September 30, 1978

Assume the h_n are real and consider the Schur recursion relation

$$1)\qquad \begin{pmatrix} 1 & -h_n \\ -h_n & 1 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix}$$

which is satisfied by the image of $\begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix}$ under any of the representations in, out or more generally any homomorphism $\mathcal{H} \longrightarrow V$ compatible with $\mathcal{U}$ in $\mathcal{H}$ and z in V (V might be a space of functions on a subset of $\mathbb{C}^*$).
Introduce the change of variable

$$2)\qquad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

and we get

$$3)\qquad \begin{pmatrix} 1-h_n & 0 \\ 0 & 1+h_n \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}.$$

Eliminate y:
$$\left(\frac{z-1}{2}\right) y_{n-1} = (1-h_n)x_n - \left(\frac{z+1}{2}\right)x_{n-1}$$
and get

$$4)\qquad (1+h_n)(1-h_{n+1})\,x_{n+1} - (z+1)x_n + z x_{n-1} = 0$$

If $z \neq 1$, then solutions of 4) and 3) are in 1-1 corresp.

Given the numbers $t_n = (1+h_n)(1-h_{n+1})$ the h_n are not determined. To find them take a solution of

$$t_n x_{n+1} - 2x_n + x_{n-1} = 0$$

and put $1-h_n = \dfrac{x_{n-1}}{x_n}$. The condition that $-1 < h_n < 1$ is equivalent to x_n not changing sign.

Example: $t_n = 1$ all n.

$$x_{n+1} - 2x_n + x_{n-1} = 0$$

has the general solution

$$x_n = A + Bn$$

which doesn't change sign far out.

$$1 - h_n = \frac{x_{n-1}}{x_n} = \frac{A + Bn - B}{A + Bn} = 1 - \frac{B}{A + Bn}$$

This gives

$$h_n = \begin{cases} \dfrac{1}{n+c} & c \text{ constant} \\ 0 & (\text{case } c = \infty) \end{cases}.$$

Let us suppose that suppose that we have a Dirac system such that the associated x-equation has $t_n = 1$ for $|n|$ large. Then any solution of the x-equation has the form (assuming $z \neq 1$)

$$x_n = A + Bz^n$$

for either $n \gg 0$ or $n \ll 0$. Then

$$\left(\frac{z-1}{2}\right) y_n = (1 - h_{n+1}) x_{n+1} - \left(\frac{z+1}{2}\right) x_n$$

$$= -h_{n+1} x_{n+1} + A \frac{1-z}{2} + B \frac{z-1}{2} z^n$$

$$y_n = -\frac{2h_{n+1} x_{n+1}}{z-1} - A + Bz^n$$

So

$$u_n = \frac{x_n + y_n}{2} = Bz^n - \frac{h_{n+1}}{z-1}(A + Bz^{n+1})$$

$$v_n = \frac{x_n - y_n}{2} = A + \frac{h_{n+1}}{z-1}(A + Bz^{n+1})$$

I want to calculate the scattering for the Dirac system. I look at the solution

$$k\begin{pmatrix}0\\T\end{pmatrix} = k\,in\begin{pmatrix}e^-_{in}\\ e^-_{out}\end{pmatrix} \xleftarrow{\;n\to-\infty\;} \begin{pmatrix}z^{-n}u_n\\ v_n\end{pmatrix} = in\begin{pmatrix}u^{-n}\tilde{p}_n\\ \tilde{q}_n\end{pmatrix} \xrightarrow{\;n\to+\infty\;} in\begin{pmatrix}c_{out}\\ e_{in}\end{pmatrix} = \begin{pmatrix}R\\ 1\end{pmatrix}$$

where k is a positive constant $= T(0)$ [redacted] $= \prod (1-|h_n|^2)^{1/2}$.

[redacted] This tells me to look for a solution of the x-equation with asymptotic behavior

$$kT\cdot 1 + 0\cdot z^n \longleftrightarrow 1 + R\cdot z^n .$$

Thus to find R what we do is to compute the solution with

$$1 \longleftrightarrow A + Bz^n$$

and then $\qquad R = \dfrac{B}{A}$.

Compute the scattering matrix for the x-equation, assuming all $t_n = 1$ except for t_0. Suppose
[redacted]

$$C + Dz^n \longleftrightarrow A + Bz^n$$

one has
$$\begin{aligned} x_n &= C + Dz^n && \text{for } n \le 0 \\ &= A + Bz^n && \text{for } n \ge 0 \end{aligned}$$

since the key equation is $\quad t_0 x_1 = (1+z)x_0 - zx_{-1}$

and coming from $n<0$ the change occurs with x_1.

$$A + B = C + D$$

$$t_0(A + Bz) = (1+z)(C+D) - z(C + Dz^{-1}) = C + Dz$$

So $\begin{pmatrix} 1 & 1 \\ zt_0 & t_0 \end{pmatrix}\begin{pmatrix} B \\ A \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ z & 1 \end{pmatrix}\begin{pmatrix} D \\ C \end{pmatrix}$ and

$$\begin{pmatrix} B \\ A \end{pmatrix} = \frac{1}{t_0(1-z)}\begin{pmatrix} t_0 & -1 \\ -zt_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ z & 1 \end{pmatrix}\begin{pmatrix} D \\ C \end{pmatrix}$$

$$= \frac{1}{1-z}\begin{pmatrix} 1 & -1/t_0 \\ -z & \frac{1}{t_0} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ z & 1 \end{pmatrix}\begin{pmatrix} D \\ C \end{pmatrix}$$

$$= \begin{pmatrix} \frac{z/t_0 - 1}{z-1} & \frac{1/t_0 - 1}{z-1} \\ \frac{(1-1/t_0)z}{z-1} & \frac{z-1/t_0}{z-1} \end{pmatrix}\begin{pmatrix} D \\ C \end{pmatrix}$$

So the (transfer form of the) scattering matrix for any of the Dirac systems belonging to the x-equation should be essentially

$$\tilde{T}_{\infty,-\infty} = \text{const.}\begin{pmatrix} \frac{1}{T} & \frac{R}{T} \\ \frac{R}{T} & \frac{1}{T} \end{pmatrix} = \begin{pmatrix} \frac{z/t_0 - 1}{z-1} & \frac{1/t_0 - 1}{z-1} \\ \frac{(1-1/t_0)z}{z-1} & \frac{z-1/t_0}{z-1} \end{pmatrix}$$

Thus $R = \dfrac{1/t_0 - 1}{z - 1/t_0}$ $\quad T = \dfrac{z-1}{z-1/t_0}\cdot\text{const}$ $\quad$ Now

$$\det\left(\tilde{T}_{\infty,-\infty}\right) = \frac{1}{(z-1)^2}\left[\frac{z^2}{t_0} - z - \frac{z}{t_0^2} + \frac{1}{t_0} + z\left(1 - \frac{2}{t_0} + \frac{1}{t_0^2}\right)\right]$$

$$\frac{z^2 - 2z + 1}{t_0} = \frac{1}{t_0}$$

so the constant is $\sqrt{1/t_0}$. So

$$R = \frac{(1/t_0) - 1}{z - (1/t_0)} \qquad T = \sqrt{\frac{1}{t_0}}\,\frac{z-1}{z-(1/t_0)}$$

Note that we have to have $0 < t_0 < 1$ in order that T be analytic in the disk. Check abs. values

$$|R|^2 + |T|^2 = \frac{(1/t_0 - 1)^2 + \frac{1}{t_0}(z-1)(z^{-1}-1)}{(z-\frac{1}{t_0})(z^{-1}-\frac{1}{t_0})} = \frac{(\frac{1}{t_0})^2 - 2/t_0 + 1 + \frac{1}{t_0}(1 - z - z^{-1} + 1)}{1 - \frac{1}{t_0}(z+z^{-1}) + \frac{1}{t_0^2}} = 1.$$

The next stage in the computation will be to compute the transfer matrices $\tilde{T}_{-\infty,0}$, $\tilde{T}_{0,\infty}$ for the Dirac system under consideration. We have for the solution $x_n = A + Bz^n \quad n \geq 0$ that

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} B - \frac{h_1}{z-1}(A+Bz) \\ A + \frac{h_1}{z-1}(A+Bz) \end{pmatrix} = \boxed{\underbrace{\begin{pmatrix} 1 - \frac{h_1 z}{z-1} & -\frac{h_1}{z-1} \\ \frac{h_1 z}{z-1} & 1 + \frac{h_1}{z-1} \end{pmatrix}}_{\tilde{T}_{0,\infty}}} \begin{pmatrix} B \\ A \end{pmatrix}$$

So we have

Note that $\det \tilde{T}_{0,\infty} = 1 + \frac{h_1}{z-1} - \frac{h_1 z}{z-1} - \frac{h_1^2 z}{(z-1)^2} + \frac{h_1^2 z}{(z-1)^2} = 1 - h_1$

which checks with

$$\det \tilde{T}_{0,\infty} = (1-h_1^2)(1-h_2^2) \cdots\cdots$$
$$= (1-h_1)\underbrace{(1+h_1)(1-h_2)}_{t_1 = 1}\underbrace{(1+h_2)(1-h_3)}_{t_2 = 1} \cdots\cdots$$

Work next from the other end: $x_n = C + Dz^n \quad n \leq 0$.

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} D - \frac{h_1}{z-1}x_1 \\ C + \frac{h_1}{z-1}x_1 \end{pmatrix}$$ incorrect

But $t_0 x_1 = (z+1)x_0 - zx_{-1} = (z+1)(C+D) - z(C+Dz^{-1}) = C + Dz$

$$x_1 = \frac{1}{t_0}(C+Dz)$$

so $$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} D - \frac{(h_1/t_0)}{z-1}(C+Dz) \\ C + \frac{(h_1/t_0)}{z-1}(C+Dz) \end{pmatrix} = \begin{pmatrix} 1 - \frac{(h_1/t_0)z}{z-1} & -\frac{(h_1/t_0)}{z-1} \\ \frac{(h_1/t_0)z}{z-1} & 1 + \frac{(h_1/t_0)}{z-1} \end{pmatrix} \begin{pmatrix} D \\ C \end{pmatrix}$$

$$x_n = C + Dz^n \qquad n \leqslant 0, \qquad x_1 = \frac{1}{t_0}(C+Dz)$$

$$\frac{z-1}{2}\, y_0 = (1-h_1)\,x_1 - \frac{z+1}{2}\,x_0 = \frac{1-h_1}{t_0}(C+Dz) - \frac{z+1}{2}(C+D)$$

$$y_0 = \frac{2(1-h_1)}{(z-1)}\frac{1}{t_0}(C+Dz) - \frac{z+1}{z-1}(C+D)$$

$$u_0 = \frac{x_0+y_0}{2} = \frac{1}{z-1}\left(\frac{1-h_1}{t_0}\right)(C+Dz) + \overbrace{\frac{1}{2}\left(1-\frac{z+1}{z-1}\right)}^{-\frac{1}{z-1}}(C+D)$$

$$= \frac{1}{z-1}\left\{\left(\frac{1-h_1}{t_0}z - 1\right)D + \left(\frac{1-h_1}{t_0} - 1\right)C\right\}$$

$$v_0 = \frac{x_0-y_0}{2} = \frac{-(1-h_1)}{(z-1)\,t_0}(C+Dz) + \frac{z}{z-1}(C+D)$$

$$= \frac{1}{z-1}\left\{\left(-\frac{1-h_1}{t_0}z + z\right)D + \left(-\left(\frac{1-h_1}{t_0}\right) + z\right)C\right\}$$

so

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{z-1}\begin{pmatrix} \frac{1-h_1}{t_0}z - 1 & \frac{1-h_1}{t_0} - 1 \\ \left(-\frac{1-h_1}{t_0}+1\right)z & -\left(\frac{1-h_1}{t_0}\right) + z \end{pmatrix}\begin{pmatrix} D \\ C \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 - \frac{kz}{z-1} & \frac{-k}{z-1} \\ \frac{kz}{z-1} & 1 + \frac{k}{z-1} \end{pmatrix}}_{\tilde{T}_{0,-\infty}}\begin{pmatrix} D \\ C \end{pmatrix}$$

where $1-k = \dfrac{1-h_1}{t_0}$. We have

$$\det\left(\tilde{T}_{0,-\infty}\right) = 1-k = \frac{1-h_1}{t_0}$$

which checks with

$$\frac{1}{t_0} = \det\left(\tilde{T}_{\infty,-\infty}\right) = \det\left(\tilde{T}_{\infty,0}\right)\det\left(\tilde{T}_{0,-\infty}\right) = \frac{1}{1-h_1}\cdot\frac{1-h_1}{t_0} \quad \checkmark$$

One can check that

$$\tilde{T}_{0,\infty}\,\tilde{T}_{\infty,-\infty} = \tilde{T}_{0,-\infty}$$

The interesting point of the calculation is that all these matrices have the same form

$$\tilde{T}_{\infty,-\infty} = \begin{pmatrix} 1 - \dfrac{\ell z}{z-1} & \dfrac{-\ell}{z-1} \\[2ex] \dfrac{\ell z}{z-1} & 1 + \dfrac{\ell}{z-1} \end{pmatrix} \qquad \ell = 1 - \frac{1}{t_0}$$

and that taking the above product for h_1 and ℓ yields the matrix with parameter $h_1 + \ell - h_1\ell$. So in this case

$$h_1 + 1 - \frac{1}{t_0} - h_1\left(1 - \frac{1}{t_0}\right) = 1 - \frac{1}{t_0} + \frac{h_1}{t_0} = 1 - \left(\frac{1-h_1}{t_0}\right) = k.$$

which is the paranalter for $\tilde{T}_{0,-\infty}$.

The purpose of this calculation was to exhibit examples of non-uniqueness.

Recall the general formulas

$$T_{\infty,-\infty} = \begin{pmatrix} \left(\frac{1}{T}\right)^{-} & \frac{R}{T} \\[1ex] \left(\frac{R}{T}\right)^{-} & \frac{1}{T} \end{pmatrix} = T_{\infty,0}\,T_{0,-\infty}$$

$$= \begin{pmatrix} \left(\frac{1}{T_+}\right)^{-} & -\left(\frac{R_+}{T_+}\right)^{-} \\[1ex] -\frac{R_+}{T_+} & \left(\frac{1}{T_+}\right)^{-} \end{pmatrix} \begin{pmatrix} \left(\frac{1}{T_-}\right)^{-} & \frac{R_-}{T_-} \\[1ex] \left(\frac{R_-}{T_-}\right)^{-} & \frac{1}{T_-} \end{pmatrix}$$

$$\frac{1}{T} = \frac{1 - R_- R_+}{T_- T_+} \qquad \text{or} \qquad T = \frac{T_- T_+}{1 - R_- R_+}$$

$$\frac{R}{T} = \frac{R_- - \overline{R}_+}{T_- \overline{T}_+}$$

What I am trying to prove is that I can't find R_+, R_- so that, when T is defined by these formulas, one has $|T| \geq \varepsilon$ on S^1.

To begin with suppose R_+, R_- are analytic for $|z| \leq 1$, but that R_-R_+ has the value 1 on S^1. Let's first show that T_+, T_- are analytic for $|z| \leq 1$. $1 - |R_+|^2$ is real analytic ≥ 0 on S^1 hence it has finitely many zeroes each of even multiplicity. Suppose as a first case that $1 - |R_+|^2$ has only one zero at $\zeta = 1$ with multiplicity 2. Then one can divide it by

$$\left|\frac{z-1}{2}\right|^2 = \left|\frac{e^{i\theta}-1}{2}\right|^2 = \sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

So we have
$$1 - |R_+|^2 = \left|\frac{z-1}{2}\right|^2 g$$
where g is real analytic and >0 on S^1. Then $g = |f|^2$ where
$$f(z) = \exp\left\{ \int \frac{\zeta + z}{\zeta - z}\left(\frac{1}{2}\log g\right)\frac{d\theta}{2\pi} \right\}$$
is analytic invertible for $|z| \leq 1$. So now it's clear that T_+ will be $-\left(\frac{z-1}{2}\right)f(z)$, and hence analytic for $|z| \leq 1$. The general case should be similar.

Suppose that R_-R_+ takes the value 1 at $z = \zeta \in S^1$; w.m.a. $\zeta = 1$ and also that $R_- = R_+ = 1$ at $z = 1$. I think it should be true that R_-R_+ has order 1 at $z = 1$.

$$R_-R_+(z) = 1 + \alpha_1(z-1) + \alpha_2(z-1)^2 + \cdots$$

Because $|R_-R_+| \leq 1$ for $|z| \leq 1$ it should follow that $\alpha_1 = ia$ with $a > 0$. Clear.

But now note that at a point where $1 - R_-R_+$ vanishes it has order 1, but T_+, T_- also vanish, hence T must also vanish at this point.

October 3, 1978.

Problem: Can one exhibit non-uniqueness for an R such that $|R| \leq 1-\varepsilon$. Equivalently can we find R_+, R_- such that if

$$T = \frac{T_- T_+}{1 - R_- R_+}$$

then $|T| \geq \varepsilon > 0$ and this is not true for T_-, T_+.

We saw this was impossible for R_-, R_+ analytic for $|z| \leq 1$. In effect any zero on S^1 of $1 - R_- R_+$ is necessarily simple, and T_-, T_+ both vanish there, so T must also.

Here's an improvement in this argument

$$|T|^2 = \frac{(1-|R_-|^2)(1-|R_+|^2)}{|1-R_-R_+|^2} = \frac{(1-|R_-|^2)(1-|R_+|^2)}{1-|R_-R_+|^2} \cdot \frac{1-|R_-R_+|^2}{|1-R_-R_+|^2}$$

The second factor we recognize as the radial limit of the positive harmonic function in the disk

$$\mathrm{Re}\left(\frac{1+R_-R_+}{1-R_-R_+}\right).$$

Hence this function is $2\pi \frac{d\nu}{d\theta}$ where $d\nu$ is the measure associated to this harmonic function. Hence

$$\int \frac{1-|R_-R_+|^2}{|1-R_-R_+|^2} \frac{d\theta}{2\pi} \leq \int d\nu \underset{\substack{\uparrow \\ \text{because } R_+(0)=0}}{=} 1$$

with equality iff $d\nu$ is abs. cont. w.r.t. $d\theta$.

The first factor can be rewritten, putting $1-|R_-|^2 = a$, $1-|R_+|^2 = b$ as

$$\frac{ab}{1-(1-a)(1-b)} = \frac{ab}{a+b-ab} = \frac{1}{\frac{1}{a}+\frac{1}{b}-1}$$

and since $a \leq 1$, $b \leq 1$ we see

$$\frac{ab}{a+b-ab} \leq \min\{a, b\}.$$

Thus we have

$$|T|^2 \frac{d\theta}{2\pi} \leq (1 - |R_-|^2)\, d\nu$$

so integrating over a small interval Δ of S^1 and using $|T| \geq \varepsilon$ we get the estimate

$$(*) \qquad |\Delta| \leq \text{const} \cdot \max_{\Delta} (1 - |R_-|^2).$$

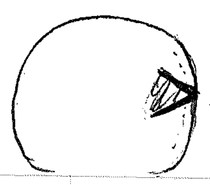
This is enough to show that R_- can't be differentiable as a function on S^1, because the derivative of $1 - |R_-|^2 \geq 0$ at a point where $|R_-| = 1$ is zero.

Conclusion: If $|T| \geq \varepsilon$, then we have $(*)$ showing that R_- can't be differentiable at a point on S^1 where it has the abs. value 1. Similarly for R_+.

Next we construct an example of R_-, R_+ such that $|T| \geq \varepsilon$ but T_+, T_- are not bounded away from zero. We take $R_- = R_+$. Then

$$|T| = \frac{1 - |R_+|^2}{|1 - R_+^2|} = \frac{1 + |R_+|}{|1 + R_+|} \cdot \frac{1 - |R_+|}{|1 - R_+|}$$

The first factor will cause no trouble provided we keep R_+ away from -1. To keep the second factor bdd away from zero, we allow R_+ to take the value 1 but we keep it in a sector, so that the distance

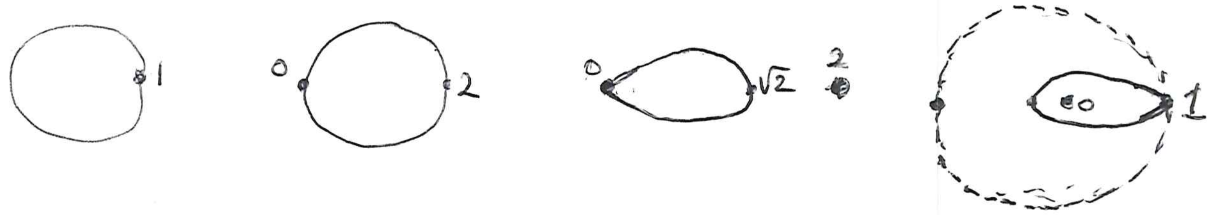


1

of R_+ from S^1 is bounded below by the distance to 1 times a constant.

Example of such an R_+.

$$z \longmapsto \frac{1-z}{2} \longmapsto \left(\frac{1-z}{2}\right)^{1/2} \longmapsto 1-\left(\frac{1-z}{2}\right)^{1/2}$$



$$\therefore \quad R_+(z) = 1-(1-z)^{1/2}$$

Such an R_+ gives us an R with $\|R\| \le 1-\varepsilon$ but it might happen that the R_-, R_+ factorization for R is part of the canonical Schur system associated to R. In the constructing the canonical system we show there is a unique solution to the requirements

$$RB+\bar{A} \quad \text{analytic in disk}$$
$$RA+\bar{B} \quad \text{"} \quad \text{"} \quad \text{"}$$

with $A \in 1+zH_+$, $B \in zH_+$; here $A = \frac{1}{T_+}$, $B = \frac{R_+}{T_+}$ up to an innocent scalar factor, Thus we know that R_+ is determined by R, when $\frac{1}{T_+} \in L^2$. So what is T_+ in the above case?

Put $z = e^{i\theta}$ and say $\theta > 0$.

$$1-z = 1-\left(1+i\theta-\frac{\theta^2}{2}\right) = -i\theta+\frac{\theta^2}{2} = -i\theta\left(1+\frac{\theta}{2}i\right)$$

$$\sqrt{1-z} = e^{-i\pi/4}\sqrt{\theta}\left(1+\frac{i}{4}\theta\right)$$

$$|T_+|^2 = 1-|R_+|^2 = 1-\left(1-\sqrt{1-z}\right)\left(1-\sqrt{1-z^{-1}}\right) = \sqrt{1-z}+\sqrt{1-z^{-1}}-\sqrt{1-z}\sqrt{1-z^{-1}}$$

$$= e^{-i\pi/4}\sqrt{\theta}\left(1+\frac{i}{4}\theta\right)+e^{i\pi/4}\sqrt{\theta}\left(1-\frac{i}{4}\theta\right)-\theta\left(1+\frac{\theta^2}{16}\right)$$

$$= \left(e^{-i\pi/4}+e^{i\pi/4}\right)\sqrt{\theta}+O(\theta)$$

$$\therefore \quad |T_+| \approx (\text{pos. const})\, \Theta^{1/4} \quad \blacksquare$$

and hence $\frac{1}{T_+}$ will be L^2. In fact even instead of the square root we put

$$R(z) = 1 - (1-z)^a \qquad a < 1$$

Then we get

$$|T_+|^2 \approx \left(e^{-i\frac{\pi}{2}a} + e^{i\frac{\pi}{2}a}\right) \theta^a$$

hence $\int \frac{1}{|T_+|^2} d\theta$ behaves like $\int \frac{d\theta}{\theta^a}$ which converges.

<u>Conclusion</u>: The factorization constructed is essentially the canonical factorization.

In general suppose $|T| \geq \varepsilon$. Then from

$$T = \frac{T_+ T_-}{1 - R_- R_+}$$

we get

$$\left|\frac{T}{T_+}\right|^2 = \frac{1 - |R_-|^2}{|1 - R_- R_+|^2} \leq \frac{1 - |R_- R_+|^2}{|1 - R_- R_+|^2}$$

because $|R_-|^2 \geq |R_- R_+|^2 \implies 1 - |R_-|^2 \leq 1 - |R_- R_+|^2$. However we have seen the latter is integrable, so we have

$$\varepsilon^2 \int \frac{1}{|T_+|^2} d\theta \leq \int \left|\frac{T}{T_+}\right|^2 d\theta < \infty.$$

showing that $\frac{1}{T_+} \in L^2$. This should be exactly what is needed to prove uniqueness for Schur systems having $\|R\|_\infty < 1$.

October 12, 1978. (Becky is 12)

Let $R \in H_\infty$ satisfy $\log(1-|R|^2) \in L^1$ so that there is ~~an~~ unique outer function T with $T(0)>0$ and $|T|^2 = 1-|R|^2$. Let $\mathcal{H}$ be obtained by completing $fe_{out} + ge_{in}$ with norm determined from R in the usual way. I know that the kernel of the "out" representation is isomorphic to L^2 in the following way:

$$\psi : L^2 \xrightarrow{\sim} \mathrm{Ker}(out)$$
$$1 \longmapsto e_{out}^- = \frac{1}{\bar{T}}(e_{in} - \bar{R}e_{out}).$$

In more detail, one has an orthogonal decomposition

$$fe_{out} + ge_{in} = (f+g\bar{R})e_{out} + g(e_{in} - \bar{R}e_{out})$$

$$\|fe_{out} + ge_{in}\|^2 = \|f+g\bar{R}\|^2 + \int |g|^2(1-|R|^2)\, d\theta/2\pi$$

which shows that $\mathcal{H}$ is the orth direct sum of $L^2 e_{out}$ and $L^2(d\mu)\cdot(e_{in} - \bar{R}e_{out})$ where $d\mu = (1-|R|^2)d\theta/2\pi$. Thus there exists an element e_{out}^- given by the above formula; also $u^n e_{out}^-$ forms an orthonormal basis for Ker out.

Next we have

$$L^2 \xrightarrow[\sim]{\psi} \mathrm{Ker}(out) \xrightarrow{in} L^2$$
$$1 \longmapsto \tfrac{1}{\bar{T}}(e_{in} - \bar{R}e_{out}) \longmapsto T$$

so by the lemma on 119, if $\mathcal{H}_{-\infty} = \cap \mathcal{H}_n = (out, in)^{-1}(0 \times H_+)$, then $\psi^{-1}\mathcal{H}_{-\infty} = \{ f \in L^2 \mid Tf \in H_+ \} = H_+$. If $\tilde{g}_{-\infty} = pr_{\mathcal{H}_{-\infty}}(e_{in})$, then

$$\tilde{g}_{-\infty} = pr_{\mathcal{H}_{-\infty}}(e_{in} - \bar{R}e_{out}) = pr_{\psi H_+}(\psi \bar{T}) = \psi\left(pr_{H_+}(\bar{T})\right)$$

$= \varphi(\bar{T}(0)) = \bar{T}(0)\, e_{out}^-$. Thus we get

$$e_{out}^- = pr_{\mathcal{H}_{-\infty}}(e_{in}) \big/ \text{norm.} = q_{-\infty}.$$

Since $\tilde{q}_{-\infty} \neq 0$ we must also have $\tilde{q}_n \neq 0$ for all n and so the question arises as to whether $\lim_{n\to+\infty} q_n = e_{in}$. Now $\lim_{n\to\infty} \tilde{q}_n$ should be the orthogonal projection of e_{in} onto

$$\mathcal{H}_\infty = \overline{\bigcup_n \mathcal{H}_n}$$

so we need only prove $e_{in} \in \mathcal{H}_\infty$.

Recall $e_{in}^- = \frac{1}{T}(e_{out} - R e_{in}) = \lim_{n\to+\infty} U^{+n} p_{-n}$

$\in \bigcap_n U^n \mathcal{H}_{-n} \left(= \bigcap U^n \mathcal{H}_{-n,0} = \bigcap \mathcal{H}_{0,-n} = \mathcal{H}_{0,-\infty} \right) \subset \mathcal{H}_0$

Hence $U^k e_{in}^- \subset U^k \mathcal{H}_0 \subset \mathcal{H}_k \subset \mathcal{H}_\infty$. ~~Thus $\mathcal{H}_\infty$ contains $L^2 e_{in}^-$ = Ker in and hence is the preimage under in of in $\mathcal{H}_\infty$.~~ for $k \geq 0$. Better

$$U^k e_{in}^- = \lim_{n\to\infty} U^{k+n} p_{-n} \in \bigcap_n U^{k+n} \mathcal{H}_{-n} = \bigcap_n \mathcal{H}_{k,-n-k}$$

$$= \mathcal{H}_{k,-\infty} \subset \mathcal{H}_k$$

Thus $\mathcal{H}_\infty$ contains $L^2 \cdot e_{in}^- = \text{Ker in}$, and so $\mathcal{H}_\infty = \text{in}^{-1}(\text{in}\, \mathcal{H}_\infty)$. But $\mathcal{H}_\infty$ also contains e_{out}^- and is stable under U, so it contains $H_+ e_{out}^-$. Thus $\text{in}(\mathcal{H}_\infty)$ contains $H_+ T$ and as $\text{in}(\mathcal{H}_\infty)$ is closed, and T is outer we see $\text{in}(\mathcal{H}_\infty) \supset H_+$. Since $\mathcal{H}_\infty \subset \text{in}^{-1}(H_+)$

we therefore find

$$\mathcal{H}_\infty = in^{-1}(H_+)$$

and so $e_{in} \in \mathcal{H}_\infty$ as was to be proved.

This proves

Prop: Let $\mathcal{H}$ be constructed from R satisfying $\log(1-|R|^2) \in L^1$. Then by orthogonal projection:

$$p_n = pr_{\mathcal{H}_n}(U^n e_{out})/_{\text{norm}} \qquad q_n = pr_{\mathcal{H}_n}(e_{in})/_{\text{norma}}$$

we get a Schur system with reflection coefficients R. Moreover we have the Szegö formula

$$\prod (1-|k_n|^2) = \exp \int \log(1-|R|^2)\, d\theta/_{2\pi} .$$

The last step comes from the fact that we know

$$T(0) = \|\tilde{q}_\infty\| = \prod_{n \le N} k_n \, \|\tilde{q}_N\| \xrightarrow[N\to\infty]{} \prod_{n\in\mathbb{Z}} k_n \cdot \underbrace{\|e_{in}\|}_{1}$$

Continuous case: suppose $|R(k)| \leq 1$ $k \in \mathbb{R}$ given, one can form $\mathcal{H}$ by completing pairs $\langle f, g \rangle \in L^2 \times L^2$ in the norm

$$\|\langle f,g\rangle\|^2 = \|f\|^2 + \|g\|^2 + 2\,\mathrm{Re}\,(Rf, g)$$

Here L^2 denotes square integrable $f(k)$ with $\|f\|^2 = \int |f|^2\, dk/2\pi$. Write $f e_{out} + g e_{in}$ for $\langle f, g\rangle$ so that one has isom. embeddings

$$i_{in},\ i_{out} : L^2 \longrightarrow \mathcal{H}$$

given by $i_{out}(f) = f e_{out}$, etc., and one can define projections $in =$ [redacted] i_{in}^*, as usual. Put

$$\mathcal{H}_x = (out, in)^{-1}(e^{ikx} H_- \times H_+)$$

$$= (e^{ikx} H_+ e_{out} + H_- e_{in})^\perp$$

We try to construct $g_x = pr_{\mathcal{H}_x}(e_{in})$ formally

$$g_x = \bar{A}_x e_{in} - e^{ikx} B_x e_{out}$$

with $A_x \in 1 + H_+$, $B_x \in H_+$. It should be orthogonal to $(e^{ikx} H_+ e_{out} + H_- e_{in})$, leading to the equations

$$\bar{R}\bar{A}_x - e^{ikx} B_x \perp e^{ikx} H_+$$

or
$$\begin{cases} \bar{R}_x \bar{A}_x - B_x \perp H_+ \\ \bar{A}_x - R_x B_x \perp H_- \end{cases} \qquad R_x = e^{ikx} R$$
and

Translate these equations into integral equations

Suppose $\quad A_x = H + \int_{y\geq 0} e^{iky}\alpha_x(y)\,dy \qquad e^{ikx}B_x = \int_{y\geq x} e^{iky}\beta_x(y)\,dy$

Then we want

$$R A_x - [redacted]\; e^{-ikx}\overline{B_x} \perp e^{-ikx} H_- \qquad \text{or}$$

$$[redacted] \int e^{-iky}\hat{R}(y)\,dy + \int_{z\geq 0} e^{ik(y+z)}\hat{R}(y)\alpha_x(z)\,dy\,dz - \int_{y\geq x} e^{-iky}\overline{\beta_x(y)}\,dy$$

is to be $\perp$ to $e^{-ikx}H_-$, i.e. without [redacted] e^{-iky} for $y \geq x$.

$$\int_{z\geq 0} e^{ik(y+z)}\hat{R}(y)\alpha_x(z)\,dy\,dz = \int_{z\geq 0} e^{-iky}\hat{R}(-y-z)\alpha_x(z)\,dy\,dz$$

So you get

$$\boxed{\overline{\beta_x(y)} = \hat{R}(-y) + \int_{z\geq 0}\hat{R}(-y-z)\alpha_x(z)\,dz \qquad \text{for } y \geq x}$$

Similarly

$$\bar{A}_x - Re^{ikx}B_x = 1 + \int_{y\geq 0} e^{-iky}\overline{\alpha_x(y)}\,dy - \int_{z\geq x} e^{ik(y+z)}\hat{R}(y)\beta_x(z)\,dy\,dz$$

$$y \longmapsto -y-z$$

is to be $\perp$ H_-, i.e. without e^{-iky} for $y \geq 0$ so we get

$$\boxed{\overline{\alpha_x(y)} = \int_{z\geq x}\hat{R}(-y-z)\beta_x(z)\,dz \qquad \text{for } y \geq 0}$$

It might be nicer to work with

$$B_x = \int e^{iky}\tilde{\beta}_x(y)\,dy \qquad \text{so } \tilde{\beta}_x(y) = \beta_x(x+y)$$

whence the integral equations become more symmetric

$$\boxed{\begin{aligned} \overline{\tilde\beta_x(y)} &= \hat R(-y-x) + \int_{z\geq 0} \hat R(-y-x-z)\,\alpha_x(z)\,dz \qquad \text{for } y\geq 0 \\ \overline{\alpha_x(y)} &= \int_{z\geq 0} \hat R(-y-x-z)\,\tilde\beta_x(z)\,dz \qquad \text{for } y\geq 0 \end{aligned}}$$

Next point is to analyze the solutions. My first idea (p. 131) was to write the equations in the form

$$B_x = P_+(\bar R_x) + P_+\bar R_x(\bar A_x - 1)$$

$$\bar A_x - 1 = P_- R_x B_x$$

and to solve these equations for $B_x, A_x - 1 \in H_+$. For this to work one must assume $P_+(\bar R_x) = \Gamma_x 1 \in H_+$. This condition guarantees $\bar\Gamma_x\Gamma_x$ is differentiable as follows:

$$\begin{aligned} \frac{d}{dx}\bar\Gamma_x\Gamma_x u &= \frac{d}{dx} P_- R\, e^{ikx} P_+ e^{-ikx} R u \\ &= P_- R\left\{-(Ru, e^{ikx})\cdot e^{ikx}\right\} \\ &= -(\bar\Gamma_x 1)\cdot(u, R_x) = -(\bar\Gamma_x 1)\cdot(u, \bar\Gamma_x 1) \end{aligned}$$

for $u \in H_-$

Hence $$\left\| \frac{\bar\Gamma_x\Gamma_x - \bar\Gamma_0\Gamma_0}{x} u \right\| = \left\| \frac{1}{x}\int_0^x (\bar\Gamma_y 1)\cdot(u, \Gamma_y 1)\,dy \right\|$$

$$\leq \frac{1}{x}\int_0^x \|\bar\Gamma_y 1\|\,\|\bar\Gamma_y 1\|\,\|u\|\,dy \leq \left(\frac{1}{x}\int_0^x \|\bar\Gamma_y 1\|^2\,dy\right)\|u\|$$

This shows that $x \to \bar{\Gamma}_x \Gamma_x$ is differentiable in norm at $x = 0$, hence at all x. (unclear – see below)

But in addition to $\Gamma_x 1 \in H_+$ one wants to be able to differentiate it in x. Thus

$$e^{-ikx} \frac{d}{dx} e^{ikx} P_+ e^{-ikx} \bar{R} = -(\bar{R}, e^{ikx}) = -\text{[redacted]}\ \hat{\bar{R}}(x) = -\overline{\hat{R}(-x)}$$

and hence it seems we have to know that $\hat{R}(x)$ exists for all x and is say continuous (or smooth). This means R has to decay as $k \to \pm\infty$.

~~However the examples have~~ $\cancel{R = O(\tfrac{1}{k})}$ ~~which means~~ $\cancel{\hat{R}}$ ~~has a discontinuity at~~ $\cancel{x = 0}$~~, hence something peculiar happens when~~ $\cancel{x = 0}$~~. Thus we have a paradox to be resolved.~~

[redacted] Derivation of the DE.

$$B_x = P_+ \bar{R}_x \bar{A}_x \qquad \bar{A}_x = 1 + P_- R_x B_x$$

Put $\Gamma_x = P_+ \bar{R}_x$. Then [redacted]

$$B_x = \Gamma_x \bar{A}_x \qquad \bar{A}_x = 1 + \bar{\Gamma}_x B_x = 1 + \bar{\Gamma}_x \Gamma_x \bar{A}_x$$

so

$$\bar{A}_x = (1 - \bar{\Gamma}_x \Gamma_x)^{-1} 1. \qquad B_x = (1 - \Gamma_x \bar{\Gamma}_x)^{-1} \Gamma_x 1$$

Assume for the moment that the usual formulas for differentiation are [redacted] valid

$$0 = \frac{d}{dx} (1 - \bar{\Gamma}_x \Gamma_x) \bar{A}_x = (1 - \bar{\Gamma}_x \Gamma_x) \frac{d\bar{A}_x}{dx} + (\bar{\Gamma}_x 1)(\bar{A}_x, R_x)$$

so we get

$$\frac{d\bar{A}_x}{dx} = -\bar{h}(x)\,\bar{B}_x \qquad \text{where} \quad \bar{h}(x) = (\bar{A}_x, R_x)$$

Also

$$e^{-ikx}\frac{d}{dx}e^{ikx}B_x = e^{-ikx}\frac{d}{dx}e^{ikx}P_+ e^{-ikx}\bar{A}_x$$

$$= -(\bar{A}_x, R_x) + \Gamma_x \frac{d\bar{A}_x}{dx}$$

$$= -(\bar{A}_x, R_x) - \bar{h}(x)\Gamma_x \bar{B}_x$$

$$= -\bar{h}(x)(1 + \Gamma_x \bar{B}_x) = -\bar{h}(x) A_x$$

This shows that if

$$q_x = \bar{A}_x e_{in} - e^{ikx} B_x e_{out} \qquad p_x = e^{ikx} A_x e_{out} - \bar{B}_x e_{in}$$

then

$$\frac{dq_x}{dx} = -\bar{h}\bar{B}_x e_{in} + \bar{h} e^{ikx} A_x e_{out} = \bar{h}(x) p_x$$

$$e^{ikx}\frac{d}{dx}e^{-ikx}p_x = e^{ikx}\left(\frac{dA_x}{dx}e_{out} - \frac{d}{dx}(e^{-ikx}\bar{B}_x)e_{in}\right)$$

$$= -h\, e^{ikx} B_x e_{out} + h \bar{A}_x e_{in} = h(x) q_x$$

or

$$\frac{d}{dx}\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} ik & h(x) \\ \bar{h}(x) & 0 \end{pmatrix}\begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

Finally we have

?

$$\bar{h}(x) = (\bar{A}_x, R_x) = \sum_{n\geq 0}((\bar{\Gamma}_x\Gamma_x 1)^n, R_x) = \sum_{n\geq 0}((\bar{\Gamma}_x\Gamma_x)^n 1, \bar{\Gamma}_x 1)$$

$$= \left(\sum_{n\geq 0}\Gamma_x(\bar{\Gamma}_x\Gamma_x)^n 1, 1\right) = (B_x, 1)$$

? Even simpler: $\overline{h(x)} = (\bar{A}_x, R_x) = (\bar{A}_x, P_- R_x) = (\bar{A}_x, \bar{\Gamma}_x 1)$

$= (\Gamma_x \bar{A}_x, 1) = (B_x, 1)$. ?

So what remains is to establish this rigorously. What do we need to prove A_x is differentiable? We need to show that

$$\lim_{\varepsilon \to 0} \frac{\bar{\Gamma}_{x+\varepsilon}\Gamma_{x+\varepsilon} - \bar{\Gamma}_x \Gamma_x}{\varepsilon} = -(\bar{\Gamma}_x 1)(\ ,\bar{\Gamma}_x 1)$$

where the limit has to be taken in norm. However we know that for any $u \in H_-$, $\bar{\Gamma}_x \Gamma_x u$ is differentiable as a function of x, with derivative $-(\bar{\Gamma}_x 1)(u, \bar{\Gamma}_x 1)$ which is continuous in x. Hence the FTC shows that

$$\bar{\Gamma}_\varepsilon \Gamma_\varepsilon u - \bar{\Gamma}_0 \Gamma_0 u = \int_0^\varepsilon -(\bar{\Gamma}_x 1)(u, \bar{\Gamma}_x 1)\, dx$$

$$\left[\frac{\bar{\Gamma}_\varepsilon \Gamma_\varepsilon - \bar{\Gamma}_0 \Gamma_0}{\varepsilon} - \left\{-(\bar{\Gamma}_0 1)(\ , \bar{\Gamma}_x 1)\right\}\right] u = \frac{1}{\varepsilon}\int_0^\varepsilon \left[\begin{matrix} -(\bar{\Gamma}_x 1)(u, \bar{\Gamma}_x 1) \\ +(\bar{\Gamma}_0 1)(u, \bar{\Gamma}_0 1)\end{matrix}\right] dx$$

and the point is that from the continuity of $\bar{\Gamma}_x 1$ in x one can see the norm of the operator

$$(\bar{\Gamma}_x 1)(?, \bar{\Gamma}_x 1) - (\bar{\Gamma}_0 1)(?, \bar{\Gamma}_0 1)$$

approaches 0, etc.

IMPORTANT: $(\bar{A}_x, R_x)$ makes sense because R decays fast, however $(B_x, 1)$ doesn't have an immediate meaning. In fact if we have

$$B_x = \int_0^\infty e^{iky}\,\tilde{\beta}_x(y)\,dy$$

then there are two possible interpretations for

$$(B_x, 1) = \hat{B}_x(0) = \quad 0 \quad \text{or} \quad \tilde{\beta}_x(0).$$

The calculation at the bottom of page 267 is (initially) meaningless because one doesn't have

$$(1, R_x) = (1, P_- R_x)$$

as the latter is undefined.

Similarly we have trouble making sense out of

$$(p_x, q_x) = (e^{ikx} A_x e_{out} - \bar{B}_x e_{in},\; e_{in})$$

$$= (A_x, \bar{R}_x) - (\bar{B}_x, 1).$$

Deift - Trubowitz trace formula: Given

$$-u'' + qu = k^2 u$$

with $q(x)$ decaying fast as $|x| \to \infty$, let $\phi(x,k)$, $\psi(x,k)$ be the solutions with the asymptotic behavior

$$e^{-ikx} \overset{\phi}{\longleftrightarrow} A(k)e^{-ikx} + B(k)e^{ikx}$$

$$\overset{\psi}{\longleftrightarrow} e^{-ikx}$$

Thus

$$\underbrace{\frac{1}{A(k)}}_{T(k)} e^{-ikx} \longleftrightarrow e^{-ikx} + \underbrace{\frac{B(k)}{A(k)}}_{R(k)} e^{ikx}$$

or

$$T(k)\phi(x,k) = \psi(x,-k) + R(k) \text{ [redacted] } \psi(x,k)$$

(Digress to derive Marchenko equation: One knows that

$$\phi(x,k) = e^{-ikx} + \int_{y \le x} e^{-iky} v(x,y)\,dy$$

or

$$e^{ikx}\phi(x,k) \in 1 + H_+$$

Assuming no bound states one also knows that $T(k)$ $\in 1 \oplus H_+$ [redacted], hence putting

(weaker: is bounded analytic for Im $k>0$)

$$m(x,k) = e^{-ikx}\psi(x,k) \in 1 + H_+$$

we get the equation

$$\text{[redacted]} \quad m(x,-k) + R(k)e^{2ikx} m(x,k) \text{ [redacted] } \in 1 + H^+$$

which upon taking F.T. gives Marchenko's equation.)

Next form the Green's function

$$G_\lambda(x,y) = \frac{\phi(x_<,k)\,\psi(x_>,k)}{W(\phi,\psi)} = \frac{\phi(x_<,k)\,\psi(x_>,k)}{A(k)}$$

which is the kernel representing $(\lambda - L)^{-1}$, $Lu = -u'' + qu$. Here $\lambda \in \mathbb{C} - \mathbb{R}_{\geq 0}$ and $k = \sqrt{\lambda}$, $\operatorname{Im} k > 0$. [redacted] The asymptotic behavior of $G_\lambda(x,$ [redacted] $)$ for large $|k|$ can be calculated in terms of q. Use WKB which is valid thru $\frac{1}{k^2}$ terms:

$$\psi(x,k) = \left(1 - \frac{q}{k^2}\right)^{-1/4} e^{\int\sqrt{q-k^2}}$$

$$= \left(1 + \frac{q}{4k^2}\right) e^{\int ik\sqrt{1-\frac{q}{k^2}}}$$

$$= \left(1 + \frac{q}{4k^2}\right) e^{ikx + \int_\infty^x \frac{q}{2ik}} + o\left(\frac{1}{|k|^2}\right)e^{ikx}$$

$$\phi(x,k) = \left(1 + \frac{q}{4k^2}\right) e^{-ikx - \int_{-\infty}^x \frac{q}{2ik}} + o\left(\frac{1}{|k|^2}\right)e^{-ikx}$$

$$A(k) = e^{-\int_{-\infty}^\infty \frac{q}{2ik}}$$

$$\therefore \quad G_k(x,x) = \frac{\phi(x,k)\,\psi(x,k)}{A(k)} = \left(1 + \frac{q}{4k^2}\right)^2 = 1 + \frac{q(x)}{2k^2} + o\left(\frac{1}{|k|^2}\right)$$

Now from this follows

$$\frac{2i}{\pi}\int_{\frown a} k\left(G_k(x,x) - 1\right)dk \underset{a\to\infty}{\overset{\text{as}}{=}} \frac{2i}{\pi}\int \frac{q(x)}{2k}\,dk = q(x)$$

$$\frac{(2i)(-\pi i)}{\pi}$$

Because no bound states we get this is

$$q(x) = \lim_{a\to\infty} \frac{2i}{\pi}\int_{-a}^{a} k\big(G_k(x,x)-1\big)\,dk$$

$$\underbrace{\qquad}_{k\{[\psi(+x,-k)+R(k)\psi(x,k)]\psi(x,k)\,-1\}}$$

$$\boxed{q(x) = \frac{2i}{\pi}\int_{-\infty}^{\infty} k\,R(k)\,\psi(x,k)^2\,dk}$$

because $k\,\psi(+x,-k)\,\psi(x,k)$ is an odd function of k.

Problems: Can you derive this formula from your theory of the Dirac equation? Key long-range question: Is there a good theory of the Dirac equation with bound states – one that explains arbitrary $R(k)$ not subject to the condition $\overline{R(k)} = R(-k)$

Marchenko equation with bound states,

$$G_\lambda(x,y) = \frac{\phi(x_<,k)\,\psi(x_>,k)}{W}$$

$$e^{-ikx} \xleftrightarrow{\ \phi(x,k)\ } A(k)e^{-ikx} + B(k)e^{ikx}$$

$$\xrightarrow{\ \psi(x,k)\ } \qquad e^{ikx}$$

$$\therefore \quad W = A(k)\,2ik$$

One has $\quad \delta(x,y) = \frac{1}{2\pi i}\oint G_\lambda(x,y)\,d\lambda$

Each bound state with $\lambda = -\beta^2$ contributes a simple pole to δ with residue

$$C_\beta\,\psi(x,i\beta)\,\psi(y,i\beta)$$

where $C_\beta = \|\psi(\ ,i\beta)\|^{-2}$. Hence deforming the contour to

and letting $k^2 = \lambda$ to evaluate the continuous part gives the completeness relation

$$\delta(x,y) = \sum_\beta C_\beta\,\psi(x,i\beta)\psi(y,i\beta) \qquad \left(\frac{2k\,dk}{-2\pi i\ \ 2ik} = \frac{dk}{2\pi}\right)$$

$$+ \int_{-\infty}^{\infty} \left[\psi(x_<,-k) + R(k)\psi(x_<,k)\right]\psi(x_>,k)\,dk/2\pi$$

Since $\quad \psi(x,k) = e^{ikx} + \int_x^\infty v_x(z)e^{ikz}\,dz$, one gets for $y > x$

the relation

$$0 = \sum_\beta C_\beta \psi(x, i\beta) e^{-\beta y} + \int_{-\infty}^{\infty} \{\psi(x,-k) + R(k)\psi(x,k)\} e^{iky} dk/2\pi$$

$$\psi(x,i\beta) e^{-\beta y} = \left(e^{-\beta x} + \int_{z \geq x} e^{-\beta z} v_x(z) dz\right) e^{-\beta y}$$

$$= e^{\beta(-x-y)} + \int_{z \geq x} e^{\beta(-z-y)} v_x(z) dz$$

$$\int R(k)\psi(x,k) e^{iky} dk/2\pi = \int dk/2\pi \int e^{iku} \hat{R}(u) du \left(e^{ikx} + \int_{z \geq x} e^{ikz} v_z(x) dz\right) e^{iky}$$

$$= \hat{R}(-x-y) + \int_{z \geq x} \hat{R}(-z-y) v_z(x) dz$$

$$\int \psi(x,-k) e^{iky} dk/2\pi = \int \int_{z \geq x} e^{-ikz} v_x(z) dz \, e^{iky} dk/2\pi = v_x(y)$$

So if you put

$$\Omega(x) = \hat{R}(-x) + \sum_\beta C_\beta e^{-\beta x}$$

Then we get the Marchenko equation:

$$v_x(y) + \Omega(x+y) + \int_{z \geq x} v_x(z) \Omega(z+y) dz = 0 \qquad y > x$$

Curious - this is a standard integral equation for v_x without any conjugations. One uses the identity $\overline{\psi(x,k)} = \psi(x,-k)$ to get rid of conjugation.