August 26, 1978:

Let $d\nu$ be a probability measure on S^1. Recall how to obtain the associated Schur system. Let $p_0, p_1, \ldots$ be the sequence of polys. obtained by orthonormalizing $1, z, z^2, \ldots, \ldots$, and let $q_n = z^n p_n^*$. p_n is defined for $n <$ card Supp $d\nu$. If p_n is defined, then choosing k_n so that $zp_{n-1} - k_n p_n$ has degree $< n$, one gets a poly orthogonal to $z, \ldots, z^{n-1}$ which hence has to be a multiple of q_{n-1}. Hence we have

$$zp_{n-1} = k_n p_n - h_n q_{n-1} . \qquad q_{n-1} = k_n q_n - \bar{h}_n z p_{n-1}$$

Since $(p_n, q_{n-1}) = 0$, we get $k_n^2 + |h_n|^2 = 1$ and so

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \underbrace{\frac{1}{\sqrt{1-|h_n|^2}} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}}_{\Theta(h_n)} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Also
$$k_n (p_n, q_n) = (zp_{n-1} + h_n q_{n-1}, q_n) = h_n (q_{n-1}, q_n)$$
$$(q_{n-1}, q_n) = (k_n q_n - \bar{h}_n z p_{n-1}, q_n) = k_n > 0$$
$$\Longrightarrow \quad (p_n, q_n) = h_n$$

On the other hand if p_{n-1} is defined but p_n isn't, we know zp_{n-1} coincides in $L^2(d\nu)$ with a poly of degree $\leq n-1$ orthogonal to $z, \ldots, z^{n-1}$, hence there is an h_n with

$$zp_{n-1} = - h_n q_{n-1}$$

In this case $|h_n| = 1$ and $n =$ card (Supp $d\nu$).

So in the above way we can associate to any probability measure $d\nu$ on S^1 a Schur sequence $\underline{1} = h_0, h_1, \ldots\ldots$ which is a sequence of numbers of modulus < 1 terminated by a number of modulus 1 if it is finite.

Question: Can you prove that the map $d\nu \longmapsto \{h_n\}$ is bijective?

<u>August 28, 1978</u>

Given $d\nu$ a probability measure on S^1 with infinite support, let p_n, q_n be the associated sequence of orthonormal polys. and $h_n = (p_n, q_n) = -(\zeta p_{n-1}, q_{n-1})$.

We consider the port with $\mathcal{H}_0 = L^2(S^1, d\nu)$, $U =$ mult. by ζ^{-1}, $e_{out} = \zeta$, $e_{in} = 1$. Let's compute the associated Schur system.

$$\mathcal{H}_{-1} = \mathcal{D}_V = \{\zeta\}^\perp$$

$$\mathcal{H}_{-2} = \{\zeta, \zeta^2\}^\perp$$

$$\mathcal{H}_{-n} = \{\zeta, \dots, \zeta^n\}^\perp$$

$$e_{out,-n} = pr_{\mathcal{H}_n}\left(U^{-n} e_{out}\right) / \text{norm}$$

$$= pr_{\{\zeta, \dots, \zeta^n\}^\perp}\left(\zeta^{n+1}\right) / \text{norm}$$

$$= \zeta p_n$$

$$e_{in,-n} = pr_{\mathcal{H}_n}(e_{in}) / \text{norm} = pr_{\{\zeta, \dots, \zeta^n\}^\perp}(1) / \text{norm}$$

$$= q_n$$

$$h_{-n} = (e_{out,-n}, e_{in,-n}) = (\zeta p_n, q_n) = -h_{n+1}$$

Hence the associated Schur system is

$$\begin{pmatrix} \zeta p_n \\ q_n \end{pmatrix} = \Theta(-h_{n+1}) \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta p_{n+1} \\ q_{n+1} \end{pmatrix}$$

and so we know the response function is given by

$$R(z) = \theta(-h_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\theta(-h_2)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots$$

On the other hand we can compute $R(z)$ directly. We need a vector $\perp$ to $(1-\bar{z}\,\mathcal{U})\mathcal{D}_V$, namely $(1-\bar{z}\mathcal{U}^*)^{-1}e_{out}$ $= (1-z\zeta)^{-1}\zeta$. Then

$$R(z) = \frac{\left((1-z\zeta)^{-1}\zeta, \overset{\overset{1}{\shortparallel}}{e_{in}}\right)}{\left((1-z\zeta)^{-1}\zeta, \underset{\underset{\zeta}{\shortparallel}}{e_{out}}\right)} = \frac{\int \frac{\zeta\, d\nu}{1-z\zeta}}{\int \frac{d\nu}{1-z\zeta}}$$

and

$$\frac{1+zR(z)}{1-zR(z)} = \int \frac{1+z\zeta}{1-z\zeta}\, d\nu.$$

$$\mathrm{Re}\left\{\frac{1+zR(z)}{1-zR(z)}\right\} = \int \frac{1-|z|^2}{|1-z\zeta|^2}\, d\nu$$

Recall that as $z \longrightarrow z_0 \in S^1$ radially that

$$\frac{1-|z|^2}{|1-z\zeta|^2}\,\frac{d\theta}{2\pi} \longrightarrow \delta(\zeta, z_0^{-1})\, d\theta$$

hence we can recover $d\nu$ from R.

It runs against the grain to put U = mult. by ζ^{-1} in $L^2(d\nu)$ because then the spectrum of U is situated at the conjugate of where it should be. So let us derive new formulas:

$$\mathcal{H}_0 = L^2(S^1, d\nu), \quad U = \text{mult. by } \zeta, \quad e_{out} = \zeta^{-1}, \quad e_{in} = 1.$$

$$\mathcal{H}_{-n} = \{\zeta^{-1}, \zeta^{-2}, \ldots, \zeta^{-n}\}^{\perp}$$

$$e_{in,-n} = pr_{\mathcal{H}_{-n}}(e_{in})/\text{norm.} = pr_{\{\zeta^{-1}, \zeta^{-2}, \ldots, \zeta^{-n}\}^{\perp}}(1)/\text{norm}$$

$$= \zeta^{-n} pr_{\{\zeta^{n-1}, \ldots, 1\}^{\perp}}(\zeta^{n})/\text{norm} = \zeta^{-n} p_n(\zeta)$$

$$e_{out,-n} = pr_{\mathcal{H}_{-n}}(U^{-n} e_{out})/\text{norm} = pr_{\{\zeta^{-1}, \ldots, \zeta^{-n}\}^{\perp}}(\zeta^{-n-1})/$$

$$= \zeta^{-n-1} pr_{\{\zeta^{n}, \ldots, \zeta\}^{\perp}}(1)/\text{norm} = \zeta^{-n-1} q_n(\zeta)$$

$$\underset{\sim}{h}_{-n} = (e_{out,-n}, e_{in,-n}) = (q_n, \zeta p_n) = -\overline{h}_{n+1}$$

Check

$$\begin{pmatrix} \zeta^{-n-1} q_n \\ \zeta^{-n} p_n \end{pmatrix} \overset{?}{=} \theta(-\overline{h}_{n+1}) \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta^{-n-2} q_{n+1} \\ \zeta^{-n-1} p_{n+1} \end{pmatrix}$$

$$\begin{pmatrix} q_n \\ \zeta p_n \end{pmatrix} \overset{?}{=} \theta(-\overline{h}_{n+1}) \begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} \qquad \text{OK.}$$

So we get the formulas

$$R(z) = \frac{\int \frac{\zeta^{-1} d\nu}{1 - z\zeta^{-1}}}{\int \frac{d\nu}{1 - z\zeta^{-1}}} = \Theta(-\bar{h}_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \Theta(-\bar{h}_2)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots$$

$$\frac{1 + zR(z)}{1 - zR(z)} = \int \frac{1 + z\zeta^{-1}}{1 - z\zeta^{-1}} d\nu$$

$$\mathrm{Re}\left\{ \quad " \quad \right\} = \int \frac{1 - |z|^2}{|1 - z\zeta^{-1}|^2} d\nu \longrightarrow 2\pi \frac{d\nu}{d\theta}(\zeta) \quad \text{a.e.}$$

as $z \to \zeta$ radially

The next stage is to understand scattering. The idea here is that assuming $\prod(1 - |h_n|^2) > 0$ we know that the limits

$$\lim_{n \to \infty} \begin{pmatrix} V^n e_{out,-n} \\ e_{in,-n} \end{pmatrix} = \begin{pmatrix} e^-_{in} \\ e^-_{out} \end{pmatrix}$$

exist in $\mathcal{H}_0$, and when we have a unitary extension U of V we can then get out$^-$, in$^-$ representations. Calculate for U = mult by ζ in $L^2(d\nu)$:

$$\begin{pmatrix} V^n e_{out,-n} \\ e_{in,-n} \end{pmatrix} = \begin{pmatrix} \zeta^{-1} q_n(\zeta) \\ \zeta^{-n} p_n(\zeta) \end{pmatrix} \longrightarrow \begin{pmatrix} \zeta^{-1}\varphi(\zeta) \\ \overline{\varphi(\zeta)} \end{pmatrix}$$

where $\varphi(\zeta) = \lim_{n \to \infty} q_n(\zeta) = pr_{\{\zeta, \zeta^2, \ldots\}}(1)/\text{norm}.$

Suppose we assume that $\sum |h_n| < \infty$. We have

$$\begin{pmatrix} e_{out,-n} \\ e_{in,-n} \end{pmatrix} = \theta(\underset{\sim}{h}_{-n}) \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_{out,-n-1} \\ e_{in,-n-1} \end{pmatrix}$$

$$\begin{pmatrix} U^n e_{out,-n} \\ e_{in,-n} \end{pmatrix} = \theta(\underset{\sim}{h}_{-n} U^n) \begin{pmatrix} U^{n+1} e_{out,-n-1} \\ e_{in,-n-1} \end{pmatrix}$$

or

$$\begin{pmatrix} U^n e_{out,-n} \\ e_{in,-n} \end{pmatrix} = \underbrace{\theta(-\underset{\sim}{h}_{-n+1} U^{n-1}) \cdots \theta(-\underset{\sim}{h}_0)}_{T_{-n,0}(U)} \begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}$$

Assuming that $\sum |h_n| < \infty$ it should follow that the limit

$$T_{-\infty,0}(\cancel{Z}) = \lim_{n \to \infty} T_{-n,0}(\cancel{Z})$$

exists in the Banach algebra of continuous matrix-valued functions on S^1. (Perhaps even over the Banach algebra of continuous function with L^1 Fourier series.)

Applying out, in to

$$\begin{pmatrix} \dfrac{e_{out} - R e_{in}}{T} \\ \dfrac{e_{in} - \bar{R} e_{out}}{\bar{T}} \end{pmatrix} = \begin{pmatrix} \bar{e}_{in} \\ \bar{e}_{out} \end{pmatrix} = T_{-\infty,0}(U) \begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}$$

yields

$$\begin{pmatrix} \bar{T} & 0 \\ 0 & T \end{pmatrix} = T_{-\infty,0}(\cancel{z}) \begin{pmatrix} 1 & R \\ \bar{R} & 1 \end{pmatrix}$$

or

$$T_{-\infty,0}(z) = \frac{1}{1-|R|^2} \begin{pmatrix} \bar{T} & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} 1 & -R \\ -\bar{R} & 1 \end{pmatrix}$$

$$\boxed{T_{-\infty,0}(z) = \begin{pmatrix} \frac{1}{T} & -\frac{R}{T} \\ -\frac{\bar{R}}{\bar{T}} & \frac{1}{\bar{T}} \end{pmatrix}}$$

Finally we can look at the limiting relation in $L^2(d\nu)$:

$$\begin{pmatrix} \zeta^{-1}\varphi(\zeta) \\ \overline{\varphi(\zeta)} \end{pmatrix} = T_{-\infty,0}(\zeta) \begin{pmatrix} \zeta^{-1} \\ 1 \end{pmatrix}$$

or

$$\zeta^{-1}\varphi(\zeta) = \frac{\zeta^{-1}}{T(\zeta)} - \frac{R(\zeta)}{T(\zeta)}$$

or

$$\boxed{\varphi(\zeta) = \frac{1 - \zeta R(\zeta)}{T(\zeta)}}$$

I expect this formula to hold in general when the terms make sense, i.e. when $\prod (1 - |h_n|^2) > 0$.

$$\mathrm{Re}\left\{\frac{1+zR(z)}{1-zR(z)}\right\} = \int \frac{1-|z|^2}{|1-z\zeta^{-1}|^2}\, d\nu \longrightarrow 2\pi \frac{d\nu}{d\theta}(\zeta) \quad \text{a.e.}$$

as $z \to \zeta$ radially

But
$$\operatorname{Re}\left\{\frac{1+zR(z)}{1-zR(z)}\right\} = \frac{\operatorname{Re}\left\{1+zR(z)-\overline{zR(z)}-|zR(z)|^2\right\}}{|1-zR(z)|^2}$$
$$= \frac{1-|zR(z)|^2}{|1-zR(z)|^2}$$

Suppose ζ_0 is a point on S^1 such that $zR(z)$ is bounded away from 1 near ζ_0. Then because we know $R(z)$ has radial limits a.e. we have

$$d\nu(\zeta) = \frac{1-|R(\zeta)|^2}{|1-\zeta R(\zeta)|^2}\,\frac{d\theta}{2\pi} = \frac{1}{|\varphi(\zeta)|^2}\,\frac{d\theta}{2\pi}$$

which is consistent with the formula
$$|\varphi(\zeta)|^2 \, d\nu(\zeta) = \frac{d\theta}{2\pi}$$

Suppose $d\nu = \rho(\zeta)\frac{d\theta}{2\pi}$ where ρ is analytic. Then

$$\frac{1+zR(z)}{1-zR(z)} = \int \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}\rho(\zeta)\frac{d\theta}{2\pi}$$

is the analytic function on $|z|\leq 1$ with value 1 at $z=0$ and real part ρ on S^1. In particular we see that $R(z)$ is analytic for $|z|\leq 1$. Also we see that $zR(z)$ does not take ~~on~~ the value 1 on the unit circle. Finally from

$$Re\left\{\frac{1+\xi R(\xi)}{1-\xi R(\xi)}\right\} = \frac{1-|R(\xi)|^2}{|1-\xi R(\xi)|^2} = \rho(\xi)$$

for $|\xi|=1$, we see that $\rho(\zeta)=0 \iff |R(\zeta)|=1$.

The problem now is to see if we can characterize R analytic for $|z|\leq 1$ in terms of growth of Schur parameters.

$$\begin{pmatrix} U^n e_{out,-n} \\ e_{in,-n} \end{pmatrix} = \theta(\underset{\sim}{h}_{-n} U^n) \begin{pmatrix} U^{n+1} e_{out,-n-1} \\ e_{in,-n-1} \end{pmatrix}$$

$$\begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix} = \underbrace{\theta(\underset{\sim}{h}_0)\,\theta(\underset{\sim}{h}_{-1} U) \cdots \theta(\underset{\sim}{h}_{-n+1} U^{n-1})}_{T_{0,-n}(U).} \begin{pmatrix} U^n e_{out,-n} \\ e_{in,-n} \end{pmatrix}$$

If
$$T_{0,-n}(z) = \begin{pmatrix} \bar{A}_n & B_n \\ \bar{B}_n & A_n \end{pmatrix}$$

then
$$\begin{pmatrix} \bar{A}_{n+1} & B_{n+1} \\ \bar{B}_{n+1} & A_{n+1} \end{pmatrix} = \begin{pmatrix} \bar{A}_n & B_n \\ \bar{B}_n & A_n \end{pmatrix} \begin{pmatrix} 1 & h_{-n} z^n \\ \bar{h}_{-n} z^{-n} & 1 \end{pmatrix} \frac{1}{k_{-n}}$$

so
$$A_{n+1} = (A_n + \bar{h}_{-n} z^n \bar{B}_n) \cdot \text{const}$$
$$B_{n+1} = (B_n + h_{-n} z^n \bar{A}_n) \cdot \text{const}$$

For $n = 1$, $T_{0,-1}(z) = \theta(\underset{\sim}{h}_0)$ is constant. Assume by induction $A_n, B_n \in F_{n-1}\,\mathbb{C}[z]$ it follows $A_{n+1}, B_{n+1} \in F_n(\mathbb{C}[z])$. Hence $A_n, B_n \in F_{n-1}\,\mathbb{C}[z]$ for all n.

Suppose that $|h_{-n}| \leq C r^n$ with $0 < r < 1$, and let's try to compute a bound for the sup norm of A_n, B_n. Suppose we have
$$\|A_n\|, \|B_n\| \leq m_n$$
Then
$$\|A_{n+1}\| \leq \|A_n\| + |h_{-n}|\,|z|^n \|\bar{B}_n\| \leq m_n\left(1 + |h_{-n}|\,|z|^n\right)$$

or even better

$$|A_{n+1} - A_n| = |h_{-n}| \cdot |z|^n \cdot \|\bar{B}_n\| \leq |h_{-n}| |z|^n \cdot m_n$$

so we get

$$m_{n+1} = m_n \left(1 + |h_{-n}| |z|^n\right).$$

Thus if I assume that $\prod \left(1 + |h_{-n}| |z|^n\right) < \infty$ I know the m_n are bounded. Consequently one concludes that

$$|A_N - A_n| \leq \sum_{k=n}^{N-1} |A_{k+1} - A_k|$$

goes to zero because

$$|A_{k+1} - A_k| \leq \text{const} \cdot |h_{-n} z^n|$$

$\therefore$

<u>Prop</u>: If the h_{-n} go to zero exponentially, i.e. $h_{-n} = O(r^n)$ where $0 < r < 1$, then A_∞, B_∞ exist and are analytic for $|z| \leq 1$. In general A_∞, B_∞ are analytic for $|z| < 1$.

The above isn't quite correct because we have dropped the $k_n = (1 - |h_{-n}|^2)^{1/2}$ factors. But it does apply for the unnormalized transfer matrices:

$$\begin{pmatrix} 1 & h_{-n} z^n \\ \bar{h}_{-n} z^{-n} & 1 \end{pmatrix}$$

Note also that one has

$$R(z) = \frac{B_\infty(z)}{A_\infty(z)}$$

If $\sum_{n=0}^{\infty} |h_{-n}| < \infty$, then $\lim A_n$, $\lim B_n$ exist in the sup norm for $|z| \leq 1$, and hence $T_{0,-\infty}(z)$ is continuous on S^1. Furthermore from

$$\begin{pmatrix} 1 & R \\ \bar{R} & 1 \end{pmatrix} = T_{0,-\infty}(z) \begin{pmatrix} \bar{T} & 0 \\ 0 & T \end{pmatrix}$$

we conclude that $A_\infty = \frac{1}{T}$ $B_\infty = \frac{R}{T}$, hence we see that T can't vanish on S^1, hence $|R| < 1$ on S^1.

It would be nice to characterize those response functions R belonging to sequences $\{h_{-n}\}_{n=0}^{\infty}$ which are summable. Would it be possible that we get those $R(z)$ whose Taylor coefficients form a summable sequence and such that $|R| < 1$ [redacted] on S^1?

~~Suppose that $R(z)$ is analytic for $|z| \leq 1$ and $|R| < 1$ on $|z| = 1$. If we know $h_0, h_{-1}, \ldots$ is the sequence of Schur parameters for R:~~

$$\cancel{R(z) = \Theta(h_0) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \Theta(h_{-1}) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots}$$

~~Then $R(\rho z)$ has the Schur parameters $\{h_0, h_{-1}\rho, h_{-2}\rho^2, \ldots\}$~~

~~If $R(z)$ has associated Schur sequence $\{h_{-n}\}_{n=0}^{\infty}$, then $R(\rho z)$ has the Schur sequence $h_{-n}\rho^n$ as one sees from the formula~~

$$\cancel{R(z) = \lim_{n\to\infty} \Theta(h_0)\,\Theta(h_{-1} z) \cdots \Theta(h_{-n} z^n)(0)}$$

Question: If one suppose $|R| \leq 1-\varepsilon$ on S^1, then the map
$$\mathcal{H} \xrightarrow{(out,\,in)} L^2 \times L^2$$
is a topological isomorphism. Is it possible to obtain bounds on h_n by arguing that there must be a bound on how much an invertible transformation can deform angles?

Example. Suppose $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a invertible bounded transformation between Hilbert spaces. If x, y are orthogonal unit vectors in $\mathcal{H}_1$ can we estimate the angle between Tx, Ty in terms of $\|T\|$ and $\|T^{-1}\|$. Can suppose $\mathcal{H}_1$ is 2-diml, and since only the form $(T\alpha, T\beta) = (T^*T\alpha, \beta)$ matters, we can suppose $T^*T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ on $\mathcal{H}_1 = \mathbb{C}^2$, with $\lambda > \mu > 0$. Then $\|T\| = \lambda^{1/2}$, $\|T^{-1}\| = \mu^{-1/2}$. Then the possible choices for x, y are columns in an arbitrary unitary matrix in $U(2)$. Changing x, y by scalars of modulus 1 doesn't affect the angles.

$$\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} \alpha & -\bar\beta \\ \zeta\beta & \zeta\bar\alpha \end{pmatrix} \qquad \begin{matrix} |\zeta| = 1 \\ |\alpha|^2 + |\beta|^2 = 1. \end{matrix}$$

$$\|Tx\|^2 = \lambda|\alpha|^2 + \mu|\beta|^2$$
$$\|Ty\|^2 = \lambda|\beta|^2 + \mu|\alpha|^2$$
$$(Tx, Ty) = \lambda\,\alpha(-\beta) + \mu\,\zeta\beta(\bar\zeta\alpha) = -(\lambda-\mu)\alpha\beta$$

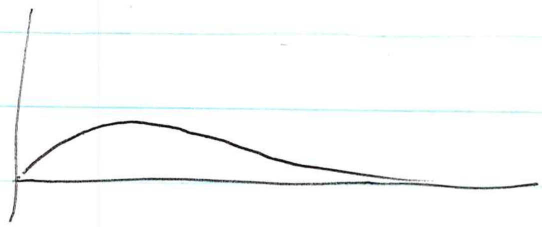
The angle between Tx, Ty is given by

$$\cos\theta = \frac{(\lambda-\mu)|\alpha\beta|}{\sqrt{\left(\lambda|\alpha|^2+\mu|\beta|^2\right)\left(\lambda|\beta|^2+\mu|\alpha|^2\right)}}$$

and we are interested in the maximum value of this under the condition $|\alpha|^2+|\beta|^2=1$. So if $|\beta| = x|\alpha|$ and we cancel we want to maxcimumize

$$f(x) = \frac{\left(1-\frac{\mu}{\lambda}\right)x}{\sqrt{\left(1+\frac{\mu}{\lambda}x^2\right)\left(x^2+\frac{\mu}{\lambda}\right)}}$$

This function has a graph:



so there is a maximum value which evidently depends only on the ratio $\frac{\mu}{\lambda}$.

Rough estimate: If $|\alpha|^2+|\beta|^2=1$, then

$$2|\alpha\beta| \leq |\alpha|^2+|\beta|^2 = 1$$

and

$$\lambda|\alpha|^2+\mu|\beta|^2 \geq \mu \qquad \lambda|\beta|^2+\mu|\alpha|^2 \geq \mu$$

hence

$$\cos\theta \leq \frac{(\lambda-\mu)/2}{\mu}$$

Better estimate

$$\frac{d}{dx}\cdot\frac{x}{\sqrt{(1+cx^2)(1+c^{-1}x^2)}} = \frac{x}{\sqrt{}}\left\{\frac{1}{x}-\frac{1}{2}\frac{2cx}{1+cx^2}-\frac{1}{2}\frac{2c^{-1}x}{1+c^{-1}x^2}\right\}$$

$$= \frac{1}{\sqrt{}}\left\{\frac{(1+cx^2)(1+c^{-1}x^2) - cx^2(1+c^{-1}x^2) - c^{-1}x^2(1+cx^2)}{x(1+cx^2)(1+c^{-1}x^2)}\right\}$$

$= \text{junk}\cdot(1-x^4)$

So the maximum occurs with $x = 1$, i.e. when $|\alpha| = |\beta|$. So the maximum value is

$$\boxed{\cos(\theta_{max}) = \frac{\lambda - \mu}{\lambda + \mu}}$$

Let's return to the construction of orthogonal polys wrt $d\nu$. Put

$$c_n = \int z^n \, d\nu$$

and $\tilde{p}_n(z) = z^n - \sum_{j \in [0,n)} a_{nj} z^j$. The orthog. relations are

$$(\tilde{p}_n, z^k) = c_{n-k} - \sum_{j \in [0,n)} a_{nj} c_{j-k} = 0 \qquad \text{for } k \in [0,n)$$

Moreover we have

$$h_n = (p_n, q_n) = p_n(0)(1, q_n) = \frac{p_n(0)}{q_n(0)}(q_n, q_n)$$
$$= \frac{\tilde{p}_n(0)}{\tilde{q}_n(0)} = \tilde{p}_n(0) = a_{n0}$$

Consequently I want to estimate the size of the coefficients a_{n0} as $n \to \infty$. Hence I want to estimate the norm of the inverse of the matrix (c_{j-k}) for $j,k \in [0,n)$.

Here is an interpretation of this matrix. Form

$$\sum_{m \in \mathbb{Z}} \bar{c}_m z^n = \rho(z) \qquad \text{where } d\nu = \rho \frac{d\theta}{2\pi}$$

If $\sum_{j=0}^{n-1} x_j z^j$ is a poly of degree $n-1$, then

$$\sum_{m\in\mathbb{Z}} \bar{c}_m z^m \sum_{j\in[0,n)} x_j z^j = \sum_{k\in\mathbb{Z}} z^k \sum_{j\in[0,n)} \bar{c}_{k-j}\, x_j$$

so projecting onto F_{n-1} = the space of polys of degree $\leq n-1$ gives

$$\sum_{k\in[0,n)} z^k \sum_{j\in[0,n)} \bar{c}_{k-j}\, x_j$$

Hence we can interpret the linear transformation

$$(x_j)_{j\in[0,n)} \longmapsto \Big(\sum_{j\in[0,n)} \bar{c}_{k-j}\, x_j\Big)_{k\in[0,n)}$$

as

$$f \longmapsto pr_{F_{n-1}}(\rho f)$$

September 9, 1978

The problem: Given a response function $R(z)$ analytic for $|z| \le 1$ and of modulus < 1 we want to prove the associated sequence of Schur parameters is exponentially decaying. Recall that the $\tilde{q}_n = pr_{\mathcal{H}_n}(e_{in})$ are given by

$$\tilde{q}_n = (1-\beta_n)e_{in} - z^n \alpha_n e_{out}$$

where $\alpha_n \in zH_+$, $\beta_n \in z^{-1}H_-$ are determined by the equations

$$-\beta_n = P_- z^n R \alpha_n$$
$$\alpha_n = P_+ z^{-n} \bar{R}(1-\beta_n)$$

with $P_- = pr_{z^{-1}H_-}$, $P_+ = pr_{zH_+}$. This gives

$$(1 - P_- z^{+n} R P_+ z^{-n} \bar{R})(1-\beta_n) = 1$$

What I tried to do is to show that $\beta_n \to \beta_{-\infty}$ fast as $n \to -\infty$, where $\beta_{-\infty}$ satisfies the limiting equation

$$(1 - P_- |R|^2)(1-\beta_{-\infty}) = 1$$
$$\Rightarrow \quad P_-(1-|R|^2)(1-\beta_{-\infty}) = 0$$
$$\Rightarrow \quad 1-\beta_{-\infty} \text{ is multiple of } \frac{1}{\bar{T}}.$$

However this is too hard because [redacted] one has

$$P_- z^{+n} R P_+ z^{-n} \bar{R} \longrightarrow P_- |R|^2$$

strongly but not uniformly. Somehow the convergence has to be seen from a viewpoint which makes $n = -\infty$

trivial.

The new idea is to work from the other end. (no longer supposing R analytic in the disk) Look at the same equations and what they say about the approach to $n=\infty$. The operator

$$[\text{redacted}] \quad P_+ z^{-n}\bar{R} \;:\; H_{[\text{redacted}]-} \longrightarrow zH_+$$

[redacted] has for its matrix relative to the usual bases the entries

$$\left(P_+ z^{-n}\bar{R}\, z^{+i},\; z^j\right).$$

If $\bar{R} = \sum r_k z^k$, then

$$P_+ z^{-n+i} \sum r_k z^k = \sum_{-n+i+k>0} r_k z^{k-n+i}$$

$j = k-n+i$

$k = j+n-i$

Here [redacted] $i \leq 0$ so the coefficients

$$\left(P_+ z^{-n}\bar{R} z^i,\; z^j\right) = r_{[\text{redacted}]\,n+j-i}$$

are essentially the Fourier coefficients of $\bar{R}$ in degrees $> n$. It should therefore be possible to [redacted] show the norm of the operator $P_+ z^{-n}\bar{R}$ goes to zero roughly as fast as the smoothness of $\bar{R}$.

Suppose R is a function on S^1 of modulus ≤ 1 and we try to solve for $\beta_n \in z^{-1}H_-$, $\alpha_n \in zH_+$ so that

$$\tilde{g}_n = (1-\beta_n)e_{in} - z\alpha_n e_{out} \in \mathcal{H}_n$$

or equivalently

$$\begin{cases} -\beta_n = P_- z^n R \alpha_n \\ \alpha_n = P_+ z^{-n}\bar{R}(1-\beta_n). \end{cases}$$

Eliminating α_n gives the equation

$$-\beta_n = P_- z^n R P_+ z^{-n} \bar{R}(1-\beta_n)$$

or

$$\left(1 - P_- z^n R P_+ z^{-n}\bar{R}\right)(-\beta_n) = P_- z^n R P_+ z^{-n} \bar{R}$$

We have to consider the operator

$$\Gamma_n = P_+ z^{-n}\bar{R} P_- : z^{-1}H_- \longrightarrow zH_+$$

and its adjoint $\Gamma_n^* = P_- z^n R P_+$. One has for $f \in z^{-1}H_-$

$$\Gamma_n f = P_+ z^{-n}\bar{R} P_- f = P_+ z^{-n-1}\bar{R}(zf) = P_+\left(P_+(z^{-n-1}\bar{R})\cdot zf\right)$$

because

$$\left\{z^{-n-1}\bar{R} - P_+(z^{-n-1}\bar{R})\right\}\cdot zf \in H_-^{\infty}\cdot H_- \subset H_-$$

(Assume that $P_+(z^{-n-1}\bar{R}) \in L^\infty$). Since P_+ is a projection which has norm 1 we have

$$\|\Gamma_n\| \leq \|P_+(z^{-n-1}\bar{R})\|_\infty \qquad \text{(sup norm)}$$

It appears now that α_n will converge slower to zero than β_n. We have

$$\alpha_n = P_+ z^{-n}\bar{R}\left(1 + P_- z^n R\alpha\right)$$

or

$$\left(1 - \Gamma_n \Gamma_n^*\right)\alpha_n = P_+\left(z^{-n}\bar{R}\right)$$

So now what can we say about the behavior of $P_+(z^{-n}\bar{R})$ as $n \to +\infty$? This should depend on smoothness of R.

Suppose to fix the ideas that R has absolutely convergent Fourier transform. Then

$$\bar{R} = \sum_{k\in\mathbb{Z}} a_k z^k \qquad \text{with} \qquad \sum_{k\in\mathbb{Z}} |a_k| < \infty$$

Put

$$c(n) = \sum_{k\geq n} |a_k|$$

so that $c(n) \downarrow 0$. Then

$$\left\| P_+ z^{-n}\bar{R}\right\|_\infty = \left\| \sum_{k\geq n} a_k z^{k-n}\right\|_\infty \leq c(n+1)$$

I'm trying to estimate the convergence of α_n to zero, and to begin I want to work in L^2. from the estimate

$$\|\Gamma_n\| \leq \|P_+ z^{-n-1}\bar{R}\|_\infty \leq c(n+2)$$

and the Neumann series

$$\alpha_n = \sum_{k\geq 0} \left(\Gamma_n\Gamma_n^*\right)^k \left(P_+ z^{-n}\bar{R}\right)$$

we get
$$\|\alpha_n\| \leq \sum_{k\geq 0} c(n+2)^{2k} \|P_+ z^{-n}\bar{R}\| = \frac{1}{1-c(n+2)^2} \|P_+ z^{-n}\bar{R}\|$$

In fact from $\quad \alpha_n + \Gamma_n\Gamma_n^* \alpha_n = P_+ z^{-n}\bar{R} \quad$ we get
$$\|\alpha_n - P_+ z^{-n}\bar{R}\| = \|\Gamma_n\Gamma_n^*\alpha_n\| \leq c(n+2)^2\|\alpha_n\| \leq \frac{c(n+2)^2}{1-c(n+2)^2}\|P_+ z^{-n}\bar{R}\|$$

This shows that α_n has the same behavior as $P_+ z^{-n}\bar{R}$ as $n \to \infty$. Note that the assumption $|R| \leq 1$ implies $|a_k| \leq 1$ for all k, hence
$$\|P_+ z^{-n}\bar{R}\|^2 = \sum_{k\geq n} |a_k|^2 \leq c(n+1)$$
but this is a stupid estimate. One has
$$\sum_{k\geq n} |a_k|^2 \leq \Big(\sum_{k\geq n} |a_k|\Big)^2 = c(n+1)^2$$

hence $\quad \|P_+ z^{-n}\bar{R}\| \leq c(n+1)$, or even simpler the L^2 norm of a function is bounded by the L^∞ norm. Consequently we see that we have the estimate
$$\|\alpha_n\| = O(c(n+1)) \quad \text{as } n \to \infty.$$

Since
$$-\beta_n = \Gamma_n^*\alpha_n$$

we get
$$\|\beta_n\| \leq \|\Gamma_n\| \, \|\alpha_n\| \leq c(n+2)\, O(c(n+1))$$
$$= O(c(n+1)^2)$$

Finally let us look at the Schur parameters.

Notice that so far we haven't used [redacted] that $|R| \leq 1$ except to get the initial Hilbert space. So the above calculations show that one gets [redacted] the existence of p_n, q_n for n large. In effect $\tilde{q}_n \neq 0$ and hence we can normalize it to get q_n. [redacted]
Recall

$$h_n = (p_n, q_n) = \left(\frac{\tilde{p}_n}{\|\tilde{p}_n\|}, \frac{\tilde{q}_n}{\|\tilde{q}_n\|} \right) = \frac{1}{\|\tilde{q}_n\|^2} (\tilde{p}_n, \tilde{q}_n)$$

since $\|\tilde{p}_n\| = \|\tilde{q}_n\|$. The basic recursion relation is

[redacted]

$$\begin{array}{ccc} & \mathcal{H}_n & \\ \mathcal{H}_{n-1} & & U\mathcal{H}_{n-1} \end{array}$$

$$U\tilde{p}_{n-1} = \tilde{p}_n - (\tilde{p}_n, q_n) q_n = \tilde{p}_n - h_n \tilde{q}_n$$

$$\tilde{q}_{n-1} = \tilde{q}_n - (\tilde{q}_n, p_n) p_n = \tilde{q}_n - \bar{h}_n \tilde{p}_n$$

$$\|\tilde{q}_n\|^2 = \|\tilde{q}_{n-1}\|^2 + |h_n|^2 \|\tilde{p}_n\|^2$$

$$\Longrightarrow \quad \|\tilde{q}_{n-1}\|^2 = (1 - |h_n|^2) \|\tilde{q}_n\|^2$$

So one sees easily that

$$\prod_{n \geq N} (1 - |h_n|^2) > 0$$

which implies that h_n is an ℓ^2-sequence. It also implies that h_n can be estimated via $(\tilde{p}_n, \tilde{q}_n)$. All this happens when p_n, q_n exist.

[redacted]

In the present case we know that $\alpha_n, \beta_n \to 0$ hence $\tilde{q}_n \to e_{in}$ and so $\|\tilde{q}_n\| \longrightarrow 1$. And

$$(\tilde{p}_n, \tilde{q}_n) = (pr_{\mathcal{H}_n}(U^n e_{out}), \tilde{q}_n) = (U^n e_{out}, (1-\beta)e_{in} - (z^n\alpha_n)e_{out})$$

$$= (z^nR, 1-\beta_n) - \underbrace{\cancel{(z^n, z^n\alpha_n)}}_{=0} \quad \text{as } \alpha_n \in zH_+$$

Now $(z^nR, -\beta_n) = (P_- z^nR, -\beta_n) = O(c(n+1)^3)$

I am trying to prove that $(\tilde{p}_n, \tilde{q}_n)$ is a summable sequence. Since $(z^nR, 1) = \bar{a}_{+n}$ is a summable sequence we want to show $(P_- z^nR, -\beta_n)$ is summable.

Example. The sequence $c(n)^3$ need not be summable. Integral analogue:

$$c(x) = \int_x^\infty f(t)\,dt. \qquad \text{Say } f(t) = t^{-p} \qquad p > 1$$

Then $c(x) = \dfrac{x^{1-p}}{p-1}$ so $c(x)^3 \sim x^{3-3p}$ which is L^1 for $3-3p < -1$ or $p > \frac{4}{3}$.

Example: Let us suppose that a given coefficient $a_m = 0$ in $\bar{R} = \sum a_k z^k$. Then because we have for $f \in z^{-1}H_-$

$$\Gamma_n f = P_+(z^{-n}\bar{R} f) = P_+(P_+(z^{-n}\bar{R}) f)$$

$$\| $$

$$P_+(z^{-n-1}\bar{R}\, zf) = P_+(z P_+(z^{-n-1}\bar{R}) f)$$

I was hoping $\Gamma_m = z\Gamma_{m+1}$ or something like it, but this doesn't seem to be the case. Notice that if we assume $\overline{R(z)}$ is analytic for $|z| < 1$, i.e. $a_k = 0$ for $k < 0$, then we do not seem to get that $h_n = 0$ for $n<0$. In effect we have for $n<0$ that $P_+(z^{-n}\bar{R}) = z^{-n}\bar{R}$, but

$$\Gamma_n f = P_+(z^{-n}\bar{R}f)$$ [redacted]

as an operator is not $zP_+(z^{-n-1}\bar{R}f)$. Consequently it would seem, and perhaps we can get an example with h_n real, to show that h_n [redacted] $\neq 0$ for $n<0$ is possible.

[redacted]

The conjecture is [redacted] that when $\bar{R} = \sum' a_k z^k$ $\sum |a_k| < \infty$, that $\{h_n\}$ is [redacted] summable at least in the direction $n \to +\infty$. I am beginning to think this unreasonable. The idea would be to find an example with $\bar{R}$ analytic in the disk but $h_n = O(\frac{1}{n})$ as $n \to -\infty$. Then the sequence $z^N \bar{R}$ would show you couldn't estimate $\sum_{n \geq 0} |h_n|$ in terms of $\sum |a_k|$.

What can be accomplished with the estimates we have?

Assume that R is C^r. From

$$\left(\frac{1}{i}\frac{d}{d\theta}\right)^\nu R = \sum a_k k^\nu e^{ik\theta}$$

we see that $a_k = O\left(\frac{1}{|k|^r}\right)$. Then for $r \geq 2$

$$c(n) = \sum_{k \geq n} |a_k| = O\left(\sum_{k \geq n} \frac{1}{k^r}\right) = O\left(\frac{1}{n^{r-1}}\right)$$

~~$h_n = O(z^n R, 1-\ldots) = \ldots$~~

and
$$(\tilde{p}_n, \tilde{q}_n) = \underbrace{(z^n R, 1)}_{\bar{a}_n = O\left(\frac{1}{n^r}\right)} - \underbrace{(z^n R, \beta_n)}_{O\left(c(n+1)^3\right) = O\left(\frac{1}{n^{3r-3}}\right)}$$

and $r \geqslant 2 \Rightarrow 3r-3 > r$. Thus $h_n = O\left(\frac{1}{n^r}\right)$.

Similarly it is clear that R analytic on S^1 $\Rightarrow$ h_n decays exponentially.

The problem: Suppose given $\{h_n\}_{n\in\mathbb{Z}}$ such that we have scattering as $n\to\infty$, which means that the limits

$$e_{out} = \lim_{n\to\infty} z^n p_n$$
$$e_{in} = \lim_{n\to\infty} q_n$$

exist. Then we can ask about whether the system p_n, q_n coincides with the one constructed from the reflection coefficient R in the standard way. In other words we are really concerned with the uniqueness of the Schur parameters with a given reflection coefficients.

Restrict attention to the case $h_0 = 1$ in which case the Hilbert space $\mathcal{H}$ is $L^2(d\nu)$ and the p_n, q_n are the orthonormal polys. Scattering occurs when $\prod_{n=1}^{\infty}(1-|h_n|^2) > 0$ and by Szegö's thm. this occurs when $\log\frac{d\nu}{d\theta} \in L^1$. A necessary condition for reconstruction is that $d\nu$ be absolutely continuous w.r.t. $d\theta$. When scattering occurs

$$q_n \longrightarrow e_{in} = \varphi \in L^2(d\nu)$$

and $\qquad |\varphi|^2 d\nu = \frac{d\theta}{2\pi}$. Furthermore the reflection coefficient is

$$S = \frac{\bar{\varphi}}{\varphi}$$

(This makes sense as an element of L^∞.) If we impose the absolutely continuous requirement:

$$d\nu = \rho \frac{d\theta}{2\pi}$$

and the scattering requirement $\log \rho \in L^1$, then we have (Wiener-Kolmogorov Prediction Formula)

$$\frac{1}{\varphi} = \exp\left(\int_0^{2\pi} \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} \log\left(\rho^{1/2}\right) \frac{d\theta}{2\pi} \right).$$

With these two requirements the reconstruction problem amounts to recovering $\frac{1}{\varphi}$ which is an outer function in H^2 from its phase $S = \frac{\bar{\varphi}}{\varphi}$.

Example: 1) Assume φ analytic non-vanishing for $|z| \le 1$. Then S is analytic on S^1 and of degree 0 and we know $H_+ \cap SH_-$ is the space of sections of the holomorphic line bundle over $\mathbb{P}^1$ with clutching function S, hence $H_+ \cap SH_-$ is 1-dimensional

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The idea is that once we have the reflection coefficient S we get a filtration $\mathcal{H}_n$ on $\mathcal{H} = L^2(d\nu)$.

$$\begin{array}{ccccc}
L^2 & \underset{\sim}{\xleftarrow{\text{out}}} & L^2\left(\rho \frac{d\theta}{2\pi}\right) & \underset{\sim}{\xrightarrow{\text{in}}} & L^2(S^1) \\
1 & \longmapsto & \bar{\varphi} \qquad \varphi & \longleftarrow\!\shortmid & 1 \\
f/\bar{\varphi} & \longleftarrow\!\shortmid & f & \longmapsto & f/\varphi
\end{array}$$

$S = \mathrm{in}(e_{\mathrm{out}}) = \bar{\varphi}/\varphi$

$$\mathcal{H}_n = \left\{ f \;\middle|\; \begin{array}{l} \mathrm{in}(f) = \frac{f}{\varphi} \in H_+ \\ \mathrm{out}(f) = f/\bar{\varphi} \in z^n H_- \end{array} \right\}$$

$$\underset{\sim}{\xrightarrow{\text{in}}} \left\{ \frac{f}{\varphi} \;\middle|\; \frac{f}{\varphi} \in H_+ ,\ \bar{S}\frac{f}{\varphi} \in z^n H_- \right\}$$

$$= H_+ \cap z^n S H_-$$

Now in the good situation $\mathcal{H}_n$ coincides with span of $\{1, \dots, z^n\}$.

Note that because $z^{n-1}H_-$ is of codim 1 in z^nH_- it follows that $\mathcal{H}_{n-1}$ is of codim ≤ 1 in $\mathcal{H}_n$. If $\mathcal{H}_{n-1} = \mathcal{H}_n$, then $\mathcal{H}_n$ is stable under multiplication by z, hence $\mathrm{out}(\mathcal{H}_n) \subset z^nH_-$ is stable under mult. by z, hence $\mathrm{out}(\mathcal{H}_n) = 0$, hence $\mathcal{H}_n = 0$. The other point is that $\frac{1}{\varphi} \in H_+ \cap SH_-$ and hence $\mathcal{H}_{n-1}$ is of codim $\underline{1}$ in $\mathcal{H}_n$ for all $n \ge 0$. So we see that

$$H_+ \cap SH_- \text{ spanned by } \frac{1}{\varphi}$$

$$\iff \quad H_+ \cap z^nSH_- \text{ spanned by } \frac{1}{\varphi}, \dots, \frac{z^n}{\varphi} \quad \text{for all } n \ge 0.$$

Now go to example 1) above which shows that $H_+ \cap SH_-$ is spanned by $\frac{1}{\varphi}$ when φ is analytic non-vanishing for $|z| \le 1$.

More generally suppose that S is a continuous map of degree 0 from S^1 to S^1. Then $\frac{1}{i}\log S$ is a continuous map from S^1 to $\mathbb{R}$ unique up to an additive constant from $2\pi\mathbb{Z}$. From theory of Dirichlet problem we can find v continuous for $|z| \le 1$ and harmonic inside S^1 [redacted] with $v = \frac{1}{i}\log S$ on S^1. Let f be an analytic function in the disk with imaginary part v. Note that what I am showing is that

$$\frac{1}{i} f = \int_0^{2\pi} \frac{1 + \zeta^{-1}z}{1 - \zeta^{-1}z} \, \frac{1}{i}\log S \, \frac{d\theta}{2\pi}$$

has continuous real part for $|z| \le 1$. [redacted] So I see that $\dfrac{1}{\varphi} = e^{\frac{1}{2}f}$ is analytic for $|z|<1$

that
$$\frac{\bar\varphi}{\varphi} = e^{\frac{1}{2}f - \frac{1}{2}\bar f} = e^{iv}$$

extends continuously to S^1 and has the boundary values S on S^1. Is $\frac{1}{\varphi} \in H_+$?
$$\left|\frac{1}{\varphi}\right|^2 = e^{\operatorname{Re} f}$$

But $\operatorname{Re} f$ is the Hilbert transform of $v = \frac{1}{i}\log S$, and I don't know how nicely it behaves for an arbitrary continuous functions.

In any case if one makes smoothness assumptions on S, one can make $\operatorname{Re} f$ bounded so that $\frac{1}{\varphi}, \varphi$ are bounded analytic functions for $|z|<1$.

Lemma: If $\varphi, \frac{1}{\varphi}$ are bounded analytic functions for $|z|<1$, then $H_+ \cap S H_-$ is spanned by $\frac{1}{\varphi}$ for $S = \frac{\bar\varphi}{\varphi}$.

Proof: Let $f \in H_+ \cap S H_-$ so that
$$f = \frac{\bar\varphi}{\varphi}\,\bar g \qquad \text{with } f, g \in H_+$$

Because φ is bounded analytic we have
$$\varphi f = \bar\varphi \bar g \quad \in \quad H_+ \cap H_- = \text{constants}$$

so $f = c \cdot \frac{1}{\varphi}$, $c \in \mathbb{C}$.

Need only $\frac{1}{\varphi} \in H_+$, $\varphi \in H^\infty$

<u>Example</u>: $\frac{1}{\varphi} = z-1 \in H_+$, and we know it is outer. Then $S = \frac{z-1}{z^{-1}-1} = -z$ has degree 1 and so $H_+ \cap SH_-$ is two-dimensional.

<u>Remark</u>: If S is invertible continuous on S^1 it can be approximated uniformly by invertible analytic functions, hence $H_+ \cap SH_-$ which is the kernel of a certain Toeplitz operator will be a uniform limit of Fredholm operators. So what?

Recall that for S continuous & non-vanishing on the circle that the Toeplitz operator $T(S) = P_+ S P_+$ on H_+ is Fredholm. Proof. This operator is defined for S continuous and its norm is bounded by $\|S\|_\infty$. Modulo compact operators $[T(z), T(z^{-1})] = 0$, and hence one argues by uniform approximation that $T(S_1 S_2) \equiv T(S_1)T(S_2)$ modulo compacts. But then for S invertible $T(S)T(S^{-1}) \equiv I$ modulo compacts and so $T(S)$ is Fredholm.