June 17, 1979

energy-momentum as extra structure beyond Newton 971
Hamilton-Jacobi S-theory as a kind of fluid flow 978
particle motion as hypersurface in T^*_M ~991

Maupertuis principle: Suppose we have a Hamiltonian $H(q,p)$ independent of time. Then the stationary curves for the form $\eta = p\,dq$ on the surface $H(q,p)=E$ are the trajectories of the system.

Note that the hypersurface $H(p,q)=E$ has dimension $2n-1$. The form $d\eta = dp\,dq$ has the maximal rank $2n$ on phase space, so when restricted to the tangent plane to this hypersurface

$$\frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp = 0$$

it has a 1-dimensional kernel. Since

$$i\left(-\frac{\partial H}{\partial p}\frac{\partial}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial}{\partial p}\right)dp\,dq = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp$$

it follows that the kernel line is spanned by

$$-\frac{\partial H}{\partial p}\frac{\partial}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial}{\partial p} = -\left(\dot q\frac{\partial}{\partial q} + \dot p\frac{\partial}{\partial p}\right)$$

which is also the tangent line to the Hamiltonian flow.

The interesting point about the Maupertuis principle is that it gives you the actual paths in phase space without their time parameterization, so therefore it seems suited for the [redacted] relativity problem.

New view of mechanics. Newton explains mechanics by saying that the motion of a particle is governed by a 2nd order ODE:

$$\ddot{q} = F(t, q, \dot{q})$$

But he doesn't say much about the "force" F. The point of the Lagrange - Hamilton formulation is that usual mechanics carries extra structure beyond just being a 2nd order O.D.E. For example, the second order ODE gives us a line field in $(t, q, \dot{q})$ - space transversal to $t = \text{const}$, and which at each point projects to $dq = \dot{q}\,dt$. If the mechanics comes from a Lagrangian, then we have a 1-form

$$\eta = \frac{\partial L}{\partial \dot{q}} dq - \left(\dot{q} \frac{\partial L}{\partial \dot{q}} - L \right) dt = p\,dq - H\,dt$$

$$= L\,dt + \frac{\partial L}{\partial \dot{q}} (dq - \dot{q}\,dt)$$

whose stationary curves are the trajectories.

<u>Question:</u> Just what is involved in being able to realize a 2nd order DE in Hamiltonian form?

June 18, 1979

The problem is to understand what is involved in getting a 2nd order DE

$$\ddot{q} = F(t, q, \dot{q})$$

in Hamiltonian form. To simplify suppose F is time-independent.

A second order DE $\ddot{q} = F(q, \dot{q})$ is simply a vector field in the tangent bundle such that at a point $(q, \dot{q})$ if we project it down to M we get $\dot{q}\frac{\partial}{\partial q}$. Thus it is a partial way of lifting vectors in M to vectors in T_M. An example is provided by a connection on T_M, so we should review this.

Corresponding to the coordinate system q^i on M we get the frame $\partial_i = \frac{\partial}{\partial q^i}$ in T_M, so any vector field v can be written

$$v = v^i \frac{\partial}{\partial q^i}$$

with v^i functions of q. The connection is given by a diff. operator $D : T \longrightarrow T \otimes T^*$ hence by

$$D(\partial_i) = \theta^j_{ik}\, \partial_j\, dx^k = \partial_j \otimes \theta^j_i$$

Then

$$D(v^i \partial_i) = dv^j \partial_j + v^i D(\partial_i)$$

$$= \partial_j (dv^j + v^i \theta^j_i)$$

The section v is horizontal when $Dv = 0$.

Now suppose we has a curve $x(t)$ and a lift of this curve $v(t)$ to T. The connection allows us to measure the change in v along the curves. For v to be "parallel" or "constant" along $x(t)$ we want

$$0 = \frac{d}{dt} v = \frac{d}{dt}\left(v^j \partial_j\right)$$

$$= \frac{dv^j}{dt}\partial_j + v^j\, \theta^i_{jk}\, \partial_i \frac{dx^k}{dt}$$

or

$$0 = \frac{dv^i}{dt} + v^j\, \theta^i_{jk} \frac{dx^k}{dt}$$

A <u>geodesic</u> is a curve such that $v(t) = \frac{dx}{dt}$ is parallel along $x(t)$:

$$0 = \frac{d^2x^i}{dt^2} + \theta^i_{jk} \frac{dx^j}{dt}\frac{dx^k}{dt}$$

The connection is symmetric when $\theta^i_{jk} = \theta^i_{kj}$. Obviously the geodesics only depend on the symmetryation.

<u>Example</u>: $ds^2 = g_{ij}\, dx^i dx^j$. Geodesics are stationary paths for kinetic energy:

$$L(x, \dot{x}) = \frac{1}{2} g_{ij}\, \dot{x}^i \dot{x}^j$$

$$\frac{\partial L}{\partial \dot{x}^i} = g_{ij}\, \dot{x}_j \qquad \frac{\partial L}{\partial x^i} = \frac{1}{2}\partial_i g_{jk}\, \dot{x}^j \dot{x}^k$$

Euler DE:

$$\frac{d}{dt}\left(g_{ij}\dot{x}_j\right) = \partial_k g_{ij}\, \dot{x}_k \dot{x}_j + g_{ij}\, \ddot{x}_j = \frac{1}{2}\partial_i g_{jk}\, \dot{x}_j \dot{x}_k$$

or

$$g_{ij}\, \ddot{x}_j + \left[\partial_k g_{ij} - \frac{1}{2}\partial_i g_{jk}\right] \dot{x}_j \dot{x}_k = 0$$

Put $\Gamma_{ijk} = \frac{1}{2}\left(\partial_k g_{ij} - \partial_i g_{jk} + \partial_j g_{ki}\right)$ symmetrization of coeff. of $\dot{x}_j \dot{x}_k$

$$\Gamma^i_{jk} = g^{ia}\Gamma_{ajk} \qquad (= \theta^i_{jk} \text{ above})$$

so that the geodesic equation is

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0.$$

Example: $\frac{d^2q}{dt^2} = F\left(q, \frac{dq}{dt}\right)$ $\quad q$ is a single unknown.

Solve by putting $p = \frac{dq}{dt}$ and changing q to ind. vble.

$$\frac{d^2q}{dt^2} = \frac{dp}{dt} = \frac{dp}{dq}\frac{dq}{dt} = p\frac{dp}{dq}$$

So

$$p\frac{dp}{dq} = F(q, p).$$

We can solve this by means of an integrating factor μ.

$$-\mu F\,dq + \mu p\,dp = dH.$$

Thus we see there exists a function $H = H(q, p)$ such that H is constant on the trajectories of the original DE.

$$\frac{d}{dt}H\left(q, \frac{dq}{dt}\right) = \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p}\frac{d^2q}{dt^2} = -\mu F\frac{dq}{dt} + \mu p\frac{dp}{dt}$$

$$= -\mu\left(F - p\frac{dp}{dq}\right)\frac{dq}{dt} = 0$$

Notice that if $\mu = 1$, then we actually get Hamilton's equations

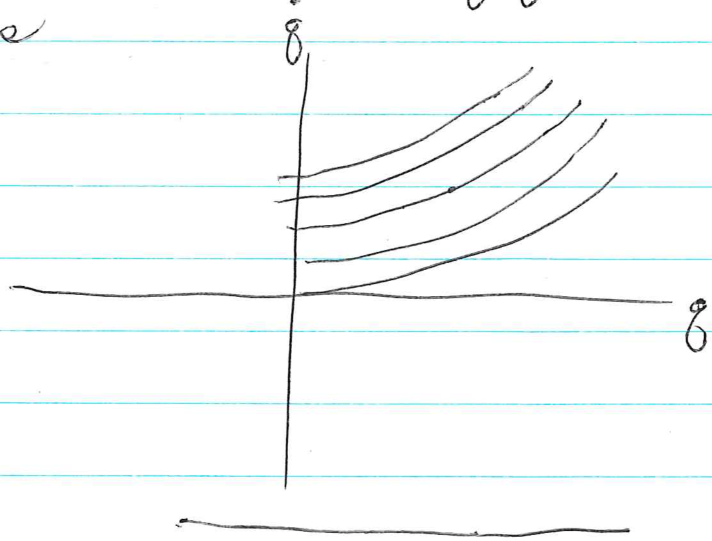
$$\frac{\partial H}{\partial p} = p = \dot{q} \quad , \quad \frac{\partial H}{\partial q} = -F = -\dot{p}.$$

However this can happen only when

$$\frac{\partial}{\partial p}(-F) = \frac{\partial}{\partial q}(p) = 0$$

which means that F depends only on q, and then $H = \frac{p^2}{2} + V(q)$.

Let's see if we can find an obstruction to expressing the DE in Hamilton form even locally. First let's work in the $q, \dot{q}$ plane and plot the trajectories



Here seems to be the method to use. Take our basic coordinates to be t, $q, \dot{q}$. Our 2nd order DE can be expressed,

$$d\dot{q} = F\,dt \quad , \quad dq = \dot{q}\,dt.$$

Because F is time-independent we can view the motion as the flow for the vector field

$$X = \dot{q}\frac{\partial}{\partial q} + F\frac{\partial}{\partial \dot{q}}$$

Locally at least we can find a function $H(q,\dot{q})$ constant on the trajectories

$$\dot{q}\frac{\partial H}{\partial q} + F\frac{\partial H}{\partial \dot{q}} = 0$$

with $dH \neq 0$ (we stay away from critical points so that $(\dot{q}, F) \neq 0$). It follows there is a unique function μ such

$$\frac{\partial H}{\partial q} = -\mu F \quad , \quad \frac{\partial H}{\partial \dot{q}} = \mu \dot{q}$$

and further $\mu \neq 0$. Now let $\Omega = \mu\, d\dot{q}\, dq$. Then

$$i(X)\Omega = i\left(\dot{q}\frac{\partial}{\partial q} + F\frac{\partial}{\partial \dot{q}}\right)\mu\, d\dot{q}\, dq = -\dot{q}\mu\, d\dot{q} + F\mu\, dq = -dH$$

so that X is the flow corresponding to H. Finally to get the equations in Hamiltonian form, we need a function p with

$$d(p\,dq) = \mu\, d\dot{q}\, dq$$

or

$$\frac{\partial p}{\partial \dot{q}} = \mu$$

and this can be solved uniquely up to an additive function of q.

Therefore in this one-dimensional case there is no local obstruction to expressing the equation in Hamiltonian form.

June 19, 1979

Yesterday we looked at 1-diml stationary motion:

$$\ddot{q} = F(q,\dot{q}).$$

This is the flow associated to the vector field

$$X = \dot{q}\frac{\partial}{\partial q} + F\frac{\partial}{\partial \dot{q}}$$

in the $q,\dot{q}$ plane. If we work locally at a regular point ($(\dot{q},F)\neq 0$), then there is an $H(q,\dot{q})$ constant on the trajectories:

$$\dot{q}\frac{\partial H}{\partial q} + F\frac{\partial H}{\partial \dot{q}} = 0$$

with $dH \neq 0$. It follows $\exists\, \mu \neq 0$ with

$$\frac{\partial H}{\partial \dot{q}} = \mu\dot{q} \qquad \frac{\partial H}{\partial q} = -\mu F$$

hence we get a non-deg. closed 2-form $\Omega = \mu\, d\dot{q}\, dq$ such that

$$i(X)\Omega = -\dot{q}\mu\, d\dot{q} + F\mu\, dq = -dH.$$

Thus X is the Hamiltonian vector field belonging to Ω and H. Then we can define p by requiring

$$d(p\,dq) = \Omega$$

or

$$\frac{\partial p}{\partial \dot{q}} = \mu$$

This means that any other choice $\tilde{p}$ for p is of the form

$$\tilde{p} = p + f(q)$$

We now ask what Hamilton's W function is for the trajectory with $H = E$. [redacted] Along the trajectory we can solve $H(q,\dot{q}) = E$ for $\dot{q}$, and hence p, as a function of q. Then we solve

$$\frac{dW}{dq} = p$$

for the function W. So once p [redacted] has been found, W is determined up to an additive constant [redacted] on each trajectory by the requirement that p is its rate of change with respect to q. Again we see that the rate of change of phase along a trajectory is not directly related to the velocity of the particle.

Once H, p have been defined we can define the Lagrangian to be

$$L(q,\dot{q}) = p\dot{q} - H$$

Then

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial p}{\partial \dot{q}}\dot{q} + p - \frac{\partial H}{\partial \dot{q}} = \mu\dot{q} + p - \mu\dot{q} = p$$

$$\frac{\partial L}{\partial q} = \frac{\partial p}{\partial q}\dot{q} - \frac{\partial H}{\partial q} = \frac{\partial p}{\partial q}\dot{q} + \mu F$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}p = \dot{q}\frac{\partial p}{\partial q} + F\frac{\partial p}{\partial \dot{q}}$$

so Lagrange's equation is satisfied

One moral from the above is that the energy + momentum of a particle undergoing a general stationary one-diml

motion don't come out of the force law. Somehow
the energy-momentum is some extra physics, or better, extra structure of mechanics.

June 21, 1979

Consider 1-dimensional motion:

$$\frac{d^2q}{dt^2} = F\left(t, q, \frac{dq}{dt}\right).$$

Trajectories are flow curves for the vector field

$$X = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + F\frac{\partial}{\partial \dot{q}}$$

in $t, q, \dot{q}$ spaces. Trajectories are curves in $(t, q, \dot{q})$ space on which the forms

$$d\dot{q} - F\,dt \quad , \quad dq - \dot{q}\,dt$$

vanish, or equivalently curves whose tangent lines are the kernel for the 2 form

$$(d\dot{q} - F\,dt)\wedge(dq - \dot{q}\,dt) = d\dot{q}\,dq - (\dot{q}\,d\dot{q} - F\,dq)\,dt$$

I'd like to be able to find a non-vanishing [redacted] function [redacted] μ such that upon multiplying this 2 form by [redacted] μ it becomes exact. Put

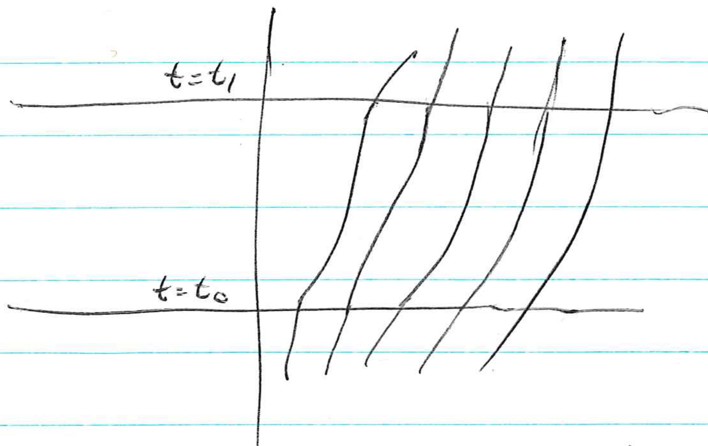
$$\Omega = \mu(d\dot{q} - F\,dt)(dq - \dot{q}\,dt) = \mu\,d\dot{q}\,dq - (\mu\dot{q}\,d\dot{q} - \mu F\,dq)\,dt$$

and assume $d\Omega = 0$. Because $i(X)\Omega = 0$ we have

$$\theta(X)\,\Omega = [d\,i(X) + i(X)\,d]\,\Omega = 0$$

and so therefore the volume induced by Ω on any space-like surface is invariant under time-evolution.

Conversely if I restrict [redacted] Ω to $t = \text{constant}$ I get the volume $\mu\, d\dot{q}\, dq$ and [redacted] so therefore from the invariance I can specify μ along $t=0$ and then determine it for other times using [redacted] time evolution



The question arises whether once μ is determined in this way, is the corresponding Ω closed? This should be clear by Stoke's theorem

$$\iiint d\Omega = \underbrace{\iint_{\text{top}} \Omega - \iint_{\text{bottom}} \Omega}_{=0 \text{ because volume invariant}} + \underbrace{\iint_{\text{sides}} \Omega}_{0 \text{ because sides contain the lines in } \operatorname{Ker}\Omega.} = 0$$

Now that Ω is closed one can express it as $d\eta$ where η is a 1-form. In general

$$\eta = a\,dt + b\,dq + [redacted]\ c\,d\dot{q}$$

and then we can solve $\dfrac{\partial f}{\partial \dot{q}} = c$ to eliminate c by replacing η by $\eta - df$. Thus we can get

[redacted] a form
$$\eta = p\,dq - H\,dt$$
with $d\eta = \Omega$. It follows that we have managed to express general one-dimensional motion in Hamiltonian form.

[redacted] Let's look at the equations more carefully, to see if we can understand momentum better.

$$\Omega = \mu(d\dot q - F\,dt)(dq - \dot q\,dt)$$
$$= \mu\, d\dot q\, dq + \mu F\, dq\, dt - \mu \dot q\, d\dot q\, dt$$
$$d\Omega = \left(\frac{\partial\mu}{\partial t} + \frac{\partial}{\partial \dot q}(\mu F) + \frac{\partial}{\partial q}(\mu \dot q)\right) d\dot q\, dq\, dt$$

Hence $d\Omega = 0$ when

$$\underbrace{\frac{\partial\mu}{\partial t} + \dot q\frac{\partial\mu}{\partial q} + F\frac{\partial\mu}{\partial\dot q}}_{X\mu} + \mu\frac{\partial F}{\partial \dot q} = 0$$
$$X\mu + \mu\frac{\partial F}{\partial\dot q} = 0$$

This is a linear PDE whose solution is of the form
$$\mu_t = \exp\left(-\int_0^t \frac{\partial F}{\partial \dot q}\,dt\right)\mu_0$$
with following interpretation: μ_0 is the value of μ for $t=0$; [redacted] to find $\mu_t(q) = \mu(t,q)$ one takes the trajectory ending at $q(t) = q$ and integrates back to $t=0$.

$$\eta = p\,dq - H\,dt$$

$$d\eta = \frac{\partial p}{\partial t}\,dt\,dq + \frac{\partial p}{\partial \dot q}\,d\dot q\,dq - \frac{\partial H}{\partial q}\,dq\,dt - \frac{\partial H}{\partial \dot q}\,d\dot q\,dt$$

Then $d\eta = \Omega$ means

$$\frac{\partial p}{\partial \dot q} = \mu \qquad -\frac{\partial p}{\partial t} - \frac{\partial H}{\partial q} = \mu F \qquad \frac{\partial H}{\partial \dot q} = \mu \dot q$$

and we can solve this by integrating the first equation for p, and then solving the last two for H.

So [redacted] it is clear there is a lot of arbitrariness in the choice of μ, p, H. [redacted] μ is determined up to a multiplicative function constant on the trajectories. Once μ given, then p is unique up to adding a function of t, q (usually one chooses $p = 0$ when $\dot q = 0$), and then μ, p given, H is unique up to adding a function of t.

Let's see if its possible to [redacted] directly obtain DE for p, H. Let $\eta = p\,dq - H\,dt$ and then compute

$$0 = i\left(\frac{\partial}{\partial t} + \dot q \frac{\partial}{\partial q} + F\frac{\partial}{\partial \dot q}\right)\left(\frac{\partial p}{\partial t}\,dt\,dq + \frac{\partial p}{\partial \dot q}\,d\dot q\,dq - \frac{\partial H}{\partial q}\,dq\,dt - \frac{\partial H}{\partial \dot q}\,d\dot q\,dt\right)$$

$$= \left(\frac{\partial p}{\partial t} + \frac{\partial H}{\partial q} + F\frac{\partial p}{\partial \dot q}\right) dq$$

$$+ \left(\frac{\partial H}{\partial \dot q} - \dot q \frac{\partial p}{\partial \dot q}\right) d\dot q$$

$$+ \left(-\dot q \frac{\partial p}{\partial t} - \dot q \frac{\partial H}{\partial q} - F\frac{\partial H}{\partial \dot q}\right) dt$$

This gives us the following equations

$$\dot q \frac{\partial p}{\partial \dot q} = \frac{\partial H}{\partial \dot q}$$

$$F\frac{\partial p}{\partial \dot q} + \frac{\partial p}{\partial t} + \frac{\partial H}{\partial q} = 0$$

(The third follows from these two provided $\dot q \neq 0$). Another way to obtain these is to eliminate μ from the equations (p. 975, line 4). Notice that these are two first order linear homogeneous DE's in the 2 unknowns p, H. Hence the solutions depend on 2 functions of 2 variables — namely the Cauchy data on a non-characteristic hypersurface.

$$\begin{pmatrix} \dot q \frac{\partial}{\partial \dot q} & \frac{\partial}{\partial \dot q} \\ \frac{\partial}{\partial t} + F\frac{\partial}{\partial \dot q} & -\frac{\partial}{\partial q} \end{pmatrix}\begin{pmatrix} p \\ H \end{pmatrix} = 0$$

Characteristic equation is

$$+\dot q \frac{\partial}{\partial \dot q}\frac{\partial}{\partial q} + \frac{\partial}{\partial t}\frac{\partial}{\partial \dot q} + F\left(\frac{\partial}{\partial \dot q}\right)^2 = 0$$

which has the factor $\frac{\partial}{\partial \dot q}$. Thus the characteristic conic is degenerate. Write the char. eqn:

$$\frac{\partial}{\partial \dot q}\underbrace{\left(\frac{\partial}{\partial t} + \dot q \frac{\partial}{\partial q} + F\frac{\partial}{\partial \dot q}\right)}_{X} = 0$$

Then a surface will be non-characteristic provided its tangent plane does not contain the vectors X and $\frac{\partial}{\partial \dot q}$.
Obvious choice is to take $\dot q = 0$. This will not

contain X provided $F \neq 0$. Hence there is a unique choice for p, H normalized so that both vanish for $\dot{q} = 0$.

Is it possible to understand p, H by thinking of a fluid flow? I want to think of a fluid being made up of particles moving according to the given DE. Recall that we want to give the velocity $v(t,q)$ of the fluid at each (t,q). This is rigged so that if $q = q(t)$ is a trajectory then

$$\frac{dq}{dt} = v(t,q)$$

and

$$\frac{d^2q}{dt^2} = \frac{d}{dt} v(t,q) = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial q}\frac{dq}{dt}$$

$$\parallel$$

$$F\left(t, q, \frac{dq}{dt}\right)$$

so we see that v satisfies

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial q} v = F(t,q,v).$$

So perhaps it would be better to say that fluid flow is described by giving $\dot{q}$ as a function of t, q in such a way that

$$\frac{\partial \dot{q}}{\partial t} + \dot{q}\frac{\partial \dot{q}}{\partial q} = F(t,q,\dot{q})$$

or equivalently $\quad X\left(\dot{q} - v(t,q)\right) = 0$

We see that a fluid flow is the same as a suitable family of trajectories. If the equations of motion is given in Lagrangian or Hamiltonian form, then according to Hilbert a fluid flow must be the same as a solution of the Hamilton - Jacobi PDE. This should be checked carefully.

So let us start with a solution $S = S(t, q)$ of the HJ equation

$$\frac{\partial S}{\partial t} + H\left(t, q, \frac{\partial S}{\partial q}\right) = 0$$

The surface $S = S(t,q)$ in (t, q, S)-space has tangent plane

$$dS = p\,dq - H\,dt \qquad \text{where } p = \frac{\partial S}{\partial q}$$

at each of its points. If we vary p to $p + \delta p$ we get the nearby plane

$$dS = (p + \delta p)\,dq - \left(H + \frac{\partial H}{\partial p}\,\delta p\right)dt.$$

Any vector (dt, dS, dq) lying in these planes for all δp infinitesimal satisfies

$$\delta p\left(dq - \frac{\partial H}{\partial p}\,dt\right) = 0 \qquad \forall\,\delta p$$

and hence $dq = \frac{\partial H}{\partial p}\,dt$, $dS = \left(p\frac{\partial H}{\partial p} - H\right)dt$ describes a canonical line in each tangent plane. This gives us a family of integral curves lying in the surface $S = S(t,q)$. Along such an integral we have

$$\frac{dp}{dt} = \frac{\partial^2 S}{\partial q^2}\frac{dq}{dt} + \frac{\partial^2 S}{\partial t\,\partial q} = -\frac{\partial H}{\partial q}$$

since if we differentiate the HJ DE with respect to q we get

$$\frac{\partial^2 S}{\partial q \partial t} + \frac{\partial H}{\partial q} + [redacted] \frac{\partial H}{\partial p}\frac{\partial^2 S}{\partial q^2} = 0$$

Consequently a solution of the Hamilton-Jacobi equation gives us a family of solutions of Hamilton's equations:

$$\dot{q} = \frac{\partial H}{\partial p} [redacted] \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

namely, it gives us the solution passing through (t,q) with $p = \frac{\partial S}{\partial q}(t,q)$.

Now notice that a solution of the [redacted] HJ equation is determined by giving [redacted] S arbitrarily for [redacted] $t=0$, and this amounts [redacted] to giving $\frac{\partial S}{\partial q}$ at $t=0$ arbitrarily, subject to requiring $\frac{\partial S}{\partial q}dq$ be exact. [redacted] In the case where $p = \dot{q}$ i.e. $H = \frac{p^2}{2} + V(t,q)$, this means we are [redacted] given an irrotational flow.

Recall the formulas for the bichar. flow belonging to

$$f(q, S, \frac{\partial S}{\partial q}) = 0$$

$$\frac{dq}{dt} = \frac{\partial f}{\partial p} \qquad \frac{dS}{dt} = p\frac{\partial f}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial f}{\partial q} - p\frac{\partial f}{\partial S}$$

Now let's apply this to the PDE satisfied by the velocity for a 1-dimensional fluid flow

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial q} = F(t,q,v)$$

Here

$f(t,q,v,r,$ [redacted] $p) = r + vp - F(t,q,v) \qquad r = \frac{\partial S}{\partial t},\ p = \frac{\partial S}{\partial q}$

so the bicharacteristic flow is given by

$$\frac{dt}{d\tau} = \frac{\partial f}{\partial r} = 1 \qquad \text{so} \qquad d\tau = dt$$

$\frac{dq}{dt} = \frac{\partial f}{\partial p} = v$ [redacted]

$$\frac{dv}{dt} = r\frac{\partial f}{\partial r} + p\frac{\partial f}{\partial p} = r + pv = F(t,q,v)$$

for trajectories of interest $f=0$

$$\frac{dr}{dt} = -\frac{\partial f}{\partial t} - r\frac{\partial f}{\partial v} = -\frac{\partial F}{\partial t} - rp + r\frac{\partial F}{\partial v}$$

$$\frac{dp}{dt} = -\frac{\partial f}{\partial q} - p\frac{\partial f}{\partial v} = \frac{\partial F}{\partial q} - p^2 + p\frac{\partial F}{\partial v}$$

The first three equations show that the bicharacteristic flow coincides with the fluid flow.

June 22, 1940 (39 years old)

The problem is still to get some feeling for wavefronts and energy-momentum. Given the equation of motion

$$\frac{d^2 q}{dt^2} = F\left(t, q, \frac{dq}{dt}\right)$$

equivalently the vector field

$$X = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + F\frac{\partial}{\partial \dot{q}}$$

in $(t, q, \dot{q})$ space one seeks p, H functions of $(t, q, \dot{q})$ such that

$$\eta = p\,dq - H\,dt$$

satisfies a) $i(X)\,d\eta = 0$
b) $\frac{\partial p}{\partial \dot{q}}$ non-singular

$\Big\} \Rightarrow$ $d\eta$ has max rank $2n$ and X spans its kernel

a) is a first order linear homogeneous PDE for p, H which we have seen can be solved when $n = 1$. Note that there are $n+1$ unknowns (p, H) and $2n+1$ equations

$$i\left(\frac{\partial}{\partial t}\right)(dp\,dq - dH\,dt) = \frac{\partial p}{\partial t}dq + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial \dot{q}}d\dot{q}$$

$$i\left(\frac{\partial}{\partial q^i}\right)(\qquad) = \frac{\partial p}{\partial q}dq - dp - \frac{\partial H}{\partial q}dt = -\frac{\partial p_i}{\partial \dot{q}}d\dot{q} - \frac{\partial p_i}{\partial t}dt - \frac{\partial H}{\partial q_i}dt$$

$$i\left(\frac{\partial}{\partial \dot{q}_i}\right)(\qquad) = \frac{\partial p_j}{\partial \dot{q}_i}dq_j - \frac{\partial H}{\partial \dot{q}_i}dt$$

These get killed by $i(X)$:

$$\dot{q}\left(\frac{\partial p}{\partial t} + \frac{\partial H}{\partial q}\right) + F\frac{\partial H}{\partial \dot{q}} = 0$$

$$+\left(\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i}\right) + F_j\frac{\partial p_i}{\partial \dot{q}_j} = 0$$

$$-\frac{\partial H}{\partial \dot{q}_i} + \dot{q}_j\frac{\partial p_j}{\partial \dot{q}_i} = 0$$

The first equation follows from the bottom $2n$ equations. So it seems that we have $2n$-equations in $n+1$ unknowns, and therefore there are integrability conditions on [redacted] F in order that a solution exist.

Supposing p, H exist we can consider solutions S of the Hamilton-Jacobi PDE. Such an S gives us an n-parameter family of trajectories, [redacted] namely, if we index the trajectory by its position at $t=0$, we have the trajectory with initial momentum $p = \frac{\partial S}{\partial q}(0,q)$. The trajectory passing thru q at time t has momentum $p = \frac{\partial S}{\partial q}(t,q)$. Therefore we have a fluid flow with

$$v(t,q) = \frac{\partial H}{\partial p}(t,q,p(t,q))$$

[redacted] (or equivalently $p(t,q) = \frac{\partial L}{\partial \dot{q}}(t,q,v(t,q))$.) Check:

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + \frac{\partial p}{\partial q}\frac{\partial H}{\partial p} \quad = -\frac{\partial H}{\partial q}$$

holds for each t, q. But

$$\frac{\partial S}{\partial t} + H\left(t, q, \frac{\partial S}{\partial q}\right) = 0$$

$$\underbrace{\frac{\partial^2 S}{\partial q \partial t}}_{=\frac{\partial p}{\partial t}} + \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\underbrace{\frac{\partial^2 S}{\partial q^2}}_{\frac{\partial p}{\partial q}} = 0$$

so it's OK[redacted].

What I have learned is that solutions of the HJ equation describe fluid flow such that [redacted] at each time the momentum field is irrotational, i.e. the gradient of some function.

Our next goal is to understand relativistic motion. In this case a time coordinate is not singled out. Instead of $(t, q, \dot{q})$ space we work in the bundle of time-like tangent lines over space-time. If we do single out space and time coordinates we can identify this bundle with $(t, q, \dot{q})$ space, with the restriction that $|\dot{q}| < 1$.

How do we describe trajectories? These are time-like curves, which I would like to obtain as stationary curves for a suitable form η. Maybe η should sit up in the projective tangent bundle. In any case notice that a time-like tangent line contains a unique vector u with $u \cdot u = 1$ and $u_0 > 0$, so that η sits on this unit tangent bundle.

The next thing we need is energy-momentum.

I need a model. So let's consider on q, p space a Hamiltonian $H(q,p)$ and restrict ourselves to the hypersurface

$$H(q,p) = \text{const.}$$

Then the canonical form $\eta = p\,dq$ gives stationary curves in this hypersurface, but without a time parametrization. More accurately, if the hypersurface is given, then one gets stationary curves, and if H is given, these stationary curves come with a time parametrization.

If coordinates are chosen on space-time: (t, q) then the bundle of tangent lines which are time-like becomes $(t, q, \dot{q})$ space with the restriction $|\dot{q}| < 1$ when time-like is defined by the Minkowski metric. More generally time-like could be defined by requiring $dt \neq 0$ when restricted to the line. In this case the line contains a unique vector

$$\frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q}$$

and there is no restriction on $\dot{q}$. So the moral is that all relativistic calculations can be done in the $(t, q, \dot{q})$ - coordinates.

Next idea is that energy-momentum is the 1-form

$$\eta = p\,dq - H\,dt$$

whose stationary curves give the trajectories. Here p, H are functions of $(t, q, \dot{q})$ satisfying the conditions

a) $\dfrac{\partial p}{\partial \dot{q}}$ non-singular

b) $\dfrac{\partial H}{\partial \dot{q}_j} = \dfrac{\partial p_i}{\partial \dot{q}_j}\dot{q}_i$

To see the reason for the last requirement form

$$L = p\dot{q} - H$$

Then $$\frac{\partial L}{\partial \dot{q}} = p + \frac{\partial p}{\partial \dot{q}}\dot{q} - \frac{\partial H}{\partial \dot{q}} = p$$

and $H = p\dot{q} - L$, so that the stationary curves for η are going to be described by Lagrange equations.

Next idea is to try to get things invariantly defined, independent of (t, q). One method would be to take a 1-form η on the bundles of lines such that $d\eta$ has maximal rank and such that the kernel line for $d\eta$ is compatible with the canonical line. Perhaps there's another possibility based upon the cotangent bundle. The idea is that if I start with $H(t,q,p)$ on the cotangent bundle with its canonical form $p\,dq$, then I can define [redacted] the isomorphism of (t,q,p)-space with $(t,q,\dot{q})$ space via the Hamilton equation

$$\dot{q} = \left(\frac{\partial H}{\partial p}\right)_{t,q}.$$

(You assume this gives [redacted] an isomorphism[redacted], so

$$\frac{\partial \dot{q}}{\partial p} = \frac{\partial^2 H}{\partial p^2} \quad \text{non-singular}$$

It follows that $\dfrac{\partial H}{\partial \dot{q}} = \dfrac{\partial H}{\partial p}\dfrac{\partial p}{\partial \dot{q}} = \dot{q}\dfrac{\partial p}{\partial \dot{q}}$ so that the formalism works.)

June 23, 1979

I am trying to understand relativistic motion of a single particle. The world curve of such a particle is a curve in space-time = (t,q) space, such that the tangent line to the curve is time-like, i.e. contains a unique vector of the form

$$\frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} \qquad |\dot{q}| < c = 1.$$

The motion will be described by a Lagrangian function $L(t,q,\dot{q})$, or better, an energy-momentum 1-form

$$p\,dq - H\,dt = L\,dt + \frac{\partial L}{\partial \dot{q}}(dq - \dot{q}\,dt)$$

defined on the bundle of time-like tangent lines over space-time.

An important point to note is that if we give simply a Hamiltonian function

$$H(t,q,p)$$

on (t,q,p) space such that $\dfrac{\partial^2 H}{\partial p^2}$ is non-singular, then we can define

$$\dot{q} = \frac{\partial H}{\partial p}$$

and, at least locally, transport $p\,dq - H\,dt$ on (t,q,p)-space to $(t,q,\dot{q})$-space, and so obtain a Lagrangian, etc.

So the idea I have is that I should be able to invariantly define (t,q,p)-space with the form $\eta = p\,dq - H\,dt$.

The obvious candidate is to take the cotangent bundle of space-time and to cut down to a non-singular hypersurface. [redacted] Let us denote the canonical form by

$$E\,dt + p\,dq$$

and suppose the hypersurface is given by

$$H(t, q, p) + E = 0$$

(Universal property of the cotangent bundle $T^* \to X$: Given a submersion $f: Y \to X$ [redacted] and a 1-form η on Y vanishing for tangent vectors along the fibres of f, there is a unique map $Y \to T^*$ over X inducing η from the canonical form $p\,dq$. In effect in local coords (q, r) on Y one has

$$\eta = a\,dq + b\,dr$$

and $b=0$, since η vanishes on the fibres, so $\eta = a\,dq$ and the map $Y \to T^*$ is given by $p = a$)

An interesting point is that the above formulation for relativistic motion makes no reference to the metric on space-time. The only thing that matters is the hypersurface in the cotangent bundle. We shall need some examples to understand this [redacted] better. Suppose we [redacted] go back to motion in the electromagnetic field, or better, motion in a stationary magnetic field viewed non-relativistically.

Let the stationary magnetic field be described by the vector potential $A = A(q)$. The Lagrangian, etc. are

$$L = \frac{1}{2} m \dot{q}^2 + e A \cdot \dot{q}$$

$$p = m\dot{q} + eA$$

$$H = p\dot{q} - L = (m\dot{q} + eA)\dot{q} - \frac{1}{2} m \dot{q}^2 - eA\dot{q}$$

$$= \frac{1}{2} m \dot{q}^2 = \frac{1}{2m}\left(p - eA\right)^2$$

So on the cotangent bundle to 3-space, i.e. (q,p) space, we have the canonical form $\eta = p\,dq$ and the hypersurface

$$H(q,p) = \frac{1}{2m}(p - eA)^2 = \text{constant}$$

In the relativistic setting one expects similar formulas. Precisely one has the same Hamiltonian, so that

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{1}{m}(p - eA).$$

But the relativistic velocity has length 1, hence the hypersurface we work with is

$$\frac{1}{m^2}(p - eA)^2 = 1$$

or

$$H(q,p) = \frac{m}{2}$$

Note: In order to define $\dot{q}$ one needs H not just the hypersurface $H =$ constant. Since $\dot{q}^2 = 1$ this seems perhaps to single out a choice for H at least [redacted] near the hypersurface. [redacted]

For example suppose we consider the hypersurface $|p| = m$. Then

$$\dot{q} = \frac{\partial |p|}{\partial p} = \frac{p}{|p|}$$

will satisfy $|\dot{q}| = 1$. On the other hand if we take $H(q,p) = c|p|^\alpha$, then

$$\dot{q} = c\frac{\partial |p|^\alpha}{\partial p} = c\alpha |p|^{\alpha-1}\frac{p}{|p|}$$

will have

$$|\dot{q}| = c\alpha|p|^{\alpha-1} = \frac{\alpha}{|p|}c|p|^\alpha$$

$$= \frac{\alpha}{m}cm^\alpha$$

on the same hypersurface. So c, α, m can be adjusted in many ways to make this 1. The point is that at $|p| = m$ both $|p|$ and $c|p|^\alpha$ [redacted] have the same derivative.

Interesting calculation. Consider motion in a stationary magnetic field. Translating things to the tangent bundle we have the hypersurface $H = \frac{m}{2}$ is given by $\dot{q}^2 = 1$. The canonical 1-form is

$$\eta = p\,dq = (m\dot{q} + eA)dq$$

and the trajectories are obtained by looking for the kernel curves in $\dot{q}^2 = 1$ for

$$d\eta = m\,d\dot{q}\,dq + e\,dA\,dq$$

Notice that $A\,dq$ is the canonical interpretation of A as a 1-form, and $dA\,dq$ is the field B interpreted as a 2-form. What's striking about this is that B gives a closed 2 form Ω on the unit tangent bundle and that the choice of A is essentially the same as choosing η with $d\eta = \Omega$.

June 24, 1979

Yesterday we understood the way to think about particle motion: One has a manifold M called space-time and one is given a hypersurface in the cotangent bundle of M transversal to the fibres. If we describe this hypersurface in the form $H(q,p)=0$, then Hamilton's equations give the trajectories for the particles

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

(Here t denotes a "proper time" parameter which in general is not intrinsic.) Solutions of the Hamilton-Jacobi [redacted] equation

$$H\left(q, \frac{\partial S}{\partial q}\right) = 0$$

describe the geometric wave fronts. (Mathematically, one assumes that $\frac{\partial H}{\partial p} \neq 0$; this expresses transversality to the fibres. Then $\frac{dq}{dt} \neq 0$, so we get a definite tangent line in M.)

This is the model for particle motion, and [redacted] it seems to cover[redacted] relativistic motion, as well as Newtonian motion. From this viewpoint it is hard to understand why "the Lagrangian[redacted] formulation is [redacted] used in field theory instead of the Hamiltonian one because the latter is not Lorentz-invariant."

The next step is to understand quantization. There seem to be first + second quantizations. Take a simple example: Let's consider 2-diml space-time with coordinates (t, q) and suppose given an external electromagnetic field of compact support. Then

$$\eta = p_\mu dx^\mu = -E dt + p dq$$

$$A_\mu dx^\mu = -\varphi dt + A dq$$

Here φ, A are functions of t, q of compact support. The Hamiltonian is

$$\tilde{H} = \frac{1}{2m}(p_\mu - eA_\mu)(p^\mu - eA_\mu)$$

$$= \frac{1}{2m}\left[(-E + e\varphi)^2 - (p - eA)^2\right]$$

~~Maybe it would be easier to consider 2-diml space with a stationary magnetic field of compact support. Then~~

$$\cancel{\eta = p dq}$$

The hypersurface we want is $\tilde{H} = \frac{m}{2}$, i.e.

$$(E - e\varphi)^2 - (p - eA)^2 = m^2$$

or

$$E = e\varphi + \sqrt{(p - eA)^2 + m^2}$$

(As a check let's use this to define a usual Hamiltonian

$$H(t, q, p) = e\varphi + \sqrt{(p - eA)^2 + m^2}$$

Then

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p - eA}{\sqrt{(p - eA)^2 + m^2}} \quad \text{or}$$

$$p - eA = \frac{m\dot{q}}{\sqrt{1 - \dot{q}^2}} \quad \text{or} \quad p = \frac{m\dot{q}}{\sqrt{1 - \dot{q}^2}} + eA$$

which checks. The equation of motion is

$$\dot{p} = -\frac{\partial H}{\partial q} = -\left[e\frac{\partial \varphi}{\partial q} + \frac{p-eA}{\sqrt{(p-eA)^2+m^2}}\left(-e\frac{\partial A}{\partial q}\right)\right]$$

or

$$\frac{d}{dt}\left(\frac{m\dot{q}}{\sqrt{1-\dot{q}^2}}\right) = e\left[-\frac{\partial \varphi}{\partial q} - \frac{dA}{dt} + \dot{q}\frac{\partial A}{\partial q}\right] = e\Bigg[\underbrace{-\frac{\partial \varphi}{\partial q} - \frac{\partial A}{\partial t}}_{\text{electric field}}\Bigg]$$

Note: no magnetic effect for 1-dimensional motion)

The Euclidean analogue is to take Euclidean space with coordinates [redacted] q and a 1-form $A\,dq$. The Hamiltonian is

$$\tilde{H} = \frac{1}{2m}\left(p_\mu - eA_\mu\right)^2$$

and the hypersurface of interest is

$$\sum_\mu \left(p_\mu - eA_\mu\right)^2 = m^2$$

which leads to

$$\dot{q} = \frac{\partial \tilde{H}}{\partial p} = \frac{p-eA}{m}$$

so that $|\dot{q}| = 1$, i.e. trajectories are parameterized via arc-length.

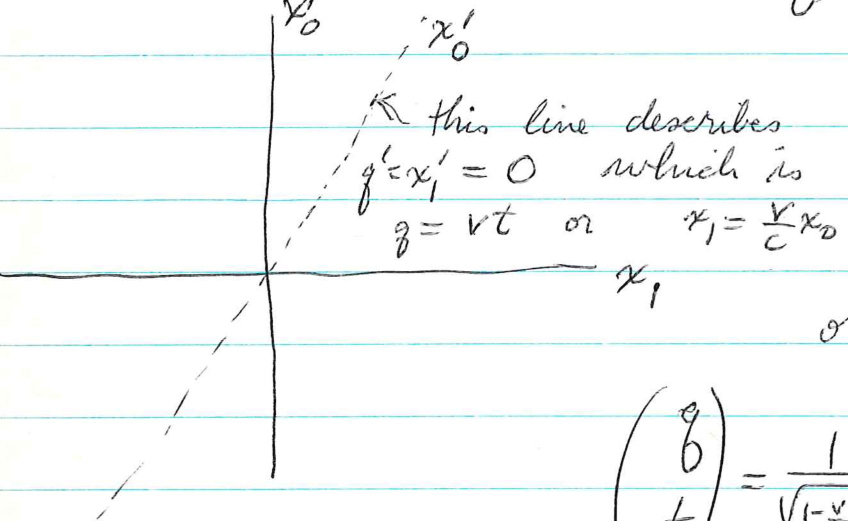
The problem is now how to quantize when you are given M and the hypersurface in T^*_M. Quantization consists of [redacted] some sort of wave theory with wavefronts described by the characteristic hypersurfaces. It would be nice to be able to use the [redacted] Hörmander theory, that is, to concoct a PDE whose characteristics are correct. However, one difficulty is that [redacted] characteristics for a PDE are homogeneous cones in T^*, not hypersurfaces like $|p^{[redacted]} - eA| = m$, which is a displaced sphere. So the question becomes whether there is a modification of characteristic theory which allows such hypersurfaces.

Question: Does the Hormander theory [redacted] allow one to quantize locally? So for example, given a symplectic manifold can I construct some sort of linear space on which functions act.

June 25, 1979

The program is [redacted] to understand the motion of a charged particle in an external electro-magnetic field from all possible angles. To keep things simple, we work with 2-space-time dimensions with coordinates $(t, q) = (x^\mu)$. The first thing to do is to put in the velocity of light so as to obtain the non-relativistic approximation.

One puts $(x^\mu) = (ct, q)$. The Lorentz transf. [redacted]



is

$$\begin{pmatrix} q \\ ct \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} q' \\ ct' \end{pmatrix}$$

or

$$\begin{pmatrix} q \\ t \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} q' \\ t' \end{pmatrix} \xrightarrow[c \to \infty]{\text{as}} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q' \\ t' \end{pmatrix}$$

The basic 1-form over [redacted] space-time is

$$\eta = -p_\mu dx^\mu = -E\,dt + p\,dq$$

hence $p_0 = +\frac{E}{c}$ and $p_1 = -p$. So we get the basic energy relation where m is the rest mass:

$$p_0^2 - p_1^2 = \frac{E^2}{c^2} \text{ [redacted] } p^2 = m^2 c^{\text{[redacted]} 2}$$

or

$$E^2 = c^2 p^2 + m^2 c^4$$

$$E = mc^2 \sqrt{1 + \frac{p^2}{m^2 c^2}} \sim mc^2 + \frac{p^2}{2m} + \cdots$$

To a free particle with energy-momentum vector $(p_\mu) = (\frac{E}{c}, -p)$ we associate the plane wave

$$\psi = e^{-\frac{i}{\hbar} p_\mu x^\mu} = e^{-\frac{i}{\hbar}(Et - pq)}$$

so that as usual the frequency/2π is $\omega = \frac{E}{\hbar}$ and the wave number is $k = \frac{p}{\hbar}$.

The "electromagnetic potential" 1-form is

$$\alpha = A_\mu dx^\mu = -\varphi dt + A dq$$

so that $A_\mu = (-\frac{\varphi}{c}, A)$. However in order to have a good limit as $c \to \infty$, we want the basic 1-form to be

$$\alpha = -\frac{1}{c} A_\mu dx^\mu = -\varphi dt + A dq$$

hence $A_\mu = (+\varphi, -cA)$. The reason is so that if we make the replacement $p_\mu \mapsto p_\mu - \frac{e}{c} A_\mu$, then the basic energy relation is

$$\left(\frac{E}{c} - \frac{e}{c} A_0\right)^2 - \left(-p - \frac{e}{c} A_1\right)^2 = m^2 c^2$$

$$E = eA_0 + \sqrt{c^2 (p + \tfrac{e}{c} A_1)^2 + m^2 c^4}$$

or

$$E = +e\varphi + mc^2 \sqrt{1 + \frac{(p - eA)^2}{m^2 c^2}}$$

$$\sim +e\varphi + \frac{(p - eA)^2}{2m} + mc^2 .$$

Derive the particle motion using Hamilton's equations with

$$H(t,q,p) = e\varphi + mc^2\left(1+\left(\frac{p-eA}{mc}\right)^2\right)^{1/2}$$

$$\dot q = \frac{\partial H}{\partial p} = mc^2\,\frac{1}{\cancel{2}}\left(1+\left(\frac{p-eA}{mc}\right)^2\right)^{-1/2}\cancel{2}\,\frac{p-eA}{m^2c^2}$$

$$\dot q = \left(1+\left(\frac{p-eA}{mc}\right)^2\right)^{-1/2}\frac{p-eA}{m}$$

so

$$\frac{\dot q}{c} = \left(\qquad\right)^{-1/2}\frac{p-eA}{mc} = \sin\theta \quad \text{where} \quad \tan\theta = \frac{p-eA}{mc}$$

hence

$$\frac{p-eA}{mc} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{\dot q/c}{\sqrt{1-(\dot q/c)^2}} \quad \text{and so}$$

$$\boxed{\frac{p-eA}{m} = \frac{\dot q}{\sqrt{1-(\dot q/c)^2}}} \quad \text{or} \quad p = \frac{m\dot q}{\sqrt{\quad}} + eA$$

$$\dot p = -\frac{\partial H}{\partial q} = -\left[e\frac{\partial\varphi}{\partial q} + mc^2\,\frac{1}{\cancel{2}}\left(1+\left(\frac{p-eA}{mc}\right)^2\right)^{-1/2}\cancel{2}\,\frac{p-eA}{m^2c^2}(-e)\frac{\partial A}{\partial q}\right]$$

$$\dot p = -e\frac{\partial\varphi}{\partial q} + e\dot q\frac{\partial A}{\partial q}$$

$$\dot p - \frac{d}{dt}(eA) = -e\frac{\partial\varphi}{\partial q} + e\left(\dot q\frac{\partial A}{\partial q} - \frac{dA}{dt}\right)$$

or

$$\boxed{\frac{d}{dt}\left(\frac{m\dot q}{\sqrt{1-(\dot q/c)^2}}\right) = -e\left(\frac{\partial\varphi}{\partial q}+\frac{\partial A}{\partial t}\right)} \quad \text{relativistic}$$

If we let $c \to \infty$, we get simply

$$\boxed{m\ddot q = -e\left(\frac{\partial\varphi}{\partial q}+\frac{\partial A}{\partial t}\right)} \quad \text{non-relativistic}$$

We see that the particle motion depends only on

$$d(-\varphi\,dt + A\,dq) = \left(\frac{\partial\varphi}{\partial q}+\frac{\partial A}{\partial t}\right)dt\,dq.$$

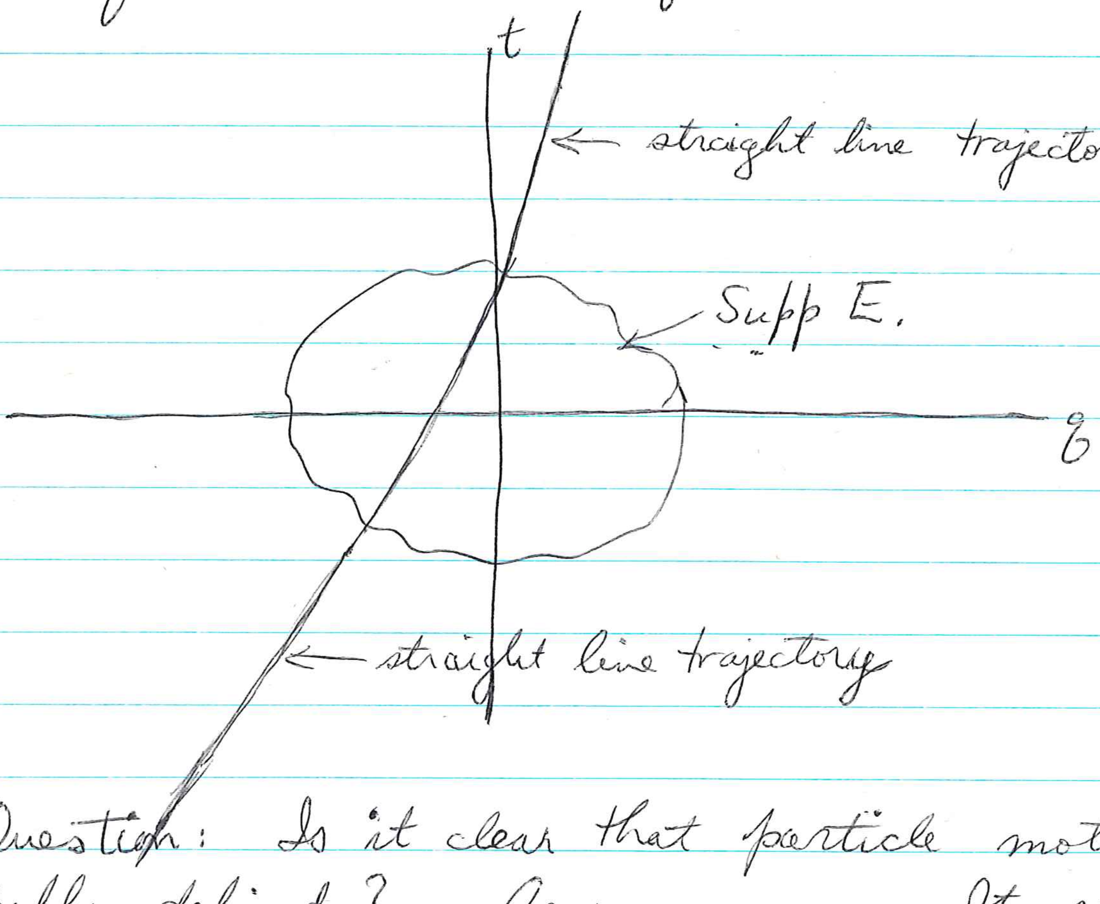
Changing the form $-\varphi\,dt + A\,dq$ by $d\chi$ is called a "gauge" transformation, and it not supposed to alter things in any essential way. In this one space-dimension situation, [redacted] the only thing that matters for the particle motion is the electric field

$$E(t,q) = -\frac{\partial\varphi}{\partial q} - \frac{\partial A}{\partial t}$$

which we assume to be of compact support. Outside of Supp E, the trajectories are straight lines

$$q = \dot{q}t + q_0$$

with $\dot{q}$ constant and $|\dot{q}| < c$.



Question: Is it clear that particle motion is globally defined? Assume so. It should follow from the fact the force $F = eE$ is bounded and [redacted] hence

$$\frac{dE}{dt} = [redacted]\ Fv$$

is bounded (because $|v| < c$), hence the energy change over the finite time E acts is bounded, so $E = \frac{mc^2}{\sqrt{1-(\frac{v}{c})^2}}$ remains finite, which keeps $|v|$ away from c. (Review the formulas: [redacted]

[redacted] $$E = \frac{mc^2}{\sqrt{1-(\frac{v}{c})^2}} \qquad p = \frac{mv}{\sqrt{1-(\frac{v}{c})^2}}$$

so $$\left(\frac{E}{c}\right)^2 - p^2 = \frac{m^2c^2 - m^2v^2}{1-(\frac{v}{c})^2} = m^2c^2$$

so $$E^2 = c^2p^2 + m^2c^4$$

$$E\frac{dE}{dt} = c^2 p\frac{dp}{dt} \qquad \frac{dE}{dt} = \frac{c^2p}{E}\frac{dp}{dt} = vF.)$$

It[redacted] follows that the classical scattering operator is well-defined. It associates to any line coming [redacted] from $t \ll 0$ an outward going line for $t \gg 0$.