Return to the discrete case. Suppose given R analytic of modulus <1 for $|z|<1$, and form $\mathcal{H}$ as usual,

$$\mathcal{H}_n = (\text{out}, \text{in})^{-1}(z^n H_- \times H_+)$$

$$p_n = pr_{\mathcal{H}_n}(U^n e_{out}) \qquad\qquad p_n = U^n e_{out} \quad n \geq 0$$

$$q_n = pr_{\mathcal{H}_n}(e_{in}) \qquad\qquad q_n = e_{in} \quad n \geq 0$$

$$h_n = (p_n, q_n) \qquad\qquad h_n = 0 \quad n > 0$$

Recursion relation

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \underbrace{\frac{1}{\sqrt{1-|h_n|^2}} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}}_{\Theta(h_n)} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

I saw before that $\sum_{n=-\infty}^{0} |h_n|^2 < \infty \iff \log(1-|R|^2) \in L^1$

Suppose this is the case. Then I showed that the limits

$$\boxed{\begin{aligned} \lim_{n\to-\infty} q_n &= q_{-\infty} = \frac{e_{in} - \bar{R} e_{out}}{\bar{T}} = e_{out}^- \\ \lim_{n\to-\infty} U^{-n} p_n &= t = \frac{e_{out} - R e_{in}}{T} = e_{in}^- \end{aligned}}$$

exist in $\mathcal{H}$. Moreover $T = \text{in}(q_{-\infty})$ is the outer function in H_+ with $T(0) > 0$ and $|T|^2 = 1 - |R|^2$, and we have a direct sum decomp.

$$\begin{array}{ccccc} L^2(S^1) & \xleftarrow{\ \text{out}^-\ } & \mathcal{H} & \xrightarrow{\ \text{out}\ } & L^2(S^1) \\ g\bar{T} & \longleftarrow\!\!\shortmid & f e_{out} + g e_{in} & \longmapsto & f + g\bar{R} \end{array}$$

$$1 \longmapsto e_{out}^- = \frac{e_{in} - \bar{R} e_{out}}{\bar{T}}$$

Now I want to consider solutions

$$\vec{u}(n,z) = \begin{pmatrix} u_1(n,z) \\ u_2(n,z) \end{pmatrix}$$

of the Schur system:

$$\vec{u}(n,z) = \theta(h_n)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\vec{u}(n-1,z).$$

Here $u_i(n,z)$ is a function (or more generally distribution on S^1). The solutions form a space S which is isomorphic to the space of pairs of functions on S^1: For any solution

$$\vec{u}(n,z) = \begin{pmatrix} z^n u_1(0,z) \\ u_2(0,z) \end{pmatrix} \qquad n \geq 0$$

and there is a unique solution with a prescribed value for $\vec{u}(0,z)$. We put

$$\begin{aligned} in(\vec{u}) &= u_1(0,z) \\ out(\vec{u}) &= u_2(0,z). \end{aligned} \qquad (\text{why?})$$

The problem I am concerned with is to show that any solution of the Schur system has asymptotic behavior as $n \to -\infty$. I take the solution

$$\vec{u}(n,z) = in\begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

As $n \to -\infty$, $q_n \to q_{-\infty}$ in $\mathcal{H}$, hence $in(q_n) \to in(q_{-\infty}) = T$ in $H_+ \subset L^2$. Also $u^{-n} p_n \to t \Rightarrow z^{-n}\, in(p_n) \to in(t) = 0$ $\Rightarrow in(p_n) \to 0$ in $H_+ \subset L^2$. Hence we have the asymptotic behavior

$$\begin{pmatrix} 0 \\ T \end{pmatrix} \xleftarrow[\text{as } n\to\infty]{L^2 \text{ conv.}} \text{in}\begin{pmatrix} p_n \\ q_n \end{pmatrix} \xrightarrow[\text{at } n=0]{} \begin{pmatrix} R \\ 1 \end{pmatrix}$$

Similarly

$$\begin{pmatrix} z^n\overline{T} \\ 0 \end{pmatrix} \longleftarrow \text{out}\begin{pmatrix} p_n \\ q_n \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ \overline{R} \end{pmatrix}$$

To be more precise $z^{-n}\,\text{out}(p_n) \to \overline{T}$ in H_- or $\|\text{out}(p_n) - z^n\overline{T}\| \to 0$. Similarly one finds that

$$\begin{pmatrix} z^n\cdot\alpha \\ 1 \end{pmatrix} \longleftarrow \text{out}^-\begin{pmatrix} p_n \\ q_n \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \overline{T} \end{pmatrix}$$

$$\begin{pmatrix} z^n\cdot 1 \\ \beta \end{pmatrix} \longleftarrow \text{in}^-\begin{pmatrix} p_n \\ q_n \end{pmatrix} \longrightarrow \begin{pmatrix} T \\ 0 \end{pmatrix}$$

where

$$\beta = \text{in}^-(q_{-\infty}) = \text{in}^-\left(\frac{e_{in} - \overline{R}e_{out}}{\overline{T}}\right) = -\frac{\overline{R}}{\overline{T}}T$$

$$\alpha = \text{out}^-(t) = \text{out}^-\left(\frac{e_{out} - Re_{in}}{T}\right) = -\frac{R}{T}\overline{T}$$

Let's consider the solution $\vec{u}$ with $\vec{u}(0) = \begin{pmatrix} f \\ 0 \end{pmatrix}$

i.e.

$$\vec{u}(n) = \frac{f}{T}\,\text{in}^-\begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

Now we know that

$$z^{-n}\,\text{in}^-(p_n) \longrightarrow 1 \quad \text{in } L^2$$

so in what sense and under what conditions can we conclude that

$$z^{-n}u_1(n) = \frac{f}{T}z^{-n}\,\text{in}^-(p_n) \quad \text{converges to } \frac{f}{T}\ ?$$

Claim: If $\frac{f}{T} \in L^2$, then one has convergence in L^1.

In effect if $f_n \to f$ in L^2 and $g \in L^2$, one has

$$\int |(f_n - f)g| \frac{d\theta}{2\pi} \leq \|f_n - f\| \cdot \|g\|$$

by the Schwarz inequality.

Recall that a function defines a distribution only when it is L^1.

Next notice that $\frac{f}{T} \in L^2$ is exactly the condition that there exist and element v in $\mathcal{H}$ with

$$\mathrm{in}(v) = f$$
$$\mathrm{out}(v) = 0$$

One has $v = g\, e_{\mathrm{out}^-}$ for some $g \in L^2$ for any $v \in \operatorname{Ker}\, \mathrm{out}$. Also we have

$$\mathrm{in}(g\, e_{\mathrm{out}^-}) = gT.$$

So we are led to the following

<u>Conjecture</u>: Among the solutions of the Schur system are solutions $\vec{u}$ where $\vec{u}(0) \in \begin{pmatrix} \mathrm{in} \\ \mathrm{out} \end{pmatrix}(\mathcal{H})$. For these solutions one has L^1 limits as $n \to -\infty$.

$\begin{pmatrix} \mathrm{in} \\ \mathrm{out} \end{pmatrix}(\mathcal{H})$ is the sum of $\left\{ \begin{pmatrix} gT \\ 0 \end{pmatrix} \middle| g \in L^2 \right\}$ and $\left\{ \begin{pmatrix} fR \\ f \end{pmatrix} \middle| f \in L^2 \right\}$

because $\begin{pmatrix} \mathrm{in} \\ \mathrm{out} \end{pmatrix}(g\, e_{\mathrm{out}^-} + f e_{\mathrm{out}}) = \begin{pmatrix} gT \\ 0 \end{pmatrix} + \begin{pmatrix} fR \\ f \end{pmatrix}$

However for $\begin{pmatrix} fR \\ f \end{pmatrix}$ we can use the solution $f\,\mathrm{in}\begin{pmatrix} p_n \\ q_n \end{pmatrix}$ and again because f, $\mathrm{in}\,p_n \in L^2$ it follows that $f\,\mathrm{in}(p_n) \in L^1$ converges to f in L^1.

Prop. Let $\mathcal{H}_n$ be a decreasing [redacted] sequence of closed subspaces of a Hilbert space $\mathcal{H}$, and g an element of $\mathcal{H}$ such that $\tilde{g}_n = pr_{\mathcal{H}_n}(g) \neq 0$ for all n, [redacted] but $\tilde{g}_\infty = pr_{\cap \mathcal{H}_n}(g) = 0$. Put $g_n = \tilde{g}_n / \|\tilde{g}_n\|$. Then $g_n \to 0$ weakly.

Proof. [redacted] If $f \in \mathcal{H}$, then $(f, g_n) = (pr_{\mathcal{H}_n \ominus \mathcal{H}_\infty} f, g_n) \to 0$ because $\|g_n\| = 1$ and $pr_{\mathcal{H}_n \ominus \mathcal{H}_\infty} f \to 0$.

Application: In case $\prod_{n=-\infty}^{0} (1 - |h_n|^2) = 0$, one has $q_n \to 0$, $u^{-n} p_n \to 0$ weakly.

From the recursion relations $\mathrm{in}\begin{pmatrix} p_n \\ q_n \end{pmatrix}$, $\mathrm{out}\begin{pmatrix} p_n \\ q_n \end{pmatrix}$ are in L^∞ for all n, [redacted] because $R, \bar{R}$ are. So if $f \in L^2$, then for all $g \in L^\infty$ we have

$$\lim_{n \to -\infty} (f\,\mathrm{in}(q_n), g) = \lim_{n \to -\infty} (\mathrm{in}(q_n), g\bar{f}) = \begin{cases} 0 & \log(1-|R|^2) \notin L^1 \\ (f, g) & \text{"} \quad \in \text{"} \end{cases}$$

In other words, $f\,\mathrm{in}(q_n)$ has a (weak) limit in the sense of distributions, for any $f \in L^2$, and in the bad case the limit is zero.

Suppose given h_n with $|h_n| < 1$ for all $n \in \mathbb{Z}$ such that $\sum |h_n|^2 < \infty$. We then consider [redacted] solutions $\vec{u} = \begin{pmatrix} u_1(n,z) \\ u_2(n,z) \end{pmatrix}$ of the recursion relation

$$\vec{u}(n) = \theta(h_n)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\vec{u}(n-1)$$

which I can write in the form

$$\begin{pmatrix} z^{-n}u_1(n) \\ u_2(n) \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix}\begin{pmatrix} z^{-n+1}u_1(n-1) \\ u_2(n-1) \end{pmatrix}$$

What interests me is the limiting values for the solutions as $n \to \pm\infty$.

~~For $n \neq 0$ we have~~

$$\cancel{\underbrace{\frac{1}{k_{n+1}}\begin{pmatrix} 1 & -h_{n+1}z^{-n-1} \\ -\bar{h}_{n+1}z^{n+1} & 1 \end{pmatrix}\cdots\frac{1}{k_0}\begin{pmatrix} 1 & -h_0 \\ -\bar{h}_0 & 1 \end{pmatrix}}_{M_{n+1}}\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} z^{-n}u_1(n) \\ u_2(n) \end{pmatrix}}$$

~~Suppose~~ $\cancel{M_n = \begin{pmatrix} A_n & B_n \\ \bar{B}_n & \bar{A}_n \end{pmatrix}}$

$$\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \underbrace{\frac{1}{k_0}\begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix}\frac{1}{k_{-1}}\begin{pmatrix} 1 & h_{-1}z \\ \bar{h}_{-1}z^{-1} & 1 \end{pmatrix}\cdots\frac{1}{k_{-n+1}}\begin{pmatrix} 1 & h_{-n+1}z^{n-1} \\ \bar{h}_{-n+1}z^{-n+1} & 1 \end{pmatrix}}_{\begin{pmatrix} A_{n-1} & B_{n-1} \\ \bar{B}_{n-1} & \bar{A}_{n-1} \end{pmatrix}}\begin{pmatrix} z^n u_1(-n) \\ u_2(-n) \end{pmatrix}$$

Show by induction that A_{n-1} is a poly of degree $n-1$ in z^{-1} and B_{n-1} is a poly of degree $n-1$ in z.

$$\begin{pmatrix} A_{n-1} & B_{n-1} \\ \bar{B}_{n-1} & \bar{A}_{n-1} \end{pmatrix} \begin{pmatrix} 1 & h_{-n} z^{n} \\ \bar{h}_{-n} z^{-n} & 1 \end{pmatrix} \frac{1}{k_{-n}} = \begin{pmatrix} A_n & B_n \\ \bar{B}_n & \bar{A}_n \end{pmatrix}$$

Clear. As $n \to +\infty$ we get at least formally.

$$\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{\bar{T}} & \frac{R}{T} \\ \frac{\bar{R}}{\bar{T}} & \frac{1}{T} \end{pmatrix} \lim_{n \to -\infty} \begin{pmatrix} z^{-n} u_1(n) \\ u_2(n) \end{pmatrix} .$$

<u>Question:</u> Among the solutions with $u_1(0), u_2(0) \in L^2$ is there a dense set for which $\lim_{n\to-\infty} \begin{pmatrix} z^{-n}u_1(n) \\ u_2(n) \end{pmatrix}$ exists in L^2?

If $f \in L^\infty$, then the solution

$$\begin{pmatrix} 0 \\ fT \end{pmatrix} \xleftarrow{n \to -\infty} f\,\mathrm{in} \begin{pmatrix} p_n \\ q_n \end{pmatrix} \xrightarrow{n=0} \begin{pmatrix} fR \\ f \end{pmatrix}$$

has an L^2 limit as $n \to -\infty$. Similarly for $f\,\mathrm{out}\begin{pmatrix} p_n \\ q_n \end{pmatrix}$. So the question comes down to whether $\begin{pmatrix} fR \\ f \end{pmatrix} + \begin{pmatrix} gT \\ 0 \end{pmatrix}$ with $f, g \in L^\infty$ are dense in $L^2 \times L^2$. This is clear because L^∞ is dense in L^2 and because T is bdd. and $\neq 0$ a.e. so that TL^2 is dense in L^2.

Next consider the limit as $n \to +\infty$. One has

$$\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \frac{1}{k_1} \begin{pmatrix} 1 & -h_1 z^{-1} \\ -\bar{h}_1 z & 1 \end{pmatrix} \begin{pmatrix} z^{-1} u_1(1) \\ u_2(1) \end{pmatrix}$$

$$= \frac{1}{k_1} \begin{pmatrix} 1 & -h_1 z^{-1} \\ -\bar{h}_1 z & 1 \end{pmatrix} \cdots\cdots \frac{1}{k_n} \begin{pmatrix} 1 & -h_n z^{-n} \\ -\bar{h}_n z^{n} & 1 \end{pmatrix} \begin{pmatrix} z^{-n} u_1(n) \\ u_2(n) \end{pmatrix}$$

$$\doteq \begin{pmatrix} A_n & B_n \\ \overline{B_n} & \overline{A_n} \end{pmatrix} \begin{pmatrix} z^{-n} u_1(n) \\ u_2(n) \end{pmatrix}$$

Induction shows that A_n is a poly of degree n in z, B_n is a poly of degree n in z^{-1} without constant term.

$$\begin{pmatrix} A_n & B_n \\ \overline{B_n} & \overline{A_n} \end{pmatrix} \begin{pmatrix} 1 & -h_{n+1} z^{-n-1} \\ -\overline{h}_{n+1} z^{n+1} & 1 \end{pmatrix} \frac{1}{k_{n+1}} = \begin{pmatrix} A_{n+1} & B_{n+1} \\ \overline{B_{n+1}} & \overline{A_{n+1}} \end{pmatrix}$$

So in the limit we get at least formally

$$\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{T_1} & \overline{\left(\frac{R_1}{T_1}\right)} \\ \frac{R_1}{T_1} & \frac{1}{\overline{T_1}} \end{pmatrix} \lim_{n\to\infty} \begin{pmatrix} z^{-n} u_1(n) \\ u_2(n) \end{pmatrix}$$

By analogy with the case $n \to -\infty$, we should be able to show L^2 convergence of the limit as $n \to +\infty$ when the initial values are

$$f\begin{pmatrix} \overline{R_1} \\ 1 \end{pmatrix} + g\begin{pmatrix} \overline{T_1} \\ 0 \end{pmatrix}$$

and $f, g \in L^\infty$. Simpler to work with

$$f\begin{pmatrix} 0 \\ T_1 \end{pmatrix} + g\begin{pmatrix} \overline{T_1} \\ 0 \end{pmatrix}$$

and the density is clear.

So the point is that any initial data $\begin{pmatrix} fT\overline{T}_1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ fT_1\overline{T} \end{pmatrix}$ with $f \in L^\infty$ will have asymptotic limits at both ends.

Example: Let $d\mu$ be a probability measure (with infinite support) on S^1 and construct the sequence of orthonormal polys $1 = p_0, p_1, \cdots$ as usual. Hence if $q_n = z^n \overline{p_n}$, then one gets a recursion relation

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \Theta(h_n) \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad n = 1, 2, \ldots$$

Let us suppose that $\sum |h_n|^2 < \infty$. In this case we know that $\lim\limits_{n\to\infty} q_n = q_\infty$ exists in $L^2(d\mu)$ and that it is the projection of 1 perpendicular to $z H^+(d\mu)$ normalized to be a unit vector. We also know that $\lim\limits_{n\to\infty} z^{-n} p_n = v$ exists and that it is the projection of 1 perp. to $z^{-1} H^-(d\mu)$ normalized to be a unit vector.

Suppose we put $e_{in} = q_\infty$, $e_{out} = v$. I know that $d\mu = d\mu_a + d\mu_s$, abs. cont. + singular parts w.r.t. Lebesgue measure. In fact

$$|v|^2 d\mu_a = |q_\infty|^2 d\mu_a = \frac{d\theta}{2\pi}.$$

So the cyclic subspaces of $L^2(d\mu)$ generated by e_{in} and e_{out} coincide with $L^2(d\mu_a)$. There has to an outer function ρ with $d\mu_a = |\rho|^2 \frac{d\theta}{2\pi}$ and then one has

$$S e_{in} = e_{out} \qquad S = \rho / \bar{\rho} \quad (\text{or } \bar{\rho}/\rho)$$

This S function depends only $d\mu_a$, so therefore as we vary $d\mu_s$ we get many different $\{h_1, h_2, \ldots\}$ belonging to the same S.

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$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \Theta(h_n)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$ Assume h_n real

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{2} = \begin{pmatrix} 1+h & h+1 \\ 1-h & h-1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{2} = \begin{pmatrix} 1+h & 0 \\ 0 & 1-h \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\frac{1}{2}\right) = \begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix}$$

So

$$\begin{pmatrix} p_n+q_n \\ p_n-q_n \end{pmatrix} = \begin{pmatrix} \frac{1+h_n}{k_n} & 0 \\ 0 & \frac{1-h_n}{k_n} \end{pmatrix}\begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix}\begin{pmatrix} p_{n-1}+q_{n-1} \\ p_{n-1}-q_{n-1} \end{pmatrix}$$

Note that $\alpha_n = \dfrac{1+h_n}{k_n} > 0$ and $\dfrac{1-h_n}{k_n} = \alpha_n^{-1}$.

$$\alpha_n = \left(\frac{1+h_n}{1-h_n}\right)^{1/2} = 1 + h_n + O(h_n^2)$$

Put $x_n = p_n + q_n$ $\quad y = p_n - q_n$ so that

(1) $$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \alpha_n & 0 \\ 0 & \alpha_n^{-1} \end{pmatrix}\begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

$$\frac{z-1}{2}y_{n-1} = \frac{x_n}{\alpha_n} - \left(\frac{z+1}{2}\right)x_{n-1}$$

$$\left(\frac{z-1}{2}\right)\alpha_n y_n = \left(\frac{z-1}{2}\right)^2 x_{n-1} + \left(\frac{z+1}{2}\right)\left(\frac{z-1}{2}\right)y_{n-1}$$

$$\alpha_n\left\{\frac{x_{n+1}}{\alpha_{n+1}} - \left(\frac{z+1}{2}\right)x_n\right\} = \left(\frac{z-1}{2}\right)^2 x_{n-1} + \left(\frac{z+1}{2}\right)\left\{\frac{x_n}{\alpha_n} - \left(\frac{z+1}{2}\right)x_{n-1}\right\}$$

(2) $$\frac{\alpha_n}{\alpha_{n+1}} x_{n+1} - \left(\alpha_n + \frac{1}{\alpha_n}\right)\left(\frac{z+1}{2}\right) x_n + z x_{n-1} = 0$$

Conversely given a solution of (2) with $z \neq 1$ we get a solution of (1) by putting

$$\left(\frac{z-1}{2}\right) y_n = \frac{x_{n+1}}{\alpha_{n+1}} - \left(\frac{z+1}{2}\right) x_n$$

Find the equation satisfied by y:

$$\frac{z-1}{2} x_{n-1} = \alpha_n y_n - \left(\frac{z+1}{2}\right) y_{n-1}$$

$$\frac{x_n}{\alpha_n} = \left(\frac{z+1}{2}\right) x_{n-1} + \left(\frac{z-1}{2}\right) y_{n-1}$$

$$\frac{1}{\alpha_n}\left\{\alpha_{n+1} y_{n+1} - \left(\frac{z+1}{2}\right) y_n\right\} = \left(\frac{z+1}{2}\right)\left\{\alpha_n y_n - \left(\frac{z+1}{2}\right) y_{n-1}\right\} + \left(\frac{z-1}{2}\right)^2 y_{n-1}$$

(3) $$\frac{\alpha_{n+1}}{\alpha_n} y_{n+1} - \left(\alpha_n + \frac{1}{\alpha_n}\right)\left(\frac{z+1}{2}\right) y_n + z y_{n-1} = 0$$

Solutions of (1) and (3) for $z \neq 1$ are in one-one correspondence. [I expected there to be a problem also at $z = -1$ but the reason this doesn't occur is that the z here corresponds to z^2 of my previous use (June 8, 1978)]

Note $\alpha_n + \frac{1}{\alpha_n} = \frac{2}{k_n}$. Let's make a change of variable in (2) which makes it in a more standard form. Put $\rho_n \tilde{x}_n = x_n$ where ρ_n is such that

$$\left(\alpha_n + \frac{1}{\alpha_n}\right)\frac{\rho_n}{\rho_{n-1}} = \frac{2}{k_n}\frac{\rho_n}{\rho_{n-1}} = 2 \qquad \text{or} \qquad \rho_n = k_n \rho_{n-1}$$

$$\frac{\alpha_n}{\alpha_{n+1}} \frac{\rho_{n+1}}{\rho_{n-1}} = \frac{\alpha_n}{\alpha_{n+1}} k_{n+1} k_n = (1+h_n)(1-h_{n+1})$$

Then ② becomes:

$$(2') \qquad (1+h_n)(1-h_{n+1})\, \tilde{x}_{n+1} - 2\left(\frac{z+1}{2}\right) \tilde{x}_n + z\tilde{x}_{n-1} = 0.$$

Now an important point is that because we make the assumption that $\sum |h_n|^2 < \infty$ one has that $\lim_{n\to\pm\infty} \rho_n$ exists. [redacted]

This is to be contrasted with the fact that $\prod (1+h_n)$ won't exist necessarily.

Consider

$$\begin{pmatrix} (1-h_n)\,\tilde{x}_n \\ (1+h_n)\,\tilde{y}_n \end{pmatrix} = \begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix} \begin{pmatrix} \tilde{x}_{n-1} \\ \tilde{y}_{n-1} \end{pmatrix}$$

$$\left(\frac{z-1}{2}\right)\tilde{y}_{n-1} = (1-h_n)\tilde{x}_n - \left(\frac{z+1}{2}\right)\tilde{x}_{n-1}$$

$$\frac{z-1}{2} \cdot \left\{ (1+h_n)\tilde{y}_n = \frac{z-1}{2}\tilde{x}_{n-1} + \frac{z+1}{2}\tilde{y}_{n-1} \right\}$$

$$(1+h_n)\left\{(1-h_{n+1})\,\tilde{x}_{n+1} - \left(\frac{z+1}{2}\right)\tilde{x}_n\right\} = \left(\frac{z-1}{2}\right)^2 \tilde{x}_{n-1} + \frac{z+1}{2}\left\{(1-h_n)\tilde{x}_n - \left(\frac{z+1}{2}\right)\tilde{x}_{n-1}\right\}$$

$$(1+h_n)(1-h_{n+1})\,\tilde{x}_{n+1} - (z+1)\tilde{x}_n + z\tilde{x}_{n-1} = 0$$

Similarly

$$(1-h_n)(1+h_{n+1})\,\tilde{y}_{n+1} - (z+1)\tilde{y}_n + z\tilde{y}_{n-1} = 0$$

Note: Let $\tilde{x}_n$ satisfy

$$t_n \tilde{x}_{n+1} - 2\tilde{x}_n + \tilde{x}_{n-1} = 0$$

and put

$$1 - h_n = \frac{\tilde{x}_{n-1}}{\tilde{x}_n}.$$

Then

$$t_n \frac{\tilde{x}_{n+1}}{\tilde{x}_n} - 2 + \frac{\tilde{x}_{n-1}}{\tilde{x}_n} = t_n \frac{1}{1-h_{n+1}} - 2 + 1 - h_n = 0$$

or

$$t_n = (1+h_n)(1-h_{n+1})$$

Assuming $t_n > 0$ and $\tilde{x}_n > 0$ for all n we have

$$1 - h_n > 0 \implies h_n < 1$$

$$1 + h_n = \frac{t_n}{1-h_{n+1}} > 0 \implies h_n > -1$$

Finally if we assume that $\prod_{n=1}^{\infty} t_n$ converges, it follows that

$$\prod_{k=0}^{n} t_k = (1+h_0)(1-h_1)(1+h_1)(1-h_2)\cdots(1-h_{n+1})$$

$$= (1+h_0) \prod_{k=1}^{n} (1-h_k^2) \; (1-h_{n+1})$$

remains $> \varepsilon$ as $n \to \infty$. Since $1 - h_{n+1} < 2$ it follows that $\prod_{k=1}^{n} (1-h_k^2)$ does not decrease to zero.

Let's work with the unnormalized system

$$\tilde{p}_n - h_n \tilde{q}_n = u\tilde{p}_{n-1}$$
$$\tilde{q}_n - \bar{h}_n \tilde{p}_n = \tilde{q}_{n-1}$$

or $\begin{pmatrix} 1 & -h_n \\ -\bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{p}_{n-1} \\ \tilde{q}_{n-1} \end{pmatrix}$

Suppose we drop the ~'s, and assume h_n real.

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -h \\ -h & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{2} = \begin{pmatrix} 1-h & 1-h \\ 1+h & -1-h \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\frac{1}{2}\right) = \begin{pmatrix} 1-h & 0 \\ 0 & 1+h \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\frac{1}{2}\right) = \begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix}$$

Put $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} p_n + q_n \\ p_n - q_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} p_n \\ q_n \end{pmatrix}$

$$\begin{pmatrix} 1-h_n & 0 \\ 0 & 1+h_n \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

$$\frac{z-1}{2} y_{n-1} = (1-h_n)x_n - \frac{z+1}{2} x_{n-1}$$

$\frac{z-1}{2}\cdot$ $$(1+h_n)\, y_n = \frac{z-1}{2} x_{n-1} + \frac{z+1}{2} y_{n-1}$$

$$(1+h_n)\left\{(1-h_{n+1})x_{n+1} - \frac{z+1}{2} x_n\right\} = \left(\frac{z-1}{2}\right)^2 x_{n-1} + \frac{z+1}{2}\left\{(1-h_n)x_n - \frac{z+1}{2} x_{n-1}\right\}$$

$$\boxed{(1+h_n)(1-h_{n+1})\, x_{n+1} - (z+1)\, x_n + z\, x_{n-1} = 0}$$

Yesterday I saw that conversely given a recursion relation

$$(*) \qquad t_n x_{n+1} - (z+1) x_n + z x_{n-1} = 0$$

with $t_n > 0$ and with $\prod t_n$ convergent, provided that there is a solution of

$$t_n x_{n+1} - 2x_n + x_{n-1} = 0$$

with all $x_n > 0$, then we get a sequence of h_n with $-1 < h_n < 1$ such that $t_n = (1+h_n)(1-h_{n+1})$ by putting

$$1 - h_n = \frac{x_{n-1}}{x_n}$$

The convergence of $\prod t_n$ is equivalent to $\sum h_n^2 < \infty$.

I want to use the above to construct examples of different $\{h_n\}$ with the same scattering. Consider the simplest example where all $t_n = 1$ except for t_0. Solutions of $x_{n+1} - (z+1)x_n + z x_{n-1} = 0$ are given by

$$x_n = Az^n + B$$

except for $z = 1$ where one also has the solution $x_n = n$.

Consider the solution of $(*)$ with $x_n = 1$ for $n \leq 0$. One has

$$t_0 x_1 - (z+1) + z = 0 \qquad x_1 = \frac{1}{t_0}$$

so if $x_n = Az^n + B$ for $n \geq 0$ we have

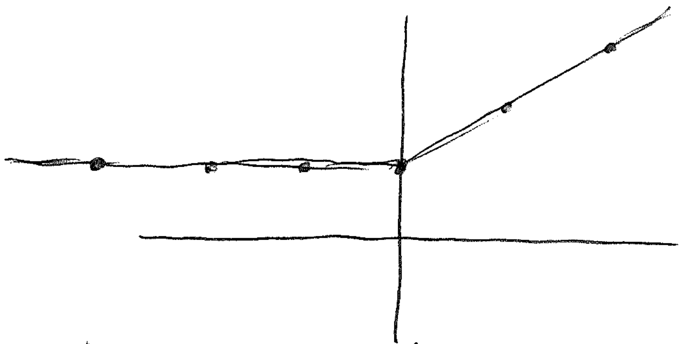
$$1 = A + B \qquad A = \frac{\begin{vmatrix} 1 & 1 \\ \frac{1}{t_0} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ z & 1 \end{vmatrix}} = \frac{1 - \frac{1}{t_0}}{1-z}$$

$$\frac{1}{t_0} = Az + B \qquad B = \frac{\begin{vmatrix} 1 & 1 \\ z & 1/t_0 \end{vmatrix}}{1-z} = \frac{\frac{1}{t_0} - z}{1-z}$$

$$x_n = \frac{1-\frac{1}{t_0}}{1-z} + \frac{\left(\frac{1}{t_0}-z\right)z^n}{1-z} = \frac{1-z^{n+1}}{1-z} + \left(\frac{1}{t_0}\right)\frac{z^n-1}{1-z}$$

As $z \to 1$ we get

$$x_n = (n+1) + \frac{1}{t_0}(-n) = n\left(1-\frac{1}{t_0}\right)+1$$



We need $t_0 > 1$ so that x_n stays > 0.

Hence taking the h_n belonging to this $\{x_n\}$ we get

$$1-h_n = \frac{x_{n-1}}{x_n} = \frac{1+n\alpha-\alpha}{1+n\alpha} = 1-\frac{\alpha}{1+n\alpha} \qquad \alpha = 1-\frac{1}{t_0}$$

so $h_n = O(\frac{1}{n})$ is square-summable.

To calculate the scattering for the x_n, we have

$$1 \longleftrightarrow Az^n + B$$

$$\frac{1}{B} \longleftrightarrow \frac{A}{B}z^n + 1$$

so $$\bar{R} = \frac{A}{B} = \frac{1-\frac{1}{t_0}}{\frac{1}{t_0}-z}.$$

Next I want to calculate the scattering for the Schur system.

$$x_n = p_n + q_n = \alpha z^n + \beta$$

$$\left(\frac{z-1}{2}\right)y_{n-1} = (1-h_n)x_n - \frac{z+1}{2}x_{n-1}$$

$$= -h_n x_n + \alpha\left(z^n - \frac{z+1}{2}z^{n-1}\right) + \beta\left(1-\frac{z+1}{2}\right)$$

$$y_{n-1} = \left(\frac{2}{z-1}\right)(-h_n)(\alpha z^n+\beta) + \alpha z^{n-1} - \beta$$

$$y_n = p_n - q_n = \alpha\left(z^n - \frac{2h_{n+1}}{z-1}z^{n+1}\right) + \beta\left(-1-\frac{2h_{n+1}}{z-1}\right)$$

$$p_n = \alpha\left(z^n - \frac{h_{n+1}}{z-1}z^{n+1}\right) + \beta\left(-\frac{h_{n+1}}{z-1}\right)$$

$$q_n = \alpha\left(\frac{h_{n+1}}{z-1}z^{n+1}\right) + \beta\left(1+\frac{h_{n+1}}{z-1}\right)$$

I am interested in the case where for $n \leq 0$ we have $x_n = \frac{1}{B}$ so here $\alpha = 0$, $\beta = \frac{1}{B}$ and

$$\begin{cases} p_n = 0 \\ q_n = \dfrac{1}{B} = \dfrac{1-z}{\frac{1}{t_0}-z} \end{cases} \qquad n \ll 0$$

and for $n \geq 0$ we have $x_n = \overset{\beta}{1} + \overset{\alpha}{\frac{A}{B}} z^n$, so

$$q_n = \beta + \frac{h_{n+1}}{z-1}\left(\frac{A}{B}z^{n+1}+1\right)$$

But $x_n = \frac{A}{B}z^n + 1 = \left(\frac{1-\frac{1}{t_0}}{\frac{1}{t_0}-z}\right)z^n + 1 = \frac{1}{\frac{1}{t_0}-z}\underbrace{\left\{\left(1-\frac{1}{t_0}\right)z^n + \left(\frac{1}{t_0}-z\right)\right\}}_{\text{vanishes at } z=1.}$

Hence one sees that $\frac{x_n}{z-1}$ is regular at $z=1$.

Better

$$\begin{cases} p_n = \alpha z^n - \dfrac{h_{n+1}}{z-1}(x_{n+1}) \\ q_n = \beta + \dfrac{h_{n+1}}{z-1}(x_{n+1}) \end{cases}$$

and

$$\frac{x_{n+1}}{z-1} = \frac{1}{z-1\cdot B}\left(\frac{1-z^{n+1}}{1-z} + \left(\frac{1}{t_0}\right)\frac{z^n-1}{z-1}\right) = O(\sqrt{n}) \text{ in } L^2 \text{ sense}$$

Since $h_n = O(\frac{1}{n})$ the terms $\frac{h_{n+1} x_{n+1}}{z-1}$ go to zero in L^2 as $n \to \infty$. Thus the limits are independent of the choice of the $\{h_n\}$.