December 19, 1978

One basic problem I haven't made any progress on is to understand [redacted] the significance of the operator $1 - G_k^+ V$ on the whole Hilbert space $\mathcal{H}$. What I have is the operator Ω^+ on $\mathcal{H}$ satisfying

$$\Omega^+ H_0 = H \Omega^+$$

and hence which induces a map

$$\Omega_k^+ : \operatorname{Ker}(H_0 - k) \longrightarrow \operatorname{Ker}(H-k).$$

which is inverse to $1 - G_k^+ V$. Here k perhaps should be real, but first I ought to see [redacted] how much can be put in a distribution setting.

So return to the case of the perturbation $U = U_0(1+V)$ where U_0 is the shift on ℓ^2 and V has finite support. Then we work not just with ℓ^2 but with [redacted] $C_0(\mathbb{Z}) = \{$compact support sequences$\}$ (analogue of $C_0^\infty(\mathbb{R})$) and $C(\mathbb{Z}) =$ all sequences (analogues of distributions). Recall

$$\Omega^\pm = \lim_{t \to \pm\infty} U^t U_0^{-t}$$

make[redacted] sense on $C_0(\mathbb{Z})$; are they defined on $C(\mathbb{Z})$? Try defining by duality: $f \in C$, $g \in C_0$

$$\lim_{t\to\infty} \left(U^t U_0^{-t} f, g\right) = \lim_{t\to\infty} \left(f, U_0^t U^{-t} g\right)$$

For this to exist for all $f \in C$ one would require $U_0^t U^{-t} g$

to converge in C_0, i.c. with fixed support. So you obviously run into trouble with bound states.

But if $\exists$ no bound states $\lim\limits_{t\to\infty} U_0^t U^{-t} g$ is not of compact support in general. For example take $g = \delta_n$ $n \gg 0$. Then the trajectory $U^t g$ represents an outgoing pulse wave and $\lim\limits_{t\to\infty} U_0^t U^{-t} g$ says what it looked like initially. Thus if the scattering matrix $S(z)$ is not monomial, this last limit has infinite support extending to the right of δ_n. We can say, however:

$$\Omega^+ f = \lim_{t\to+\infty} U^t U_0^{-t} f$$ exists for f supported in a left half-line

$$\Omega^- f = \lim_{t\to-\infty} U^t U_0^{-t} f$$ exists for f supported in a right half-line

So one sees that Ω^+ is an operator on those sequences f supported in a left-half-line. The Fourier transforms of such f which have at most exponential growth are Laurent series convergent in a neighborhood of $z = \infty$, with perhaps poles at $z = \infty$.

December 20, 1978:

The problem of the significance of the operator $1 - G_k^+ V$ remains. Yesterday I began to look at this discretely:

U_0 = shift on sequences: $U_0\{a_n\} = \{a_{n-1}\}$, $U = U_0(I+V)$. The operator

$$\Omega = \lim_{t\to\infty} U^{-t} U_0^t$$

is defined on sequences which are zero for $n \ll 0$, and under the assumption that U is unitary, it is probable that Ω is defined for sequences decaying exponentially as $n \to -\infty$, for example, the sequences

$$\varphi_\zeta = \{\zeta^{-n}\} \qquad |\zeta| < 1.$$

[redacted] Fourier (or Laplace) transform sets up a 1-1 correspondence between sequences $\{a_n\}$ supported in $\{n \geq 0\}$ [redacted] of exponential growth and functions analytic in a nbd. of zero:

$$\{a_n\}_{n\geq 0} \longmapsto \sum a_n z^n$$

$$\oint f(z) z^{-n} \frac{dz}{2\pi i z} \longleftarrow f$$

The inversion formula shows that the sequences φ_ζ for $|\zeta| < 1$ are dense in some sense in the space of sequences on which Ω is defined. Therefore Ω will be described when we [redacted] tell how to compute $\Omega\varphi_\zeta$. [redacted]

$$(U_0 \varphi_\zeta)(n) = \zeta^{-n+1} = \zeta \cdot \varphi_\zeta(n)$$

It's more or less clear from $\Omega\varphi_\zeta = \lim_{t\to\infty} \zeta^t U^{-t} \varphi_\zeta$ that to the right of the perturbation $\Omega\varphi_\zeta = \varphi_\zeta$, so consequently $\Omega\varphi_\zeta = \psi_\zeta$ should be the eigenvector for U coinciding with φ_ζ to the far right. Find it as usual:

$$U\psi_\zeta = \zeta\psi_\zeta$$

$$\shortparallel$$

$$(U_0 + U_0 V)\psi_\zeta$$

$$\underbrace{(U_0-\zeta)}_{\text{inv. on } \ell^2}(\psi_\zeta - \varphi_\zeta) = (U_0-\zeta)\psi_\zeta = \underbrace{-U_0 V\psi_\zeta}_{\text{in } \ell^2}$$

$$\therefore \quad \psi_\zeta = \varphi_\zeta - (U_0-\zeta)^{-1} U_0 V \psi_\zeta$$

So if $G_\zeta^+ = (U_0-\zeta)^{-1}U_0 = (1-\zeta U_0^{-1})^{-1}$ on ℓ^2, then we get

$$(1+G_\zeta^+ V)\psi_\zeta = \varphi_\zeta$$

It might be better to work with ℓ^2 quantities:

$$\psi_\zeta - \varphi_\zeta = -G_\zeta^+ V(\psi_\zeta - \varphi_\zeta + \varphi_\zeta)$$

$$\boxed{(1+G_\zeta^+ V)(\psi_\zeta - \varphi_\zeta) = -G_\zeta^+ V\varphi_\zeta}$$

This is an equation to be solved in ℓ^2. You can do it by Fredholm alternative, the point being that

$$(U_0 - \zeta)(1 + G_\zeta^+ V) = U - \zeta$$

so that $\operatorname{Ker}(1 + G_\zeta^+ V) \subset \operatorname{Ker}(U - \zeta) = 0$ if U unitary and $|\zeta| < 1$.

So we are back at the same problem: the operator $1 + G_\zeta^+ V$ is nicely related to the ζ-part of Ω, but we don't have an understanding of $1 + G_\zeta^+ V$ on the whole space.

Picture of G_ζ^+. Recall

$$(G_\zeta^+ f)(n) = \left((1 - \zeta U_0^{-1})^{-1} f\right)(n) =$$

$$= f(n) + \zeta f(n+1) + \zeta^2 f(n+2) + \cdots$$

$$= \sum_{n' \geq n} \zeta^{n'-n} f(n')$$

$$\therefore \quad G_\zeta^+(n, n') = \begin{cases} 0 & n' < n \\ \zeta^{n'-n} & n' \geq n \end{cases}$$

$$\begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 & & \cdot^{\cdot^{\cdot}} \\ & 1 & \zeta & \zeta^2 & \zeta^3 & \\ & & 1 & \zeta & \zeta^2 & \zeta^3 \\ & & & 1 & \zeta & \zeta^2 \\ & 0 & & & 1 & \zeta \\ & & & & & 1 \end{pmatrix}$$

<u>Idea:</u> Our earlier study of scattering on the line provides a way of going from scattering back to the potential which produces it. Thus it produces a quasi-canonical perturbation from a scattering coefficient $S(k)$. In this case it might be the case that an interpretation of $1 - G_k^+ V$ is possible.

So consider potential scattering on a half-line $x \geq 0$. We need to construct a free situation $U_0(t)$.

<u>December 21, 1978:</u>

Consider a Dirac system on $[0, \infty)$:

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & h \\ \bar{h} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad u_1(0) = u_2(0)$$

This defines a self-adjoint operator on $L^2([0,\infty))^{\oplus 2}$. The unperturbed case occurs with $h = 0$. Solutions of the corresponding wave equation are

$$\underline{u}(x,t) = \int e^{-ikt}\begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} \alpha(k)\, dk/2\pi$$

$$= \begin{pmatrix} \hat{\alpha}(x-t) \\ \hat{\alpha}(-x-t) \end{pmatrix}$$

The idea is to identify $L^2(\mathbb{R}_{\geq 0})^{\oplus 2}$ with $L^2(\mathbb{R})$ by putting

$$u(x) = \begin{cases} u_1(x) & x \geq 0 \\ u_2(-x) & x \leq 0 \end{cases}$$

so the solutions of the free system are

$$u(x,t) = \hat{\alpha}(x-t) \qquad \text{or} \qquad u(x,k) = \alpha(k)e^{ikx}$$

For the perturbed system we get the equation

$$\frac{d}{dx}u(x) = ik\,u_1(x) + h(x)\,u_2(x)$$

$$= ik\,u(x) + h(x)\,u(-x) \qquad x \geq 0$$

$$\frac{d}{dx}u(x) = \frac{d}{dx}u_2(-x)$$

$$= [redacted]\ u_2'(-x)(-1)$$

$$= -\{\bar{h}(-x)u_1(-x) - ik\,u_2(-x)\}$$

$$= ik\,u(x) - \bar{h}(-x)\,u(-x) \qquad x \leq 0.$$

$$\therefore \quad \boxed{\frac{d}{dx}u(x) = ik\,u(x) + \tilde{h}(x)u(-x) \qquad \text{all } x}$$

where $\tilde{h}(x) = \begin{cases} h(x) & x \geq 0 \\ -\bar{h}(-x) & x \leq 0 \end{cases}$

Now drop $\sim$ from $\tilde{h}$. For h to be continuous [redacted] we need $h(0) = 0$.

Let us now recall how such a Dirac system arises by factoring a Schroed. equation. If $h = \bar{h}$, then the Dirac system can be written

$$\frac{d}{dx}\begin{pmatrix} u_1+u_2 \\ u_1-u_2 \end{pmatrix} = \begin{pmatrix} h & ik \\ ik & -h \end{pmatrix}\begin{pmatrix} u_1+u_2 \\ u_1-u_2 \end{pmatrix}$$

so that if we put $\psi = u_1 + u_2$ we get

$$\left(\frac{d}{dx} + h\right)\left(\frac{d}{dx} - h\right)\psi = -k^2\psi \qquad \text{or}$$

1) $$\left\{-\frac{d^2}{dx^2} + \overset{V}{\text{[redacted]}}\right\}\psi = k^2\psi$$

where $V = h^2 + h'$. The boundary condition corresp. to $u_1(0) = u_2(0)$ is

2) $$\frac{d\psi}{dx}(0) = h(0)\psi(0)$$

[redacted] Conversely given a Schroedinger equation 1) on $(0,\infty)$ with a boundary condition 2), if there is [redacted] no negative spectrum, then "the" solution ϕ of 1) for $k=0$ satisfying 2) doesn't vanish [redacted] and

$$h = \phi'/\phi$$

satisfies $h' = V - h^2$, so we get a Dirac system.

The simplest case appears to be where [redacted] the Schroed. problem is symmetric around 0 and given on all of $\mathbb{R}$. Then $V(x) = V(-x)$ and $h(0) = 0$. Then $h = \phi'/\phi$ is an odd fn. because ϕ is even.

So consider now the basic equation

(*) $$\left(\frac{d}{dx} - ik\right)\psi(x) = h(x)\,\psi(-x) \qquad \text{on } \mathbb{R}$$

and compute the scattering. Take $\varphi = e^{-ikx}$ and look for ψ satisfying (*) such that $\psi - \varphi \in L^2$.

~~[illegible]~~

$$\Big[e^{-ikx}(\psi(x)\ \text{[redacted]} - \varphi(x))\Big]_{-\infty}^{\cancel{\infty}\,x} = \int_{-\infty}^{x} e^{-ikx'} h(x')\,\psi(-x')\,dx'$$

$$\psi(x) = e^{ikx} + \int_{-\infty}^{x} e^{ik(x-x')} h(x')\psi(-x')\,dx'$$

As a check note that the appropriate Green's function for $\frac{d}{dx} - ik$ and $\operatorname{Im} k > 0$ is

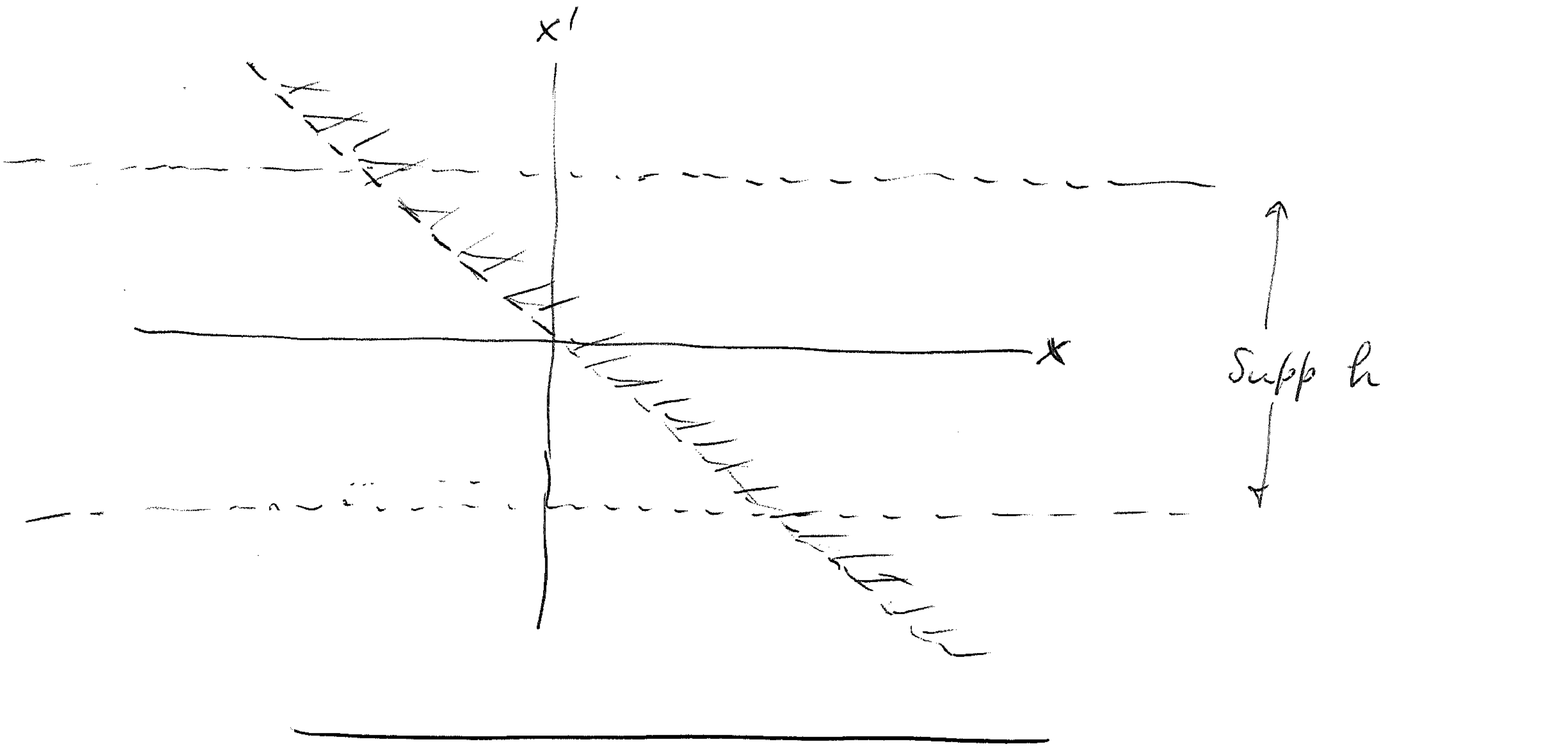
$$G_k(x,x') = \begin{cases} e^{-ik(x-x')} & x > x' \\ 0 & [redacted]\ x < x' \end{cases}$$

[redacted] The integral equation should be written in the following more familiar form

$$\boxed{\psi(x) = e^{ikx} + \int_{-x}^{\infty} e^{ik(x+x')} h(-x')\psi(x')\,dx'}$$

with the kernel

$$K(x,x') = \begin{cases} e^{ik(x+x')} h(-x') & [redacted]\ x+x' \geq 0 \\ 0 & x+x' \leq 0 \end{cases}$$



Still the basic problem remains: to understand the LS operator $1 - G_k^+ V$.

Suppose k such that $\det(1 - G_k^+ V) = 0$. Recall that we compute this determinant on the support of V. By the Fredholm theory there is a non-zero ψ defined on $\mathrm{Supp}(V)$ with

$$\psi = G_k^+ V \psi \qquad \text{on Supp } V$$

and then we can use the right-side to define ψ off Supp V. So we conclude that $\det(1 - G_k^+ V) = 0$ signifies the existence of a $\psi \neq 0$ such that

$$\text{(A)} \qquad \psi = G_k^+ V \psi$$

When $\mathrm{Im}(k) > 0$, the right-side is in L^2, but not necessarily for $\mathrm{Im}(k) < 0$.

Since

$$\left(\frac{d}{dx} - ik\right)\left(1 - G_k^+ V\right) = \left(\frac{d}{dx} - ik - V\right)$$

any solution of $(*)$ satisfies

$$\text{(B)} \qquad \left(\frac{d}{dx} - ik - V\right)\psi = 0$$

Conversely if ψ satisfies (B), then $\psi - G_k^+ V \psi$ is a multiple of e^{ikx}.

$$\left(\frac{d}{dx}-ik\right)\left(1-G_k^+V\right)=\left(\frac{d}{dx}-ik-V\right)$$

$$\Rightarrow \qquad \mathrm{Ker}\left(1-G_k^+V\right) \subset \mathrm{Ker}\left(\frac{d}{dx}-ik-V\right).$$

But $\psi = G_k^+V\psi$ means

$$\psi(x) = \int_{-\infty}^{x} e^{ik(x-x')}\, h(x')\,\psi(-x')\,dx'$$

$$= 0 \qquad \text{for} \qquad x < \mathrm{Supp}(h).$$

~~[illegible]~~ Thus if $\psi \in \mathrm{Ker}\left(\frac{d}{dx}-ik-V\right)$, then from the first formula we know $\psi - G_k^+V\psi \in \mathrm{Ker}\left(\frac{d}{dx}-ik\right)$, so that $\psi - G_k^+V\psi$ is a multiple of e^{ikx}. We get the zero multiple by putting in the boundary condition $\psi = 0$ for $x <$ Supp h.

Recall that in the case of the original Dirac system

$$\frac{d}{dx}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}ik & h\\ \bar{h} & -ik\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

we have for the solution $\phi(x,k)$ starting at $\binom{1}{1}$ that

$$\phi(x,k)=\begin{pmatrix}B(k)\,e^{ikx}\\ A(k)\,e^{-ikx}\end{pmatrix} \qquad B(k) = \overline{A(\bar{k})}$$

for $x > \mathrm{Supp}(h)$. Thus the extra boundary condition besides $u_1(0) = u_2(0)$ needed to describe solns. of $\psi = G_k^+V\psi$ is that $u_2 = 0$ for $x \gg 0$. It's more or less clear that in this case

$$\det\left(1-G_k^+V\right) = A(k)$$

up to some constant.

The next idea will be to recognize if there is some way of getting $\det(1 - G_k^+ V)$ into the form of a characteristic polynomial. Can we recognize different operators of the form $1 - G_k^+ V$ leading to the same determinant?

For example, compare a Schroedinger DE with the Dirac DE obtained by factorization. Take up the situation at the bottom of 405 where we have a Schroedinger equation with $V(x) = V(-x)$

$$\left(\frac{d^2}{dx^2} - V + k^2\right)\psi = 0$$

$$\frac{d\psi}{dx}(0) = 0$$

which is factorable: $V = h^2 + h'$ where $h = \phi'/\phi$, $\phi'' = V\phi$, $\phi'(0) = 0$, $\phi(0) = 1$. The Green's function for $\frac{d^2}{dx^2} + k^2$ on $[0, \infty)$ with $\psi'(0) = 0$ is

$$G_k^+(x, x') = \frac{\cos kx_< \, e^{ikx_>}}{ik}$$

and so our basic determinant is

$$\det\left(I - G_k^+ V\right) = 1 - \int_0^\infty \frac{\cos kx \;\; e^{ikx}}{ik} V(x)\,dx + \ldots$$

How does this compare with

$$\det\left(I - K_{\text{on page 407}}\right) = 1 - \int_0^\infty e^{2ikx} h(-x)\,dx + \ldots\ldots ?$$

The former seems to have a pole at $k = 0$, but not the latter

Check: Let $\phi(x,k^2)$ be the solution of Schroedinger with [redacted] $\phi(0)=1$, $\phi'(0)=0$. Then [redacted] for large x we have

$$\phi(x,k^2) = \tfrac{1}{2}\{A(k)e^{-ikx} + B(k)e^{ikx}\}.$$

[redacted] We should have (if preceding guesses are OK)

$$\det(1-G_k^+V) = \frac{W(\phi,\, e^{ikx})_{x\gg 0}}{W(\cos kx,\, e^{ikx})} = \frac{\frac{1}{2}A(k)\,2ik}{\frac{1}{2}\;2ik}$$

$$= A(k).$$

On the other hand $\phi = \dfrac{\phi_1+\phi_2}{2}$ where $\underline{\phi} = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ satisfies Dirac eqn. with initial values $\begin{pmatrix}1\\1\end{pmatrix}$. Consequently $A(k)$ should be an entire fn. ??

This difficulty already occurs on the line when we established that if

$$e^{-ikx} \xleftrightarrow{\;\phi(x,k)\;} A(k)e^{-ikx} + B(k)e^{ikx}$$

then

$$\det(1-G_k^+V) = A(k).$$

$$\parallel$$

$$1 - \int_{-\infty}^{\infty}\frac{V(x)}{2ik}\,dx + \frac{1}{(2ik)^2}\iint_{x_1\geq x_2} dx_1dx_2\,V(x_1)V(x_2)\left(1-e^{2ik(x_1-x_2)}\right) - \cdots$$

Each term in the expansion of the determinant has a first order pole at $k=0$.

We can ask the same question on the line — namely comparing the determinants of a Schroedinger DE with a Dirac factoring:

Compute Green's function for D-system

$$\begin{pmatrix} \frac{d}{dx} - ik & 0 \\ 0 & \frac{d}{dx} + ik \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$G_k^+(x, x') = \begin{pmatrix} e^{ik(x-x')}\eta(x-x') & 0 \\ 0 & -e^{-ik(x-x')}\eta(-x+x') \end{pmatrix}$$

where η is the Heaviside function. Thus

$$K = G_k^+ V = \begin{pmatrix} 0 & e^{ik(x-x')}\eta(x-x')\,h(x') \\ -e^{-ik(x-x')}\eta(-x+x')\,h(x') & 0 \end{pmatrix}$$

$$\operatorname{tr} K = 0 \qquad \operatorname{tr}(K^2) = \iint \begin{pmatrix} e^{ik(x-y)}\eta(x-y)\,h(y)\cdot(-1)e^{-ik(y-x)}\eta(-y+x)\,h(x) & 0 \\ 0 & -e^{-ik(x-y)}\eta(-x+y)h(y)e^{ik(y-x)}\eta(y-x)h(x) \end{pmatrix}$$

$$\therefore \operatorname{tr}(K^2) = -2\iint_{x>y} e^{2ik(x-y)}\,h(x)\,h(y)\,dx\,dy$$

But $\operatorname{tr}(\Lambda^2 K) = \frac{1}{2}\sum_{i,j} \begin{vmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{vmatrix} = \frac{1}{2}\left\{(\operatorname{tr} K)^2 - \operatorname{tr} K^2\right\}$

So

$$\operatorname{tr}(\Lambda^2 K) = \iint_{x>y} e^{2ik(x-y)}\,h(x)\,h(y)\,dx\,dy$$

So the first term for Dirac determinant is

$$1 + \iint_{x>y} e^{2ik(x-y)} h(x) h(y)\, dx\, dy + \cdots$$

whereas the first term for the Schroedinger det. is

$$1 - \int_{-\infty}^{\infty} \frac{V(x)}{2ik}\, dx + \quad = 1 - \int_{-\infty}^{\infty} \frac{h(x)^2}{2ik} dx$$

These two are related by an integration by parts

$$\iint_y^{\infty} dx \quad e^{2ik(x-y)} h(x) h(y) = \int dy \left[\frac{e^{2ik(x-y)}}{2ik} h(x) h(y) \right]_y^{\infty}$$

$$- \int dy \int_{x>y}^{\infty} dx \quad \frac{e^{2ik(x-y)}}{2ik} h'(x) h(y)$$

$$= - \int dy \frac{h(y)^2}{2ik} - \iint_{x>y} \frac{e^{2ik(x-y)}}{2ik} h'(x) h(y)\, dx\, dy$$

So therefore maybe it is reasonable to expect the Dirac and Schroed. determinants to coincide.

Question: Can we prove the equality of these determinants? Notice the Schroedinger determinant looks like it might have a pole at $k=0$, although in fact we know it doesn't because there is a tunnel solution.

December 23, 1978:

Let's show that [redacted] the two determinants have the same zeroes. Start with a solution of $\underline{\underline{u}} = \underline{\underline{G}}_k^+ \underline{\underline{V}}\,\underline{\underline{u}}$ i.e.

$$u_1(x) = \int_{-\infty}^{x} e^{ik(x-x')} h(x') u_2(x')dx'$$

$$u_2(x) = -\int_{x}^{\infty} e^{-ik(x-x')} h(x') u_1(x')dx'$$

Then

$$\frac{du_1}{dx} = iku_1 + hu_2 \qquad \text{and} \quad u_1 = 0 \qquad x \ll 0$$

$$\frac{du_2}{dx} = -iku_2 + hu_1 \qquad \text{and} \quad u_2 = 0 \qquad x \gg 0$$

Consequently $\psi = \frac{1}{2}(u_1+u_2)$ satisfies Schroedinger

$$\left(\frac{d}{dx}+h\right)\left(\frac{d}{dx}-h\right)\psi = -k^2\psi$$

and

$$\psi = \text{multiple of } e^{ikx} \qquad x \gg 0$$

$$\text{" \quad " } \quad e^{-ikx} \qquad x \ll 0$$

Thus $\psi - G_k^+ V\psi = \varphi$ will satisfies $(\Delta + k^2)\varphi = 0$ and since

$$G_k^+ V\psi = \int \frac{e^{ik|x-x'|}}{2ik} V(x')\psi(x')dx'$$

$$= \text{mult of } e^{-ikx} \qquad x \gg 0$$

$$\text{" \quad " } \quad e^{-ikx} \qquad x \ll 0$$

it follows that φ has the same property, which implies $\varphi = 0$.

Conversely from $\qquad \psi = \frac{1}{2}(u_1+u_2)$

$$\frac{1}{ik}\left(\frac{d}{dx}-h\right)\psi = \frac{1}{2}(u_1-u_2)$$

we have
$$u_1 = \psi + \frac{1}{ik}\left(\frac{d}{dx} - h\right)\psi$$
$$u_2 = \psi - \frac{1}{ik}\left(\frac{d}{dx} - h\right)\psi .$$

Hence if $\psi = G_k^+ V \psi$, then ψ satisfies Schroedinger and the outgoing radiation boundary conditions, so we know u_1, u_2 satisfy the Dirac system with the appropriate boundary conditions.

The interesting question is whether the two operators $1 - G_k^+ V$ and $1 - \mathbb{G}_k^+ \mathbb{V}$ are conjugate in a suitable way that implies their determinants are equal.

Suppose we proceed formally:

$$\mathbb{K} = \mathbb{G}_k^+ \mathbb{V} = \begin{pmatrix} \left(\frac{d}{dx} - ik\right)^{-1} & 0 \\ 0 & \left(\frac{d}{dx} + ik\right)^{-1} \end{pmatrix} \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \left(\frac{d}{dx} - ik\right)^{-1} h \\ \left(\frac{d}{dx} + ik\right)^{-1} h & 0 \end{pmatrix}$$

$$\log \det (1 - \mathbb{K}) = -\sum_{m=1}^{\infty} \frac{1}{m} (\operatorname{tr} \mathbb{K}^m)$$ odd terms vanish

$$= \frac{1}{2}\left(-\sum_{1}^{\infty} \frac{1}{m} \operatorname{tr}(\mathbb{K}^{2m}) \right) = \frac{1}{2} \log \det (1 - \mathbb{K}^2)$$

Put $Q_+ = \left(\frac{d}{dx} - ik\right)^{-1} h$, $Q_- = \left(\frac{d}{dx} + ik\right)^{-1} h$. Then

$$\mathbb{K}^2 = \begin{pmatrix} 0 & Q_+ \\ Q_- & 0 \end{pmatrix} \begin{pmatrix} 0 & Q_+ \\ Q_- & 0 \end{pmatrix} = \begin{pmatrix} Q_+ Q_- & 0 \\ 0 & Q_- Q_+ \end{pmatrix}$$

It is clear that

$$\det(1-Q_+Q_-) = \det(1-Q_-Q_+)$$

because using the log formula

$$\begin{aligned} tr(Q_+Q_-)^m &= tr\, Q_+Q_- \cdots Q_+Q_- \\ &= tr\, Q_- \cdots Q_+Q_-Q_+ \\ &= tr\,(Q_-Q_+)^m \end{aligned}$$

Thus we have

$$\begin{aligned} \det(1-\mathbb{K}) = \det(1-Q_+Q_-) &= \det(1-Q_-Q_+) \\ &= \det\left(1-\left(\tfrac{d}{dx}+ik\right)^{-1} h \left(\tfrac{d}{dx}-ik\right)^{-1} h\right) \end{aligned}$$

What I want to do is to show this coincides with

$$\det(1-G_k^+V) = \det\left(1-\left(\tfrac{d}{dx}+ik\right)^{-1}\left(\tfrac{d}{dx}-ik\right)^{-1}(h'+h^2)\right).$$

First we compare kernels:

$$\left\{\left(\frac{d}{dx}+ik\right)^{-1}\left(\frac{d}{dx}-ik\right)^{-1}\right\}(x,x') = \int dy\,(-1)e^{-ik(x-y)}\eta(-x+y)\, e^{ik(y-x')}\eta(y-x')$$

$$= e^{-ik(x+x')}\int_{\max(x,x')}^{\infty}(-1)e^{2iky}\,dy = e^{-ik(x+x')}\frac{e^{2ik\max(x,x')}}{2ik}$$

$$= \frac{e^{ik|x-x'|}}{2ik} \qquad \text{This checks}$$

So $\left(\frac{d^2}{dx^2}+k^2\right)^{-1}(h'+h^2)$ has the kernel

$$\frac{e^{-ik|x-x'|}}{2ik}\left(h'(x')+h(x')^2\right)$$

whereas $\left(\frac{d}{dx}+ik\right)^{-1} h \left(\frac{d}{dx}-ik\right)^{-1} h$ has the kernel

$$\int -e^{-ik(x-y)}\, \eta(-x+y)\, h(y)\, e^{ik(y-x')}\, \eta(y-x')\, h(x')\, dy$$

$$= e^{-ik(x+x')} \int_{\max(x,x')}^{\infty} (-1)\, e^{2iky}\, h(y)\, dy\ h(x')$$

These kernels are obviously different especially the former because it has a first order term in h which has to disappear.

$$\det\left(1-\left(\frac{d}{dx}+ik\right)^{-1}h\left(\frac{d}{dx}-ik\right)^{-1}h\right) \stackrel{?}{=} \det\left(1-(\Delta+k^2)^{-1}(h'+h^2)\right)$$

Proceed formally:

$$\det\left(1-(\Delta+k^2)^{-1}(h'+h^2)\right) = \frac{\det(\Delta-V+k^2)}{\det(\Delta+k^2)}$$

$$= \det\begin{pmatrix}\frac{d}{dx}-h & ik \\ ik & \frac{d}{dx}+h\end{pmatrix} \Big/ \det(\Delta+k^2)$$

$$= \det\begin{pmatrix}\frac{d}{dx}-ik & h \\ h & \frac{d}{dx}+ik\end{pmatrix} \Big/ \det\begin{pmatrix}\frac{d}{dx}-ik & 0 \\ 0 & \frac{d}{dx}+ik\end{pmatrix} \qquad \text{conjugation with } \begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}$$

$$= \det\begin{pmatrix}1 & \left(\frac{d}{dx}-ik\right)^{-1}h \\ \left(\frac{d}{dx}+ik\right)^{-1}h & 1\end{pmatrix}$$

$$= \det\left(1-\left(\frac{d}{dx}+ik\right)^{-1}h\left(\frac{d}{dx}-ik\right)^{-1}h\right)$$

More succintly

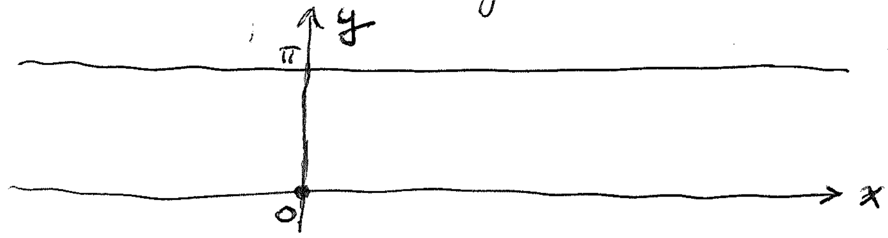
$$\det\left(1-(\Delta+k^2)^{-1}(h'+h^2)\right) = \frac{\det(\Delta-V+k^2)}{\det(\Delta+k^2)}$$

$$= \det\begin{pmatrix}\frac{d}{dx}-h & ik \\ ik & \frac{d}{dx}+h\end{pmatrix} \Big/ \det\begin{pmatrix}\frac{d}{dx} & ik \\ ik & \frac{d}{dx}\end{pmatrix}$$

$$= \det\left\{1-\begin{pmatrix}\frac{d}{dx} & ik \\ ik & \frac{d}{dx}\end{pmatrix}^{-1}\begin{pmatrix}-h & 0 \\ 0 & h\end{pmatrix}\right\}$$

And now you finish by conjugation. The key steps in this formal argument are the above three equalities.

Obstacles in a wave-guide:



[redacted] To solve $(\Delta + k^2)\psi = 0$ inside with $\psi = 0$ on walls. Typical Dirichlet problem.

Free case – no obstacle present: Expand ψ in a sine series in y:

$$\psi(x,y) = \sum_1^\infty \psi_n(x) \sin ny$$

then

$$\left(\frac{d^2}{dx^2} + k^2 - n^2\right)\psi_n = 0$$

so ψ_n is a linear comb. of $e^{\pm i k_n x}$ where

$$k_n^2 = k^2 - n^2$$

Think of k in the UHP and take $k_n \in$ UHP.
In practice the operating frequency k satisfies

$$0 < k^2 - 1 \qquad k^2 - 2^2 < 0$$

or

$$1 < k < 2$$

so that only k_1 is real. Thus the lowest mode propagates and the higher ones attenuate.

Next we want the Green's function for the wave-guide, that is, we want to solve the inhomogeneous equation:

$$(\Delta + k^2)\psi = f$$

Expand f:

$$f = \sum_1^\infty f_n(x) \sin ny \qquad f_n = \int_0^\pi f \sin ny \, dy/(\pi/2)$$

Then to solve

$$\left(\frac{d^2}{dx^2} + k^2 - n^2\right)\psi_n = f_n$$

we use the appropriate Green's function:

$$\psi_n = \int \frac{e^{ik_n|x-x'|}}{2ik_n} f_n(x')\,dx'$$

Then

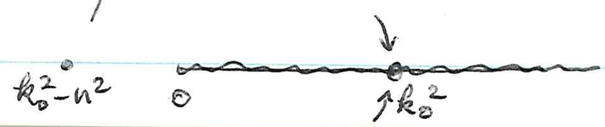
$$\psi(x,y) = \int_{-\infty}^{\infty}\int_0^\pi \sum_1^\infty \frac{e^{ik_n|x-x'|}}{2ik_n} \frac{\sin ny \sin ny'}{(\pi/2)} f(x',y')\, dy'dx'$$

so the Green's function for the waveguide is

$$\boxed{G_k^+(x,y,x',y') = \sum_1^\infty \frac{e^{ik_n|x-x'|}}{2ik_n} \frac{\sin ny \sin ny'}{(\pi/2)}}$$

The $+$ indicates that we start with k in the UHP and take $k_n = \sqrt{k^2-n^2}$ in the UHP and then we try to analytically continue.

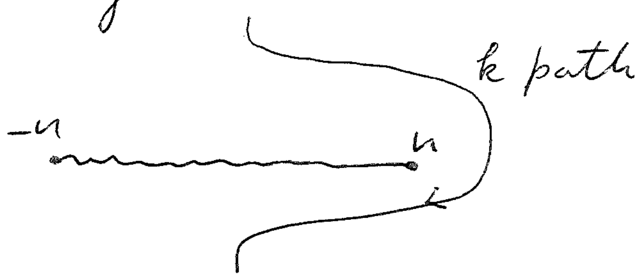
This analytic continuation is very interesting because of the branches of $k_n = \sqrt{k^2-n^2}$. Let $k_0 > 0$ and let us compare what happens to k_n as $k \to k_0$ and $k \to -k_0$ from the UHP. If $k_0 < n$, then



so that $k_n^2 \to k_0^2 - n^2$ away from the branch cut for $\sqrt{z}$, so that the two approaches yield the same value for k_n which is on $i\mathbb{R}_{>0}$. On the other hand if $k_0 > n$, then $k_n^2 \longrightarrow k_0^2 - n^2$ from opposite sides of the cut, so we get opposite signs for the two limiting values of k_n.

So it is clear that analytic continuation from the UHP leads to different Greens functions in the LHP. All are given by the boxed formulas on the preceding page with a suitable interpretation of k_n. The key point is how we continue $k_n = \sqrt{k^2 - n^2}$: If we cross $\mathbb{R}$ between n and $-n$ the cut goes

so if $k = ia$, then $k_n = i\sqrt{a^2+n^2}$ stays in UHP. If on the other hand we cross $\mathbb{R}$ outside of $[-n, n]$, then the cut goes



and if $k = ia$ then $k_n = \begin{cases} i\sqrt{a^2+n^2} & a>0 \\ -i\sqrt{a^2+n^2} & a<0 \end{cases}$ so k_n crosses into the LHP.

Thus depending on where k crosses $\mathbb{R}$ we get a range of k_n changing sign, and hence a corresponding Green's function.

Once the type of Green's function has been specified (and to simplify suppose we cross R through $1<|k|<2$ so that only k_1 changes to the LHP) then we have [redacted] a notion of outgoing wave defined. For example

$$e^{ik_1 x} \sin [redacted]\, y \qquad x \to +\infty$$

represents an outgoing wave; if $\operatorname{Im} k > 0$ then $\operatorname{Im} k_1 > 0$ and this wave decays and if $\operatorname{Im} k < 0$, then $\operatorname{Im} k_1 < 0$ and this wave grows.

Look at potential scattering. Then for φ a free equation solution, then a solution of

$$\psi = \varphi + G_k V \psi$$

is the same as a solution of

$$(\Delta + k^2)\psi = V\psi$$

such that $\psi - \varphi$ consists of only outgoing waves. [redacted] Furthermore

$$\psi = G_k V \psi \iff (\Delta + k^2)\psi = V\psi \text{ and } \psi \text{ is outgoing.}$$

Thus the Green's function [redacted] gives the boundary condition at ∞.

Scattering by a sphere: $r=a$. To solve $(\Delta+k^2)\psi=0$ for $r>a$ with $\psi=0$ on $r=a$. Because of spherical symmetry we can expand ψ using spherical harmonics

$$\psi=\sum_{l,m}\psi_{lm}(r)\,Y_{lm}$$

and then ψ_{lm} satisfies

$$\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}-\frac{l(l+1)}{r^2}+k^2\right)\psi_{lm}(r)=0$$

$$\psi_{lm}(a)=0.$$

or

$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+k^2\right)\left(r\psi_{lm}(r)\right)=0$$

(*) The DE
$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-1\right)u=0$$

is equivalent to Bessel's DE modified of order $l+\frac{1}{2}$, but it will be instructive to directly calculate its [redacted] solutions. [redacted] Put $u=e^{-r}v$ and you get

$$\left(\left(\frac{d}{dr}-1\right)^2-\frac{l(l+1)}{r^2}-1\right)v=0$$

$$\left(\frac{d^2}{dr^2}-2\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)v=0$$

Expand $v=\sum a_n r^{-n}$ formally:

$$\sum_{n\geq 0}(-n)(-n-1)a_n r^{-n-2}+\sum_{n\geq 1}2na_n r^{-n-1}-\sum_{n\geq 0}l(l+1)a_n r^{-n-2}$$

$$2(n+1)\,a_{n+1} = \underbrace{\{\ell(\ell+1) - n(n+1)\}}_{\ell^2 - n^2 + \ell - n} a_n$$

$$= (\ell-n)(\ell+n+1)\,a_n$$

$$a_n = \frac{(\ell-n+1)(\ell+n)}{2n}\,a_{n-1}$$

Thus the series stops with [redacted] $n=\ell$. We get

$$a_n = \frac{(\ell-n+1)\cdots(\ell-1)\ell\;(\ell+n)\cdots(\ell+1)}{2^n\, n!}$$

$$a_n = \frac{(\ell-n+1)\cdots\cdots(\ell-1)\,\ell\,(\ell+1)\cdots \text{[redacted]}\,(\ell+n)}{2^n\; n!}$$

Denote by $Q_\ell(r)$ the function

$$Q_\ell(r) = \text{[redacted]} \sum_{n=0}^{\ell} \frac{(\ell-n+1)\cdots\cdots(\ell+n)}{2^n\, n!\;r^n} \;;$$

it is a polynomial in $\frac{1}{r}$ of degree ℓ with constant term 1. Two independent solutions of

$$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + k^2\right) u = 0$$

are $\quad u = e^{-ikr} Q_\ell(ikr) \;, \quad e^{+ikr} Q_\ell(-ikr)$.

December 27, 1978:

Continue with scattering by sphere $r=a$. Separating $(\Delta+k^2)\psi=0$ in [redacted] spherical coords leads to simple solutions

$$\frac{u(r)}{r}\,Y_{lm}$$

where u satisfies the DE

$$(*)_\ell \qquad \left\{\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}+k^2\right\}u=0 .$$

This DE has the independent solutions $(k\neq 0)$

$$h_\ell(ikr)\ ,\quad h_\ell(-ikr)$$

where

$$h_\ell(r)=e^{-r}\sum_{n=0}^{\ell}\frac{(\ell-n+1)\cdots(\ell)(\ell+1)\cdots(\ell+n)}{n!\ \ 2^n}\,r^{-n}$$

$$=e^{-r}\left(1+\frac{\ell(\ell+1)}{2r}+\cdots\cdots [redacted] +\frac{(2\ell)!}{\ell!\,(2r)^\ell}\right)$$

What is the analogue of LS? Recall that analytically continuing the Green's function from the UHP amounts to choosing outgoing waves. Thus

$$\psi=\varphi+G_k^+V\psi \qquad \text{with} \quad (\Delta+k^2)\varphi=0$$

$$\iff (\Delta+k^2)\psi=V\psi \quad \text{and} \quad \psi-\varphi \text{ is outgoing}$$

Proof $(\Longrightarrow)$ obvious

$(\Longleftarrow)$ [redacted]

$(\Delta+k^2)\,(\psi-\varphi-G_k^+V\psi)=0$ and $\psi-\varphi-G_k^+\psi$ has zero incoming part, hence it must be zero. The reason there are no purely outgoing waves for $\Delta + k^2$ is that the appropriate boundary condition for $(*)_\ell$ is $u(0) = 0$ which is not satisfied by $h_\ell(-ikr)$.

Take $\ell = 0$. Then φ is a multiple of

$$\frac{e^{ikr} - e^{-ikr}}{2ik} = \frac{\sin kr}{k}$$

and ψ is a multiple of

$$\frac{\sin k(r-a)}{k} = \frac{e^{ikr}e^{-ika} - e^{-ikr}e^{ika}}{2ik}$$

Solving LS in this case, i.e. finding the ψ which is $\Omega^+\varphi$ amounts to finding the ψ with the same incoming part as φ:

$$\frac{\sin kr}{k} \xmapsto{\ \Omega^+\ } \frac{e^{-2ika}e^{ikr} - e^{-ikr}}{2ik}$$

similarly

$$\text{"} \xmapsto{\ \Omega^-\ } \frac{e^{ikr} - e^{2ika}e^{-ikr}}{2ik}$$

and so

$$\boxed{S_0(k) = e^{2ika}}.$$

Notice in this case that there are no poles to $\Omega^\pm$.

Take general ℓ: Then φ has to be regular at $r = 0$, like $r^{\ell+1}$, so it must be a multiple

of
$$h_\ell(ikr) - (-1)^\ell h_\ell(-ikr).$$

Note that $h_\ell(ikr) \sim \dfrac{(2\ell)!}{\ell!(2ikr)^\ell}$ as $r \to 0$, so we have to cancel the poles.

Now ψ has to vanish at $r = a$ and so it must be a multiple of
$$\underbrace{h_\ell(ikr)}_{\sim e^{-ikr}} - \frac{h_\ell(ika)}{h_\ell(-ika)} h_\ell(-ikr)$$
$\sim e^{-ikr}$ which is incoming.

Thus
$$h_\ell(ikr) - (-1)^\ell h_\ell(-ikr) \xmapsto{\Omega^+} h_\ell(ikr) - \frac{h_\ell(ika)}{h_\ell(-ika)} h_\ell(-ikr)$$
$$\xmapsto{\Omega^-} (-1)^\ell \frac{h_\ell(-ika)}{h_\ell(ika)} h_\ell(ikr) - (-1)^\ell h_\ell(-ikr)$$

and so
$$\boxed{S_\ell(k) = (-1)^\ell \frac{h_\ell(-ika)}{h_\ell(ika)}}$$

Singularities of Ω^+ are given by those k such that
$$h_\ell(-ika) = 0$$
i.e. when k is such that $\exists$ an outgoing solution
$$h_\ell(-ikr) \sim e^{ikr}$$
satisfying the boundary condition.

Examples: $l=1$. $h_1(r)=e^{-r}\left(1+\frac{1}{r}\right)$

$$h_1(-ikr)=e^{ikr}\left(1-\frac{1}{ikr}\right)$$

$$\therefore \quad h_1(-ika)=0 \iff k=\frac{1}{ia}$$

$l=2$ $\quad h_2(r)=e^{-r}\left(1+\frac{2\cdot 3}{2r}+\frac{4!}{2!\,4r^2}\right)=e^{-r}\left(1+\frac{3}{r}+\frac{3}{r^2}\right)$

$$h_2(r)=0 \iff r^2+3r+3=0$$

$$\iff r=\frac{-3\pm\sqrt{9-12}}{2}=-\frac{3}{2}\pm\frac{\sqrt{3}}{2}i$$

$$h_2(-ika)=0 \iff -ika=-\frac{3}{2}\pm\frac{\sqrt{3}}{2}i$$

$$\iff k=\frac{i}{a}\left(-\frac{3}{2}\pm\frac{\sqrt{3}}{2}i\right)$$ neg. imag part.

Following what Victor said about Lax-Phillips look at the zeroes of $h_\ell(-ika)$ with k purely imaginary, i.e. look at the real zeroes of $h_\ell(r)$ which satisfies

$$\left(\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}-1\right)h_\ell(r)=0.$$

This is a Schroedinger DE with potential >0. Consequently $h_\ell(+r)=e^{-r}(1+\cdots)$ can't vanish for $r>0$, and $h_\ell(-r)=e^{r}(1-\cdots)$ can only vanish once. Since

$$h_\ell(-r)=e^{r}\left(1-\cdots\cdots+\frac{(2\ell)!}{\ell!\,2^\ell}(-1)^\ell\frac{1}{r^\ell}\right)$$

$$\to(-1)^\ell(+\infty) \quad \text{as } r\to 0+$$

so that $h_\ell(-r)$ has no real zeroes for ℓ even and exactly one real zero for ℓ odd.

Discuss modification necessary in two-dimensions.
In this case $(\Delta+k^2)\psi = \left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + k^2\right)\psi = 0$
so expanding as
a Fourier series $\psi = \sum_{n\in\mathbb{Z}} \psi_n(r)e^{-in\theta}$ we see
ψ_n satisfies Bessels DE

$$(*)_n \qquad \left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{n^2}{r^2} + k^2\right) \text{[redacted]}\ u = 0.$$

Since $r^{1/2}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}r^{-1/2} = \left(\frac{d}{dr}+\frac{1}{2r}\right)\left(\frac{d}{dr}-\frac{1}{2r}\right) = \frac{d^2}{dr^2} + \left(-\frac{1}{4}+\frac{1}{2}\right)\frac{1}{r^2}$
this can be written

$$\left(\frac{d^2}{dr^2} - \frac{(n^2-\frac{1}{4})}{r^2} + k^2\right) r^{1/2}u = 0$$

from which it is clear that for large r

$$u = \text{lin. comb. of } \frac{e^{\pm ikr}}{\sqrt{r}}$$

Following preceding procedure, use homogenity to concentrate on the solution to

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{n^2}{r^2} - 1\right) u = 0$$

with $\quad u \sim \frac{e^{-r}}{\sqrt{r}}$

Recall that by steepest descent

$$K_n(r) = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} t^{n-1}dt \sim e^{-r}\int_{-\infty}^{\infty} e^{-r\frac{\varepsilon^2}{2}}d\varepsilon = e^{-r}\sqrt{\frac{2\pi}{r}}$$

$$\frac{(1+\varepsilon)+(1-\varepsilon+\varepsilon^2)}{2} = 1+\frac{\varepsilon^2}{2}$$

So what we want here is

$$h_n(r) = \frac{1}{\sqrt{2\pi}} K_n(r) \sim \frac{e^{-r}}{\sqrt{r}}$$

Two independent solutions of $(*)_n$ are then

$$h_n(-ikr) \quad , \quad h_n(ikr)$$

where the former is initially defined in the UHP and the latter in the LHP. $h_n(-ikr) \sim \dfrac{e^{ikr}}{\sqrt{-ikr}}$ is outgoing.

(Note that for $\mathrm{Im}\, k > 0$, $r^{1/2} h_n(-ikr) \sim \text{const}\; e^{ikr}$ decays as $r \to +\infty$, and it satisfies the Schrod. equation

$$\left(\frac{d^2}{dr^2} - \frac{(n^2 - 1/4)}{r^2} + k^2 \right) \psi = 0$$

hence it is impossible for $h_n(-ikr)$ to vanish for real $r > 0$, for otherwise we could conclude $k^2 \in \mathbb{R}$. This would force k to be purely imaginary, say $k = ip$, $p > 0$ and then

$$h_n(-ikr) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-pr\left(\frac{t+t^{-1}}{2}\right)} t^{n-1}\, dt$$

can't vanish because the integrand is > 0.)

simpler proof uses Re of integrand > 0.

Note there is some trouble with analytically continuing $h_n(-ikr)$ to the LHP because of the log singularity at $-ikr = 0$. So therefore we choose to analytically continue across the positive real k-axis. This defines $h_n(-ikr)$ for $k \notin \mathbb{R}_{\leq 0}$ and we can look for those k such that $h_n(-ika)=0$.

Green's function for $\Delta + k^2$ source at origin is independent of θ, so satisfies

$$\left(\frac{1}{r}\frac{d}{dr}\, r\frac{d}{dr} + k^2\right) u = 0 \qquad \text{Bessel of order 0.}$$

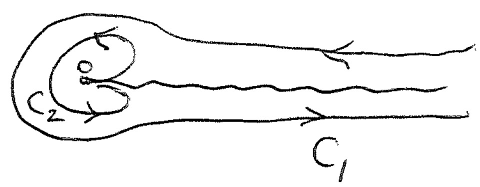
$$\left(\frac{d^2}{dr^2} + \frac{1/4}{r^2} + k^2\right)\left(r^{1/2}u\right) = 0$$

For $\operatorname{Im} k > 0$ it has to decay and hence it is a multiple of

$$K_0(-ikr) \sim \frac{e^{ikr}}{\sqrt{-ikr}}\sqrt{2\pi}$$

To determine the multiple we need the asymptotic behavior of $K_0(r)$ as $r \to 0$.

$$K_s(r) = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)}\, t^s\, \frac{dt}{t}$$



$$\int_{C_1} - \int_{C_2} = \left(e^{2\pi i s} - 1\right) K_s \qquad \frac{rt}{2} = u \quad t = \frac{2u}{r}$$

$$\int_{C_1} = \int_{C_1} e^{-u-\frac{r^2}{4u}} \left(\frac{2}{r}\right)^s u^s \frac{du}{u} = \left(\frac{2}{r}\right)^s \sum_{n\geq 0} \int_{C_1} e^{-u}\left(-\frac{r^2}{4}\right)^n \frac{1}{n!}\, u^{-n+s}\frac{du}{u}$$

$$= \left(\frac{r}{2}\right)^{-s} \sum_{n\geq 0} \frac{1}{n!}\left(-\frac{r^2}{4}\right)^n \left(e^{2\pi i s} - 1\right) \frac{\Gamma(s-n)\,\Gamma(s)}{\Gamma(s-n)(s-n)\cdots(s-1)}$$

Since K_s is symmetric under $s \mapsto -s$

$$K_s(r) = \left(\frac{r}{2}\right)^{-s}\Gamma(s) \sum_{n\geq 0} \frac{1}{n!}\left(\frac{r^2}{4}\right)^n \frac{1}{(-s+1)\cdots(-s+n)} + \text{same for } -s \mapsto s$$

$$\therefore \quad K_s(r) = \Gamma(s) I_{-s}(r) + \Gamma(-s) I_s(r)$$

where
$$I_s(r) = \left(\frac{r}{2}\right)^s \sum_{n \geq 0} \frac{1}{n!\,(s+1)\cdots(s+n)} \left(\frac{r}{2}\right)^{2n}$$

So now we want to take the limit as $s \to 0$

$$K_s = \left(\Gamma(s) - \frac{1}{s}\right) I_{-s} + \left(\Gamma(-s) + \frac{1}{s}\right) I_s + \frac{1}{s}(I_s - I_{-s})$$

Now
$$\lim_{s\to 0} \Gamma(s) - \frac{1}{s} = \lim_{s \to 0} \frac{\Gamma(s+1) - 1}{s} = \Gamma'(1)$$

$$\lim_{s\to 0} \Gamma(-s) + \frac{1}{s} = \Gamma'(1)$$

$$\lim_{s \to 0} \frac{1}{s}(I_s - I_{-s}) = 2 \frac{d}{ds} I_s \Big|_{s=0}$$

$$= 2\left\{ \log\left(\frac{r}{2}\right) I_0 + \sum_{n \geq 0} \frac{1}{n!} \frac{d}{ds}\left(\frac{1}{(s+1)\cdots(s+n)}\right)\Big|_{s=0} \left(\frac{r}{2}\right)^{2n} \right\}$$

$$(s+1)\cdots(s+n) \frac{d}{ds} \frac{1}{(s+1)\cdots(s+n)} = -\left(\frac{1}{s+1} + \cdots \frac{1}{s+n}\right) \to -\left(1 + \cdots + \frac{1}{n}\right)$$

So
$$K_0 = 2\left\{ \Gamma'(1) I_0 - \log\left(\frac{r}{2}\right) I_0 + \sum_{n \geq 1} \frac{1}{(n!)^2}\left(1 + \cdots + \frac{1}{n}\right)\left(\frac{r}{2}\right)^{2n} \right\}$$

Now recall
$$\frac{1}{\Gamma(s)} = e^{+\gamma s}\, s \prod_1^\infty \left(1 + \frac{s}{n}\right) e^{-s/n}$$

$$-\frac{\Gamma'(s)}{\Gamma(s)} = +\gamma + \frac{1}{s} + \sum_1^\infty \left(\frac{1}{s+n} - \frac{1}{n}\right) \xrightarrow{s \to 1} \gamma$$

$$\therefore \quad \Gamma'(1) = -\gamma \qquad \gamma \leftarrow \left(\sum_1^N \frac{1}{n} - \log N\right)$$

$$\therefore \quad \boxed{K_0 = 2\left\{ -\left(\log\left(\frac{r}{2}\right) + \gamma\right) I_0(r) + \sum_1^\infty \frac{1}{(n!)^2}\left(1 + \cdots + \frac{1}{n}\right)\left(\frac{r}{2}\right)^{2n} \right\}}$$

This tells me that as $r \to 0$

$$K_0(r) \sim -2 \log r$$

so $K_0(-ikr) \sim -2 \log r$. The Green's function should satisfy

$$1 \longleftarrow \int_{r=\varepsilon} \frac{\partial u}{\partial r}\, \varepsilon\, d\theta \xrightarrow{\text{for } K_0} 2\pi\varepsilon \frac{2}{\varepsilon} = 4\pi$$

Thus

$$G(r) = +\frac{1}{4\pi} K_0(-ikr) \sim -\frac{1}{2\pi} \log(r)$$

December 28, 1978: Easier:

$$K_0(r) = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} \frac{dt}{t} = 2\int_1^\infty e^{-\frac{r}{2}t} \underbrace{e^{-\frac{r}{2}t^{-1}} t^{-1}}_{t^{-1} + \frac{e^{-r/2\, t^{-1}} - 1}{t^{-1}} t^{-2}} dt$$

$$= 2\int_1^\infty e^{-\frac{r}{2}t} \frac{1}{t}\, dt + 2\int_1^\infty e^{-\frac{r}{2}t} \left(\frac{e^{-r/2\, t^{-1}} - 1}{t^{-1}}\right) \underbrace{t^{-2} dt}_{\text{integrable}}$$

$$= 2\int_1^\infty e^{-\frac{r}{2}t} \frac{dt}{t} + o(1)$$

$$= 2\int_{r/2}^\infty e^{-t} \frac{dt}{t} + o(1)$$

$$= 2\int_{r/2}^1 \left(\frac{1}{t} + \frac{e^{-t} - 1}{t}\right) dt + O(1)$$

$$= -2 \log\left(\frac{r}{2}\right) + O(1) \quad \text{as } r \to 0$$

$$= -2 \log(r) + O(1)$$