Electrical circuits. Connect up linear components: capacitances, resistors, inductors in a box with 2-terminals. Such a gadget is called a 1-port. Connect it up to a generator which produces a given voltage at a given frequency. (Notice that the frequency depends on how fast the armature turns; [redacted] according to Faraday's law $\nabla \times E = \frac{\partial B}{\partial t}$ [redacted] one gets a fixed [redacted] voltage [redacted] independent of current flow provided the field magnetic ~~strength~~ and the rotation rate are fixed.) Then a current flows. [redacted]

For a fixed frequency the set of pairs (E, I) consisting of applied voltage and resulting current forms [redacted] a line [redacted] in $\mathbb{C}^2$. I see I have to understand complex numbers.

So we have to understand sinusoidal variation in voltage. [redacted] If ω is the frequency this is of the form

$$E(t) = a_1 \cos \omega t + a_2 \sin \omega t \qquad a_1, a_2 \in \mathbb{R}$$

$$= \operatorname{Re}\, (a_1 + i a_2)\, e^{-i\omega t}$$

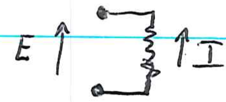
So in this way we have a 1-1 correspondence between complex numbers $a_1 + i a_2$ and sinusoidal functions of frequency ω. So we think of a voltage of frequency ω as a complex function $E e^{-i\omega t}$. Similarly the current will be of the form $I e^{-i\omega t}$ and

$$Z = \frac{E}{I}$$

is a complex number called the impedance of the 1-port (at the frequency ω).

Basic components are as follows.

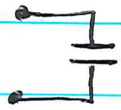
1) Resistor. Here



$$E = IR$$

where $R \geq 0$, so $\boxed{Z(\omega) = R}$ is independent of ω.

2) Capacitor. Let Q be the charge stored in it.



$$CE = Q \qquad \text{basic law with } C \geq 0.$$

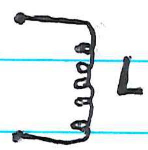
$$\therefore \frac{dE}{dt} = C^{-1} I$$

so if $E = E(\omega) e^{-i\omega t}$ then

$$I = \frac{1}{C^{-1}}(-i\omega) E(\omega) e^{-i\omega t}$$

so that $\boxed{Z(\omega) = \frac{C^{-1}}{-i\omega}}$ $\quad C \geq 0$

3) Inductor.



Physical law says that if $I(t)$ is the current flowing through, that $-L\frac{dI}{dt}$ is the induced EMF, hence this must be balanced by the applied EMF. So

$$E = L\frac{dI}{dt}$$

or

$$E(\omega) e^{-i\omega t} = L\, I(\omega)(-i\omega) e^{-i\omega t}$$

or

$$\boxed{Z(\omega) = L \cdot (-i\omega)} \qquad L \geq 0$$

January 18, 1978

Notation: One uses the basic representation for voltage

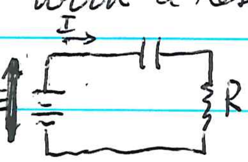
$$E(t) = \int_{-\infty}^{\infty} \hat{E}(i\omega)\, e^{i\omega t} \frac{d\omega}{2\pi}$$

For, if we put $i\omega = s$, then $\frac{d\omega}{2\pi} = \frac{ds}{2\pi i}$ so we have

$$E(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [redacted]\ \hat{E}(s)\, e^{st}\, ds$$

which is the inversion formula for the Laplace transform. Notice then that $E(t) = 0$ for $t < 0$ corresponds to $\hat{E}(s)$ being analytic in the right half-plane. Also poles of $\hat{E}(s)$ in the left half-plane give rise to decaying solutions.

Example: Charge a [redacted] capacitor in series with a resistor:



$$Q = CE$$

Here the applied volage is the Heaviside fn. $H(t)$

$$H(t) = \frac{Q(t)}{C} + R \cdot I(t) = \frac{Q(t)}{C} + R\frac{dQ}{dt}$$

Laplace transform equation is

$$\frac{1}{s} = \frac{1}{C} Q(s) + Rs\, Q(s)$$

or

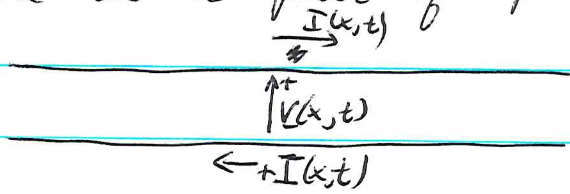
$$Q(s) = \frac{\frac{1}{s}}{\frac{1}{C} + Rs} = \frac{C}{s + RCs^2} = \frac{C}{s(1+RCs)} = [redacted]$$

$$= \frac{R^{-1}}{s(s+\frac{1}{RC})} = [redacted] \left(\frac{1}{s} - \frac{1}{s+\frac{1}{RC}}\right) C$$

So

$$Q(t) = C\, H(t) \left\{ 1 - e^{-\frac{t}{RC}} \right\}$$

We are going to consider lossless transmission lines. Picture these as a pair of parallel wires



running along the x axis. Let $V(x,t)$ denote the voltage at position x and time t. View the top wire as connected to the + terminal of a voltmeter. Let $I(x,t)$ be the current moving to the right in the top wire and to the left in the bottom wire.

Look at a small interval $x, x+dx$ of the line

It appears like a small capacitor. The charge accumulated on this capacitor in time dt is

$$-\left[I(x+dx,t) - I(x,t)\right]dt$$

This storage of charge has to be produced by a net voltage change between the plate which is

$$E(x, t+dt) - E(x,t)$$

The capacitance will be $C\,dx$, so we get

$$C\,dx\left[E(x,t+dt) - E(x,t)\right] = -\left[I(x+dx,t) - I(x,t)\right]dt$$

So you get the equation

$$\frac{\partial I}{\partial x} = -C\frac{\partial E}{\partial t}$$

On the other hand the current sets up a magnetic field between the two wires, and a changing [redacted] mag. field will induce an EMF. So

$$E(x+dx,t) - E(x,t) = -L\,dx\,\frac{\partial I}{\partial t}$$

which gives
$$\frac{\partial E}{\partial x} = -L\frac{\partial I}{\partial t}$$

So we get the equation

$$\boxed{\begin{aligned} \frac{\partial E}{\partial t} &= -\frac{1}{C}\frac{\partial I}{\partial x} \\ \frac{\partial I}{\partial t} &= -\frac{1}{L}\frac{\partial E}{\partial x} \end{aligned}}$$

which leads to the wave equation

$$\frac{\partial^2 E}{\partial t^2} = \frac{1}{CL}\frac{\partial^2 E}{\partial x^2}$$ also for I

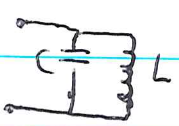
hence waves travel with the [redacted] speed

$$\boxed{\text{speed} = \frac{1}{\sqrt{LC}}}$$

[redacted]

Impedance Functions have the fundamental property [redacted]

[redacted] $\mathrm{Re}(s) > 0 \implies \mathrm{Re}\, Z(s) > 0$

e.g. 

$$Z(s) = \frac{1}{\frac{1}{Cs} + Ls} = \frac{Cs}{1+LCs^2}$$

and $$\frac{1}{s+i\omega_0} + \frac{1}{s-i\omega_0} = \frac{2s}{s^2+\omega_0^2}$$

One has the following representation for the impedance function of a lossless 1-port

$$Z(s) = \int \frac{d\mu(\hat{\omega})}{s - i\hat{\omega}}$$

$d\mu$ finite measure on $\mathbb{R}$, symmetric.

$$= \int \frac{1}{2}\left\{\frac{1}{s-i\hat{\omega}} + \frac{1}{s+i\hat{\omega}}\right\} d\mu(\hat{\omega})$$

$$= s\int \frac{d\mu(\hat{\omega})}{s^2+\hat{\omega}^2}$$

since $s = i\omega$ we have

$$Z(s) = \int \frac{d\mu(\hat{\omega})}{i\omega - i\hat{\omega}} = \frac{1}{i}\underbrace{\int \frac{d\mu(\hat{\omega})}{\omega - \hat{\omega}}}_{m(\omega)}$$

[redacted] Thus $$\frac{iE(s)}{I(s)} = m(\omega)$$

so if we rewrite the equations for a transmission line at frequency ω [redacted] in terms of $\begin{pmatrix} iE \\ I \end{pmatrix}$ we get

$$i\omega E = -\frac{1}{C}\frac{dI}{dx} \qquad -C\omega(iE) = \frac{dI}{dx}$$

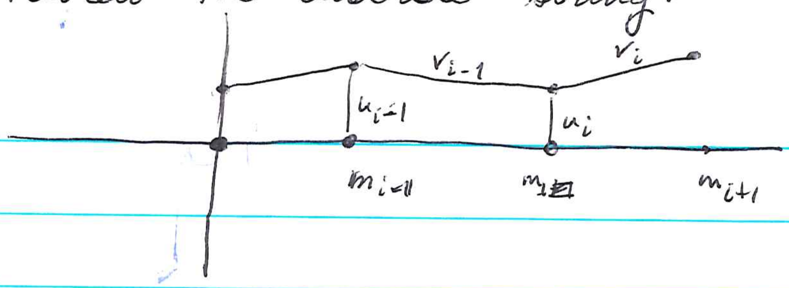
$$i\omega I = -\frac{1}{L}\frac{dE}{dx} \qquad L\omega I = \frac{d(iE)}{dx}$$

or

$$\frac{d}{dx}\begin{pmatrix} iE \\ I \end{pmatrix} = \omega\begin{pmatrix} 0 & L \\ -C & 0 \end{pmatrix}\begin{pmatrix} iE \\ I \end{pmatrix} = \omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix}\begin{pmatrix} iE \\ I \end{pmatrix}$$

which is in deBranges form.

Review the discrete string:



m_i = mass of ith particle, l_i = distance between ith & $(i+1)$th particle. Put

$$u_i = \text{displacement of } i\text{th particle}$$

$$v_i = \frac{1}{\lambda}\,\text{slope of } i\text{th string} = \frac{u_{i+1}-u_i}{\lambda l_i}$$

Equations are

$$-\lambda m_i u_i = \underset{\displaystyle v_i}{\underset{\parallel}{\frac{u_{i+1}-u_i}{\lambda l_i}}} - \underset{\displaystyle v_{i-1}}{\underset{\parallel}{\frac{u_i - u_{i-1}}{\lambda l_{i-1}}}}$$

$$u_{i+1} = u_i + \lambda l_i v_i$$

$$v_i = v_{i-1} - \lambda m_i u_i$$

which lead to

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\lambda m_i & 1 \end{pmatrix} \begin{pmatrix} u_i \\ v_{i-1} \end{pmatrix}$$

$$\begin{pmatrix} u_{i+1} \\ v_i \end{pmatrix} = \begin{pmatrix} 1 & \lambda l_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

so

$$\begin{pmatrix} u_{n+1} \\ v_n \end{pmatrix} = \begin{pmatrix} 1 & \lambda l_n \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -\lambda m_n & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & \lambda l_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -\lambda m_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $\begin{matrix} u_0 = 1 \\ v_0 = 0 \end{matrix}$ is the standard initial state for the string. What would be the corresponding network? The l_i have to be interpreted as inductance and the m_i as capacitances.

$$\begin{pmatrix} iE_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -C_1\omega & 1 \end{pmatrix} \begin{pmatrix} iE_0 \\ I_0 \end{pmatrix}$$

[redacted]

$$E_1 = E_0$$

$$I_1 = -C_1 \omega i E_0 + I_0 = -C_1 s E_0 + I_0$$

or

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -C_1 s & 1 \end{pmatrix} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

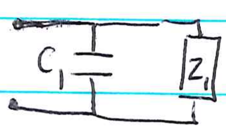
or

$$\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C_1 s & 1 \end{pmatrix} \begin{pmatrix} E_1 \\ I_1 \end{pmatrix}$$

or

$$Z_0 = \frac{Z_1}{C_1 s Z_1 + 1} = \frac{1}{\frac{1}{\frac{1}{C_1 s}} + \frac{1}{Z_1}}$$

so that Z_0 is the parallel connection of a capacitance C_1 with Z_1

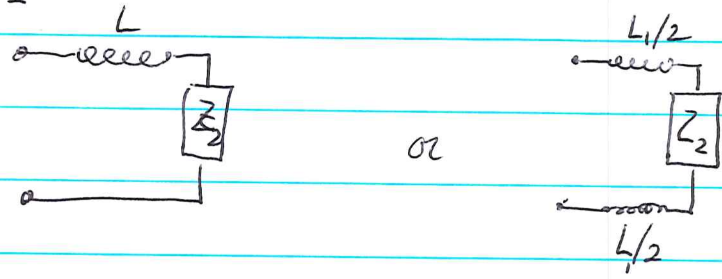


$$iE_2 = iE_1 + L_1 \frac{s}{i}$$

Next $\begin{pmatrix} iE_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} 1 & L_1\omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} iE_1 \\ I_1 \end{pmatrix}$ or $\begin{pmatrix} E_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} 1 & -L_1 s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_1 \\ I_1 \end{pmatrix}$

$$[redacted]\quad Z_1 = \begin{pmatrix} 1 & +L_1 s \\ 0 & 1 \end{pmatrix} (Z_2) = Z_2 + L_1 s Z_1$$

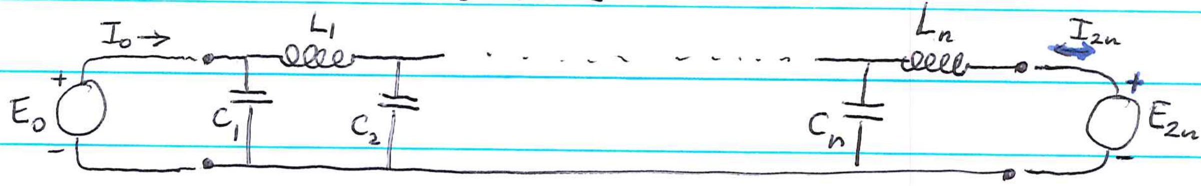
so Z_1 is Z_2 connected in series with an inductance L_1:



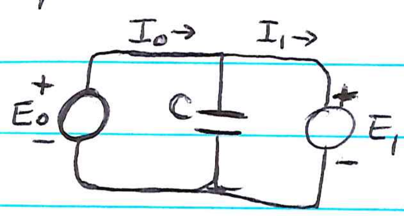
So we end up with the [redacted] formula

$$\begin{pmatrix} E_{2n} \\ I_{2n} \end{pmatrix} = \begin{pmatrix} 1 & -L_n s \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & -L_1 s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -C_1 s & 1 \end{pmatrix} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

associated to the network



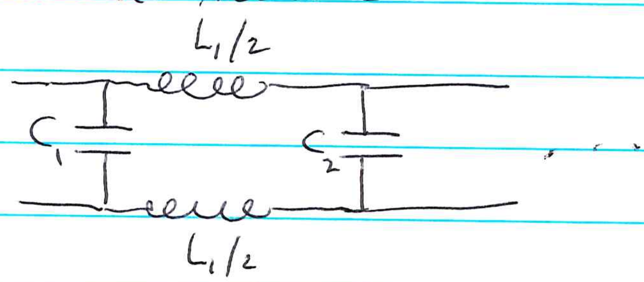
For example:



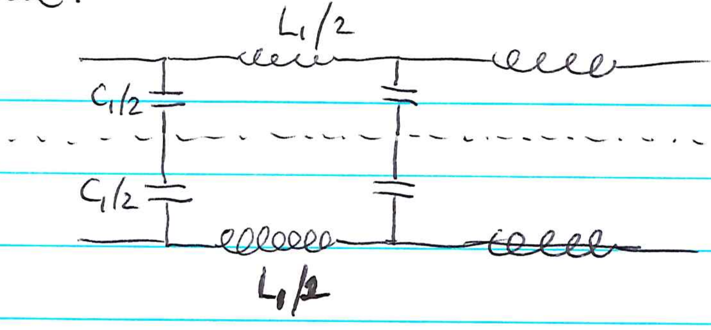
Then $E_0 = E_1$ and $I_c =$ [redacted] $I_0 - I_1 = (Cs) E_0$ so

$$I_0 = I + C_1 s E_0$$

Instead of the above network which is grounded, one can use a "balanced" network:

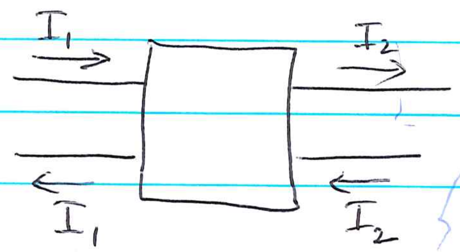


But this is equivalent to reflecting a grounded network:



It seems therefore that a transmission line can always be viewed as a single line with a return ground.

I am a little suspicious of 2 ports in general because the typical assumption about them is that when they occur in a circuit the current flow is:



This is maybe OK if there is no possibility for feedback. One way of forcing this behavior is to use 1:1 perfect transformers.

January 19, 1978

Scattering for Dirac system $\quad \frac{d}{dx}u = \begin{pmatrix} i(\lambda-a) & \bar{p} \\ p & -i(\lambda-a) \end{pmatrix} u$

or

$$\underbrace{\begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix}}_{P} \frac{du}{dx} + \underbrace{\begin{pmatrix} a & i\bar{p} \\ -ip & a \end{pmatrix}}_{Q} u = \lambda u$$

(Q arbitrary hermitian $\ni$ $\operatorname{tr} P^{-1}Q = 0$). We suppose Q has compact support, and let $u^+_{>0}$, $u^-_{>0}$ be the solutions with behavior

$$u^+_{>0} = \begin{pmatrix} e^{i\lambda x} \\ 0 \end{pmatrix} \qquad u^-_{>0} = \begin{pmatrix} 0 \\ e^{-i\lambda x} \end{pmatrix}$$

for $x \gg 0$ and let $u^{\pm}_{<0}$ be defined similarly for $x \ll 0$. Put

$$u^+_{>0} = A u^+_{<0} + B u^-_{<0}$$

i.e

$$x \gg 0 \quad \begin{pmatrix} e^{i\lambda x} \\ 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} Ae^{i\lambda x} \\ Be^{-i\lambda x} \end{pmatrix} \quad x \ll 0$$

Using the fact that $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} \bar{u}_2 \\ \bar{u}_1 \end{pmatrix}$, $\lambda \longmapsto \bar{\lambda}$ is a symmetry of the Dirac equation, and that under this symmetry

$$\begin{pmatrix} e^{i\lambda x} \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ \overline{e^{i\bar{\lambda}x}} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-i\lambda x} \end{pmatrix}$$

we see this symmetry interchanges $u^{\pm}_{>0}$ and also $u^{\pm}_{<0}$. So we have

$$u^-_{>0} = B^{\#} u^+_{<0} + A^{\#} u^-_{<0}$$

or in matrix form.

$$\left(u^+_{>0} \quad u^-_{>0}\right) = \left(u^+_{<0}, u^-_{<0}\right)\begin{pmatrix} A & B^\# \\ B & A^\# \end{pmatrix}$$

Taking the Wronskian we see

$$AA^\# - BB^\# = 1 \qquad \text{or} \qquad |A|^2 - |B|^2 = 1 \quad \text{for } \lambda \in \mathbb{R}.$$

The scattering matrix relates the incoming states $u^+_{<0}, u^-_{>0}$ to the outgoing states $u^-_{<0}, u^+_{>0}$.

$$\underset{\substack{\uparrow \\ \text{its transmission}}}{\frac{1}{A} u^+_{>0}} = \underset{\substack{\uparrow \\ \text{incoming} \\ \text{wave}}}{u^+_{<0}} + \underset{\substack{\uparrow \\ \text{its reflection}}}{\frac{B}{A} u^-_{<0}}$$

$$\left(u^+_{<0} \quad u^-_{<0}\right) = \left(u^+_{>0} \quad u^-_{>0}\right)\begin{pmatrix} A^\# & -B^\# \\ -B & A \end{pmatrix}$$

$$\therefore \quad u^-_{<0} = -B^\# u^+_{>0} + A u^-_{>0}$$

or

$$\underset{\substack{\uparrow \\ \text{its transm.} \\ \text{part}}}{\frac{1}{A} u^-_{<0}} = \underset{\substack{\uparrow \\ \text{its ref.} \\ \text{part}}}{-\frac{B^\#}{A} u^+_{>0}} + \underset{\substack{\uparrow \\ \text{incoming} \\ \text{wave from the right}}}{u^-_{>0}}$$

$$\therefore \quad u^+_{<0} = \frac{1}{A} u^+_{>0} - \frac{B}{A} u^-_{<0}$$

$$u^-_{>0} = +\frac{B^\#}{A} u^+_{>0} + \frac{1}{A} u^-_{<0}$$

So again

$$S = \begin{pmatrix} \frac{1}{A} & -\frac{B}{A} \\ \frac{B^\#}{A} & \frac{1}{A} \end{pmatrix} \qquad \text{or its transpose}$$

$S(\lambda)$ is unitary for λ real
A, B are entire functions of λ
A doesn't vanish in the UHP. Its zeroes in the lower half-plane govern the rate exponential decay of waves.

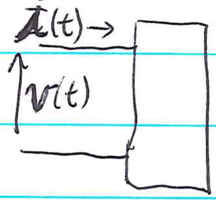
Notice that the diagonal entries of S are equal and analytic in the UHP. Hence from the relation

$$\frac{1}{|A|^2} + \left|\frac{B}{A}\right|^2 = 1$$

if we are given the reflection coefficient $\frac{B}{A}$ we know the modulus of the transmission coefficient, hence we know A up to an multiplicative constant of modulus 1.

January 20, 1978.

I want to understand power flow in a circuit. The basic definition is that in



the power ($= \frac{dE}{dt}$, E = energy) flowing into the circuit at time t is

$$
\begin{aligned}
P(t) &= v(t)\,i(t) \\
&= \mathrm{Re}(Ve^{i\omega t})\,\mathrm{Re}(Ie^{i\omega t}) \\
&= \frac{1}{4}(Ve^{i\omega t} + \bar{V}e^{-i\omega t})(Ie^{i\omega t} + \bar{I}e^{-i\omega t}) \\
&= \frac{1}{4}(VIe^{2i\omega t} + \bar{V}I + V\bar{I} + \bar{V}\bar{I}e^{-i\omega t})
\end{aligned}
$$

Assuming $\omega \neq 0$, the average power flowing is

$$P_{av} = \frac{1}{4}(\bar{V}I + V\bar{I}) = \frac{1}{2}\mathrm{Re}(V\bar{I})$$

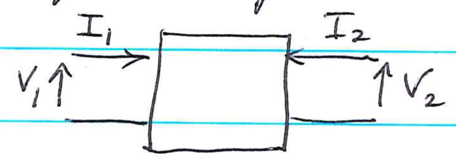
[redacted] n-port: This assigns to each frequency ω an n-dimensional subspace of $\mathbb{C}^n \times \mathbb{C}^n$ consisting of the admissible (V, I) for the frequency ω. Often this subspace can be described as the graph of an impedance matrix Z, so that

$$V = ZI$$

describes the voltages belonging to given currents. Also one frequently has an [redacted] admittance matrix Y:

$$I = YV$$

Consider for example a 2-port.



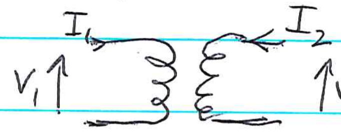
Then usual one has relations

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

To measure $\begin{pmatrix} z_{11} \\ z_{21} \end{pmatrix}$ one wants $I_2 = 0$, hence one opens the left port sends a current I_1 in and measures V_1, V_2. Hence the z_{ij} are called <u>open-circuit impedances</u>.

Similarly if $I = YV$, then the y_{ij} are called <u>short-circuit admittances</u>.

Some circuits don't have Y, Z matrices for example an ideal transformer which is described by



$$V_2 = nV_1$$
$$I_2 = -\tfrac{1}{n} I_1$$

A basic condition on an n-port is that it can [redacted] only dissipate energy, not create it. (called "passive") This means that if (V,I) is a possible state then

$$Re\,(I^* V) \geq 0$$

for any real frequency ω. Assuming Z exists this means

$$Re\,(I^* Z I) \geq 0$$

for all $I \in \mathbb{C}^n$. [redacted]

$$Re\,(I^* Z I) = \text{[redacted]}\ \tfrac{1}{2}\left(I^* Z I + I Z^* I\right) = I^* Z_H I$$

where $Z_H = \dfrac{Z+Z^*}{2}$ is the hermitian part of Z.

Recall the Laplace transform representation:

$$v(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \text{[redacted]}\ V(s)\, e^{st}\, ds$$

and similarly for i. Now

$I(s)$ analytic for $Re(s) > 0$

$\Rightarrow\ i(t) = 0 \qquad t < 0$

$\Rightarrow\ v(t) = 0 \qquad t < 0$ by causality

$\Rightarrow\ V(s) = Z(s)\,I(s)$ analytic for $Re(s) > 0$.

Thus causality implies $Z(s)$ is analytic in the right half-plane.

I have seen that immittance (=impedance or admittance) matrices for an n-port need not exist. There is something called the scattering matrix which always exists.

Motivation for scattering parameters. Consider a transmission line with $L = C = 1$:

$$\frac{dV}{dx} = -i\omega I$$

$$\frac{dI}{dx} = -i\omega V$$

Then $$\frac{d}{dx}(V+I) = -i\omega I - i\omega V = -i\omega(V+I)$$

$$\frac{d}{dx}(V-I) = -i\omega(I-V) = i\omega(V-I)$$

hence $$V+I = (V_0+I_0)\cdot e^{-i\omega x}$$

$$V-I = (V_0-I_0)\cdot e^{+i\omega x}$$

and taking into account time dependence we have

$$(v+i)(x,t) = (V_0+I_0)\, e^{i\omega(t-x)}$$

which is a wave travelling to the right, and

$$(v-i)(x,t) = (V_0-I_0)\, e^{i\omega(t+x)}$$

which is a wave travelling to the left.

Therefore a natural set of parameters to use is

$a = \frac{1}{2}(V+I)$ incident voltage

$b = \frac{1}{2}(V-I)$ reflected voltage

and so $V = a+b$ $\quad$ $(a+b) = Z(a-b)$

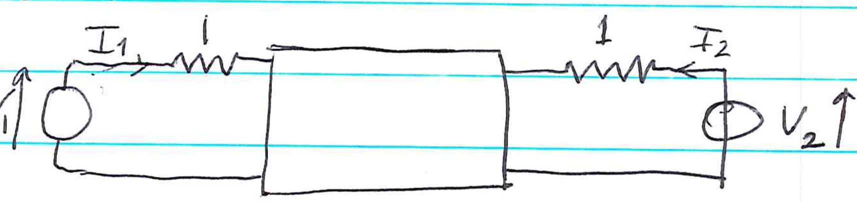
$I = a-b$ $\quad$ $(Z+1)b = (Z-1)a$

so if the scattering matrix is defined by $b = Sa$ then

$$S = \frac{Z-1}{Z+1}$$

Lossless case: Here $Re(I^* Z I) = 0$ for all input I which implies that $Z = -Z^*$ is skew-hermitian. Hence S is unitary; it is essentially the Cayley transform of Z.

Why S exists: Connect a generator of voltage V_i in series with a resistance of 1 ohm at the ith port for each i



The current response exists and is unique in this case, and we have (assuming Z exists to compute):

$$V - I = ZI$$

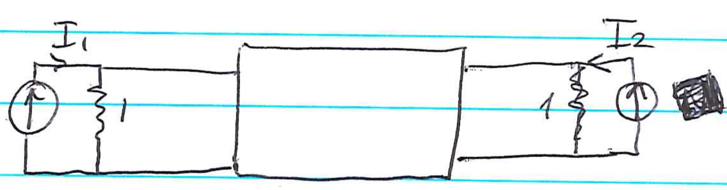
or

$$V = (Z+1)I.$$

Thus we see $\frac{1}{Z+1}$ exists always, so

$$-\frac{2}{Z+1} + 1 = \frac{Z-1}{Z+1} = S$$

exists always. Similarly we see that by considering current sources connected in parallel with 1 ohm resistors



there exists a unique voltage response

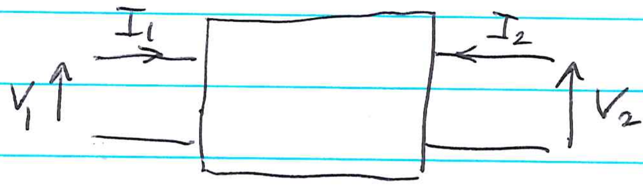
$$V = Z(I - V)$$

$$(Z+1)V = ZI$$

hence $\frac{Z}{1+Z}$ exists. Unfortunately this doesn't show that $1-Z$ is invertible which would show S was always invertible.

January 21, 1978:

Consider a 2-port having an impedance matrix Z



$$V_1 = z_{11} I_1 + z_{12} I_2$$
$$V_2 = z_{21} I_1 + z_{22} I_2$$

To obtain the chain matrix we consider V_1, [redacted] I_1 as independent variables and V_2, I_2 [redacted] as dependent

$$I_2 = \frac{1}{z_{12}}\left(V_1 - z_{11} I_1\right)$$

$$V_2 = z_{21} I_1 + z_{22}\frac{1}{z_{12}}\left(V_1 - z_{11} I_1\right)$$

$$\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} \frac{z_{22}}{z_{12}} & -\frac{z_{11}z_{22} - z_{12}z_{21}}{z_{12}} \\ \frac{1}{z_{12}} & -\frac{z_{11}}{z_{12}} \end{pmatrix}\begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

For the purposes of composition (cascade connections) one wants the output variables to be $\begin{pmatrix} V_2 \\ -I_2 \end{pmatrix}$ so that the chain matrix is

$$\begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{z_{22}}{z_{12}} & -\frac{|z|}{z_{12}} \\ -\frac{1}{z_{12}} & \frac{z_{11}}{z_{12}} \end{pmatrix}}_{A} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

The determinant of the chain matrix is

$$\det(A) = \frac{z_{11}z_{22} - (z_{11}z_{22} - z_{12}z_{21})}{z_{12}^2} = \frac{z_{21}}{z_{12}}$$

hence $\det(A) = 1 \iff Z$ is symmetric. For Z to be symmetric means the 2-port is "reciprocal", which is the case if it is built up out of RLC, but not gyrators.

Next consider power flow. Newcomb (Linear Multiport Synthesis) takes the viewpoint that physically realizable signals are C^∞ fns. of time with support in a right half-line $t \geq a$. Also he include exponentials e^{st} with $Re(s) > 0$ as physical by a limiting process. Suppose then we have admissible behavior for our 2 port

$$v(t) = Re\left(V e^{st}\right)$$

$$i(t) = Re\left(I e^{st}\right)$$

with $Re(s) = \sigma > 0$. Then the total energy flow into the circuit up to time t is

$$\int_{-\infty}^{t} v(\hat{t})^t \, i(\hat{t})\, d\hat{t} = \int_{-\infty}^{t} \frac{1}{4}\Big(V^t I e^{2s\hat{t}} + \bar{V}^t I e^{(s+\bar{s})\hat{t}} + V^t \bar{I} e^{(s+\bar{s})\hat{t}} + \bar{V}^t \bar{I} e^{2\bar{s}\hat{t}} \Big) d\hat{t}$$

$$= \frac{1}{4}\left(V^t I \, \frac{e^{2st}}{2s} + \bar{V}^t I \, \frac{e^{(s+\bar{s})t}}{s+\bar{s}} + V^t \bar{I} \, \frac{e^{(s+\bar{s})t}}{s+\bar{s}} + \bar{V}^t \bar{I} \, \frac{e^{2\bar{s}t}}{2\bar{s}}\right)$$

$$= \frac{1}{4}\left(\mathrm{Re}\left(V^t I \, \frac{e^{2st}}{s}\right) + \mathrm{Re}\left(\bar{V}^t I \, \frac{e^{2\sigma t}}{\sigma}\right)\right)$$

$$= \frac{e^{2\sigma t}}{4}\left\{ \mathrm{Re}\left(\frac{V^t I e^{2i\omega t}}{s}\right) + \frac{1}{\sigma}\mathrm{Re}\left(V^* I\right)\right\}$$

This has to be ≥ 0. For $\omega \neq 0$ the first term oscillates, so we conclude that

$$\mathrm{Re}(V^* I) \geq 0$$

for any s with $\mathrm{Re}(s) > 0$ (The case $\omega = 0$ is handled by a limiting argument).

Putting in $V = ZI$ you get

$$\mathrm{Re}(I^* Z^* I) \geq 0$$

hence the hermitian part of Z, $\frac{Z+Z^*}{2}$ has to be ≥ 0 for $\mathrm{Re}(s) > 0$, hence also for $\mathrm{Re}(s) = 0$ when it makes sense.

So for a reciprocal n-port $Z(s)$ is a symmetric complex matrix whose real hermitian part is ≥ 0. So

$$iZ(s)$$

which relates iV to I is a symmetric complex matrix whose imaginary part is ≥ 0. Recall that Siegel's upper half plane consists of symmetric complex matrices $X + iY$ with X, Y real symmetric such that $Y > 0$. We see therefore that the impedance matrix for a reciprocal n-port is just a holomorphic map of $Re(s) > 0$ into the Siegel upper half plane. (Maybe we should make some non-degeneracy hypothesis.)

[redacted] Suppose our n-port is loss-less. This means that for ω real the energy flow into the network is zero. I have seen this means

$$Re(V^* I) = Re(I^* Z I) = (I^* Z_H I) = 0$$

hence $Z_H = 0$ and so Z is skew-hermitian. Thus $iZ(s)$ will be a real symmetric matrix for $s \in i\mathbb{R}$ in the loss-less reciprocal case.

Let return to our 2-port with the chain matrix

$$\begin{pmatrix} iV_2 \\ -I_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{z_{22}}{z_{12}} & -i\frac{|z|}{z_{12}} \\ \frac{i}{z_{12}} & \frac{z_{11}}{z_{12}} \end{pmatrix}}_{\tilde{A}} \begin{pmatrix} iV_1 \\ I_1 \end{pmatrix}$$

If Z is reciprocal & loss-less it is $i \cdot$ real symm. matrix, hence the chain matrix in the above form is real and of determinant $+1$, (for real ω).

Note that

$$(-i\bar{V}_2 \;\; -\bar{I}_2)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} iV_2 \\ -I_2 \end{pmatrix} = (-i\bar{V}_2 \;\; -\bar{I}_2)\begin{pmatrix} I_2 \\ iV_2 \end{pmatrix} = -i(\bar{V}_2 I_2 + V_2 \bar{I}_2)$$

$$(-i\bar{V}_1 \;\; \bar{I}_1)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} iV_1 \\ I_1 \end{pmatrix} = (-i\bar{V}_1 \;\; \bar{I}_1)\begin{pmatrix} -I_1 \\ iV_1 \end{pmatrix} = i(\bar{V}_1 I_1 + \bar{I}_1 V_1)$$

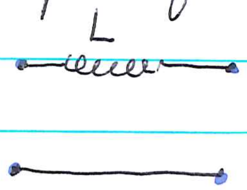
Hence if I put $u_{in} = \begin{pmatrix} iV_1 \\ I_1 \end{pmatrix}$ $u_{out} = \begin{pmatrix} iV_2 \\ -I_2 \end{pmatrix}$ I have

$$\text{Power in} = \frac{1}{2}\,\mathrm{Re}\left(\bar V_1 I_1 + \bar V_2 I_2\right)$$

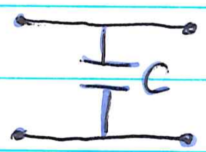
$$= \frac{1}{4}\left[\frac{1}{i} u_{in}^* P u - \frac{1}{i} u_{out}^* P u_{out}\right] \geq 0$$

Consequently $\mathrm{Re}(s) > 0 \implies \tilde A$ expands UHP.

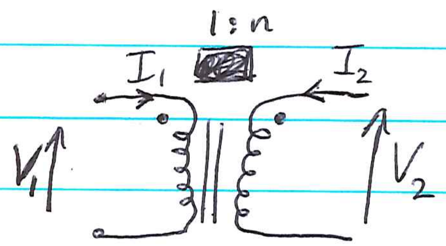
Examples of chain matrices



$$\begin{pmatrix} 1 & L\omega \\ 0 & 1 \end{pmatrix}$$



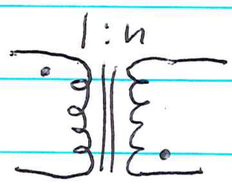
$$\begin{pmatrix} 1 & 0 \\ -C\omega & 1 \end{pmatrix}$$



$$V_2 = n V_1$$
$$-I_2 = +\frac{1}{n} I_1$$

$$\begin{pmatrix} n & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$$

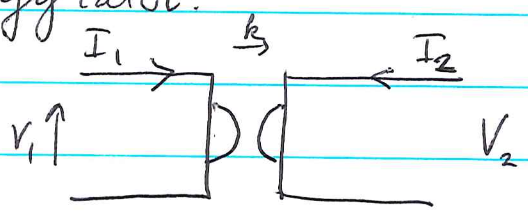
dots indicate $n > 0$



here $n < 0$ so $V_2 = -n V_1$

$$-I_2 = -\frac{1}{n} I_1$$

Gyrator:



$$\begin{pmatrix} iV_2 \\ -I_2 \end{pmatrix} = \begin{pmatrix} -kiI_1 \\ +i\frac{1}{k} iV_1 \end{pmatrix} = \begin{pmatrix} 0 & -ki \\ \frac{i}{k} & 0 \end{pmatrix} \begin{pmatrix} iV_1 \\ I_1 \end{pmatrix}$$

$$Z_{in} = \frac{k^2}{Z_{load}}$$

Impedance matrix for gyrator is

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

so this network is not reciprocal (hence $\tilde{A}$ not of determinant 1).

Interesting Point: If we view a 2 port in terms of its chain matrix defined by

$$\begin{pmatrix} iV_2 \\ -I_2 \end{pmatrix} = \tilde{A} \begin{pmatrix} iV_1 \\ I_1 \end{pmatrix}$$

then in the lossless reciprocal case $\tilde{A}$ has determinant 1, is real-valued for ω real, and for $\mathrm{Re}\, s > 0$ ($\mathrm{Im}\, \omega < 0$) it expands the UHP. So it looks like a Nevanlinna matrix (would be one if it were entire in ω).

January 22, 1978

Consider a linear mechanical system consisting of n particles of masses $m_1, \dots, m_n$ such that the force F_i on the ith particle is a linear function of displacements $y_1, y_2, \dots, y_n$:

$$F_i(y) = -\sum_{j=1}^{n} a_{ij} y_j$$

In order for there to exist a potential $V(y)$ for these forces, i.e. such that $F_i = \frac{\partial V}{\partial y_i}$ one must have

$$-a_{ij} = \frac{\partial F_i}{\partial y_j} = \frac{\partial F_j}{\partial y_i} = -a_{ji}$$

in which case:

$$T = \text{Kin energy} = \frac{1}{2}\sum m_i \dot{y}_i^2$$

$$V = \text{Pot. energy} = \frac{1}{2}\sum a_{ij} y_i y_j$$

and the equations of motion are

$$m_i \ddot{y}_i = -\sum_{j=1}^{n} a_{ij} y_j .$$

Since this system is linear, superposition of solutions is valid, [redacted] and since it has constant coefficients one looks for exponential solutions

$$y = v e^{i\omega t}$$

$$-m\omega^2 v = -Av$$

$$(m\omega^2 - A)v = 0.$$

[redacted] To simplify suppose $m = 1$. Then ω^2 has to be an eigenvalue of A, v an eigenvector. If $v_1, \dots, v_n$ are independent eigenvectors (these $\exists$ because A is real symmetric) and $\omega_1^2, \dots, \omega_n^2$ are the corresp. eigenvalues, then we get $2n$ linear independent solutions

$$v_j e^{i\omega_j t}, \quad v_j e^{-i\omega_j t}$$

for the equation. If $A \geq 0$, then $\omega_j^2 \geq 0$, so $\omega_j \in \mathbb{R}$.

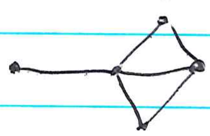
Generically the eigenvalues are distinct [redacted] and one can always perturb into this situation as follows. Let D be a diagonal matrix with unequal eigenvalues. [redacted] Then $A + tD$ has distinct eigenvalues for t large, hence the discriminant of its characteristic poly is [redacted] $\neq 0$ for t large, hence also for t very small > 0.

Introduce the graph with vertices the particles and edges for each i, j, $i \neq j$, with $a_{ij} \neq 0$. If the graph is linear and connected

I know already that the eigenvalues are [redacted] simple.

~~[illegible]~~

Suppose the graph has a free edge



I would like to understand if there is a characteristic impedance at this free edge at any frequency. So what I am trying to work in is the concept of an excitation. The eigenvalues of A represent the natural vibrations of the system.

There seem to be 2 ways to proceed. First you might try applying an external force to the particles

$$\ddot{y}_i = -\sum_{j=1}^{n} a_{ij} y_j + F_i(t) .$$

$$\ddot{y} = -Ay + F(t)$$

If $F(t) = F_0 e^{i\omega t}$ and $y = v e^{i\omega t}$ we want to solve

$$-\omega^2 v = -Av + F_0$$

or

$$(A - \omega^2) v = F_0$$

so we want to avoid $\omega^2 =$ eigenvalue of A.

Secondly, you might introduce a new vertex y_0 connected to the rest. Then the equations become

$$\ddot{y}_i = -\sum_{j=1}^{n} a_{ij} y_j - a_{i0} y_0 \qquad i = 1, \ldots, n$$

Now force y_0 to be $y_0 = \alpha e^{i\omega t}$. This leads to the same sort of motions.

So it seems that when there is a free edge, forcing the end vertex to undergo a vibration $e^{i\omega t}$ has the same effect as applying an external force [redacted] $\propto e^{i\omega t}$ to the end of the free vertex, which one removes. To be more precise suppose one is given a system described by $y_1, \dots, y_n$ and the matrix A. To y_1 one applies the external force $e^{i\omega t}$. There is a unique steady-state solution at the frequency ω obtained by solving

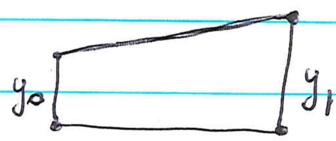
$$(A - \omega^2) v = e_1$$

So one look at the [redacted] behavior of y_1

$$y_1(t) = \gamma e^{i\omega t}$$

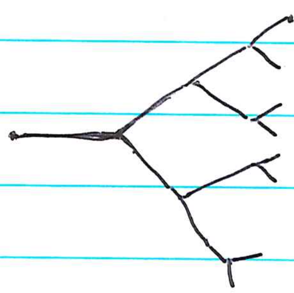
and then γ is essentially the impedance of the system at the vertex 1.

Think of a string segment over the free edge



The slope of the string has to be $-e^{i\omega t}$. Then the impedance gives you the effect of this requirement on y_0.

Maybe things are clearer electrically. [redacted]
The idea is that if I have a tree I can fit arbitrary



impedances at the right ends; these impedances [redacted] are numbers $Z \in RHP$. Then I get an impedance $Z(s)$ at the left end and I can hope that the limit exists as the graph spreads and gives a <u>meromorphic function of s</u> [redacted].

Given $Lu = P\dfrac{du}{dx} + Qu = \lambda Ru$ self-adjoint 2 dim'l system with $R > 0$, $S(\lambda)$ the propagation matrix between $x = 0$ and $x = \ell$. Suppose

$$tr(P^{-1}Q) = tr(P^{-1}R) = 0$$

hence $S(\lambda)$ has $\det = 1$. Since $P^* = -P$, the matrix [redacted] $R^{-1}P$ is skew-adjoint wrt the inner product defined by R:

$$(R[redacted]R^{-1}P)x, y) = (Px, y) = -(x, Py) = -(Rx, R^{-1}Py).$$

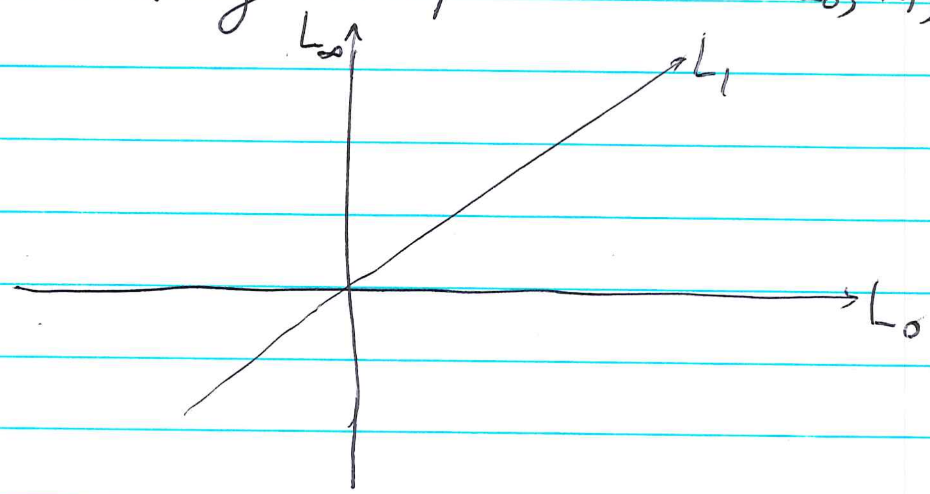
Hence the eigenvalues of $R^{-1}P$ are purely imaginary; the same holds for $P^{-1}R$. Since $tr(P^{-1}R) = 0$, the eigenvalues of $R^{-1}P$ are $ia, -ia$ and hence $\det(R^{-1}P) = a^2 > 0$. Assume x chosen so that $a = 1$. Hence $R^{-1}P$ is skew-hermitian wrt [redacted] (Rx, y) and has eigenvalues $\pm i$, permitting one to decompose the space of values at each point x into orthogonal lines in a canonical way.

Let V_0 be the space of values at $x = 0$. V_0 is a 2-dim'l complex vector space which carries a [redacted] hermitian bilinear form $\frac{1}{i}(Pv, w)$ of signature $(+,-)$ as well as the inner product (Rv, w). The former reduces its structure to $U(1,1)$ and the latter to $U(2)$, so the structure of V_0 reduces to $U(1) \times U(1)$.

Better viewpoint. Let V_{in} denote the possible boundary values at $x=0$ and V_{out} the boundary values at $x=l$. These are 2-dimensional complex vector spaces each carrying a hermitian form of signature $(+,-)$.

Suppose V is a 2-diml vector space over $\mathbb{C}$ with a hermitian form $\langle\,,\rangle$ of type $(+,-)$. Look in $\mathbb{P}^1(V)$ at the set of isotropic lines for $\langle\;\rangle$. It is a "circle" with a definite choice of interior (where $\langle v,v\rangle > 0$). Hence choosing three points in the correct order on this circle we can move it to $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$.

To be specific suppose the points on the circle are represented by isotropic lines L_0, L_1, L_∞



Then $L_0 \oplus L_\infty \xrightarrow{\sim} V$ and L_1 will be the graph of an isomorphism between L_0 and L_∞. Choose a non-zero vector e_1, let $e_2 \in L_\infty$ be the unique vector such that $e_1+e_2 \in L_1$. Then in terms of this basis for V we have

$$\langle e_1, e_1\rangle = 0 = \langle e_2, e_2\rangle$$

$$0 = \langle e_1+e_2, e_1+e_2\rangle = \langle e_1,e_2\rangle + \langle e_2,e_1\rangle = 2\,\mathrm{Re}\,\langle e_1,e_2\rangle$$

so $\langle e_1,e_2\rangle = ia$. If the choice of e_1 is changed to

ζe_1, then e_2 goes to ζe_2 and [redacted]

$$\langle \zeta e_1, \zeta e_2 \rangle = |\zeta|^2 \langle e_1, e_2 \rangle$$

so we can suppose $a = \pm 1$. In fact, because L_0, L_1, L_∞ is supposed to go counterclockwise around the interior of the circle, one of these signs holds. If

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then $\langle e_1, e_2 \rangle = \frac{1}{i}(Pe_1, e_2) = \frac{1}{i}(e_2, e_2) = -i$

so $a = -1$. At this stage the choice of e_1 is determined up to a ζ of modulus 1, and hence the basis [redacted] for V is unique up to the scalar ζ.

[redacted]

Another version: Let $GL_2(\mathbb{C})$ act on the set of hermitian forms of sign. $(+,-)$ on $\mathbb{C}^2$. A hermitian form is uniquely represented (Bv, w) with $B = B^*$. The action of $T^{-1} \in GL_2(\mathbb{C})$ is $B \longmapsto T^* B T$. Under the action of $T \in U_2$ we can reduce B to a diagonal real matrix, and then we can use T diagonal to get $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and then to

$$B = \frac{1}{i} P = \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}$$

Let $T \in$ stabilizer of $\frac{1}{i}P$: $\quad T^* \frac{1}{i} P T = \frac{1}{i} P$ or $T^* P T = P$

This implies $\overline{\det T} \cdot \det T = 1$, so $|\det T| = 1$. If we multiply T by a scalar diagonal matrix ζI, then with $|\zeta| = 1$

$$\det(\zeta T) = \zeta^2 \det T$$

and so we can suppose $\det T = 1$. Then compute

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = T^{-1} = P^{-1}T^*P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \bar a & \bar c \\ \bar b & \bar d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \bar b & \bar d \\ -\bar a & -\bar c \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \bar d & -\bar b \\ -\bar c & \bar a \end{pmatrix}$$

whence $T \in SL_2(\mathbb{R})$. Consequently the stabilizer [redacted] in $GL_2(\mathbb{C})$ of [redacted] the hermitian form assoc. to $\frac{1}{i}P$ is

$$SL_2(\mathbb{R}) \underset{\{\pm I\}}{\times} S^1$$

Prop:
$$GL_2(\mathbb{C}) \Big/ SL_2(\mathbb{R}) \underset{\{\pm I\}}{\times} S^1 \xrightarrow{\sim} \begin{array}{l}\text{hermitian forms} \\ \text{on } \mathbb{C}^2 \text{ of signature } (+,-)\end{array}$$

$$T^{-1} \longmapsto T^*\left(\tfrac{1}{i}P\right)T$$

Suppose we consider the structure of hermitian form of type $(+,-)$ together with a volume element on $\mathbb{C}^2$. Then

$$GL_2(\mathbb{C}) \Big\backslash \left(GL_2(\mathbb{C}) / SL_2(\mathbb{R})S^1 \times GL_2(\mathbb{C}) / SL_2(\mathbb{C}) \right)$$

$$= SL_2(\mathbb{R})S^1 \Big\backslash GL_2(\mathbb{C}) / SL_2(\mathbb{C}) = \det(SL_2(\mathbb{R})S^1) \backslash \mathbb{C}^*$$

$$= S^1 \backslash \mathbb{C}^* = \mathbb{R}_{>0}.$$

So the action is not transitive unless we restrict attention to B with $|\det B| = 1$. [redacted] Thus we want to let $SL_2(\mathbb{C})$ act on hermitian B with $|\det B| = 1$, which means $\det B = -1$, since $\det B < 0$. [redacted] The stabilizer of $B = \frac{1}{i}P$ is evidently $SL_2(\mathbb{R})$.

It seems therefore that given a V with $\langle , \rangle$, there is a S^1-orbit of volumes determined in V such that to give one of these amounts to putting an underlying real structure on V. Might be better to say that given a "real" ray of volumes and $\langle , \rangle$ you get a real structure in V.

<u>January 23, 1978</u>: Symmetry

I can picture a 2-port as consisting of a pair V_{in}, V_{out} of 2-diml complex vector spaces equipped with hermitian forms $\langle \; \rangle$ of signature $(+,-)$. In addition one is given a 2-diml subspace

$$\Gamma \subset V_{in} \times V_{out}$$

which in good cases is the graph of a chain matrix $A : V_{in} \to V_{out}$. A will depend on the frequency ω, however in the "reciprocal" case $\Lambda^2 A : \Lambda^2 V_{in} \to \Lambda^2 V_{out}$ will be independent of ω. Hence I can choose a ray in $\Lambda^2 V_{in}$ and transport it to one in $\Lambda^2 V_{out}$. This gives one $SL_2(\mathbb{R})$ structures in both V_{in} and V_{out}, hence if I choose bases I get a matrix realization of A such that $A(\omega) \in SL_2(\mathbb{R})$ for $\omega \in \mathbb{R}$. If I choose a different ray in $\Lambda^2 V_{in}$, I can move to the original one by multiplying with a scalar of modulus 1. You do this both in V_{in} and V_{out}, so $A(\omega)$ gets conjugated by a scalar, hence it doesn't change.

Suppose given $\Gamma_A \subset V_{in} \times V_{out}$ which is symmetrical i.e. such that there exists σ of order 2 on $V_{in} \times V_{out}$ interchanging the factors and preserving Γ_A. Then

$$(v, Av) \in \Gamma \Rightarrow (\sigma Av, \sigma v) \in \Gamma \Rightarrow A\sigma Av = \sigma v$$

so $(\sigma A)^2 = 1$. Actually it seems to be better to think of a symmetry as being an isomorphism $\sigma : V_{in} \xrightarrow{\sim} V_{out}$ such that $(\sigma^{-1}A)^2 = I$. (We define σ to be σ^{-1} on V_{out}; then $(v, Av) \in \Gamma \overset{?}{\Rightarrow} (\sigma^{-1}Av, \sigma v) \in \Gamma$ Yes because $A\sigma^{-1}A = \sigma$.)

~~[illegible]~~

If $(\sigma^{-1}A)^2 = 1$, then the eigenvalues of $\sigma^{-1}A$ are ± 1. If both are $+1$, then $\sigma^{-1}A = I$ so

$$A = \sigma$$

so effectively A is the identity matrix. Similarly [redacted] if $\sigma^{-1}A = -1$, then $A = -\sigma$. (No Possibility of nilpotence because the roots of $x^2 - 1$ are simple). These cases aren't too interesting because A essentially [redacted] coincides with the symmetry [redacted] σ.

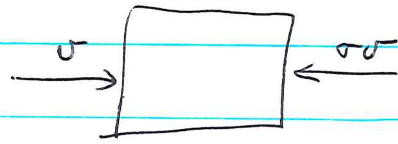
The interesting case then [redacted] is when $\sigma^{-1}A$ has the eigenvalues $+1, -1$. Note that $\Lambda^2(\sigma^{-1}A)$ in this case is -1.

<u>Wait:</u> We still have to take into account the power forms. Let $\langle , \rangle$ denote power in at V_{in} and power out at V_{out} so that for ω real

$$\langle v, v \rangle = \langle Av, Av \rangle$$

and for [redacted] $\operatorname{Im} \omega < 0$, $\quad \langle v, v \rangle \geq \langle Av, Av \rangle$

Now σ has to reverse power. If one has an input v with power in $\langle v, v\rangle$, then σv is the symmetrical input



and it has to have the same power in, hence

$$\langle \sigma v, \sigma v\rangle = -\langle v, v\rangle$$

So it is impossible for $A = \sigma$ or $A =$ [redacted] σ and therefore one always has eigenvalues $+1, -1$ for $\sigma^{-1}A$.

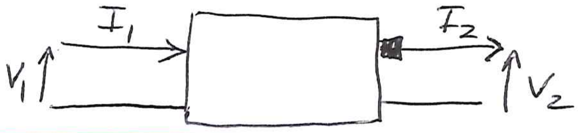
(From now on you want think of [redacted] a port V as being the complexification of a 2-dimensional real vector space with a symplectic form P [redacted] (v, w). This form is then extended to a skew-hermitian form on V and then the power form is $\langle v, w\rangle = \frac{1}{i} P(v, w)$.)

It is clear that before one can talk [redacted] about symmetric 2 ports one has to be given [redacted] the form of σ. Once V_{in}, V_{out}, σ are given one can then discuss those A's which are [redacted] symmetric. Let us identify V_{in} and V_{out} by $A(0)$. The idea is that for a transmission line at frequency zero one has

$$\text{[redacted]} \quad \frac{\partial v}{\partial x} = \frac{\partial i}{\partial x} = 0$$

so that the transfer matrix is the identity. Notice that we are using de Branges normalization [redacted] here.

If one does this then $V_{in} = V_{out}$ splits into $+$ and $-$ pieces under σ, so one might as well suppose one has the standard voltage-current picture



Here an element of V_{in} is a vector $\begin{pmatrix} iV_1 \\ I_1 \end{pmatrix}$.
Its power in is

$$\frac{1}{i}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} iV_1 \\ I_1 \end{pmatrix}, \begin{pmatrix} iV_1 \\ I_1 \end{pmatrix}\right) = \frac{1}{i}\left(\begin{pmatrix} -I_1 \\ iV_1 \end{pmatrix}, \begin{pmatrix} iV_1 \\ I_1 \end{pmatrix}\right)$$

$$= \frac{1}{i}\left\{ (-I_1)(-i\bar{V}_1) + (iV_1)\bar{I}_1 \right\}$$

$$= I_1\bar{V}_1 + \bar{I}_1 V_1 = 2\,\mathrm{Re}(V_1\bar{I}_1)$$

σ takes $\begin{pmatrix} iV_1 \\ I_1 \end{pmatrix}$ to $\begin{pmatrix} iV_1 \\ -I_1 \end{pmatrix}$. $\qquad \sigma : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Suppose we've made this choice of σ. Then A is symmetric when $\sigma^{-1}A\,\sigma^{-1}A = I$

$$\sigma A \sigma^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}$$

So symmetry amounts to the diagonal entries of the chain matrix being equal.