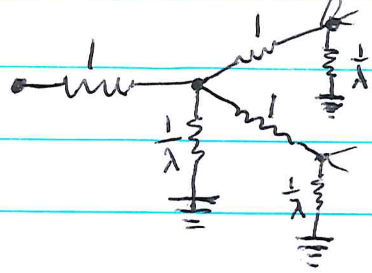
February 14, 1978:

Consider again $\Delta u = \lambda u$ on the modular tree or rather a branch of it



The impedance calculation done yesterday amounts to looking for a solution depending exponentially on the distance from the left vertex, i.e.

$$u = \alpha^d$$

It is, so to speak, a radial solution, what one gets by using the retraction onto an apartment associated to a chambre. The condition that this be a solution is

$$1 + 2\alpha^2 - 3\alpha = \lambda\alpha$$

and for it to be an ℓ^2 solution means that

$$1^2 + |\alpha|^2 + 2|\alpha|^4 + 4|\alpha|^6 + \cdots < \infty$$

i.e. $2|\alpha|^2 < 1$ or $|\alpha| < \frac{1}{\sqrt{2}}$. So if we rewrite the above

$$\frac{1}{\alpha} - 3 + 2\alpha = \lambda$$

or $$\frac{\sqrt{2}}{(\sqrt{2}\alpha)} - 3 + \sqrt{2}(\sqrt{2}\alpha) = \lambda \qquad \frac{1}{2}\left\{(\sqrt{2}\alpha) + \frac{1}{(\sqrt{2}\alpha)}\right\} = \frac{3+\lambda}{2\sqrt{2}}$$

we see the unit circle $|\sqrt{2}\alpha| < 1$ gets mapped isomorphically onto the region in the λ plane which is the complement of the slit $-1 \leq \frac{3+\lambda}{2\sqrt{2}} \leq 1$.

For each λ outside the slit we can construct the corresponding Green's function with unit source at y

$$G(x,y,\lambda) = \text{const}\cdot \alpha^{d(x,y)}$$

where the constant c is chosen so that

$(\Delta - \lambda) G(\,,y,\lambda) = -\delta_y$, hence

$$c(3\alpha - 3 - \lambda) = -1$$

$$c = \frac{-1}{3\alpha - \left(\frac{1}{\alpha} + 2\alpha\right)} = \frac{-1}{\alpha - \frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha} - \alpha}$$

Check: Take $\lambda = 0$ whence $\alpha = \frac{1}{2}$ and $c = \frac{1}{2 - \frac{1}{2}} = \frac{2}{3}$.

So

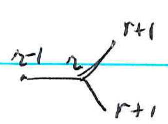
$$G(x,y,\lambda) = \frac{1}{\frac{1}{\alpha} - \alpha}\, \alpha^{d(x,y)}$$

This should be the kernel for the operator $(\lambda - \Delta)^{-1}$. However, at the moment I don't know if this kernel gives a bounded operator on ℓ^2, although this seems likely.

Spherical functions: Fix a vertex O and consider functions on the vertices which are radial, i.e. depend only on the distance r from O. Let f be a radial eigenfunction for Δ with eigenvalue λ. Then at O we have

$$\Delta f(0) = 3f(1) - 3f(0) = \lambda f(0)$$

and at other values of r we have



$$f(r-1) + 2f(r+1) - 3f(r) = \lambda f(r)$$

One sees from these equations that there is a unique radial eigenfunction with given value $f(0)$ [redacted] and eigenvalue λ. Moreover it has the form

$$f(r) = c_1 \alpha_1^r + c_2 \alpha_2^r$$

where α_1, α_2 are the roots of the characteristic equation.

$$\alpha^{-1} + 2\alpha = (\lambda+3)$$

Now [redacted] it should be clear that bounded eigenfunctions, in fact polynomial growth eigenfunctions, do not exist except for λ in the cut $[-3-2\sqrt{2}, -3+2\sqrt{2}]$. For if we had an eigenfunction $u \neq 0$, choose the origin to be a point where $u \neq 0$, then average over the compact group of autos. of the tree preserving O and you get a radial eigenfunction $f(r) \neq 0$ with polynomial growth. But there can't be any [redacted] if λ is off the cut, but one would have to have $f(r) = c_2 \alpha_2^r$ where α_2 is the small root (in modulus $< \frac{1}{\sqrt{2}}$). This would mean that

$$(\lambda+3) f(0) = (\lambda+3) c_2$$ [redacted] and also

$$(\lambda+3) f(0) = 3 f(1) = 3 c_2 \alpha_2$$

so $\alpha_2 = \lambda + 3$ [redacted] $= \alpha_2^{-1} + 2\alpha_2$ or

$$-\alpha_2 = \alpha_2^{-1} \qquad \alpha_2^2 = -1 \qquad \text{or } |\alpha_2| = 1$$

which is impossible.

Consider $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ on the UHP. We have

$$y^{-1/2}\left(\Delta + \tfrac{1}{4}\right) y^{1/2} = y^2 \frac{\partial^2}{\partial x^2} + y^2\left(\frac{\partial}{\partial y} + \frac{1/2}{y}\right)^2 + \frac{1}{4} = y^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y}$$

$$= y^2\frac{\partial^2}{\partial x^2} + \left(y\frac{\partial}{\partial y}\right)^2$$

Thus
$$\Delta + \frac{1}{4} = y^2\frac{\partial^2}{\partial x^2} + y^{1/2}\cdot\left(y\frac{\partial}{\partial y}\right)^2\cdot y^{-1/2}$$
so

$$\left(\left(\Delta+\tfrac{1}{4}\right)u, u\right) = \int \left(y^2\frac{\partial^2 u}{\partial x^2} + y^{1/2}\left(y\frac{\partial}{\partial y}\right)^2\left(y^{-1/2}u\right)\right) u \,\frac{dx\,dy}{y^2}$$

$$= \int \frac{\partial^2 u}{\partial x^2}\, u\, dx\,dy + \int \left(y\frac{\partial}{\partial y}\right)^2\left(y^{-1/2}u\right)\cdot\left(y^{-1/2}u\right)\frac{dx\,dy}{y}$$

If u has compact support we can integrate by parts to get

$$= -\int\left(\frac{\partial u}{\partial x}\right)^2 dx\,dy - \int\left[\left(y\frac{\partial}{\partial y}\right)\left(y^{-1/2}u\right)\right]^2\frac{dx\,dy}{y} \qquad y\frac{\partial}{\partial y}\, y \cdot \frac{1}{y}$$

$$\leq 0$$

Consequently $\Delta + \frac{1}{4} \leq 0$.

Formula for spherical functions in the UHP case:
Recall that in geodesic polar coordinates the Laplacian for the upper half plane is

$$\Delta = \frac{1}{\sinh(r)}\frac{\partial}{\partial r}\left(\sinh r\frac{\partial}{\partial r}\right) + \frac{1}{\sinh^2 r}\frac{\partial^2}{\partial\theta^2}$$

A radial eigenfunction satisfies the DE

$$\frac{1}{\sinh r}\frac{d}{dr}\left(\sinh r\frac{du}{dr}\right) = \lambda u$$

This should be compared with

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \lambda u$$

which is Bessel's DE of order 0 essentially: Put $\lambda = -k^2$, so

$$\left(r\frac{d}{dr}\right)^2 u + k^2 r^2 u = 0.$$

The only solution regular at $r=0$ is $J_0(kr)$ up to scalar factors. The same should be true for the $\sinh r$ case so one sees there is a unique-up-to-scalar-factors radial eigenfunction.

To find it we can start with the eigenfunction y^s for Δ and average it over the rotation group

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^* y = \frac{y}{|\cos\theta - z\sin\theta|^2}$$

hence we are interested in

$$y^s \int_0^\pi |\cos\theta - z\sin\theta|^{-2s}\, d\theta$$

Restrict this to the y axis and recall $y = e^r$ on this axis.

$$y^s \int_0^\pi \left[(\cos\theta)^2 + y^2(\sin\theta)^2\right]^{-2s} d\theta$$

$$= \text{const} \int_0^\pi \left[\frac{1}{y}\frac{(1+\cos 2\theta)}{2} + y\frac{(1-\cos 2\theta)}{2}\right]^{-2s} d\theta$$

$$= \text{const} \int_0^{2\pi} (\cosh r - \sinh r \cos\theta)^{-2s}\, d\theta$$

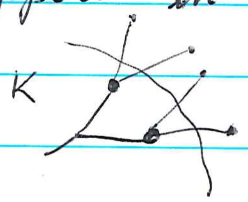
which agrees with what's in Helgason's book except for the exponent $-2s$.

February 15, 1978

X the tree, K finite subtree. One has an exact sequence in real cohomology

$$\begin{array}{ccccccc} H^0(X) & \longrightarrow & H^0(X-K) & \xrightarrow{\delta} & H^1(X,X-K) & \longrightarrow & H^1(X) \\ \| & & \| & & & & \| \\ \mathbb{R} & & Map(\pi_0(X-K),\mathbb{R}) & & & & 0 \end{array}$$

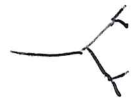
$H^1(X,X-K)$ is constructed out of 1-cochains on X with support in the set of 1-simplices not in $X-K$, i.e. which



touch K. Given such a cochain α, since X is contractible we get $\alpha = \delta f$ where $f \in C^0(X)$ is unique up to additive constants. Since $\delta f = \alpha$ vanishes on $X-K$, f is locally constant on $X-K$, hence constant on each component.

Conversely given $f \in H^0(X-K) \subset C^0(X-K)$ we can extend f to a function $\tilde{f}$ on the vertices of K, say by 0 for example. Then $\alpha = \delta\tilde{f}$ is a 1-cochain with support off $X-K$ whose class depends only on f. $\tilde{f}$ is unique up to functions on the vertices of K and it is natural in the L^2-context to minimize $\|\delta\tilde{f}\|^2$. If $\tilde{f}$ is the minimum then $(\delta\tilde{f},\delta g)=0$ for all $g \in C^0(K)$ and so $\delta^*\delta\tilde{f}=0$. Thus we are solving the Dirichlet problem, i.e. finding a function $\tilde{f}$ on $K \cup \partial K$, $\partial K =$ vertices joined to K by edges, such that $\Delta\tilde{f}=0$ in the interior. Physically we can think of f on ∂K as being applied external voltages, and then $\tilde{f}$ is the resulting voltage internally.

Now we want to let K expand but keeping $f \in H^0(X-K)$ fixed. In the limit I should get a harmonic function $\tilde{f}$ on the vertices such that $\tilde{f} \sim f$. But I have already analyzed the voltage distribution on a branch of the tree



which tends to zero as we go $\longrightarrow$, and I found the branch to have a resistance of 2 ohms. So it's more or less clear that to find the limiting voltage distribution at a point inside K I replace each branch issuing from K by a resistance of 2 ohms in series with a voltage given by the value of f on that branch.

In this manner it seems possible to associate to each locally constant function f on the space ∂X of ends of X a harmonic function $\tilde{f}$ on X having f as its boundary values. In other words we can solve the Dirichlet problem for locally constant boundary values.

Denote by $C_2^i(X) \subset C^i(X)$ the subspace of L^2-cochains. Assuming Δ bounded away from zero on $C_2^0(X)$, I would know that $\delta: C_2^0(X) \longrightarrow C_2^1(X)$ has a closed range so that the cokernel

$$H_2^1(X) = [redacted]\ C_2^1(X)/\delta C_2^0(X)$$

is a Hilbert space. Here's a simple proof that the canonical map

$$H_c^1(X) \longrightarrow H_2^1(X)$$

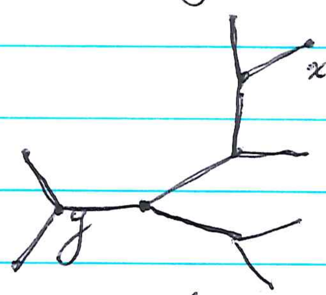
is injective. Let $\alpha \in C_c^1(X)$ be an l^2-coboundary $\alpha = \delta g$ with $g \in C_2^0(X)$. Then g is locally constant [redacted] far out, hence as it is square integrable, it must be zero far out, hence $g \in C_c^0(X)$, and so α represents 0.

~~Because $C_c^1(X)$ is dense in $C_2^1(X)$~~

February 16, 1978:

Yesterday I saw that given a locally constant function f on ∂X one could solve the Dirichlet problem: find a harmonic $\tilde{f}$ on X such that $\tilde{f}$ has the boundary value f. This should imply that for each vertex y there is a Poisson measure μ_y on ∂X associated to y, that is, which represents $f \longmapsto \tilde{f}(y)$.

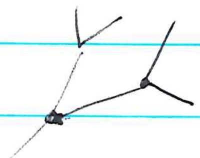
By rotational symmetry around y it's clear that since μ_y is a probability measure (the constant functions are harmonic) that for any x the subset of ends



on the other side of x from y should have measure $= 1/\text{number of vertices at distance } d(y,x) \text{ from } y$

$$= \frac{2}{3 \cdot 2^{d(y,x)}}$$

We can prove this by showing that for any harmonic u, $u(y) = $ average of $u(x)$ as x runs over the circle C_d of radius d. Clear for $d=1$. Next observe that for $d \geq 2$



$$\sum_{d(x,y)=d+1} u(x) + 2 \sum_{d(x,y)=d-1} u(x) = 3 \sum_{d(x,y)=d}' u(x)$$

so that

$$\frac{1}{2} \sum_{d(x,y)=d+1}' u(x) - \sum_{d(x,y)=d} u(x) + \underbrace{\sum_{d(x,y)=d-1}' - \frac{1}{2} \sum_{d(x,y)=d}}_{\text{zero by induction}} = 0$$

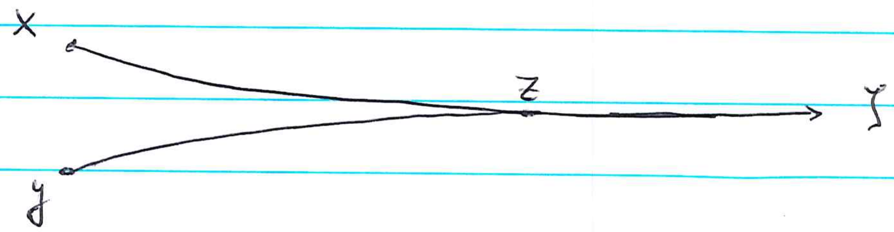
So by induction if we know the averages over C_{d-1}, C_d are the same we can get the average of C_{d+1} to be the same, this for $d \geq 2$. Finally you should check for $d = 1$.

$$\underbrace{-3\sum_{C_1} u}_{3u(y)} + \sum_{C_2} u + 3\underbrace{\sum_{C_0} u}_{u(y)} = 0 \qquad \Rightarrow \sum_{C_2} u = 6\,u(0)$$

OK.

Finally let us fix an origin O, whence we get a definite measure $d\mu_0$ on ∂X, and then compute the other Poisson measures $d\mu_y$ in terms of $d\mu_0$ and a function on ∂X.

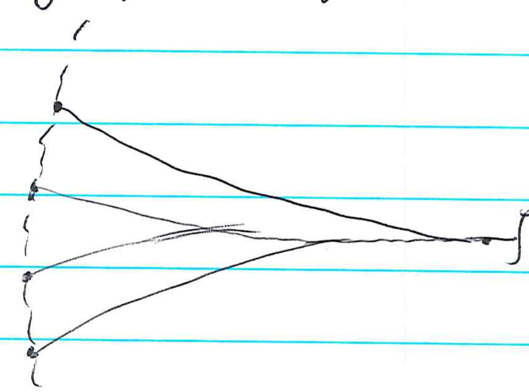
Let ξ be an end and O an origin. It makes sense to talk about vertices equally distant from ξ:



In want we can define a function

$$h_\xi(x,y) = \lim_{z \to \xi} d(x,z) - d(y,z)$$

Then you get the following picture for vertices equally distant from ξ



Moreover we get a harmonic function:

$$u_\xi(x) = \frac{1}{2^{b_\xi(x,0)}} = \lim_{z\to\xi} \frac{2^{d(0,z)}}{2^{d(x,z)}}$$

The choice of origin suffices to normalize $u_\xi(x)$ to be 1 at $x=0$.

Classical version: The harmonic function $y = \operatorname{Im}(z)$ in the UHP when considered on the disk $|w|<1$ via the transformation

$$w = \frac{z-i}{z+i} = \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}(z) \qquad z = \begin{pmatrix} i & +i \\ -1 & 1 \end{pmatrix}(w) = i\,\frac{w+1}{-w+1}$$

becomes

$$\operatorname{Im}\left(i\,\frac{1+w}{1-w}\right) = \operatorname{Re}\left(\frac{1+w}{1-w}\right) = \frac{1-|w|^2}{|1-w|^2}$$

This blows up at $w=1$ but vanishes at other points of $|w|=1$. Rotated so that the singularity occurs at ξ it becomes

$$\frac{1-|w|^2}{|\xi-w|^2}$$

The Poisson measure on $|w|=1$ belonging to $w=0$ is $\frac{d\theta}{2\pi} = \frac{dw}{2\pi i w}$. Suppose we transform it via $\begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix}$:

$|a|^2-|b|^2=1$

$$\begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix}^* \frac{dw}{2\pi i w} = \frac{1}{2\pi i}\left(\frac{aw+b}{\bar b w+\bar a}\right)^{-1} \frac{dw}{(\bar b w+\bar a)^2}$$

$$= \frac{1}{2\pi i}\,\frac{dw}{(aw+b)(\bar b w+\bar a)} = \frac{1}{2\pi i}\,\frac{e^{i\theta}\,i\,d\theta}{(ae^{i\theta}+b)(\bar b e^{i\theta}+\bar a)}$$

$$\frac{1}{|a|^2} = 1-\left|\frac{b}{a}\right|^2 \qquad = \frac{1}{2\pi}\,\frac{d\theta}{|ae^{i\theta}+b|^2} = \frac{d\theta}{2\pi}\,\frac{\left(1-\left|-\frac{b}{a}\right|^2\right)}{\left|e^{i\theta}+\frac{b}{a}\right|^2}$$

Hence the Poisson measure belonging to the point $w = w_0 = -\frac{b}{a}$ is

$$\frac{1-|w_0|^2}{|e^{i\theta}-w_0|^2}\,\frac{d\theta}{2\pi}$$

So it maybe it's clear that the Poisson kernel for the tree X with origin O is

$$\frac{1}{2^{h_\xi(x,O)}}\,d\mu_0(\xi).$$

There seems to be an interesting inner product on $H_c^1(X)$ namely the one obtained from the embedding

$$H_c^1(X) \hookrightarrow H_2^1(X)$$

Note that because C_c^1 is dense in $C_2^1(X)$, it follows the the above embedding is dense. The inner product can be described as follows. Given f locally constant on ∂X, let $\tilde{f}$ be its harmonic extension. Then $\|f\|^2 = \|\delta\tilde{f}\|^2$. Put another way, you replace a 1-cochain with compact support by its harmonic equivalent and you take the norm.

Return to Δ on UHP. Using the Dirichlet problem we can identify smooth functions on S^1 with harmonic functions on the closed disk. Mod-ing by constants this should be an irreducible representation of $PSL_2(\mathbb{R})$. Hence there should be a unique invariant inner product up to scalars.

Because we've seen that $\Delta \leq -\frac{1}{4}$ on $L^2(\text{UHP})$ the irreducible representation of functions on S^1 described above does not occur as an eigenspace of Δ. In other words if $L^2(\text{UHP})$ is written as an integral of irreducible representations then only $\lambda = s(s-1)$ occurs for $\lambda \leq -\frac{1}{4}$, i.e. $s \in \frac{1}{2} + i\mathbb{R}$. It seems that that the space of harmonic functions on the disk with Dirichlet norm mod constants is the irreducible representation belonging to $s=1$, whereas to $s=0$ belongs the trivial representation.

February 17, 1978

Dirichlet norm: On a Riemann surface let ω be a 1-form. Then we get a 1-form $\bar{\omega}$ by conjugating and a 2-form $\omega \wedge \bar{\omega}$ which can be integrated to get a number. For example if $\omega = f dz$ is of type $(1,0)$ then $\bar{\omega} = \bar{f}\, d\bar{z}$ and

$$\omega \wedge \bar{\omega} = |f|^2 dz\, d\bar{z} = |f|^2 \{(dx+idy)(dx-idy)\}$$
$$= |f|^2 (-2i)\, dx\, dy$$

so $\int \frac{i}{2}\, \omega \wedge \bar{\omega}$ is an intrinsic norm for forms of type $(1,0)$.

Take $\omega = \partial f = \frac{\partial f}{\partial z} dz$. Then

$$\left|\frac{\partial f}{\partial z}\right|^2 = \left|\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\right|^2 = \frac{1}{4}\left\{\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 + i\frac{\partial f}{\partial x}\frac{\partial \bar{f}}{\partial y} - i\frac{\partial f}{\partial y}\frac{\partial \bar{f}}{\partial x}\right\}$$

$$\left|\frac{\partial f}{\partial \bar{z}}\right|^2 =$$

Consequently for a function f on the Riemann surface the Dirichlet integral

$$\int\!\!\int \left\{ \left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 \right\} dx\,dy$$

is invariant for biholomorphic maps, because it can be expressed in [redacted] terms of

$$\int \partial f \wedge \overline{\partial f} + \int \bar\partial f \wedge \partial \bar f$$

So now I consider all smooth functions on S^1 as a space on which $G = PSL_2(\mathbb{R})$ acts. Then to each f on S^1 we can associate the unique harmonic extension $\tilde f$ of f to the disk and take its Dirichlet norm. In this way one gets an inner product which is G-invariant on the space of smooth functions on the circle modulo constants.

Notice that on S^1 functions and densities (i.e. 1-forms as [redacted] the orientation is preserved) are dual. Hence the inner product maybe gives a method of associating to [redacted] an f mod constants a density dg of measure 0. It seems what I should look for is an operator L [redacted] on the space of smooth functions mod constants (perhaps the Hilbert transform) such that the norm I am after is $\|f\|^2 = \int_{S^1} f\, dLf$.

Suppose u harmonic in the [redacted] UHP with finite Dirichlet integral and that u is real-valued.

$$\int\!\!\int \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} dx\,dy = \int\!\!\int d\left(u\frac{\partial u}{\partial x}dy - u\frac{\partial u}{\partial y}dx\right) - \int\!\!\int u \underbrace{\Delta u}_{=0}\, dx\,dy$$

$$= \int_{x=-\infty}^{\infty} u\frac{\partial u}{\partial x}dy - u\frac{\partial u}{\partial y}dx = -\int_{-\infty}^{\infty} u\frac{\partial u}{\partial y}dx$$

Now if v is a conjugate harmonic function to u, i.e. $u+iv$ is analytic, then Cauchy-Riemann equations give

$$\frac{\partial(u+iv)}{\partial(iy)} = \frac{\partial(u+iv)}{\partial x} \quad \text{or}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So we get

$$\iint\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} dxdy = \int_{-\infty}^{\infty} u\, dv$$

where v is "the" conjugate harmonic function to u.

Let's understand the operator $T: u \longrightarrow v$ on S^1!

If f is analytic, with $f = u+iv$, u, v real

$$f(z) = \sum_{n\geq 0} a_n z^n \qquad \bar{f}(z) = \sum_{n\geq 0} \bar{a}_n z^{-n} \quad \text{on } S^1$$

then

$$u = \frac{f+\bar{f}}{2} = \sum_{n\geq 0} \frac{a_n}{2} z^n + \sum_{n\leq 0} \frac{\bar{a}_{-n}}{2} z^n$$

$$v = \frac{f-\bar{f}}{2i} = \sum_{n\geq 0} \frac{1}{i}\frac{a_n}{2} z^n + \sum_{n\leq 0} i\,\frac{\bar{a}_{-n}}{2} z^n$$

Hence in general given

$$u(z) = \sum_{n\in\mathbb{Z}} c_n z^n$$

then

$$T(u) = \sum_{n>0} \frac{1}{i} c_n z^n + \sum_{n<0} i c_n z^n$$

which is well-defined modulo constants. Also it makes sense even when u is not real. Clearly T is a unitary operator in $L^2(S^1, \frac{d\theta}{2\pi})/\mathbb{C}$ with $T^2 = -1$.

Note that because $T^2 = -1$, T will be unitary with respect to any inner product for which its $+i$ and $-i$ eigenspaces are orthogonal.

Real line version. Let $u(x)$ be a smooth function of rapid descent on $\mathbb{R}$ and $\hat{u}(\xi)$ its Fourier transform:

$$u(x) = \int_{-\infty}^{\infty} e^{ix\xi}\,\hat{u}(\xi)\,\frac{d\xi}{2\pi}$$

Define its Hilbert transform by

$$Tu(x) = i\int_{-\infty}^{0} e^{ix\xi}\hat{u}(\xi)\,\frac{d\xi}{2\pi} + \frac{1}{i}\int_{0}^{\infty} e^{ix\xi}\hat{u}(\xi)\,\frac{d\xi}{2\pi}$$

Note that

$$u(x) + iTu(x) = 2\int_0^{\infty} e^{ix\xi}\hat{u}(\xi)\,\frac{d\xi}{2\pi}$$

is analytic for $Re(\xi) > 0$ and it decays exponentially as $Im(\xi) \to +\infty$. T is a singular integral operator of order 0.

Formulas for the Hilbert transform: First we want the formulas giving $f = u + iTu$. On the circle

$$u = \sum_{n\in\mathbb{Z}} c_n e^{in\theta} \qquad c_{-n} = \bar{c}_n \quad \text{if } u \text{ real}$$

$$f = c_0 + \sum_{n\geq 1} 2c_n z^n = \int_0^{2\pi} u(e^{i\theta})\left\{1 + 2\sum_{n=1}^{\infty} e^{-in\theta}z^n\right\}\frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} u(e^{i\theta})\,\frac{1+e^{-i\theta}z}{1-e^{-i\theta}z}\,\frac{d\theta}{2\pi}$$

On the line

$$f(z) = 2\int_0^\infty e^{iz\xi}\left(\int_{-\infty}^{\infty} e^{-i\hat{x}\xi}\, u(\hat{x})\,d\hat{x}\right)\frac{d\xi}{2\pi}$$

$$= \int_{-\infty}^{\infty} \hat{u}(x)\left\{2\int_0^\infty e^{i(z-\hat{x})\xi}\,\frac{d\xi}{2\pi}\right\}dx \qquad \text{for } \operatorname{Im} z > 0$$

$$= \int_{-\infty}^{\infty} \hat{u}(x)\left\{\frac{1}{\pi i}\,\frac{-1}{z-\hat{x}}\right\}dx = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{u(\hat{x})\,d\hat{x}}{z-\hat{x}}$$

Check: assuming u real

$$\operatorname{Re} f = \frac{1}{\pi}\int_{-\infty}^{\infty} u(\hat{x})\,d\hat{x}\left(\frac{\operatorname{Re}\left(+i(\bar{z}-\hat{x})\right)}{|z-\hat{x}|^2}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} u(\hat{x})\frac{y}{(x-\hat{x})^2+y^2}\,d\hat{x}$$

$$\operatorname{Im} f = \frac{1}{\pi}\int_{-\infty}^{\infty} u(\hat{x})\,\frac{x-\hat{x}}{(x-\hat{x})^2+y^2}\,d\hat{x}$$

Thus

$$Tu(x) = \lim_{y\to 0^+}\frac{1}{\pi}\int_{-\infty}^{\infty} u(\hat{x})\,\frac{x-\hat{x}}{(x-\hat{x})^2+y^2}\,d\hat{x}$$

$$= \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{u(\hat{x})\,d\hat{x}}{x-\hat{x}}$$

where P denotes the Cauchy principal value defined as follows: No problem with the definition if $u(x) = 0$ for then $u(\hat{x})$ is divisible by $x-\hat{x}$. For constant functions you define the P value to be zero. The good definition for differentiable u is

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}$$

Be careful: $f(z)$ is defined only for $\operatorname{Im} z > 0$ but will have an analytic continuation if u is analytic. The integral

$$\frac{i}{\pi}\int \frac{u(\hat{x})\,d\hat{x}}{z-\hat{x}}$$

will not represent this analytic continuation. Let us take f to be what is given by the integral. Then we have $\overline{f(z)} = -f(\bar{z})$ so that on approaching the x-axis we have

$$f^+(x) = u(x) + iTu(x)$$

$$f^-(x) = -u(x) + iTu(x)$$

Also we have

$$\frac{\partial f}{\partial x}(z) = f'(z) = \frac{1}{\pi i}\int \frac{u(\hat{x})\,d\hat{x}}{(z-\hat{x})^2}$$

and

$$\frac{\partial f^+}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial Tu}{\partial x}$$

$$\frac{\partial f^-}{\partial x} = -\frac{\partial u}{\partial x} + i\frac{\partial Tu}{\partial x}$$

since

$$\int_{-\infty}^{\infty} u\frac{\partial u}{\partial x}\,dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)dx = 0$$

it follows that we can evaluate from either side:

$$\int_{-\infty}^{\infty} u\frac{\partial}{\partial x}(Tu)\,dx = \int_{-\infty}^{\infty} u\,dx\,\frac{\partial f^{\pm}}{\partial x}\,\frac{1}{i}$$

$$= \lim_{y\to 0}\int_{-\infty}^{\infty} u(x)\,dx\int_{-\infty}^{\infty}\frac{u(\hat{x})\,d\hat{x}}{(x+iy-\hat{x})^2}\left(-\frac{1}{\pi}\right)$$

$$\text{“}=\text{”} -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x)\,u(\hat{x})\,dx\,d\hat{x}}{(x-\hat{x})^2}$$

The sign is strange.

Circle approach. Different approach to the Dirichlet norm: On a Riemann surface define $*$ on the real cotangent bundle to be the transpose of multiplication by $\frac{1}{i}$ on the tangent bundle. Thus

$$*\,dx = d\,\mathrm{Re}\,\tfrac{1}{i}z = dy$$
$$*\,dy = d\,\mathrm{Im}\,\tfrac{1}{i}z = -dx$$

Extend $*$ conjugate linearly to complex 1-forms. Then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$
$$*du = \frac{\partial \bar u}{\partial x}dy - \frac{\partial \bar u}{\partial y}dx$$
$$du \wedge *du = \left\{\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2\right\}dx\,dy$$

and so $\iint du \wedge *du$ is the Dirichlet norm. Let's use this in polar coordinates:

$$\begin{cases} *\,dr = r\,d\theta \\ *\,r\,d\theta = -dr \end{cases}$$

$$du = \frac{\partial u}{\partial r}dr + \frac{1}{r}\frac{\partial u}{\partial \theta}r\,d\theta$$
$$*du = \frac{\partial \bar u}{\partial r}r\,d\theta + \frac{1}{r}\frac{\partial \bar u}{\partial \theta}(-dr)$$

$$\iint du \wedge *du = \iint\left\{\left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2\right\}r\,dr\,d\theta$$

Suppose we're given a harmonic function (say real-valued)

$$u = \sum_{n>0} r^n(c_n e^{in\theta} + \bar c_n e^{-in\theta}) + \text{a real const}$$

The conjugate harmonic function is

$$v = \sum_{n>0} r^n\left(-ic_n e^{in\theta} + i\bar{c}_n e^{-in\theta}\right) + \text{real const}$$

I expect the Dirichlet norm of u to be

$$\iint \nabla u \cdot \nabla u \, dV = \iint \Big(\nabla\cdot(u\nabla u) - u\underset{\underset{0}{\shortparallel}}{\Delta u}\Big)\, dV$$

$$= \oint u \nabla u \cdot \hat{n}\, ds \qquad n \text{ outward normal}$$

$$= \int_0^{2\pi} u \frac{\partial u}{\partial r}\, d\theta \qquad \text{Cauchy-Riemann: } \frac{\partial u}{\partial r} = \underset{\underset{1}{\shortparallel}}{\frac{1}{r}} \frac{\partial v}{\partial \theta}$$

$$= \int_0^{2\pi} u \frac{\partial v}{\partial \theta}\, d\theta$$

$$\frac{\partial v}{\partial \theta} = \sum_{n>0} \underset{\underset{1}{\shortparallel}}{r^n}\left(nc_n e^{in\theta} + n\bar{c}_n e^{-in\theta}\right)$$

hence

$$\int_0^{2\pi} u \frac{\partial v}{\partial \theta}\, d\theta = 4\pi \sum_{n>0} n|c_n|^2 = \text{Dirichlet norm of } u.$$

Also

$$\frac{\partial u}{\partial r} = \sum_{n>0} n r^{n-1}\left(c_n e^{in\theta} + \bar{c}_n e^{-in\theta}\right)$$

$$\int_0^1 r\,dr \int_0^{2\pi} \left(\frac{\partial u}{\partial r}\right)^2 d\theta = \int_0^1 r\,dr \cdot 2\pi\, n^2 r^{2n-2}\left(|c_n|^2 + |c_n|^2\right)$$

$$= 4\pi n^2 |c_n|^2 \int_0^1 r^{2n-1}\, dr = 2\pi n |c_n|^2$$

$$\frac{\partial u}{\partial \theta} = \sum_{n>0} r^n\left(inc_n e^{in\theta} - in\bar{c}_n e^{-in\theta}\right)$$

$$\int_0^1 r\,dr \frac{1}{r^2}\int \left(\frac{\partial u}{\partial \theta}\right)^2 d\theta = \int_0^1 dr \frac{1}{r}\sum_{n>0} r^{2n} \cdot 2n^2 |c_n|^2 \cdot 2\pi = 2\pi n |c_n|^2$$

so we get again $4\pi \sum_{n>0} n|c_n|^2$ for $\iint du * du$.

February 18, 1978:

If $u = \sum' c_n e^{in\theta}$ is a function on S^1, then its unique harmonic extension to the disk is $u = \sum c_n r^{|n|} e^{in\theta}$.

$$\frac{\partial u}{\partial r} = \sum' |n| c_n r^{|n|-1} e^{in\theta} \qquad \frac{\partial u}{\partial \theta} = \sum' in c_n r^{|n|} e^{in\theta}$$

$$\int_0^1 r\,dr \int_0^{2\pi} \left(\left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2 \right) d\theta = 2\pi \int_0^1 r\,dr \left\{ \sum' |n|^2 r^{2|n|-2} |c_n|^2 + \sum' |nc_n|^2 r^{2|n|-2} \right\}$$

$$= 2\pi \sum_{n \in \mathbb{Z}}' |n| |c_n|^2$$

Write this form in terms of a kernel on S^1.

$$\iint d\theta_1 d\theta_2 \quad \sum c_n e^{in\theta_1} \sum \bar{c}_n e^{-in\theta_2} \quad \sum' |n| e^{in(\theta_2-\theta_1)} \frac{1}{2\pi}$$

So the kernel is the distribution

$$\frac{1}{2\pi} \sum_n' |n| e^{in(\theta_2-\theta_1)}$$

Now

$$\sum_{n>0}' n e^{in\theta} = z\frac{d}{dz} \sum_{n=0}^{\infty} z^n = z \frac{d}{dz} \frac{1}{1-z} = \frac{z}{(1-z)^2}$$

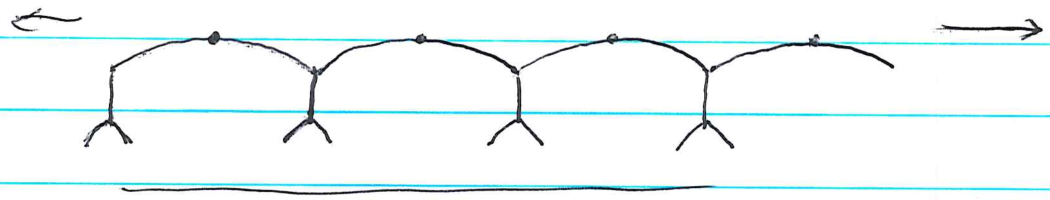
$$= \frac{e^{i\theta}}{(1-e^{i\theta})^2} = \frac{1}{(e^{i\theta/2} - e^{-i\theta/2})^2} = \frac{1}{-4\sin^2\frac{\theta}{2}}$$

for $\operatorname{Im}\theta > 0$. Hence it seems that

$$\sum_n' |n| e^{in\theta} = \frac{1}{-2\sin^2\frac{\theta}{2}} \qquad \text{for } \theta \neq 0$$

At $\theta = 0$ it is a true distribution of some sort. Notice the same peculiar negative which means the zero part is very positive.

Remark: There is a nice description of the ends of the modular tree. Suppose you define Dedekind cut (incorrectly) as a partition of the rational numbers: $\mathbb{Q} = A + \mathbb{Q} - A$ such that every member of A is less than every member of $\mathbb{Q} - A$. Then each [redacted] rational number r defines two of these cuts depending on whether $r \in A$ or $r \in \mathbb{Q} - A$, and an irrational number determines exactly one such cut. [redacted] Such a cut can be identified with an end of the modular tree. Note that at ∞ belong the two cuts $A = \phi$ and $A = \mathbb{Q}$ which correspond to the 2 ways of getting to ∞.



Notice also that if an end is fixed then the other ends form a linearly ordered set. Moreover if the 2 ends belonging to ∞ ~~are~~ removed, then $\mathbb{Z} = \langle T \rangle$ acts nicely on the [redacted] rest of the ends.