July 25, 1979 — forced oscillator, Kubo formula

More Schwinger. Consider a harmonic oscillator with "source" term:

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 - J(t)q$$

and let's rapidly review the formula for $\langle 0|S|0\rangle$. Use actual time so that

$$\frac{\partial}{\partial t}\,\delta U(t,t') = -i\left[H(t)\,\delta U(t,t') + \delta H(t)\,U(t,t')\right].$$

One has

$$\delta \log\langle 0|S|0\rangle = -i\int_{t_{in}}^{t_f} \frac{\langle 0|\,U(t_f,t)\,\delta H(t)\,U(t,t_{in})\,|0\rangle}{\langle 0|\,U(t_f,t_i)\,|0\rangle}\,dt$$

$$= +i\int \delta J\,\langle q(t)\rangle\,dt$$

where

$$\frac{d^2}{dt^2}\langle q(t)\rangle = -\omega^2\langle q(t)\rangle + J(t)$$

so that

$$\langle q(t)\rangle = \int \underbrace{G_0(t,t')}_{\frac{e^{-i\omega|t-t'|}}{-2i\omega}} J(t')\,dt'$$

Hence

$$\boxed{\log\langle 0|S|0\rangle = \frac{1}{2}\,i\int dt\int dt'\; J(t)\,G_0(t,t')\,J(t')}$$

Let's consider now $J(t) = c\,\delta(t)$. Recall that to solve for $U(0^+,0^-)$ one spreads time out around 0.

$$d\psi = -i(H\,dt)\psi = -i(H_0 - c\,\delta(t)q)\,dt\,\psi$$

Use new parameter s with $ds = \delta(t)\,dt$ for t between

0^- and 0^+; hence $0 \le s \le 1$.

$$\frac{d\psi}{ds} = icq\psi \qquad \psi = e^{icqs}\psi(0)$$

and so we see that

$$U(0^+, 0^-) = e^{icq}$$

Check: For $J(t) = c\,\delta(t)$ the above formula for $\langle 0|S|0\rangle$ gives

$$\log \langle 0|S|0\rangle = \frac{1}{2} i c^2 \frac{1}{-2i\omega} = -\frac{c^2}{4\omega}.$$

But also

$$\langle 0|S|0\rangle = \langle 0|e^{icq}|0\rangle = \frac{\int e^{icq - \omega^2 q^2}\,dq}{\int e^{-\omega^2 q^2}\,dq} = e^{-\frac{c^2}{4\omega}}.$$

Schwinger uses this as follows. He wants to find the S-matrix $\langle n|S|n'\rangle$ in the occupation number representation in the case of a general source J. One has the above formula on page 98 for $\langle 0|S|0\rangle$ in terms of J. Supposing J supported inside (t_{in}, t_f) one can add δ function sources located at $t = t_{in}$ and t_f. Let

$$\tilde{J} = J + c\,\delta(t - t_f) + c'\delta(t - t_{in})$$

Then $\langle 0|\tilde{S}|0\rangle$ will essentially be $\langle \psi|S|\psi'\rangle$ where ψ, ψ' are states of the form

$$e^{icq}|0\rangle = e^{icq} e^{-\frac{1}{2}\omega q^2}$$

Notice that $\sqrt{2\omega}\, a = \left(\frac{d}{dq} + \omega q\right)$ applied to this gives

$$e^{-\frac{1}{2}\omega q^2}\underbrace{e^{(\frac{1}{2})\omega q^2}\left(\frac{d}{dq}+\omega q\right)e^{-\frac{1}{2}\omega q^2}}_{\frac{d}{dq}}e^{icq} = ic\left(e^{icq}e^{-\frac{1}{2}\omega q^2}\right)$$

Thus $e^{icq}|0\rangle$ is an eigenvector for a with eigenvalue $\frac{ic}{\sqrt{2\omega}}$, hence it is a so-called coherent state. In the polynomial repn. $a^* = z$, $a = \frac{d}{dz}$, the eigenvectors for a are

$$e^{\lambda z} = \sum_{n\geq 0}{}' \frac{\lambda^n z^n}{n!} = \sum_{n\geq 0}{}' \frac{\lambda^n}{\sqrt{n!}}|n\rangle.$$

Consequently by the use of δ-functions sources at the initial and final times one can compute the S matrix elements between coherent states and then use this as a generating function for the S matrix elements between occupation number states.

So

$$\tilde{J} = c\,\delta(t-t_f) + J(t) + c'\,\delta(t-t_{in})$$

$$\tilde{U}(t_f^+, t_{in}^-) = e^{icq}\,U(t_f, t_{in})\,e^{ic'q}$$

$$S = e^{it_f H_0}\,U(t_f, t_{in})\,e^{-it_{in}H_0}$$

$$\tilde{S} = e^{it_f H_0}\,\tilde{U}(t_f^+, t_{in}^-)\,e^{-it_{in}H_0}$$

$$= e^{it_f H_0}e^{icq}e^{-it_f H_0}\,S\,e^{it_{in}H_0}e^{ic'q}e^{-it_{in}H_0}$$

[redacted] So I want to compute

$$e^{itH_0} e^{icq} e^{-itH_0}|0\rangle = e^{it\underbrace{(H_0 - E_0)}_{\omega z \frac{d}{dz}}} \underbrace{e^{icq}|0\rangle}_{\text{const}\cdot e^{\lambda z}} \qquad \lambda = \frac{ic}{2\omega}$$

$$= \text{const}\, e^{it\omega z \frac{d}{dz}} \sum_{n\geq 0} \frac{\lambda^n \cdot z^n}{n!} = \text{const} \sum_{n \geq 0} \frac{\lambda^n}{n!} e^{itn\omega} z^n$$

$$= \text{const}\, e^{(e^{\omega it}\lambda)z} = \text{const}\ e^{i(e^{\omega it}c)q}|0\rangle$$

To determine the constants one can proceed as follows. $e^{icq}|0\rangle = e^{icq} e^{-\frac{1}{2}\omega q^2} / \sqrt{2\pi\omega}$

so $\langle 0 | e^{icq} | 0\rangle = e^{-c^2/4\omega}$ as we saw above.

But also we have

$$e^{icq}|0\rangle = C e^{\lambda \overset{a^*}{z}}|0\rangle \qquad \lambda = \frac{ic}{2\omega}$$

hence $$\langle 0 | e^{icq}|0\rangle = C \underbrace{\langle 0 | e^{\lambda a^*}}_{\langle 0|} | 0\rangle$$

or $$C = e^{-c^2/4\omega}$$

Thus $$\boxed{e^{icq}|0\rangle = e^{-c^2/4\omega} e^{\frac{ic}{2\omega} a^*} |0\rangle.}$$

and so

$$e^{itH_0} e^{icq} e^{-itH_0} |0\rangle = C e^{i e^{\omega it} c q} |0\rangle$$

$$e^{-\frac{c^2}{4\omega}} = C \cdot e^{-e^{2it}c^2/4\omega}$$

$$C = e^{-\frac{1}{4\omega}(c^2 - e^{2\omega it} c^2)}$$

or $$e^{itH_0} e^{icq} e^{-itH_0} |0\rangle = e^{-\frac{1}{4\omega}(c^2 - e^{2\omega it} c^2)} e^{i e^{it} c q} |0\rangle$$

So it seems we get the mess

$$\langle 0|\tilde{S}|0\rangle = \langle \underbrace{e^{it_f H_0} e^{-i\bar{c}q} e^{-it_f H_0}|0\rangle}_{e^{-\frac{1}{4\omega}(\bar{c}^2 - e^{2i\omega t_f}\bar{c}^2)}\, e^{-ie^{i\omega t_f}\bar{c}q}|0\rangle} \big| S \big| e^{it_{in}H_0} e^{ic'q} e^{-it_{in}H_0}|0\rangle$$

$$= e^{-\frac{1}{4\omega}(c^2 - e^{-2i\omega t_f}c^2) - \frac{1}{4\omega}(c'^2 - e^{2i\omega t_{in}}c'^2)} \langle e^{-ie^{i\omega t_f}\bar{c}q}|0\rangle \big| S \big| e^{ie^{i\omega t_{in}}c'q}|0\rangle$$

July 26, 1979

Consider a forced harmonic oscillator

$$H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} - J(t)q$$

where $J(t)$ is periodic, say $J(t+1) = J(t)$. Let $U(t,t')$ be the propagator for the quantum-mechanical motion:

$$i\frac{\partial}{\partial t} U(t,t') = H(t)U(t,t')$$

$$U(t',t') = I.$$

It follows that $U(t+1, t'+1) = U(t,t')$ and hence

$$U(t+1,0) = U(t+1,1)U(1,0) = U(t,0)U(1,0)$$

$$U(t+n,0) = U(t,0)\,U(1,0)^n.$$

$U(1,0)$ is a so-called Floquet matrix. Its eigenvectors give rise to quasi-periodic solutions

$$\psi(t+1) = \lambda\psi(t)$$

(Check $\psi(t+1) = U(t+1,0)\psi(0) = U(t,0)\underbrace{U(1,0)\psi(0)}_{\lambda\psi(0)}$

$= \lambda\psi(t)$.)

where $|\lambda| = 1$. These are the analogues of constant energy states.

Question: Is the spectrum of $U(1,0)$ discrete?

Example: If $J=0$, then

$$U(1,0) = e^{-iH}$$

has a discrete spectrum, since H does. The eigenvalues of H are $(n+\frac{1}{2})\omega$, $n \geq 0$, so $U(1,0)$ has the eigenvalues $e^{-i(n+\frac{1}{2})\omega}$. The same example holds if J is constant, because this amounts to a different origin for the oscillator.

Actually one can ask whether $U(t,0)$ has discrete spectrum for any source J. It seems reasonable especially since $U(\beta,0)$ is supposed to be of trace class for β in the Bloch direction.

Let's review yesterdays calculations:

$$a = \frac{1}{\sqrt{2\omega}}(ip + \omega q)$$

I want to compute the matrix element

$$\langle e_{\lambda'} | U(t,0) | e_\lambda \rangle$$

where e_λ denotes the coherent states:

$$e_\lambda = \sum_{n\geq 0} \frac{\lambda^n z^n}{n!} = e^{\lambda a^*}|0\rangle$$

$$a e_\lambda = \lambda e_\lambda \qquad \left(a = \frac{d}{dz}, \, a^* = z\right).$$

Let's use the Schwinger method changing J by δJ.

$$\delta \log \langle e_{\lambda'}|U(t,0)|e_\lambda\rangle = i\int_0^t \frac{\langle e_{\lambda'}|U(t,t_1)\,\delta J(t_1)\underset{\wedge}{q}\,U(t_1,0)|e_\lambda\rangle}{\langle e_{\lambda'}|U(t,0)|e_\lambda\rangle}\,dt_1$$

$$= i\int_0^t \delta J(t_1)\langle q(t_1)\rangle\,dt_1$$

The point is that $\langle q(t_1)\rangle$ satisfies the same DE

$$\left(\frac{d^2}{dt_1^2} + \omega^2\right)\langle q(t_1)\rangle = J(t_1) \qquad 0 \leq t_1 \leq t$$

except the boundary conditions are different.

$$\frac{d}{dt_1}\langle q(t_1)\rangle = \langle p(t_1)\rangle$$

$$\frac{1}{\sqrt{2\omega}}\left(i\frac{d}{dt_1}+\omega\right)\langle q(t_1)\rangle = \left\langle\left(\frac{-ip+\omega q}{\sqrt{2\omega}}\right)(t_1)\right\rangle$$

$$= \frac{\langle e_{\lambda'}|\,U(t,t_1)\,a\,U(t_1,0)\,|e_\lambda\rangle}{\langle e_{\lambda'}|\,U(t,0)\,|e_\lambda\rangle} \underset{\text{at } t_1=0}{=} \lambda$$

Similarly

$$\frac{1}{\sqrt{2\omega}}\left(-i\frac{d}{dt_1}+\omega\right)\langle q(t_1)\rangle\Big|_{t_1=t} = \bar{\lambda}'$$

Solve the DE for $\langle q(t_1)\rangle$ first for $J=0$.

$$\langle q(t_1)\rangle = Ae^{i\omega t_1} + Be^{-i\omega t_1}$$

$$\frac{1}{\sqrt{2\omega}}\left(i\frac{d}{dt}+\omega\right)\langle q(t_1)\rangle\Big|_{t_1=0} = B\frac{1}{\sqrt{2\omega}}\left(i(-i\omega)+\omega\right)e^0 = \sqrt{2\omega}\,B = \lambda$$

$$\frac{1}{\sqrt{2\omega}}\left(-i\frac{d}{dt}+\omega\right)\langle q(t_1)\rangle\Big|_{t_1=t} = A\frac{1}{\sqrt{2\omega}}\left(-i(i\omega)+\omega\right)e^{i\omega t} = \sqrt{2\omega}\,e^{i\omega t}A = \bar{\lambda}'$$

So

$$\langle q(t_1)\rangle = \frac{1}{\sqrt{2\omega}}\left(\bar{\lambda}'e^{-i\omega t+i\omega t_1} + \lambda e^{-i\omega t_1}\right)$$

In general we have

$$\langle q(t_1)\rangle = \frac{1}{\sqrt{2\omega}}\left(\bar{\lambda}'e^{-i\omega(t-t_1)} + \lambda e^{-i\omega t_1}\right)$$

$$+ \int_0^t \underbrace{G(t,t')}_{\frac{e^{-i\omega|t_1-t'|}}{-2i\omega}} J(t')\,dt'$$

Now multiply by $i\,\delta J(t_1)\,dt_1$ and integrate; then integrate δJ and you get

$$\boxed{\begin{aligned}\log \frac{\langle e_{\lambda'} | U(t,0) | e_\lambda\rangle}{\langle e_{\lambda'} | e_\lambda\rangle} &= \frac{i}{2}\int_0^t\int_0^t dt_1\, dt_2 \; J(t_1)\, G(t_1,t_2)\, J(t_2) \\ &+ \frac{i\lambda}{\sqrt{2\omega}}\int_0^t J(t_1)\, e^{-i\omega t_1}\, dt_1 \\ &+ \frac{i\bar{\lambda}'}{\sqrt{2\omega}}\int_0^t J(t_1)\, e^{-i\omega(t-t_1)}\, dt_1\end{aligned}}$$

$$U_0(t,0) = e^{-itH_0}$$

Let's check this result by taking $J(t) = c\,\delta(t)$ and $t = 0^+$. We saw that

$$U(t,0) = e^{icq}$$

so we want to compute $\langle e_{\lambda'} | e^{icq} | e_\lambda\rangle$. Recall

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

where $[A,B]$ commutes with A, B.

$$icq = \underbrace{\frac{ic}{\sqrt{2\omega}}}_{\gamma}(a + a^*)$$

$$\begin{aligned}\langle e_{\lambda'} | e^{icq} | e_\lambda\rangle &= \langle e_{\lambda'} | e^{\gamma a^* + \gamma a} | e_\lambda\rangle \\ &= e^{-\frac{1}{2}\gamma^2 \overbrace{[a^*, a]}^{-1}} \underbrace{\langle e_{\lambda'} | e^{\gamma a^*}\, e^{\gamma a} | e_\lambda\rangle}_{\langle e_{\lambda'} | e^{\gamma\bar{\lambda}'}\, e^{\gamma\lambda} | e_\lambda\rangle}\end{aligned}$$

so

$$\frac{\langle e_{\lambda'} | e^{icq} | e_\lambda\rangle}{\langle e^{\lambda'} | e^{\lambda}\rangle} = e^{\frac{1}{2}\gamma^2 + \gamma(\bar{\lambda}' + \lambda)} \qquad \gamma = \frac{ic}{\sqrt{2\omega}}$$

which agrees with the above.

We want to be able to use the formula at the top of the page

$$\frac{1}{2}\gamma^2 = -\frac{c^2}{4\omega} = \frac{i}{2}c^2\frac{1}{-2i\omega} \quad ✓$$

in order to compute $U(t,0)$ and see its spectrum.

$$e^{-itH_0} e_\lambda = e^{-itH_0} \sum \frac{\lambda^n (a^*)^n}{n!} |0\rangle$$

$$= \sum \frac{\lambda^n}{n!} e^{-itn\omega} (a^*)^n |0\rangle$$

$$= e_{\lambda e^{-it\omega}}$$

hence

$$\langle e_{\lambda'} | U_0(t,0) | e_\lambda \rangle = \langle e_{\lambda'} | e^{-itH_0} | e_\lambda \rangle$$

$$= e^{e^{-it\omega} \lambda \bar{\lambda}'}$$

Hence we find

$$\log \langle e_{\lambda'} | U(t,0) | e_\lambda \rangle = \lambda \bar{\lambda}' e^{-it\omega} + \lambda \alpha - \bar{\lambda}' e^{-it\omega} \bar{\alpha} + \beta$$

where

$$\alpha = \frac{i}{\sqrt{2\omega}} \int_0^t J(t_1) e^{-i\omega t_1} dt_1$$

$$\beta = \frac{i}{2} \int_0^t dt_1 \int_0^t J(t_1) G(t_1, t_2) J(t_2) dt_2$$

are constants.

July 27, 1979

Yesterday we found a formula for $\langle e_{\lambda'} | U(t,0) | e_\lambda \rangle$.
Notice that $\quad e^{itH_0} e_\lambda = e^{itH_0} \sum \frac{\lambda^n}{n!} z^n = \sum \frac{\lambda^n}{n!} e^{itn\omega} z^n$
$= e_{\lambda e^{i\omega t}}$, where $H_0 = \omega a^* a$. Hence

$$\langle e_{\lambda'} | e^{-itH_0} | e_\lambda \rangle = \langle e^{itH_0} e_{\lambda'} | e_\lambda \rangle$$

$$= \langle e_{\lambda' e^{i\omega t}} | e_\lambda \rangle = e^{\lambda \cdot \bar{\lambda}' e^{-i\omega t}}$$

The formula becomes simpler if $\lambda' e^{i\omega t}$ is replaced by λ''.

$$\frac{\langle e_{\lambda'} | U(t,0) | e_\lambda \rangle}{\langle e_{\lambda'} | e^{-iH_0 t} | e_\lambda \rangle} = \frac{\langle e_{\lambda'} | e^{-itH_0} \; e^{itH_0} U(t,0) | e_\lambda \rangle}{\langle e_{\lambda' e^{i\omega t}} | e_\lambda \rangle}$$

$$= \frac{\langle e_{\lambda''} | S | e_\lambda \rangle}{\langle e_{\lambda''} | e_\lambda \rangle} \qquad S = e^{itH_0} U(t,0)$$

Thus we find

$$(*) \qquad \langle e_{\lambda''} | S | e_\lambda \rangle = \exp\left\{ i\beta + \lambda\alpha + \bar{\lambda}''\bar{\alpha} + \lambda\bar{\lambda}'' \right\}$$

$$\alpha = \frac{i}{\sqrt{2\omega}} \int_0^t J(t_1) e^{-i\omega t_1} dt_1 \qquad J \text{ real valued}$$

$$\beta = \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 \, J(t_1) G(t_1, t_2) J(t_2)$$

We ought to see if β and α are connected in some way in order that a unitary transformation S can be defined by $(*)$.

First review the way the e_λ are the point-evaluators for the holomorphic representation. Recall this representation consists of entire functions $f(z)$ with finite norm

$$\|f\|^2 = \int |f(z)|^2 e^{-|z|^2} \frac{dx\,dy}{\pi}$$

and

$$|0\rangle = 1, \quad a = \frac{d}{dz}, \quad a^* = z$$

Then

$$f(w) = \sum \frac{1}{n!} f^{(n)}(0)\, w^n = \sum \frac{1}{n!} (a^n f, 1)\, w^n$$

$$= \sum \left(f, \frac{1}{n!} \bar{w}^n z^n\right) = \left(f, e^{\bar{w} z}\right)$$

so we see that e_λ is the point evaluator at $\bar{\lambda}$. Moreover we have (interchanging w, z)

$$f(z) = \int f(w)\, e^{\bar{w} z}\, e^{-|w|^2} \frac{i\,dw\,d\bar{w}}{2\pi}$$

or

$$f = \int e_{\bar{w}}\, f(w)\, e^{-|w|^2} \frac{i}{2\pi}\, dw\, d\bar{w}$$

which expresses f [redacted] in terms of [redacted] the e_λ.

Suppose we want to define a linear operator S by giving its effect Se_λ on the coherent states. Clearly we want $(Se_\lambda)(z)$ to be analytic in both λ and z. Since

$$f = \int e_\lambda\, f(\bar{\lambda})\, dG_\lambda \qquad dG_\lambda = \text{Gaussian measure}$$

we must have

$$Sf = \int Se_\lambda\, f(\bar{\lambda})\, dG_\lambda .$$

In order to use this to define Sf we need to know that

$$Se_{\lambda'} = \int Se_\lambda\, e^{\lambda' \bar{\lambda}}\, dG_{(\lambda)}$$

which will be the case if $(Se_\lambda)(z)$ as a function of λ is in the Hilbert spaces. So its clear that we to know $(Se_\lambda)(w)$ is analytic in λ, w and separately for each variable with the other one fixed in the holomorphic function Hilbert spaces. So the formula

$$\langle e_{\lambda''} | S | e_\lambda \rangle = \exp\{c_1 + c_2\lambda + c_3\bar{\lambda}'' + c_4\lambda\bar{\lambda}''\}$$

with arbitrary constants will define an operator in the holomorphic Hilbert space.

Our next problem will be to understand when we get a unitary operator. It should be that we get the transformations coming from the metaplectic representation.

<u>Example</u>: Translation $f(z) \longmapsto f(z+a)$ can be made into a unitary operator:

$$\|f\|^2 = \int |f(z)|^2 e^{-|z|^2} dV = \int |f(z+a)|^2 e^{-z\bar{z} - z\bar{a} - \bar{z}a - a\bar{a}} dV$$

$$= \int \left| f(z+a)\, e^{-\bar{a}z - \frac{1}{2}|a|^2} \right|^2 e^{-|z|^2} dV = \|T_a f\|^2$$

where

$$\boxed{(T_a f)(z) = e^{-\bar{a}z - \frac{1}{2}|a|^2} f(z+a)}$$

Then

$$(T_a e_\lambda)(z) = e^{\lambda z + \lambda a - \bar{a}z - \frac{1}{2}|a|^2}$$

or

$$\langle e_{\lambda'} | T_a | e_\lambda \rangle = e^{-\frac{1}{2}|a|^2 + a\lambda - \bar{a}\bar{\lambda}' + \lambda\bar{\lambda}'}$$

So from this formula it is clear that the transformation $S = e^{+itH_0} U(t,0)$ is a scalar of modulus 1 times T_α where

$$\alpha = \frac{i}{\sqrt{2\omega}} \int_0^t J(t_1)\, e^{-i\omega t_1}\, dt_1$$

We should next see what this scalar is, i.e. compare $i\beta$ with $-\frac{1}{2}|\alpha|^2$.

$$+\alpha\bar{\alpha} = \frac{1}{2\omega} \int_0^t dt_1 \int_0^t dt_2 \; J(t_1) e^{-i\omega t_1} J(t_2) e^{i\omega t_2}$$

$$-2i\beta = -\cancel{2i}\, \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 \; J(t_1) \frac{e^{-i\omega|t_1-t_2|}}{-\cancel{2i}\omega} J(t_2)$$

$$= \frac{1}{2\omega} \int_0^t dt_1 \int_0^t dt_2 \; J(t_1)\, e^{-i\omega|t_1-t_2|} J(t_2)$$

Clearly both have the same real part, but $i\beta$ has a possibly non-trivial imaginary part

$$\mathrm{Im}(i\beta) = +\frac{1}{2}\, \frac{1}{2\omega} \int_0^t dt_1 \int_0^t dt_2 \; J(t_1) J(t_2) \sin \omega|t_1-t_2|$$

We should now be in a position to determine if

$$U(t,0) = e^{-itH_0} S$$

has discrete spectrum by using the explicit formulas we have in the holomorphic representation. We ignore the scalar factor and replace S by T_α. We know

$$(e^{-itH_0} f)(z) = f(e^{-i\omega t} z)$$

so

$$(U(t,0)f)(z) = (T_\alpha f)(e^{-i\omega t}z)$$

$$= f(e^{-i\omega t}z+\alpha)\, e^{-\bar\alpha e^{-i\omega t}z - \frac{1}{2}|\alpha|^2}$$

Put $\zeta = e^{-i\omega t}$ and look for eigenfunctions for $U(t,0)$:

$$f(\zeta z+\alpha)\, e^{-\bar\alpha \zeta z - \frac{1}{2}|\alpha|^2} = \mu f(z)$$

Look at the fixpts $\zeta z+\alpha = z \implies z = \frac{\alpha}{1-\zeta}$ (assume $\zeta \neq 1$).

Simpler way to proceed: Take

$$U(t,0) = e^{-itH_0}T_\alpha$$

and conjugate with T_β

$$T_\beta U(t,0) T_\beta^{-1} = e^{-itH_0}\; e^{itH_0}T_\beta e^{-itH_0} T_\alpha T_\beta^{-1}$$

Now

$$\left(e^{itH_0}T_\beta e^{-itH_0}f\right)(z) = \left(T_\beta e^{-itH_0}f\right)(e^{i\omega t}z)$$

$$= \left(e^{-itH_0}f\right)(e^{i\omega t}z+\beta)\, e^{-\bar\beta e^{i\omega t}z - \frac{1}{2}|\beta|^2}$$

$$= f(z+e^{-i\omega t}\beta)\, e^{-\bar\beta e^{i\omega t}z - \frac{1}{2}|\beta|^2}$$

$$= \left(T_{e^{-i\omega t}\beta} f\right)(z)$$

$$[(T_\alpha T_\gamma)f](z) = (T_\gamma f)(z+\alpha)\, e^{-\bar\alpha z - \frac{1}{2}|\alpha|^2}$$

$$= f(z+\alpha+\gamma)\, e^{-\bar\alpha(z+\gamma) - \frac{1}{2}|\alpha|^2}\, e^{-\bar\gamma z - \frac{1}{2}|\gamma|^2}$$

$$= f(z+\alpha+\gamma)\, e^{-(\bar\alpha+\bar\gamma)z - \frac{1}{2}|\alpha+\gamma|^2}\; e^{\frac{1}{2}(\alpha\bar\gamma+\bar\alpha\gamma) - \bar\alpha\gamma}$$

$$= e^{\frac{1}{2}(\alpha\bar\gamma - \bar\alpha\gamma)} \left(T_{\alpha+\gamma} f\right)(z)$$

And $\frac{1}{2}(\alpha\bar\gamma - \bar\alpha\gamma) = i\,\mathrm{Im}(\alpha\bar\gamma)$ so it vanishes when $\mathbb{R}\alpha = \mathbb{R}\gamma$. In particular

$$T_\beta^{-1} = T_{-\beta}$$

and we see that

$$e^{itH_0} T_\beta e^{-itH_0} T_\alpha T_\beta^{-1} = \text{scalar} \cdot T_{e^{-i\omega t}\beta + \alpha - \beta}$$

If we choose β so that

$$e^{-i\omega t}\beta + \alpha - \beta = 0 \quad \Rightarrow \beta = \frac{\alpha}{1 - e^{-i\omega t}}$$

it follows that

$$T_\beta U(t,0) T_\beta^{-1} = e^{-itH_0} \cdot \text{scalar}$$

and hence the spectrum of $U(t,0)$ is discrete. All this assumes that

$$e^{i\omega t} \neq 1.$$

Notice that the eigenvalues of $U(t,0)$ are those of e^{-itH_0} shifted around the unit circle by a fixed scalar of modulus 1.

When $e^{i\omega t} = 1$, then $e^{-itH_0} = I$ and so $U(t,0) = T_\alpha$. In this case the spectrum is continuous, in fact I think that T_α is equivalent to a shift on $L^2(\mathbb{R})$.

July 28, 1979:

Recall for the forced oscillator

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2 - J(t)\,q \qquad J \text{ compact support}$$

one has two formulas for the ground-ground amplitude

$$\langle 0|S|0\rangle = \exp \frac{i}{2}\iint J(t)\,G(t,t')\,J(t')\,dt\,dt'$$

$$\langle 0|S|0\rangle = 1 + i\int J(t_1)\langle q(t_1)\rangle\,dt_1 + \frac{i^2}{2!}\iint J(t_1)J(t_2)\langle T(q(t_1)\,q(t_2))\rangle \times dt_1\,dt_2 + \cdots$$

From the latter it follows that

$$\frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}\langle 0|S|0\rangle = i^n\langle T(q(t_1)\cdots q(t_n))\rangle.$$

But notice that if you wanted to find the coefficient of $x_1\cdots x_n$ in the Taylor series expansion of

$$e^{\frac{1}{2}\sum a_{ij}x_ix_j} \qquad a_{ij}=a_{ji}$$

you can write

$$e^{\frac{1}{2}\sum a_{ij}x_ix_j} = \prod_{i<j} e^{a_{ij}x_ix_j}\prod_i e^{\frac{1}{2}a_{ii}x_i^2}.$$

Now it is crystal clear that to get a product $x_1\cdots x_n$ where these are assumed distinct, you have to partition $1,\ldots,n$ into pairs (hence n must be even) and then take the product of the a_{ij} for each pair, then add up over all partitions. This is Wick's sum over all possible pairwise contractions, and it obviously works even for thermal averages.

I want to understand the corresponding situation for fermions, like the Dirac field. Let's consider the simpler boson situation:

$$H = \omega a^* a \mp \tilde{J} a \mp J a^*$$

where $J(t)$, $\tilde{J}(t)$ have compact support. Then

$$\delta \log \langle 0|S|0\rangle = +i \int \langle \delta\tilde{J}(t)\, a(t) + \delta J(t)\, a^*(t) \rangle \, dt$$

$$= +i \int \left[\delta\tilde{J}(t) \langle a(t)\rangle + \delta J(t) \langle a^*(t)\rangle \right] dt$$

$$\frac{d}{dt}\langle a(t)\rangle = \langle [iH, a](t) \rangle$$

$$[a, a^*] = 1$$

$$[H, a] = [\omega a^* a \mp J a^*, a] = \omega(+[a^*, a]a) \mp J[a^*, a]$$

$$= -\omega a \pm J$$

$$[H, a^*] = [\omega a^* a \mp \tilde{J} a, a^*] = \omega a^* \mp \tilde{J}$$

Thus we have

$$\left(\frac{d}{dt} + i\omega\right)\langle a(t)\rangle = +iJ$$

$$\left(\frac{d}{dt} - i\omega\right)\langle a^*(t)\rangle = -i\tilde{J}$$

since $\langle a(t)\rangle = 0$ for $t \ll 0$, $\langle a^*(t)\rangle = 0$ $t \gg 0$ we have

$$\langle a(t)\rangle = \int_{-\infty}^{t} e^{-i\omega(t-t')} (+iJ(t'))\, dt'$$

$$\langle a^*(t)\rangle = \int_{t}^{\infty} -e^{i\omega(t-t')} (-i\tilde{J}(t'))\, dt'$$

so changing the signs doesn't help anything.
Reestablish notation:

$$H = \omega a^* a + \tilde{J} a + J a^*$$

$$\delta \log \langle 0|S|0\rangle = -i \int \left[\delta \tilde{J}(t) \langle a(t)\rangle + \delta J(t) \langle a^*(t)\rangle\right] dt$$

$$\langle a(t)\rangle = \int_{-\infty}^{t} e^{-i\omega(t-t')} (-iJ(t'))\, dt'$$

$$\langle a^*(t)\rangle = \int_{t}^{\infty} e^{+i\omega(t-t')} (-i\tilde{J}(t'))\, dt'$$

So

$$\delta \log \langle 0|S|0\rangle = (-1)\int dt \left[\delta \tilde{J}(t) \int_{-\infty}^{t} e^{-i\omega(t-t')} J(t')\, dt'\right] + (-)\int dt \left[\delta J(t) \int_{t}^{\infty} e^{i\omega(t-t')} \tilde{J}(t')\, dt'\right]$$

In the second integral reverse the order of integration

$$\int_{-\infty}^{\infty} dt \int_{t}^{\infty} dt' = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{t'} dt$$

then interchange t, t' and you get

$$\delta \log \langle 0|S|0\rangle = -\int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' \left[\delta \tilde{J}(t) e^{-i\omega(t-t')} J(t') + \tilde{J}(t) e^{i\omega(t'-t)} \delta J(t')\right]$$

or integrating out the δ

$$\boxed{\log \langle 0|S|0\rangle = -\int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt'\; e^{-i\omega(t-t')}\; \tilde{J}(t) J(t')}$$

Check: Put $J = \tilde{J} = \frac{-\mathbb{J}}{\sqrt{2\omega}}$ so that $Ja + \tilde{J}a^* = -\mathbb{J} q$. You get

$$\log \langle 0|S|0\rangle = +\frac{i}{2}\iint \frac{e^{-i\omega|t-t'|}}{-i2\omega} \mathbb{J}(t)\mathbb{J}(t')$$

which agrees with our earlier result.

Let's return to the Dyson expansion

$$\langle 0|S|0\rangle = 1 - i\int \langle H_I(t_1)\rangle dt_1 + \frac{(-i)^2}{2!}\iint \langle T H_I(t_1) H_I(t_2)\rangle dt_1\, dt_2$$

where $\quad H_I = \tilde{J} a + J a^*$

This is a big expansion, think of it as a power series expansion in the variables $J(t)$, $\tilde{J}(t)$ and we can ask for the coefficient of the monomial

$$J(t_1)\cdots J(t_p)\,\tilde{J}(t_{p+1})\cdots\tilde{J}(t_n)$$

where $t_1, \ldots, t_n$ are assumed distinct. This means you have to go to the n-th term in the Dyson expansion which is

$$\frac{(-i)^n}{n!}\int\cdots\iint \langle T H_I(t_1)\cdots H_I(t_n)\rangle\, dt_1\cdots dt_n$$

Let us order times so that $t_1, \ldots, t_n$ occur in order. In other words the above integral $\frac{1}{n!}$ can be taken over any "chambre", so let's use the chambre where the given $t_1, \ldots, t_n$ are in order. Then it is clear that the coefficient is

$$(-i)^n \langle T\, a^*(t_1)\cdots a^*(t_p)\, a(t_{p+1})\cdots a(t_n)\rangle$$

or in other words

$$\frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_p)\,\delta\tilde{J}(t_{p+1})\cdots\delta\tilde{J}(t_n)}\langle 0|S|0\rangle =$$

But we've seen that $\langle 0|S|0\rangle = \exp \iint_{t'<t} J(t')\, G(t,t')\tilde{J}(t)$

where $G(t,t') = -e^{-i\omega(t-t')}$

Now if you want the coefficient of $x_1 \cdots x_n y_1 \cdots y_n$ in

$$e^{\sum x_i a_{ij} y_j} = \prod_{i,j} e^{x_i a_{ij} y_j}$$

it is the sum over all ways ($n!$ in all) of attaching each x variable to a y-variable and you multiply the corresponding a_{ij}. So again one sees how Wick's theorem holds in this case.

For later reference the formulas are

$$\langle T\, a(t)\, a^*(t') \rangle = \begin{cases} e^{-i\omega(t-t')} & t > t' \\ 0 & t < t' \end{cases}$$

$$= \theta(t-t')\, e^{-i\omega(t-t')}$$

July 29, 1979

I still haven't deciphered what Schwinger is doing with sources in the fermion situation. Again consider a space W (fin. dim'l complex Hilb space) on which we have H_0: $H_0 \varphi_k = E_k \varphi_k$ where the φ_k are orthonormal. Extend H_0 to ΛW whence

$$H_0 = \sum E_k a_k^* a_k$$

with $a_k = i(\varphi_k^*)$, $a_k^* = e(\varphi_k)$. The ground state for H_0 on ΛW is $|0\rangle = \varphi_1 \wedge \cdots \wedge \varphi_p$ where $E_1, \ldots, E_p < 0$ and the rest are > 0. For simplicity let us take $|0\rangle = 1$, i.e. assume all $E_i > 0$. In this case the Green's function for the operator

$$\frac{\partial}{\partial t} + i H_0 \qquad \text{on } W$$

is

$$G(t,t') = \begin{cases} e^{-iH_0(t-t')} & t > t' \\ 0 & t < t'. \end{cases}$$

We use the Green's function with positive frequencies for positive times and negatives frequencies for negative times. Our problem is to interpret Schwinger's formula

$$\langle 0|S|0\rangle = \exp\left\{ i \iint \bar{\eta}(t) G(t,t') \eta(t')\, dt\, dt' \right\}$$

that is, to find $H = H_0 + H_1$ which gives this formula for the ground-ground amplitude. My guess is that $\eta(t)$ should be an element of W and that $\bar{\eta}(t) \in W^*$ and

$$H_1 = e(\eta) + i(\bar{\eta})$$

so that if $\bar{\eta} = \langle \eta |$ then H_1 is self-adjoint.

Let's try computing the Dyson series

$$\langle 0|S|0\rangle = 1 - i\int \langle (e(\eta)+i(\bar{\eta}))(t)\rangle + (-i)^2 \iint_{t_1 > t_2} \langle (e(\eta)+i(\bar{\eta}))(t_1)(\qquad)(t_2)\rangle$$

$$\overset{\circ}{\|}$$

Look at the second order term

$$\sum_{k,l}' \langle (\eta_k(t_1)a_k^* + \bar{\eta}_k(t_1)a_k)\, e^{-iH_0 t_1} e^{iH_0 t_2} (\eta_l(t_2)a_l^* + \bar{\eta}_l(t_2)a_l)\rangle$$

$$= \sum_{k,l} \bar{\eta}_k(t_1)\eta_l(t_2) \langle 0| a_k e^{-iE_k t_1} e^{iE_l t_2} a_l^* |0\rangle$$

$$= \sum_k' \bar{\eta}_k(t_1)\eta_k(t_2)\, e^{-iE_k(t_1-t_2)}$$

Look at fourth order.
To get something $\neq 0$
in fourth order

$$\hat{H}_1(t) = \sum_k' \left(\eta_k(t)e^{iE_k t}a_k^* + \bar{\eta}_k(t)e^{-iE_k t}a_k\right)$$

$$a_k \quad \begin{matrix} a_l \\ a_l^* \end{matrix} \quad \begin{matrix} a_m \\ a_m^* \end{matrix} \quad a_n^*$$

there are three possibilities :

$$\langle 0|a_k a_k^* a_m a_m^*|0\rangle = 1$$

$$\langle 0|a_k a_l a_l^* a_k^*|0\rangle = 1 \qquad l \neq k$$

$$\langle 0|a_k a_l a_k^* a_l^*|0\rangle = -1 \qquad l \neq k$$

which give the following

$$\sum_{k,m} \bar{\eta}_k(t_1)\eta_k(t_2)\, e^{-iE_k(t_1-t_2)}\ \bar{\eta}_m(t_3)\eta_m(t_4)\, e^{-iE_m(t_3-t_4)}$$

$$+ \sum_{k\neq l} \bar{\eta}_k(t_1)\bar{\eta}_l(t_2)\eta_l(t_3)\eta_k(t_4)\ e^{-iE_k t_1 - iE_l t_2 + iE_l t_3 + iE_k t_4}$$

$$- \sum_{k\neq l} \bar{\eta}_k(t_1)\bar{\eta}_l(t_2)\eta_k(t_3)\eta_l(t_4)\ e^{-iE_k t_1 - iE_l t_2 + iE_k t_3 + iE_l t_4}$$

which can be written

$$F(t_1,t_2)F(t_3,t_4)+F(t_1,t_4)F(t_2,t_3)-F(t_1,t_3)F(t_2,t_4)$$

where

$$F(t_1,t_2) = \sum_k \bar\eta_k(t_1)\eta_k(t_2)e^{-iE_k(t_1-t_2)}$$

$$= \bar\eta(t_1)\,G(t_1,t_2)\,\eta(t_2)$$

It seems that the − sign on the last term fouls things up. Compute the 2nd order term in $\exp\left(i\iint \bar\eta(t_1)G(t_1,t_2)\eta(t_2)\right)$ you get (-1) times

$$\frac{1}{2!}\iint_{t_1>t_2}\iint_{t_3>t_4} F(t_1,t_2)F(t_3,t_4) = \frac{1}{2!}\left\{\int_{t_1>t_2>t_3>t_4} + \int \cdots\right\}$$

We have the possibilities six in all:

$t_1>t_2>t_3>t_4$

$t_1>t_3>t_2>t_4$

1	2	3	4
1	3	2	4
1	3	4	2
3	1	4	2
3	1	2	4
3	4	1	2

Now

$$\int_{t_3>t_1>t_4>t_2} F(t_1,t_2)F(t_3,t_4) = \int_{t_1>t_3>t_2>t_4} F(t_3,t_4)F(t_1,t_2)$$

so interchanging $1\leftrightarrow 3$, $2\leftrightarrow 4$ reduces us to 3 possibilities

$$\int_{t_1>t_2>t_3>t_4} F(t_1,t_2)F(t_3,t_4) + \int_{t_1>t_3>t_2>t_4} F(t_1,t_2)F(t_3,t_4) + \int_{t_1>t_3>t_4>t_2} F(t_1,t_2)F(t_3,t_4)$$

$$= \int_{t_1>t_2>t_3>t_4} F(t_1,t_2)F(t_3,t_4) + F(t_1,t_3)F(t_2,t_4) + F(t_1,t_4)F(t_2,t_3)$$

which differs from the expression at the top of the preceding page by a − signs.

So we see that we have to do something else in order to interpret Schwinger's source. Try the following. Let's take the basic space W and enlarge it by adjoining some extra basis elements to $W \oplus W'$. Then

$$\Lambda(W \oplus W') \cong \Lambda W' \otimes \Lambda W$$

can be interpreted as enlarging our number system from $\mathbb{C}$ to $\Lambda W'$. Now we consider the Hamiltonian

$$H = \sum E_k a_k^* a_k + \sum_k (a_k^* \eta_k + \tilde{\eta}_k a_k)$$

where $\eta_k(t)$, $\tilde{\eta}_k(t)$ are functions with values in W' interpreted as exterior multiplication operators. Now let's compute $\langle 0|S|0\rangle$ by variation

$$\delta \log \langle 0|S|0\rangle = -i \int \langle 0| (\widehat{\delta H})(t) |0\rangle \, dt.$$

I should be more careful:

$$\frac{\partial}{\partial t} U(t,t') = -i\, H(t)\, U(t,t')$$

$$\frac{\partial}{\partial t}\left(e^{iH_0 t} U(t,t')\right) = e^{iH_0 t}(iH_0 - iH)U(t,t')$$

$$= -i \underbrace{e^{iH_0 t} H_1(t) e^{-iH_0 t}}_{\hat{H}_1(t)} e^{iH_0 t} U(t,t')$$

There should be no problem with the scattering formalism, because there is nothing unusual with the Hamiltonian H. So

$$\delta \log \langle 0|S|0\rangle = -i \int \sum_k \langle (\delta\tilde{\eta}_k a_k + a_k^* \delta\eta_k)(t)\rangle \, dt$$

$$\frac{d}{dt}\langle (\delta\tilde{\eta}_k a_k)(t)\rangle = i \langle [H, \delta\tilde{\eta}_k a_k](t)\rangle$$

$$[H, \delta\tilde{\eta}_k a_k] = [H, \delta\tilde{\eta}_k] a_k + \delta\tilde{\eta}_k [H, a_k]$$

I claim that $[H, \delta\tilde{\eta}_k] = 0$. Check:

$$[a_\ell^* \eta_\ell, \delta\tilde{\eta}_k] = a_\ell^* \{\eta_\ell, \delta\tilde{\eta}_k\} - \{a_\ell^*, \delta\tilde{\eta}_k\}\eta_\ell = 0$$

etc. Also

$$[H, a_k] = \sum_\ell \left(E_\ell [a_\ell^* a_\ell, a_k] + [a_\ell^* \eta_\ell + \tilde{\eta}_\ell a_\ell, a_k]\right)$$

$$= -E_k a_k - \eta_k$$

There is a problem with interpreting $\langle\ \rangle$. What one wants to do is to use the natural basis for $\Lambda(W \oplus W')$ as a module over $\Lambda W'$ and take the 1-1 matrix element. If we do this it is clear that

$$\langle (\delta\tilde{\eta}_k a_k + a_k^* \delta\eta_k)(t)\rangle$$

$$= \delta\tilde{\eta}_k \langle a_k(t)\rangle + \langle a_k^*(t)\rangle \delta\eta_k(t)$$

because we've seen that $[H, \delta\tilde{\eta}_k] = 0$, so $\delta\tilde{\eta}_k, \delta\eta_k$ remain in W'. Now

$$\frac{d}{dt}\langle a_k(t)\rangle = -iE_k\langle a_k(t)\rangle - i\eta_k$$

$$\langle a_k(t)\rangle = 0 \qquad \text{for } t \ll 0$$

so

$$\langle a_k(t)\rangle = \int_{-\infty}^{t} e^{-iE_k(t-t')}(-i)\eta_k(t')\,dt'$$

Hence just as in the boson case we should get

$$\langle 0|S|0\rangle = \exp\left\{-\sum_k \iint_{t'<t} \tilde{\eta}_k(t)\, e^{-iE_k(t-t')}\eta_k(t')\right\}$$

July 30, 1979

Let's review path integrals. In the case of 1-dimensional motion with Hamiltonian

$$H = \frac{p^2}{2} + U(q,t)$$

we saw that the propagator is expressed as a path integral

$$\langle q' | U(t_f, 0) | q \rangle = \int [dq]\, e^{i\int_0^{t_f} L}.$$

The path integral is taken over all paths $q : [0, t_f] \to \mathbb{R}$ with $q(0) = q$, $q(t_f) = q'$, and it represents the average of the amplitude $e^{i\int_0^{t_f} L}$ where $L = \frac{1}{2}\dot{q}^2 - U(q,t)$ is the Lagrangian.

Let us now consider a perturbation situation

$$H = H_0 + V(q,t) \qquad \text{e.g. } V = -J(t)q.$$

Then

$$\langle q' | U(t_f, 0) | q \rangle = \int [dq]\, e^{i\int L_0}\, e^{-i\int V}$$

Think of this as being the integral with respect to the measure $[dq]\, e^{i\int L_0}$ of the function

$$q(t) \longmapsto e^{-i\int V(q(t),t)\,dt}$$

In this case of $V = -Jq$ it is just the Fourier transform of the measure $[dq]\, e^{+i\int L_0}$, where one thinks of J as being an element of the dual space to the space of paths.

Now if $d\mu$ is a measure on $\mathbb{R}$ say we have

$$\int x^n d\mu = \left(\frac{1}{i}\frac{d}{dJ}\right)^n \int e^{iJx} d\mu \Big|_{J=0}$$

and more generally for any polynomial

$$\int f(x) d\mu = f\left(\frac{1}{i}\frac{d}{dJ}\right) \int e^{iJx} d\mu \Big|_{J=0}.$$

So one has

$$\langle q' | U(t_f, 0) | q \rangle = \int [dq] e^{i\int L_0} e^{-i\int V(q(t),t)\,dt}$$

$$= \exp\left\{-i\int_0^{t_f} V\left(\frac{1}{i}\frac{\delta}{\delta J(t)}, t\right) dt\right\} \cdot \int [dq] e^{i\int L_0} e^{i\int Jq} \Big|_{J=0}$$

July 31, 1979

Recall for $H = \frac{p^2}{2} + \frac{(\omega q)^2}{2} - Jq$

$$\langle 0 | S^J | 0 \rangle = \exp\left\{ \frac{i}{2} \iint J(t)\, G(t,t')\, J(t')\, dt\, dt' \right\}$$

where $G(t,t') = \dfrac{e^{-i\omega|t-t'|}}{-2i\omega}$. Notice that the quadratic form

(1) $$J \longmapsto \iint J(t)\, G(t,t')\, J(t')\, dt\, dt'$$

is symmetric and that its imaginary part is positive semi-definite. This is because for J real we know S^J is unitary, hence $\langle 0 | S^J | 0 \rangle \leq 1$. But we can also see this directly

$$\mathrm{Re}\left\{ J(t)\, i\, G(t,t')\, J(t') \right\} = \mathrm{Re}\left\{ J(t) \frac{e^{-i\omega|t-t'|}}{-2\omega} J(t') \right\}$$

$$= \mathrm{Re}\left\{ -\frac{1}{2\omega} J(t)\, e^{-i\omega(t-t')} J(t') \right\}$$

$$= -\frac{1}{2\omega} \mathrm{Re}\left(J(t) e^{-i\omega t}\, \overline{J(t') e^{-i\omega t'}} \right)$$

Hence
$$\iint \mathrm{Re}\; J(t)\, i\, G(t,t')\, J(t')\, dt\, dt'$$

$$= -\frac{1}{2\omega} \left| \int J(t)\, e^{-i\omega t}\, dt \right|^2$$

Here J is a real function with compact support and

$$J \longmapsto \int J(t) e^{-i\omega t}\, dt$$

is a complex linear functional; it follows that the real part of the quadratic form (1) has rank 2.

Let's look at the Euclidean version: Here the propagator for the Bloch equation is the path integral

$$\langle q'|U(t,0)|q\rangle = \int_{\substack{q(0)=q\\ q(t)=q'}} [dq]\, e^{-\int(\frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2)}\, e^{\int Jq} .$$

Better compute $\langle 0|S|0\rangle = 1 + \int \langle 0|e^{H_0 t_1} J(t_1) q\, e^{-H_0 t_1}|0\rangle + \cdots$

$$\delta \log \langle 0|S|0\rangle = \int J(t)\langle q(t)\rangle\, dt$$

$$\frac{d}{dt}\langle q(t)\rangle = \langle [H,q](t)\rangle = \frac{1}{i}\langle p(t)\rangle$$

$[q,p]=i$

$$\frac{d}{dt}\frac{1}{i}\langle p(t)\rangle = \frac{1}{i}\left\langle \left[\tfrac{1}{2}\omega^2 q^2 - Jq, p\right](t)\right\rangle$$
$$= \omega^2\langle q(t)\rangle - J(t)$$

so
$$\langle q(t)\rangle = -\int \frac{e^{-\omega|t-t'|}}{-2\omega} J(t')\, dt'$$

so
$$\boxed{\log\langle 0|S|0\rangle = \frac{1}{2}\iint J(t)\frac{e^{-\omega|t-t'|}}{2\omega} J(t')\, dt\, dt'}$$

Euclidean case

Now we know that on L^2

$$\left(-\frac{d^2}{dt^2} + \omega^2\right)^{-1} \text{ has kernel } \frac{e^{-\omega|t-t'|}}{2\omega}$$

so therefore the quadratic form

$$J \longmapsto \iint J(t)\frac{e^{-\omega|t-t'|}}{2\omega} J(t')\, dt\, dt'$$

is positive-definite. Notice that if J is replaced by iJ it becomes negative-definite.

From the path integral theory we get for

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 + V \qquad V = V(q,t) \qquad \text{comp. supp. in } t$$

we get the formula

$$\langle 0|S|0\rangle = \exp\left\{-i\int V\left(\frac{1}{i}\frac{\delta}{\delta J(t)}, t\right)dt\right\} \exp\left\{\frac{i}{2}\iint J(t)G(t,t')J(t')\right\}\Big|_{J=0}$$

which is the basis for the perturbation expansion, Feynman diagrams, etc.

I want to take the quadratic case $V = \frac{1}{2}\varepsilon(t)q^2$ in which case I get

$$\langle 0|S|0\rangle = \exp\left\{\frac{i}{2}\int \varepsilon(t)\frac{\delta^2}{\delta J(t)^2}\right\} \exp\left\{\frac{i}{2}\iint J(t)G(t,t')J(t')\right\}\Big|_{J=0}$$

Let us look at a finite-dimensional analogue of this

$$e^{\frac{i}{2}\sum \varepsilon_n \frac{\partial^2}{\partial x_n^2}} \; e^{\frac{1}{2}\sum a_{mn}x_m x_n}\Big|_{x=0}$$

Consider the simplest possible case

$$e^{aD^2} e^{bx^2}\Big|_{x=0}$$

$$= \sum_m \frac{a^m D^{2m}}{m!} \sum_n \frac{b^n x^{2n}}{n!}\Big|_{x=0} = \sum_m \frac{a^m b^m}{m!\,m!}(2m)!$$

$$= \sum_{m\geq 0} \frac{1\cdot 3\cdots 2m-1}{m!}(2ab)^m$$

Now $$(1-u)^{-1/2} = \sum_{m\geq 0}\frac{(+\frac{1}{2})(+\frac{3}{2})\cdots(+\frac{2m-1}{2})}{m!}u^m = \sum_{m\geq 0}\frac{1\cdot 3\cdots 2m-1}{m!}\left(\frac{u}{2}\right)^m$$

Consequently

$$e^{aD^2} e^{bx^2}\Big|_{x=0} = (1-4ab)^{-1/2}$$

and furthermore the perturbation series converges only for $|4ab| < 1$.

General case: To evaluate

$$(*) \qquad e^{\frac{1}{2} D^t P D} \, e^{\frac{1}{2} x^t Q x} \Big|_{x=0}$$

where $x = (x_i)$ $D = (D_i)$ are column vectors with

$$D x^t = I$$

If we make a variable change $x = Ax'$, then

$$D x'^t A^t = I \quad \text{so} \quad A^t D x'^t = I$$

$$\text{or} \quad D' = A^t D \quad \text{and} \quad D = (A^t)^{-1} D'$$

Then

$$D^t P D = D'^t A^{-1} P (A^t)^{-1} D'$$

$$x^t Q x = x'^t A^t Q A x'$$

so we are allowed the transformation

$$Q \longmapsto A^t Q A$$

$$P^{-1} \longmapsto A^t P^{-1} A$$

i.e. the simultaneous transformation of quadratic forms. The general theory here says that at least generically we can make $P' = I$ and Q' diagonal. In this case $(*)$ becomes

$$\prod (1-q_i)^{-1/2} = \det(1-P'Q')^{-1/2} = \det(1-PQ)^{-1/2}$$

where the q_i = diag entries of Q'

so we get the formula

$$e^{\frac{1}{2}D^t P D}\, e^{\frac{1}{2}x^t Q x}\Big|_{x=0} = \left[\det(1-PQ)\right]^{-1/2}$$

This leads to the formula

$$\langle 0|S|0\rangle = \det(1+\varepsilon G)^{-1/2}$$

which we found on March 3.

Notice that the quartic interaction expression

$$e^{aD^4}\, e^{bx^2}\Big|_{x=0} = \sum \frac{a^n}{n!}\frac{b^{2n}}{(2n)!} D^{4n} x^{4n} = \sum \frac{a^n b^{2n}}{n!(2n)!}(4n)!$$

diverges since if we apply the ratio test then

$$\frac{u_{n+1}}{u_n} = \frac{a b^2 \,(4n+1)\cdots(4n+4)}{(n+1)(2n+1)(2n+2)} \longrightarrow \infty .$$

Hence some other ideas will have to be used in order to handle a quartic potential such as

$$V(q) = \text{const}\; q^4$$

In the Euclidean case where we solve Bloch's equation $\frac{\partial\psi}{\partial t} = -H\psi$ for the time evolution we get

$$\langle 0| \int [dq]\; e^{-\int \frac{1}{2}(\dot q^2 + \omega^2 q^2)}\; e^{+\int J q}|0\rangle = \exp\left\{\frac{1}{2}\int J(t)\, \underbrace{D(t,t')}\, J(t')\,dt\,dt'\right\}$$

means endpt of path weighted by $e^{-\frac{1}{2}\omega q^2}$

$$\left(-\frac{d^2}{dt^2}+\omega^2\right)^{-1/2} \longrightarrow \frac{e^{-\omega|t-t'|}}{2\omega}$$

so that if we replace J by iJ we see that the Gaussian

$$\exp\left\{-\frac{1}{2}\int J(t)\,D(t,t')\,J(t')\,dt\,dt'\right\}$$

is the Fourier transform of the path space measure.

Let's now try to understand diagrams for the perturbation expansion of

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \varepsilon q^4$$

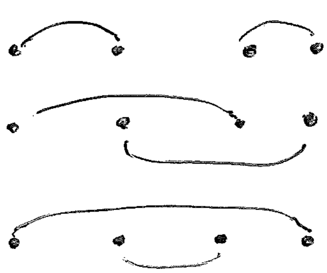
where $\varepsilon(t)$ has compact support. We have

$$\langle 0|S|0\rangle = 1 - i\int \varepsilon(t)\langle q(t)^4\rangle dt + \frac{(-i)^2}{2!}\int \varepsilon(t_1)\varepsilon(t_2)\langle T q(t_1)^4 q(t_2)^4\rangle \times dt_1\,dt_2 - \cdots$$

Recall

$$\langle T\, q(t_1)\, q(t_2)\rangle = -i\,G(t_1,t_2) = \frac{1}{2\omega}e^{-i\omega|t_1-t_2|}$$

and that $\langle T\, q(t_1)\cdots q(t_n)\rangle$ is the sum over all possible pairwise contractions. For $n=4$ we have 3 possible ways of contracting



hence $\langle q(t)^4\rangle = 3\,(-i\,G(t,t))^2 = \dfrac{3}{4\omega^2}$, hence

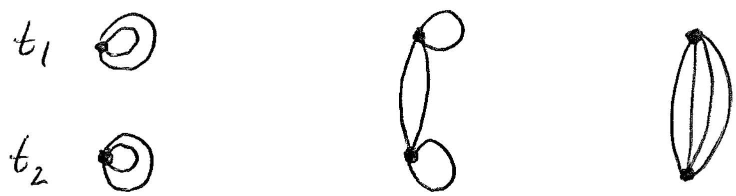
$$\langle 0|S^{(1)}|0\rangle = -i\,\frac{3}{4\omega^2}\int \varepsilon(t)\,dt$$

Next consider the 2nd order term. To compute

$\langle T\, q(t_1)^4 q(t_2)^4 \rangle$ we make $1\cdot 3\cdot 5\cdot 7 = 105$ contractions in 8 dots

$$\underbrace{\cdot\ \cdot\ \cdot\ \cdot}_{t_1} \Big| \underbrace{\cdot\ \cdot\ \cdot\ \cdot}_{t_2}$$

however $\Sigma_4 \times \Sigma_4$ acts on these leaving three types which we can represent by the diagrams



with multiplicities (= ~~index~~ of stabilizer)

$$\frac{(4!)^2}{8^2} = 9 \qquad \frac{(4!)^2}{8} = 3\cdot 24 \qquad \text{and} \qquad \frac{(4!)^2}{4!} = 24$$

(total $9 + 72 + 24 = 105$). Thus

$$T\left(q(t_1)^4 q(t_2)^4\right) = 9\left(\frac{1}{2\omega}\right)^4 + 72\left(\frac{1}{2\omega}\right)^2\left(\frac{e^{-i\omega|t_1-t_2|}}{2\omega}\right)^2 + 24\left(\frac{1}{2\omega}e^{-i\omega|t_1-t_2|}\right)^4$$

hence

$$\langle 0|S^{(2)}|0\rangle = \frac{-1}{2!}\frac{1}{(2\omega)^4}\int \varepsilon(t_1)\varepsilon(t_2)\left\{9 + 72\, e^{-i2\omega|t_1-t_2|} + 24\, e^{-i4\omega|t_1-t_2|}\right\}$$

August 1, 1979

To understand Green's functions. Suppose we have an oscillator $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$ or ~~better~~ $H = \sum \frac{1}{2} p_i^2 + \sum \frac{1}{2} q_i (\omega^2)_{ij} q_j$. Then the Green's function

$$\langle 0 | T\, q_i(t)\, q_j(0) | 0 \rangle$$

is the probability amplitude for the system starting in the ~~state~~ $q_j|0\rangle$ and being found at the later time t in the state $q_i|0\rangle$. (It would be nice in the case of lattice vibrations to interpret $q_i|0\rangle$ as the state where the i-th atom has been excited one step above the ground state. This is perhaps reasonable.)

In the case of a general $H = \frac{1}{2}p^2 + V(q)$ with discrete non-deg. energy levels, it is not clear how $q|0\rangle$ can be interpreted as an excited state.

Suppose one considers a many body problem ~~with~~ fermions:

$$H = \sum \omega_k a_k^* a_k + \sum a_n^* a_m^* V_{nmlk} a_l a_k$$

Suppose that $|0\rangle$ is the ground state for H on $\Lambda^p W$. What is the significance of the average

$$\langle T\{\psi(t)\, \psi(t')\}\rangle \; ?$$

Linear Response (Kubo):

Start with a system described by a Hamiltonian H. Assume it is initially in its ground state $|0\rangle$ and we perturb it by a small external field H_{ex}. In practice

$$H_{ex} = \varepsilon n$$

where n is a particle density operator (i.e. $n = a^* a$) and $\varepsilon = \varepsilon(t)$ is the applied field. We want to compute the change in density $\delta\langle n(t)\rangle$ resulting from the perturbation. Here

$$\langle n(t)\rangle = \langle 0| U(0,t)\, n\, U(t,0) |0\rangle .$$

We have to first order in H_{ex}

$$\delta U(t,0) = -i\int_0^t U(t,t')\, H_{ex}(t')\, U(t',0)\, dt'$$

$$\delta U(0,t) = -U(0,t)\, \delta U(t,0)\, U(0,t)$$

so

$$\delta\langle n(t)\rangle = \langle 0|\, i\int_0^t U(0,t')\, H_{ex}(t')\, U(t',t)\, n\, U(t,0)\, dt' - i\int_0^t U(0,t)\, n\, \underbrace{U(t,t')\, H_{ex}(t')\, U(t',0)}_{U(t,0)\tilde{H}_{ex}(t')}\, dt' \,|0\rangle$$

or

$$\delta\langle n(t)\rangle = i\int_0^t \langle [\tilde{H}_{ex}(t'), \tilde{n}(t)] \rangle\, dt'$$

When $H_{ex} = \varepsilon n$ this becomes

$$\delta\langle n(t)\rangle = i\int_0^t \varepsilon(t')\, \langle [n(t'), n(t)] \rangle\, dt'$$

or

$$\boxed{\delta\langle n(t)\rangle = \int_0^t -i\langle[n(t),n(t')]\rangle\ \varepsilon(t')\,dt'}$$

This expresses the linear response of the density $\langle n(t)\rangle$ to the applied field $\varepsilon(t)$. The kernel is a so-called <u>retarded</u> Green's function:

$$G^R(t,t') = -i\langle[n(t),n(t')]\rangle\,\theta(t-t')$$

Now the Feynman-Dyson series computes the time-ordered Green's function

$$G^T(t,t') = -i\langle T\,n(t)\,n(t')\rangle.$$

To relate G^T and G^R one uses the Lehmann representation.

August 2, 1979

[redacted] Perturbation expansion of the Green's function:

Begin with $H = H_0 + V$. The Green's function we want to compute is

$$\langle T q(t) q(t') \rangle .$$

Let us consider the temperature Green's function where

$$\langle A \rangle = \frac{tr\,(e^{-\beta H} A)}{tr\,(e^{-\beta H})} \qquad q(\tau) = e^{\tau H} q\, e^{-\tau H}$$

Then we are after

$$G(\tau) = \frac{tr\,(e^{-\beta H} e^{\tau H} q\, e^{-\tau H} q)}{tr\,(e^{-\beta H})}$$

We want to write this in terms of thermal averages wrt H_0. Now

$$\frac{Z}{Z_0} = \frac{tr\,(e^{-\beta H})}{tr\,(e^{-\beta H_0})} = \frac{tr\,(e^{-\beta H_0} e^{\beta H_0} e^{-\beta H})}{Z_0} = \langle U(\beta, 0) \rangle$$

where $U(\beta,\sigma) = e^{\beta H_0} e^{-\beta H} e^{\sigma H} e^{-\sigma H_0}$ is the propagator in the interaction picture. Recall we have

$$\cancel{U(\beta,\sigma) = U_0(\beta,\sigma) - \int_\sigma^\beta U_0(\beta,\tau_1) V U_0(\tau_1,\sigma)\, d\tau_1 + \int_\sigma^\beta d\tau_1 \int_\sigma^{\tau_1} d\tau_2}$$

$$U(\beta,\sigma) = I - \int_\sigma^\beta V(\tau_1)\, d\tau_1 + \frac{(-1)^2}{\text{[redacted]}} \int_\sigma^\beta d\tau_1 \int_\sigma^{\tau_1} d\tau_2\, V(\tau_1) V(\tau_2) + \cdots$$

so

$$\langle U(\beta, 0) \rangle = 1 - \int_\sigma^\beta d\tau_1 \langle V(\tau_1) \rangle + \frac{(-1)^2}{\text{[redacted]}} \int_\sigma^\beta d\tau_1 \int_\sigma^{\tau_1} d\tau_2 \langle V(\tau_1) V(\tau_2) \rangle + \cdots$$

The numerator can be written (after dividing by Z_0)

$$\mathrm{tr}\left(e^{-\beta H} e^{\tau H} q e^{-\tau H} q\right)/Z_0$$

$$= \frac{1}{Z_0}\mathrm{tr}\left(e^{-\beta H_0} e^{\beta H_0} e^{-\beta H} e^{\tau H} e^{-\tau H_0} e^{\tau H_0} q e^{-\tau H_0} e^{\tau H_0} e^{-\tau H} q\right)$$

$$= \langle U(\beta,\tau)\, q(\tau)\, U(\tau,0)\, q \rangle$$

$$= \sum_n{}' (-1)^n \int_{\beta \geq \tau_1 \geq \cdots \geq \tau_n \geq \tau} d\tau_1 \cdots d\tau_n \sum_{n'} (-1)^{n'} \int_{\tau \geq \tau_1' \geq \cdots \geq \tau_{n'}' \geq 0} d\tau_1' \cdots d\tau_{n'}' \langle V(\tau_1) \cdots V(\tau_n)\, q(\tau) \times V(\tau_1') \cdots V(\tau_{n'}')\, q \rangle$$

Now suppose you look at all terms involving p V-factors; you have one for each $n + n' = p$. Given $\beta \geq \tau_1 > \cdots > \tau_p \geq 0$ it belongs to the term where there are n τ_i's bigger than τ. So it's clear we have for the degree p contribution

$$(-1)^p \int_{\beta \geq \tau_1 > \cdots > \tau_p \geq 0} \langle T V(\tau_1) \cdots V(\tau_p)\, q(\tau)\, q \rangle \, d\tau_1 \cdots d\tau_p$$

$$= \frac{(-1)^p}{p!} \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_p \langle T V(\tau_1) \cdots V(\tau_p)\, q(\tau)\, q \rangle$$

So we get the formula

$$G(\tau) = \frac{\sum_p \frac{(-1)^p}{p!} \int_0^\beta \langle T V(\tau_1) \cdots V(\tau_p)\, q(\tau)\, q \rangle \, d\tau_1 \cdots d\tau_p}{\sum_p \frac{(-1)^p}{p!} \int_0^\beta \langle T V(\tau_1) \cdots V(\tau_p) \rangle \, d\tau_1 \cdots d\tau_p}$$

Tomorrow we want to understand why this reduces to a sum over connected diagrams.

August 3, 1979

What I am missing is a feeling for the physical significance of the 1-particle Green's function. I think in the many-body problem one is able to write the Hamiltonian

$$H = \underbrace{E_G}_{\substack{\text{ground}\\ \text{energy}}} + \underbrace{\sum \varepsilon_k A_k^* A_k}_{\substack{\text{elementary}\\ \text{excitations}}} + \text{small term}$$

and somehow the Green's function tells one about the elementary excitations.

So let's consider an interacting system of fermions with

$$H = \underbrace{\frac{p_i^2}{2} + \sum U(q_i)}_{H_0} + \underbrace{\frac{1}{2}\sum_{i\neq j} V(q_i, q_j)}_{H_1}$$

Find the eigenvectors for the 1-particle Hamiltonian

$$H_0 \varphi_k = \omega_k \varphi_k$$

and form Fock space Λ = exterior algebra on 1-particle Hilbert space with creation and annihilation operators $a_k^* = e(\varphi_k)$, $a_k = i(\varphi_k^*)$. On Λ we have

$$H_0 = \sum \omega_k a_k^* a_k$$

Now instead of the operators a_k, a_k^* it is sometimes useful to use the field operators

$$\psi(x) = \sum \varphi_k(x) a_k \qquad \psi(x)^* = \sum \overline{\varphi_k(x)}\, a_k^*.$$

(Here I assume the one particle states are scalar functions of position. If $\varphi_k(x) = (\varphi_{k\lambda}(x))$ is a vector function, e.g. λ is a spin coordinate, then we have field operators $\psi_\lambda(x)$, $\psi_\lambda(x)^*$.) If one thinks of Fock space as being the exterior algebra with basis $|x\rangle = \delta(q-x)$ for different x, then $\psi(x)$ destroys a particle at x and $\psi(x)^*$ creates a particle at x. Then in the 1-particle space

$$H_0 = \int |x\rangle\langle x| -\tfrac{1}{2}\nabla^2 + U |x'\rangle\langle x'| \quad dx\,dx'$$

so on Fock space

$$H_0 = \int \psi(x)^* \langle x| -\tfrac{1}{2}\nabla^2 + U |x'\rangle \psi(x') \; dx\,dx'$$

Now $\langle x| -\frac{1}{2}\nabla^2 + U |x'\rangle$ is sort of a diagonal matrix, which is why one sees written

$$H_0 = \int \psi(x)^* \left(-\tfrac{1}{2}\nabla_x^2 + U(x)\right) \psi(x)\, dx$$

Next let us consider the interaction

$$H_1 = \frac{1}{2} \sum_{i \neq j} V(q_i, q_j)$$

or more generally suppose we are given a two particle operator and we want to understand its extension to Fock space. Let's first look at the linear algebra.

Look at an operator V on Λ^2. Its matrix elements are

$$\langle \varphi_k \wedge \varphi_l | V | \varphi_m \wedge \varphi_n \rangle$$

so we can write V as

$$V = \frac{1}{4}\sum_{klmn} e(\varphi_k)e(\varphi_l)\langle \varphi_k \wedge \varphi_l | V | \varphi_m \wedge \varphi_n \rangle i(\varphi_n) i(\varphi_m)$$

$$= \frac{1}{4}\sum_{klmn} \langle \varphi_k \wedge \varphi_l | V | \varphi_m \wedge \varphi_n \rangle a_k^* a_l^* a_n a_m$$

at least on Λ^2. Now when you extend a 2-particle operator to Fock space just what do you do?

So suppose given $V : \Lambda^2 W \longrightarrow \Lambda^2 W$ where W is the 1-particle space. ~~One regards $\Lambda^n W$ sitting inside $W^{\otimes n}$ as the antisymmetric elements~~ ~~Think of $\Lambda^n W$ as a quotient of $W^{\otimes n}$ so that any element $\omega \in \Lambda^n W$~~

Let us think of an element $\omega \in \Lambda^n W$ as giving a function $\omega_{i_1 \cdots i_n}$ namely it's components with respect to the basis $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_n}$. Thus $\omega_{i_1 \cdots i_n}$ is a skew-symmetric tensor with

$$\langle \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_n} | \omega \rangle = \omega_{i_1 \cdots i_n}$$

and

$$\omega = \sum_{i_1 < \cdots < i_n} \omega_{i_1 \cdots i_n} \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_n}$$

For example if we use the basis $|x\rangle$, then ω gives us a skew-symmetric function $\omega(x_1, \ldots, x_n)$ of the coordinates. Moreover the element $\psi_1 \wedge \cdots \wedge \psi_n$ of $\Lambda^n W$ corresponds to the function

$$\langle x_1, \dots, x_n | \psi_1 \wedge \dots \wedge \psi_n \rangle = \det(\psi_i(x_j)).$$

Now let $V : \Lambda^2 W \longrightarrow \Lambda^2 W$ be a 2-particle operator. It's effect on

$$\omega = \sum_{m<n}' \omega_{mn} \varphi_m \wedge \varphi_n$$

is

$$V\omega = \sum_{\substack{k<l \\ m<n}} |\varphi_k \wedge \varphi_l\rangle \langle \varphi_k \wedge \varphi_l | V | \varphi_m \wedge \varphi_n \rangle \omega_{mn}$$

hence

$$(V\omega)_{kl} = \frac{1}{2} \sum_{m,n}' V_{klmn} \omega_{mn}$$

Now when we extend V to $\Lambda^N W$ we make it operate on each pair of components and then we add. So for

$$\omega = \sum_{i_1 < \dots < i_N} \omega_{i_1 \dots i_N} \varphi_{i_1} \wedge \dots \wedge \varphi_{i_N} = \frac{1}{N!} \sum_{i_1 \dots i_N}$$

we have

$$V\omega = \sum_{1 \le a < b \le N} \omega_{i_1 \dots i_N}$$

?