrt{Z}}\right)^2 = \|e_v\|^2 = e^{\bar{v}v}$$

so

$$\sqrt{Z} = e^{-\frac{1}{2}|v|^2}.$$

August 12, 1979

Change $\tilde{J}$ and J so the simple harmonic oscillator with source Hamiltonian becomes

$$H = \omega a^* a + J a + \tilde{J} a^*$$

Recall
$$(T_\alpha f)(z) = e^{-\frac{1}{2}|\alpha|^2 + \bar{\alpha} z} f(z-\alpha)$$

or
$$T_\alpha = e^{-\frac{1}{2}|\alpha|^2} e^{\bar{\alpha} a^*} e^{-\alpha a}$$
$$= e^{-\frac{1}{2}|\alpha|^2} e^{\bar{\alpha} a^* - \alpha a} e^{\underbrace{\frac{1}{2}[\bar{\alpha} a^*, -\alpha a]}_{\frac{1}{2}|\alpha|^2}}$$

or
$$\boxed{T_\alpha = e^{\bar{\alpha} a^* - \alpha a}}$$

Next notice that if $Ja + \tilde{J} a^* = \delta(t)(ca + \tilde{c} a^*)$ then
$$S = U(0^+, 0^-) = e^{-i(ca + \tilde{c} a^*)}$$

If $Ja + \tilde{J} a^* = \delta(t - t_0)(ca + \tilde{c} a^*)$, then
$$S = e^{iH_0 t} e^{-i(ca + \tilde{c} a^*)} e^{-iH_0 t} = e^{-i(ce^{-i\omega t} a + \tilde{c} e^{i\omega t} a^*)}.$$

Now one ought to be able to regard a general source $Ja + \tilde{J} a^*$ as a succession of δ-function sources, and calculate the S matrix as a product of the above types. Hence working modulo scalar operators we have for a general source

$$S = e^{-i\left(\int J(t) e^{-i\omega t} dt \; a + \int \tilde{J}(t) e^{+i\omega t} dt \; a^*\right)}$$

$$= T_\alpha \quad \text{where} \quad \alpha = i \int J(t) e^{-i\omega t} dt$$

which is the result obtained on p. 146.

Suppose we consider a 1-dimensional Schroedinger equation

$$(*) \qquad \left[-\frac{d^2}{dt^2} + V(t)\right]\phi = \omega^2\phi$$

or

$$\ddot{\phi} = -\omega^2\phi + V\phi,$$

where $V \in C_0^\infty(\mathbb{R})$. We [redacted] can view this as the equation of motion of an oscillator whose spring constant varies in time. Hence we have the Hamiltonian

$$(**) \qquad H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 - \frac{1}{2}Vq^2$$

<u>Question</u>: Are there any interesting relations between the scattering for $(*)$ and the S matrix for the perturbed oscillator $(**)$?

The viewpoint I want to adopt is that the scattering associated to $(*)$ is a 2×2 symplectic matrix and the corresponding S matrix is [redacted] the unitary matrix belonging to this symplectic matrix. Metaplectic viewpoint.

In some sense this is obvious from the Heisenberg picture. To simplify assume $\operatorname{Supp} V \subset [0,T]$. Then we have for the Heisenberg operators

$$\hat{p}(t) = U(t,0)^{-1}\,p\,U(t,0) \qquad \text{etc}$$

the equations of motion

$$\frac{d\hat{q}}{dt} = [iH, q]^\wedge = \hat{p}$$

$$\frac{d\hat{p}}{dt} = [iH, p]^\wedge = -\omega^2\hat{q} + V\hat{q}.$$

Consequently if we take the propagator matrix for (*) between 0 and T, then this is a symplectic matrix which connects $\hat{q}, \hat{p}$ at T with $\hat{q}, \hat{p}$ at 0, so $U(t,0)$ is a unitary operator compatible with this symplectic matrix. The next steps is to make this precise.

It seems also that the coherent states with avg. position & momentum zero can be intrinsically defined in terms of the span of p, q with its symplectic structure. If so, then $|0\rangle = e^{-\frac{1}{2}\omega x^2}$ has to go into a coherent state under S. <u>NO</u>: Consider the symplectic transformation

$$\begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q \\ p+cq \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

Then a state satisfying

$$(\psi, \psi) = 2 \|q'\psi\| \cdot \|p'\psi\|$$

has to be such that $\quad ip'\psi = -\lambda q'\psi \quad$ with $\lambda > 0 \quad$ or

$$\left(\frac{d}{dx} + icx\right)\psi = -\lambda x \psi \qquad \text{so } \psi = \text{const } e^{-\frac{1}{2}(\lambda + ic)x^2}$$

Notice that the unitary transformation $e^{\frac{1}{2}icq^2}$ generates the above symplectic transformation

$$e^{-\frac{1}{2}icq^2} \begin{pmatrix} q \\ p \end{pmatrix} e^{\frac{1}{2}icq^2} = \begin{pmatrix} q \\ p+cq \end{pmatrix}.$$

Also if $\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$

and $i = [q', p'] = (ad-bc)[q, p]$ so $ad - bc = 1$, then solutions of $\quad ip'\psi = -\lambda q'\psi \quad$ with $\lambda > 0$

are [redacted] easily found as follows:

$$i(cq + d\cdot p)\psi = -\lambda(aq + bp)\psi$$

$$(id + b\lambda)\,p\,\psi = -(ic + \lambda a)\,q\,\psi$$

$$(d - ib\lambda)\frac{d\psi}{dx} = -(ic + \lambda a)\,x\,\psi$$

$$\psi = \text{const}\cdot\exp\left\{-\frac{(ic+\lambda a)}{(d - ib\lambda)}\frac{x^2}{2}\right\}$$

Now $$\frac{ic + \lambda a}{d - ib\lambda} = [redacted]\, i\,\frac{c - i\lambda a}{d - i\lambda b} = [redacted]\, i\,\underbrace{\frac{a(-i\lambda) + c}{b(-i\lambda) + d}}_{\text{is in LHP}}$$

since $-i\lambda \in$ LHP and $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL_2\mathbb{R}$

so $\psi = \text{const}\ \exp\left(-\alpha\frac{x^2}{2}\right)$ with $\mathrm{Re}(\alpha) > 0$. Hence states of this form are perhaps the good coherent states.

Conjecture: The metaplectic representation can be easily described using these coherent states.

August 13, 1979

The metaplectic representation is an action of the double covering of $SL_2(\mathbb{R})$ on $L^2(\mathbb{R})$. It is obtained by identifying the quadratic elements in the q,p operators with the Lie algebra of $SL_2(\mathbb{R})$. The self-adjoint quadratic operators are $\frac{1}{2}q^2$, $\frac{1}{2}p^2$, $\frac{1}{2}(qp+pq)$ and they give rise to one parameter unitary groups on $L^2(\mathbb{R})$ and hence also on the operator algebra on $L^2(\mathbb{R})$. Moreover these one-parameter groups carry the space of operators $aq+bp$ into itself, and hence give one-parameter unitary groups in $SL_2(\mathbb{R})$. Compute:

$$\frac{d}{dt}\, e^{itX} A e^{-itX}\Big|_{t=0} = [iX, A]$$

$$\left[i\tfrac{1}{2}q^2, \begin{pmatrix} q \\ p \end{pmatrix}\right] = \begin{pmatrix} 0 \\ -q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

$$\left[i\tfrac{1}{2}p^2, \begin{pmatrix} q \\ p \end{pmatrix}\right] = \begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

$$\left[i\tfrac{1}{2}(qp+pq), \begin{pmatrix} q \\ p \end{pmatrix}\right] = \begin{pmatrix} q \\ -p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

hence

$$e^{it\frac{1}{2}(p^2+q^2)}\begin{pmatrix} q \\ p \end{pmatrix} e^{-it\frac{1}{2}(p^2+q^2)} = e^{t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}\begin{pmatrix} q \\ p \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

Notice this gives the identity transformation on $\mathbb{R}^2 = \mathbb{R}q + \mathbb{R}p$ when $t = 2\pi$. But the corresponding operator on $L^2(\mathbb{R})$ is $-I$; since

$$e^{it\frac{1}{2}(p^2+q^2)} \cdot e^{-\frac{1}{2}x^2} = e^{it\frac{1}{2}}\, e^{-\frac{1}{2}x^2}$$

Consider the Fourier transform on L^2

$$(\mathcal{F}f)(x) = \frac{1}{\sqrt{2\pi}}\int f(y)\, e^{-iyx}\, dy$$

Then
$$p\mathcal{F}f = \mathcal{F}qf \quad \text{or} \quad \mathcal{F}q\mathcal{F}^{-1} = p$$
$$q\mathcal{F}f = -\mathcal{F}pf \quad \text{or} \quad \mathcal{F}p\mathcal{F}^{-1} = -q$$

so
$$\mathcal{F}\begin{pmatrix} q \\ p \end{pmatrix}\mathcal{F}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

Notice that $\mathcal{F}$ is not the same as $e^{it\frac{1}{2}(p^2+q^2)}$ for $t = \pi/2$ since the latter squared is $-I$ whereas

$$(\mathcal{F}^2 f)(x) = f(-x),$$

e.g.
$$\mathcal{F}e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} \quad \text{whereas}$$
$$e^{i\frac{\pi}{4}(p^2+q^2)}e^{-\frac{x^2}{2}} = e^{i\frac{\pi}{4}}e^{-\frac{x^2}{2}}.$$

But one has
$$\mathcal{F}a^*\mathcal{F}^{-1} = \mathcal{F}\frac{1}{\sqrt2}(-ip+q)\mathcal{F} = \frac{1}{\sqrt2}(+iq+p)$$
$$= ia^*$$

hence
$$\boxed{\mathcal{F}(a^{*n}|0\rangle) = i^n a^{*n}|0\rangle}$$

and so we have the formula
$$\mathcal{F} = e^{i\frac{\pi}{2}\left[\frac{1}{2}(p^2+q^2)-\frac{1}{2}\right]} = e^{i\frac{\pi}{2}a^*a}.$$

Question: Is $\mathcal{F}$ an element of the metaplectic group $G = \widetilde{SL_2(\mathbb{R})}$ (double covering)?

Actually one should really check that the

the operators $\frac{1}{2}p^2$, $\frac{1}{2}q^2$, $\frac{1}{2}(pq+qp)$ are really the right things for the metaplectic reps. After all, one can add a multiple of the identity to these and still get a representation of the Lie algebra of SL_2. No, you haven't checked that these operators do give a representation of the Lie algebra. So do this:

$$\left[i\frac{pq+qp}{2}, i\frac{p^2}{2}\right] = -$$ [redacted] $$\left[pq, \frac{p^2}{2}\right] = -$$ [redacted] $$p\left[q, \frac{p^2}{2}\right] =$$ [redacted]

$$= -p^2(+i) = -2\left(\frac{i}{2}p^2\right)$$

$$\left[i\frac{pq+qp}{2}, i\frac{q^2}{2}\right] = -$$ [redacted] $$\left[pq, \frac{q^2}{2}\right] = -$$ [redacted] $$\frac{1}{i}q^2 = 2\left(\frac{i}{2}q^2\right)$$

$$\left[i\frac{p^2}{2}, i\frac{q^2}{2}\right] = -\frac{1}{4}\left([p,q^2]p + p[p,q^2]\right)$$

$$= -\frac{1}{4}\left(\frac{2}{i}qp + p\frac{2}{i}q\right) = i\left(\frac{pq+qp}{2}\right)$$

(The reason the signs are wrong is that the matrices on p. 161 [redacted] show how the basis q, p behaves, not the coordinates relative to this basis.)

I claim F is <u>not</u> in the metaplectic group, since its effect on $\binom{q}{p}$ is the matrix $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ and we know the two possible metaplectic group elements over $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, namely $e^{it\frac{1}{2}(p^2+q^2)}$ where $t = \frac{\pi}{2}, \frac{\pi}{2}+2\pi$, and these [redacted] multiply $e^{-\frac{1}{2}x^2}$ by $e^{i\frac{\pi}{4}}$, $e^{i\frac{5\pi}{4}}$ respectively [redacted].

Suppose we have a unitary operator with

$$S(q\ \ p)S^{-1} = (q\ \ p)\begin{pmatrix}a&b\\c&d\end{pmatrix} = (aq+cp\ \ \ bq+dp)$$

and let us compute $\psi = S e^{i\tau\frac{x^2}{2}}$ where $\tau \in UHP$. Then

$$SpS^{-1}Se^{i\tau\frac{x^2}{2}} = S\,\tau x e^{i\tau\frac{x^2}{2}} = \tau S q S^{-1}\psi$$

$$\begin{aligned} &\quad\;\, \| \\ &(bq+dp)\psi = \tau(aq+cp)\psi \end{aligned}$$

$$(b-a\tau)q\psi = (c\tau-d)p\psi$$

$$\frac{1}{i}\frac{d}{dx}\psi = \text{[redacted]} \left(\frac{a\tau-b}{-c\tau+d}\right)x\psi$$

$$\boxed{S e^{i\tau\frac{x^2}{2}} = \psi = \text{const.}\ e^{i\left(\frac{a\tau-b}{-c\tau+d}\right)\frac{x^2}{2}}}$$

Consequently corresponding to the symplectic matrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ on $\mathbb{R}q + \mathbb{R}p$, we have the transformation

$$\tau \longmapsto \left(\frac{a\tau - b}{-c\tau + d}\right)$$

on the UHP. Check: if $S = e^{it\frac{q^2}{2}}$ then

$$e^{it\frac{q^2}{2}}(q\ \ p)e^{-it\frac{q^2}{2}} = (q\quad p-tq) = (q\ \ p)\begin{pmatrix} 1 & -t \\ 0 & 1\end{pmatrix}$$

$$e^{it\frac{q^2}{2}}e^{i\tau\frac{x^2}{2}} = e^{i(\tau+t)\frac{x^2}{2}}$$

$$\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}\tau = \text{[redacted]}\ \ \tau + t$$

[redacted] The constant in the above boxed formula is up to sign a complicated function of $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. [redacted] It seems that the orbit [redacted]for the metaplectic group of $e^{-x^2/2}$ is free, i.e. the stabilizer is trivial. However the stabilizer of the line spanned by $e^{-x^2/2}$ is the circle group generated by $\frac{1}{2}(p^2+q^2)$. It appears that

$$S \longmapsto \langle 0|S|0\rangle \qquad |0\rangle = e^{-x^2/2}$$

is a sort of spherical function on the metaplectic group.

This suggests we can get ahold of the metaplectic group as follows. Consider all the functions of the form

$$\xi \, |\mathrm{Im}\,\tau|^{1/4} \text{ [redacted] } e^{i\tau \frac{x^2}{2}} \qquad \xi \in S^1, \qquad \tau \in UHP.$$

These are coherent states which are normalized:

$$\int \left| e^{i\tau\frac{x^2}{2}} \right|^2 dx = \int e^{\mathrm{Re}(i\tau)x^2} dx = \int e^{-(\mathrm{Im}\,\tau)x^2} dx$$

$$= \sqrt{\frac{\pi}{\mathrm{Im}\,\tau}}$$

hence $\pi^{-1/4}(\mathrm{Im}\,\tau)^{1/4} e^{i\tau\frac{x^2}{2}}$ has norm 1. The metaplectic group should act simply-transitively on the set of these functions.

Consider now

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{\varepsilon(t)}{2} q^2$$

with $\varepsilon(t)$ compactly supported, say in $[0,T]$.

$$\langle 0|S|0\rangle = \langle 0|U_0(T,0)^{-1}U(T,0)|0\rangle = \frac{\langle 0|U(T,0)|0\rangle}{\langle 0|U_0(T,0)|0\rangle}$$

$$\delta \log\langle 0|S|0\rangle = \frac{\langle 0|\delta U(T,0)|0\rangle}{\langle 0|U(T,0)|0\rangle}$$

$$= -i\int \langle (\delta H)(t)\rangle\, dt = -\frac{i}{2}\int \varepsilon(t)\langle q^2(t)\rangle\, dt$$

Recall we obtained from this the formula

$$\langle 0|S|0\rangle = \det(1+G_0\varepsilon)^{-1/2}$$

where G_0 is the Green's function for $\frac{d^2}{dt^2}+\omega^2$ i.e.

$$G_0(t,t') = \frac{e^{-i\omega|t-t'|}}{-2i\omega}$$

Let us review the diagrammatical way of handling this calculation. According to Dyson's expansion

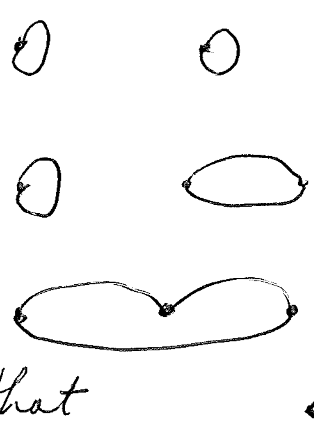
$$\langle 0|S|0\rangle = 1 - i\int dt_1 \langle H'(t_1)\rangle + \frac{(-i)^2}{2!}\int dt_1 \int dt_2 \langle T H'(t_1) H'(t_2)\rangle + \cdots$$

where $H'(t) = \frac{\varepsilon(t)}{2} q^2(t)$ (this is in the interaction picture so that $q(t) = e^{iH_0 t} q e^{-iH_0 t}$.) Using Wick's thm.

$$\langle T\{q^2(t_1)\cdots q^2(t_n)\}\rangle = \sum_{\substack{\text{all possible}\\ \text{pairwise}\\ \text{contractions.}}} \cdots\cdots\cdots$$

$$\langle T\{q(t_1)\, q(t_2)\}\rangle = -i\, G_0(t_1,t_2)$$

we see the nth order contribution to $\langle 0|S|0\rangle$ is a sum over all possible ways of making a graph with n vertices and exactly 2 edges meeting at each vertex. In 3rd order we have the following types



One knows that $\langle 0|S|0\rangle = e^{L}$ where L contains the connected graph terms. The n-th order contribution to L is given by a single n-cycle:

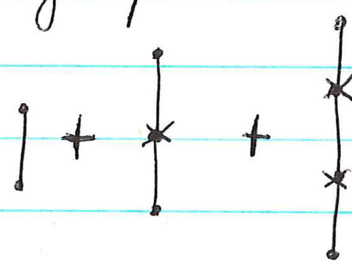
$$L_n = \underbrace{(-i)^n \frac{1}{n!}}_{\text{Dyson expansion}} \underbrace{[2(n-1)][2(n-2)]\cdots[2\cdot 1]}_{\text{total ways of doing the contractions}} \operatorname{Tr}\left(\frac{\varepsilon}{2}(-iG_0)\right)^n$$

start, then have $2(n-1)$ possibilities for first edge, etc.

$$= \frac{(-1)^n}{2}\frac{1}{n}\operatorname{Tr}(\varepsilon G_0)^n$$

Thus $$L = -\frac{1}{2}\sum_n{}' \frac{(-1)^{n-1}}{n}\operatorname{Tr}(\varepsilon G_0)^n = -\frac{1}{2}\operatorname{Tr}\log(1+\varepsilon G_0)$$

Next consider the Green's function which is the sum over connected graphs

$G(t_1,t_2) =$  $+ \cdots$

there are two possible contractions at each vertex which gets rid of the $\frac{1}{2}$ in $\frac{\varepsilon}{2}$

or $$G = G_0 + G_0(-\varepsilon)G_0 + G_0(-\varepsilon)G_0(-\varepsilon)G_0 + \cdots$$

or $$G = G_0(1+\varepsilon G_0)^{-1}$$

$$= \frac{1}{G_0^{-1}+\varepsilon} = \frac{1}{\frac{d^2}{dt^2}+\omega^2+\varepsilon}$$

Here the *irreducible self-energy part* Σ is just $-\varepsilon$

August 15, 1979

Compute $\quad e^{a\frac{D^2}{2}} e^{b\frac{x^2}{2}}$. Recall

$$\int e^{-\beta\frac{x^2}{2} - i\xi x}\, dx = \sqrt{\frac{2\pi}{\beta}}\, e^{-\frac{1}{\beta}\frac{\xi^2}{2}}$$

so

$$e^{-\beta\frac{x^2}{2}} = \int \frac{d\xi}{2\pi} e^{i\xi x} \sqrt{\frac{2\pi}{\beta}}\, e^{-\frac{1}{\beta}\frac{\xi^2}{2}}$$

$$e^{a\frac{D^2}{2}} e^{-\beta\frac{x^2}{2}} = \int \frac{d\xi}{2\pi} \underbrace{e^{a\frac{D^2}{2}} e^{i\xi x}}_{e^{-a\frac{\xi^2}{2}} e^{i\xi x}} \sqrt{\frac{2\pi}{\beta}}\, e^{-\frac{1}{\beta}\frac{\xi^2}{2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\pi}{\beta}} \int d\xi\, e^{i\xi x}\, e^{-\left(a+\frac{1}{\beta}\right)\frac{\xi^2}{2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\pi}{\beta}} \sqrt{\frac{2\pi}{\left(a+\frac{1}{\beta}\right)}}\, e^{-\frac{1}{\left(a+\frac{1}{\beta}\right)}\frac{x^2}{2}}$$

$$= \frac{1}{\sqrt{1+a\beta}}\, e^{-\frac{\beta}{1+a\beta}\frac{x^2}{2}}$$

$$\boxed{e^{a\frac{D^2}{2}} e^{b\frac{x^2}{2}} = \frac{1}{\sqrt{1-ab}}\, e^{\frac{b}{1-ab}\frac{x^2}{2}}}$$

We apply this to an oscillator:

$$H = \frac{1}{2}p^2 + \frac{1}{2}(\omega q)^2 + \frac{\varepsilon}{2}q^2 - Jq$$

with $H_0 = \frac{1}{2}p^2 + \frac{1}{2}(\omega q)^2 - Jq$. Then we have seen that

$$Z_0(J) = \iint e^{i\int L_0 + i\int Jq} = e^{\frac{i}{2}\int J(t) G_0(t,t') J(t')}$$

and that

$$Z(J) = \iint e^{i\int L + i\int Jq} = e^{i\int -\frac{\varepsilon}{2}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)^2} Z_0(J)$$

$$= e^{\int i\frac{\varepsilon}{2}\frac{\delta^2}{\delta J(t)^2}dt} \; e^{\frac{i}{2}\int J(t)\,G(t,t')\,J(t') \text{[redacted]}}$$

Use the above formula with $a = \frac{i\varepsilon}{\text{[redacted]}}$ $b = \frac{i}{\text{[redacted]}} G_0$, or more precisely, you generalize it first to matrices.

$$\frac{b}{1-ab} = b + bab + \cdots = i\left[G_0 + G_0(-\varepsilon)G_0 + \cdots\right] = iG$$

and so we get

$$Z(J) = \det\left(1+\varepsilon G_0\right)^{-1/2} e^{\frac{i}{2}\int J(t)G(t,t')J(t')}$$

The determinant factor is an interesting constant we don't expect; it arises because we've left out the appropriate determinant factor [redacted] when we use the formula

$$Z_0(J) = e^{\frac{i}{2}\int JG_0J}.$$

In more detail note that

$$\int L_0 = \int \frac{1}{2}\dot{q}^2 - \frac{1}{2}(\omega q)^2 = -\frac{1}{2}\int q\left(\frac{d^2}{dt^2} + \omega^2\right)q\, dt$$

hence the path integral

$$\iint e^{i\int L_0}\, e^{iJq}$$

is the Fourier transform of the Gaussian integral

$$q \longmapsto e^{-\text{[redacted]}\frac{1}{2}\int(+i)\,q\left(\frac{d^2}{dt^2}+\omega^2\right)q}$$

But we know that the Fourier transform of $e^{-\frac{1}{2}xAx}$ is $\dfrac{(2\pi)^{d/2}}{(\det A)^{1/2}}\, e^{-\frac{1}{2}xA^{-1}x}$

so the path integral is formally

$$\text{const.} \det\left(\frac{d^2}{dt^2}+\omega^2\right)^{-1/2} e^{\underbrace{-\frac{1}{2}\int J\,(i^{-1}G_0)^{\text{[redacted]}}\,J}_{\frac{i}{2}\int J G_0 J}}$$

Thus we have

$$Z_0(J) = \text{const} \quad \det\left(\frac{d^2}{dt^2}+\omega^2\right)^{-1/2} e^{\frac{i}{2}\int J G_0 J}$$

$$\|$$

$$Z_{\text{[redacted]}}(J) = \text{const} \quad \det\left(\frac{d^2}{dt^2}+\omega^2+\varepsilon\right)^{-1/2} e^{\frac{i}{2}\int J G J}$$

leading to the more precise result

$$\frac{Z(J)}{Z_0(J)} = \underbrace{\frac{\det\left(\frac{d^2}{dt^2}+\omega^2+\varepsilon\right)^{-1/2}}{\det\left(\frac{d^2}{dt^2}+\omega^2\right)^{-1/2}}}_{\det(1+\varepsilon G_0)^{-1/2}} \; e^{\frac{i}{2}\int J[G-G_0]J}$$