Digress to look a little at the continuous case. Recall $H_+ \subset L^2(dk/2\pi)$ consists of Fourier transforms of elements of $L^2(0,\infty;dx)$. One has

$$f(k_0) = \int_0^\infty e^{ik_0x}\check{f}(x)\,dx = (\check{f}, e^{-i\bar{k}_0x}) = (f, \widehat{e^{-i\bar{k}_0x}})$$

$$\widehat{e^{-i\bar{k}_0x}}(k) = \int_0^\infty e^{ikx-i\bar{k}_0x}\,dx = \frac{i}{k-\bar{k}_0}$$

Hence $f_a = \dfrac{i}{k-\bar{a}}$ is the point evaluator at $a \in UHP$

i.e.

$$\left(f, \frac{i}{k-\bar{a}}\right) = \int \frac{f(k)}{k-a}\frac{dk}{2\pi i} = f(a)$$

as one would expect from Cauchy's formula. Also

$$|f(a)| = |(f,f_a)| \le \|f\|\,\|f_a\|$$

$$\|f_a\|^2 = f_a(a) = \frac{1}{2\,\mathrm{Im}\,a}$$

Next suppose that R is a bounded measurable fn. on the line such that the map $f \mapsto Rf$ from L^2 to L^2 carries H_+ into H_+. Suppose $|R(k)| \le 1$ on $\mathbb{R}$. Then we can extend R analytically in the UHP by

$$R(k) = \frac{(Rf)(k)}{f(k)}$$

Take $f = f_a$ so that the denominator doesn't vanish. Also

$$|R(a)f_a(a)| = |(Rf_a, f_a)| \le \|f_a\|^2 \implies |R(a)| \le 1$$

so R is bounded by 1 in the UHP.

There is a conjugation on $\mathcal{H}$ given by

$$[\text{redacted}]\, f e_{out} + g e_{in} \,[\text{redacted}] \longmapsto \bar{g} e_{out} + \bar{f} e_{in}$$

$$\|\bar{g} e_{out} + \bar{f} e_{in}\|^2 = \|\bar{g} R + \bar{f}\|^2 + \|\bar{g}\|^2 - \|R\bar{g}\|^2$$
$$= \|f + g\bar{R}\|^2 + \|g\|^2 - \|\bar{R} g\|^2 = \|f e_{out} + g e_{in}\|^2$$

[redacted] Conjugation takes $U^n e_{out} \mapsto U^{-n} e_{in}$ and interchanges U and U^{-1}. In the port case it gives rise to a conjugation on $\mathcal{H}_0$ interchanging e_{out}, e_{in} and V, V^{-1}.

Problem: Let's formally work out the continuous version of the Schur process. The idea will be to take discrete systems

$$[\text{redacted}]\, q_{n\varepsilon} - \overline{h_{n,\varepsilon}}\, [\text{redacted}]\, p_{n\varepsilon} = [\text{redacted}] \sqrt{1-|h_{n,\varepsilon}|^2}\; q_{(n+1)\varepsilon}$$
$$[\text{redacted}]\, p_{n\varepsilon} - h_{n,\varepsilon}\, q_{n\varepsilon} = \sqrt{1-|h_{n,\varepsilon}|^2}\; [\text{redacted}]\, e^{ik\varepsilon} p_{(n+1)\varepsilon}$$

and to let $\varepsilon \to 0$ with $n\varepsilon \to x$. Now we want

$$h_{n,\varepsilon} \sim \varepsilon h_x$$

The first relation can be written

$$-\frac{1}{\varepsilon}\overline{h_{n,\varepsilon}}\, p_{n\varepsilon} = \left(\frac{\sqrt{1-|h_{n,\varepsilon}|^2}-1}{\varepsilon}\right) q_{(n+1)\varepsilon} + \frac{q_{(n+1)\varepsilon} - q_{n\varepsilon}}{\varepsilon}$$

and as $\varepsilon \to 0$ it becomes

$$-\bar{h}_x p_x = \frac{d}{dx} q_x$$

The second relation is

$$-\frac{1}{\varepsilon} h_{n,\varepsilon}\, q_{n\varepsilon} \simeq \left(\frac{\sqrt{1-|h_{n,\varepsilon}|^2}-1}{\varepsilon}\right) e^{ik\varepsilon} p_{(n+1)\varepsilon}$$

$$+ \frac{e^{ik\varepsilon}-1}{\varepsilon}\, p_{(n+1)\varepsilon} + \frac{p_{(n+1)\varepsilon}-p_{n\varepsilon}}{\varepsilon}$$

and in the limit it becomes

$$-h_x q_x = ik\, p_x + \frac{d}{dx} p_x$$

Thus we get
$$\frac{d}{dx}\begin{pmatrix} p_x \\ q_x \end{pmatrix} = \begin{pmatrix} -ik & -h_x \\ -\bar{h}_x & 0 \end{pmatrix}\begin{pmatrix} p_x \\ q_x \end{pmatrix}$$

which is essentially a Dirac system. It would seem that we maybe want to replace x by $-x$.

July 21, 1978 (Erica born July 19)

The problem is to understand orthogonal projection, that is, to make sense out of

$$g_x = pr_{\mathcal{H}_x}(e_{in})$$

Here

$$\mathcal{H}_x = (out, in)^{-1}(e^{ikx}H_- \times H_+)$$

$$= out^{-1}(e^{ikx}H_-) \cap in^{-1}(H_+)$$

A possible approach is to notice that $in^{-1}(H_+)$ is the orthogonal complement of $H_- e_{in}$, hence we have

$$\mathcal{H} \ominus \mathcal{H}_x = (e^{ikx}H_+)e_{out} + (H_-)e_{in}$$

when the latter is a closed subspace, which is the case when $|R(k)| \leq 1-\varepsilon$. So we might try to find $\alpha \in H_+$, $\beta \in H_-$ such that if we put

$$g_x = e_{in} - e^{ikx}\alpha e_{out} - \beta e_{in}$$

then g_x is (formally) orthogonal to $\mathcal{H} \ominus \mathcal{H}_x$. Thus I want

$$in(g_x) = (1-\beta) - e^{ikx}\alpha R \quad \perp H_-$$

$$out(g_x) = (1-\beta)\bar{R} - e^{ikx}\alpha \quad \perp e^{ikx}H_+$$

To solve these equations we can replace R by $e^{-ikx}R$ and so reduce to the case where $x=0$, whence the equations become

$$1-\beta-\alpha R \quad \perp \quad H_-$$
$$\bar{R}-\beta\bar{R}-\alpha \quad \perp \quad H_+$$

The 1 in the first equation can be dropped. Let $j_+ : H_+ \longrightarrow L^2$, $j_- : H_- \longrightarrow L^2$ be the inclusions. Then we get the equations

$$\beta + j_-^* R j_+ \alpha = 0$$
$$j_+^* \bar{R} j_-(1-\beta) - \alpha = 0$$

so
$$\alpha - (j_+^* \bar{R} j_-)(j_-^* R j_+)\alpha = j_+^* \bar{R}$$

If we assume that $|R| \leq 1-\varepsilon$ and that $R \in L^2$ then this equation has a unique solution α in H_+. In fact

$$j_+\alpha = P_+\bar{R} + P_+\bar{R}P_-RP_+\bar{R} + (P_+\bar{R}P_-R)^2 P_+\bar{R} + \cdots$$

where $P_\pm = j_\pm j_\pm^*$ is the projector on $H_\pm$. Also

$$-j_-\beta = P_-RP_+\bar{R} + (P_-RP_+\bar{R})^2 + \cdots$$

So therefore we have solved the projection problem under the aforementioned conditions.

To get the x variation we replace R by $R_x = e^{ikx}R$ and then have

$$g_x = (1-\beta_x)e_{in} - (e^{ikx}\alpha_x)e_{out}$$

If we put $\Gamma_x = P_+ \bar{R}_x$ then

$$1-\beta_x = 1 + \Gamma_x^*\Gamma_x 1 + (\Gamma_x^*\Gamma_x)^2 1 + \cdots = (1-\Gamma_x^*\Gamma_x)^{-1} \cdot 1$$

$$\alpha_x = \Gamma_x 1 + \Gamma_x\Gamma_x^*\Gamma_x 1 + \cdots = \Gamma_x(1-\Gamma_x^*\Gamma)^{-1} 1$$

$$\text{or } = (1-\Gamma_x\Gamma_x^*)^{-1}\Gamma_x 1$$

So

$$\boxed{\gamma_x = \left((1-\Gamma_x^*\Gamma_x)^{-1} \cdot 1\right) e_{in} - U(x)\left(\Gamma_x(1-\Gamma_x^*\Gamma_x)^{-1} 1\right) e_{out}}$$

Next find
$$p_x = pr_{H_x}(U(x)e_{out})$$
$$= U(x)e_{out} - (e^{ikx}\gamma)e_{out} - \delta e_{in}$$

with $\gamma \in H_+$, $\delta \in H_-$. Then

$$in(p_x) = e^{ikx}(1-\gamma)R - \delta \perp H_-$$
$$out(p_x) = e^{-ikx}(1-\gamma) - \delta\bar{R} \perp e^{ikx}H_+$$

or

$$P_- R_x(1-\gamma) = \delta$$
$$\gamma + P_+\bar{R}_x\delta = 0$$

$$\Gamma_x^* 1 = P_- R_x = \delta - P_- R_x P_+ \bar{R}_x \delta = (1-\Gamma_x^*\Gamma_x)\delta$$

$$\delta = (1-\Gamma_x^*\Gamma_x)^{-1}\Gamma_x^* 1 = \Gamma_x^*(1-\Gamma_x\Gamma_x^*)^{-1} 1$$

$$-\gamma = \Gamma_x\Gamma_x^*(1-\Gamma_x\Gamma_x^*)^{-1} 1$$

$$1-\gamma = (1-\Gamma_x\Gamma_x^*)^{-1} 1$$

$$\boxed{p_x = U(x)\left((1-\Gamma_x\Gamma_x^*)^{-1} \cdot 1\right) e_{out} - \left(\Gamma_x^*(1-\Gamma_x\Gamma_x^*)^{-1} \cdot 1\right) e_{in}}$$

(p_x, q_x) is a sum of four terms

① $\left(u(x)\left(\left(1-\Gamma_x\Gamma_x^*\right)^{-1}1\right) e_{out}, \left(\left(1-\Gamma_x^*\Gamma_x\right)^{-1}1\right)e_{in}\right)$

$$= \left(e^{ikx} R \left(1-\Gamma_x\Gamma_x^*\right)^{-1}\cdot 1, \underbrace{\left(1-\Gamma_x^*\Gamma_x\right)^{-1} 1}\right)$$

$1-\beta$ essentially belongs to H_- so this might be equal to

$$\overset{?}{=} \left(P_- R_x \left(1-\Gamma_x\Gamma_x^*\right)^{-1} 1, \left(1-\Gamma_x^*\Gamma_x\right)^{-1} 1\right)$$

$$= \left(\Gamma_x^* \left(1-\Gamma_x\Gamma_x^*\right)^{-1} 1, \left(1-\Gamma_x^*\Gamma_x\right)^{-1} 1\right)$$

However if we use this argument then

② $= -\left(\left(\Gamma_x^*\left(1-\Gamma_x\Gamma_x^*\right)^{-1}\cdot 1\right)e_{in}, \left(\left(1-\Gamma_x^*\Gamma_x\right)^{-1}\cdot 1\right)e_{in}\right)$

will cancel this term and we will get zero for (p_x, q_x). ?

~~Point: $\Gamma_x\Gamma_x^* = P_+ e^{ikx}\bar{R} P_- e^{ikx} R = P_+ R P_- R$ for $x \le 0$~~

~~because $P_- e^{ikx} = e^{ikx} P_-$ if $x \le 0$~~

~~Similarly $\Gamma_x^*\Gamma_x = P_- e^{ikx} R P_+ e^{-ikx}\bar{R} = \Gamma_0^*\Gamma_0$~~

~~for $x \ge 0$ because~~

<u>July 22, 1978</u>

Review: Yesterday we found a way of interpreting the formula $g_x = pr_{\mathcal{H}_x}(e_{in})$ which goes as follows. One has

$$\mathcal{H}_x = (out, in)^{-1}(e^{ikx}H_- \times H_+)$$
$$= \text{orthogonal space to } (e^{ikx}H_+)e_{out} + (H_-)e_{in}$$

so we can look for an element

$$g_x = e_{in} - (e^{ikx}\alpha)e_{out} - \beta e_{in}$$

with $\alpha \in H_+$, $\beta \in H_-$ such that g_x is (formally) orthogonal to $(e^{ikx}H_+)e_{out} + (H_-)e_{in}$. This leads to the conditions

$$in(g_x) = (1-\beta) - Re^{ikx}\alpha \overset{\text{(formally)}}{\perp} H_-$$
$$out(g_x) = \bar{R}(1-\beta) - e^{ikx}\alpha \perp e^{ikx}H_+.$$

If $P_\pm$ is the orthogonal projection on $H_\pm$, and $R_x = Re^{ikx}$, and we suppose $R \in L_2$, then we get

$$-\beta = P_- R\alpha$$
$$\alpha = P_+\bar{R}(1-\beta) = P_+(\bar{R}) - P_+\bar{R}\beta$$

Put $\Gamma_x = P_+\bar{R} : H_- \longrightarrow H_+$; then $\Gamma_x^* = P_-R$. So we get

$$\alpha = \Gamma_x 1 + \Gamma_x\Gamma_x^*\alpha$$

$$\alpha = (1 - \Gamma_x\Gamma_x^*)^{-1}\Gamma_x 1$$

This is a well-defined elt of L^2 provided we assume $|R| \leq 1-\varepsilon$ so that $\|\Gamma_x\| \leq 1-\varepsilon$, and also assume $R \in L^2$ so $\Gamma_x 1 \in L^2$.

I now want to compute how α_x changes with x.

$$\Gamma_x \Gamma_x^* = P_+ \bar{R} e^{-ikx} P_- e^{ikx} R$$

$$\Gamma_{x+\varepsilon} \Gamma_{x+\varepsilon}^* = P_+ \bar{R}_x \; e^{-ik\varepsilon} P_- e^{ik\varepsilon} R_x$$

$$e^{-ik\varepsilon} P_- e^{ik\varepsilon} \hat{f}(k) = \int_{-\infty}^{0} e^{ik(x-\varepsilon)} f(x+\varepsilon)\, dx = \int_{-\infty}^{-\varepsilon} e^{ikx} f(x)\, dx$$

$$\frac{d}{d\varepsilon} \left(e^{-ik\varepsilon} P_- e^{ik\varepsilon} \hat{f} \right)(k) \Big|_{\varepsilon=0} = -f(0) = -\int \hat{f}(k) \frac{dk}{2\pi} = -(\hat{f}, 1)$$

So

$$\frac{d}{dx} \Gamma_x \Gamma_x^* g = P_+ \bar{R}_x (-R_x g, 1) = \underbrace{(-g, \bar{R}_x)}_{\text{scalar}} \underbrace{(\Gamma_x 1)}_{\text{element of } H_+}$$

We use this to compute the derivative of β

$$\alpha = \Gamma_x 1 + \Gamma_x \Gamma_x^* \alpha$$

$$\Gamma_x^* (1 - \Gamma_x \Gamma_x^*) \alpha = \Gamma_x^* \Gamma_x 1 \qquad\qquad -\beta = \Gamma_x^* \alpha$$

$$(1 - \Gamma_x^* \Gamma_x) \beta = -\Gamma_x^* \Gamma_x 1$$

But we need to differentiate $\Gamma_x^* \Gamma_x = P_- R e^{ikx} P_+ e^{-ikx} \bar{R}$

$$e^{ik\varepsilon} P_+ e^{-ik\varepsilon} \hat{f}(k) = \int_0^{\infty} e^{ik(x+\varepsilon)} f(x+\varepsilon)\, dx = \int_{\varepsilon}^{\infty} e^{ikx} f(x)\, dx$$

$$\frac{d}{d\varepsilon} \left(e^{ik\varepsilon} P_+ e^{-ik\varepsilon} \hat{f} \right) \Big|_{\varepsilon=0} = -f(0) = -(\hat{f}, 1)$$

$$\frac{d}{dx} \Gamma_x^* \Gamma_x g = \Gamma_x^* (-\bar{R}_x g, 1) = -(g, R_x) \cdot \Gamma_x^* 1$$

So now differentiating

$$(1-\Gamma_x^*\Gamma_x)\,\beta = -\Gamma_x^*\Gamma_x 1$$

gives

$$(1-\Gamma_x^*\Gamma_x)\frac{d\beta}{dx} + (\beta, R_x)\Gamma_x^* 1 = (1, R_x)\Gamma_x^* 1$$

or

$$\boxed{\frac{d\beta}{dx} = (1-\beta, R_x)\,\underbrace{(1-\Gamma_x^*\Gamma_x)^{-1}\Gamma_x^* 1}_{\delta}}$$

see p. 129

Next $(1-\Gamma_x\Gamma_x^*)\alpha = \Gamma_x 1 = P_+ e^{-ikx}\bar{R}$, so we need to differentiate $\Gamma_x 1$.

$$\cancel{(1-\Gamma_x\Gamma_x^*)\alpha = e^{-ikx}\left(e^{ikx}P_+e^{-ikx}\bar{R}\right)}$$

$$\cancel{(1-\Gamma_x\Gamma_x^*)\frac{d\alpha}{dx} + (\alpha, \bar{R}_x)\Gamma_x 1 = -ik\Gamma_x 1 + e^{-ikx}}$$

$$\Gamma_{x+\varepsilon}1 = e^{-ik\varepsilon}e^{ik\varepsilon}P_+e^{-ik\varepsilon}\bar{R}_x$$

$$\frac{d}{dx}\Gamma_x 1 = -ik\,\Gamma_x 1 \; (\bar{R}_x, 1)$$

In general

$$\frac{d}{dx}(\Gamma_x g) = -ik(\Gamma_x g) - (g, R_x)$$

So

$$(1-\Gamma_x\Gamma_x^*)\frac{d\alpha}{dx} + (\alpha, \bar{R}_x)\Gamma_x 1 = -ik\,\Gamma_x 1 - (1, R_x)$$

?

Instead we can derive the derivative of α by starting with

$$\alpha = \Gamma_x(1-\beta)$$

$$e^{ikx}\alpha = e^{ikx}P_+ e^{-ikx}\bar{R}(1-\beta)$$

$$\frac{d}{dx}\left(e^{ikx}\alpha\right) = e^{ikx}P_+ e^{-ikx}\bar{R}\frac{d}{dx}(1-\beta)$$
$$\qquad - e^{ikx}\left(e^{-ikx}\bar{R}(1-\beta), 1\right)$$

$$e^{-ikx}\frac{d}{dx}\left(e^{ikx}\alpha\right) = \Gamma_x\left\{-(1-\beta, R_x)(1-\Gamma_x^*\Gamma_x)^{-1}\Gamma_x^* 1\right\}$$
$$\qquad - (1-\beta, R_x)$$

$$\boxed{e^{-ikx}\frac{d}{dx}\left(e^{ikx}\alpha\right) = -(1-\beta, R_x)\underbrace{(1-\Gamma_x\Gamma_x^*)^{-1}\cdot 1}_{1-\gamma \text{ on p. } 129}}$$

So from the above two boxed formulas we have

$$\boxed{\frac{d}{dx}q_x = +(1-\beta, R_x)\,p_x}$$

Similarly we can compute $\frac{d}{dx}p_x$. First we need to collect the useful formulas

$$\frac{d}{dx}\; e^{ikx}P_+ e^{-ikx} f = - e^{ikx}\left(e^{-ikx}f, 1\right)$$

$$\frac{d}{dx}\; e^{-ikx}P_- e^{ikx} f = - e^{-ikx}\left(e^{ikx}f, 1\right)$$

$$\frac{d}{dx}\left(\Gamma_x\Gamma_x^* g\right) = -(g, \bar{R}_x)\cdot\Gamma_x 1$$

$$\frac{d}{dx}\left(\Gamma_x^*\Gamma_x g\right) = -(g, R_x)\cdot\Gamma_x^* 1$$

Now $\quad p_x = \left(e^{ikx}(1-\gamma)\right)e_{out} - \delta e_{in}$

where $(1-\Gamma_x\Gamma_x^*)(1-\gamma) = 1 \quad$ so

$$-(1-\Gamma_x\Gamma_x^*)\frac{d\gamma}{dx} + (1-\gamma, \bar{R}_x)\Gamma_x 1 = 0$$

or

$$\frac{d}{dx}(1-\gamma) = -(1-\gamma, \bar{R}_x)(1-\Gamma_x\Gamma_x^*)^{-1}\Gamma_x 1$$

$$= -(1-\gamma, \bar{R}_x)\,\alpha$$

Also

$$+\delta = \Gamma_x^*(1-\gamma) = e^{ikx}\left\{e^{-ikx}P_- e^{ikx}\bar{R}(1-\gamma)\right\}$$

$$\frac{d\delta}{dx} = ik\delta - e^{ikx}\left\{e^{-ikx}\left(e^{ikx}\bar{R}(1-\gamma), 1\right)\right\} + \Gamma_x^*\frac{d}{dx}(1-\gamma)$$

$$\frac{d\delta}{dx} = ik\delta - (1-\gamma, \bar{R}_x)\underbrace{\left\{1+\Gamma_x^*\alpha\right\}}_{1-\beta}$$

$$\frac{d}{dx}p_x = \frac{d}{dx}\left\{e^{ikx}(1-\gamma)\right\}e_{out} - \frac{d\delta}{dx}e_{in}$$

$$= ikp_x - (1-\gamma, \bar{R}_x)e^{ikx}\alpha\, e_{out}$$

$$+ (1-\gamma, \bar{R}_x)(1-\beta)\,e_{in}$$

or

$$\boxed{\frac{d}{dx}p_x = ikp_x + (1-\gamma, \bar{R}_x)\,g_x}$$

So now the problem is to show that $(1-\beta, R_x), (1-\gamma, \bar{R}_x)$ are conjugate

$$(1-\gamma, \bar{R}_x) = (1, \bar{R}_x) - (\gamma, \bar{R}_x) \qquad -\gamma = \Gamma_x \Gamma_x^*(1-\Gamma_x\Gamma_x^*)^{-1}1$$

$$= (1, \bar{R}_x) + \left(\Gamma_x\Gamma_x^*(1-\Gamma_x\Gamma_x^*)^{-1}1, P_+\bar{R}_x\right)$$

$$= (R_x, 1) + \left(\Gamma_x(1-\Gamma_x^*\Gamma_x)^{-1}\Gamma_x^* 1, \Gamma_x 1\right)$$

$$(1-\beta, R_x) = (1, R_x) + (-\beta, P_- R_x)$$

$$= (1, R_x) + \left(\Gamma_x^*(1-\Gamma_x\Gamma_x^*)^{-1}\Gamma_x 1, \Gamma_x^* 1\right)$$

$$\left((1-\Gamma_x\Gamma_x^*)^{-1}\Gamma_x 1, \Gamma_x\Gamma_x^* 1\right)$$

Note that

$$(R_x, 1) = \int e^{ikx} R \frac{dk}{2\pi}$$

is the Fourier transform of R; it is an L^2 function defined a.e. The other terms $(\gamma, \bar{R}_x), (-\beta, R_x)$ are continuous functions of x, it seems.

Anyway, we have that conjugation changes $\Gamma_x = P_+\bar{R}_x$ to $P_- R_x = \Gamma_x^*$. Hence

$$\overline{1-\gamma} = \overline{(1-\Gamma_x\Gamma_x^*)^{-1}1} = (1-\Gamma_x^*\Gamma_x)^{-1}1 = 1-\beta$$

and so

$$(1-\gamma, \bar{R}_x) = (R_x, 1-\beta) = \overline{(1-\beta, R_x)}$$

Finally it remains to see if this coefficient is related to (p_x, q_x).

$$(p_x, q_x) = \left(e^{ikx}(1-\gamma)\, e_{out} - \delta e_{in} \,,\; (1-\beta) e_{in} - e^{ikx}\alpha\, e_{out}\right)$$

$$= \left((1-\gamma)R_x - \delta, 1-\beta\right) - \left(1-\gamma-\delta\bar{R}_x, \alpha\right)$$

But recall

$$in(p_x) = (1-\gamma)R_x - \delta \perp H_- \quad \text{and} \quad \beta \in H_-$$

$$e^{-ikx} out(p_x) = 1-\gamma-\delta\bar{R}_x \quad \text{and} \quad \gamma + \delta R_x \perp H_+$$

$$\text{and} \quad \alpha \in H_+$$

hence we get

$$(p_x, q_x) = ((1-\gamma)R_x, 1) - (\delta, 1) - (1, \alpha)$$

But

$$(\delta, 1) = \left(\Gamma_x^*(1-\Gamma_x\Gamma_x^*)^{-1}1, 1\right)$$

$$= \left(\Gamma_x^* 1 - \Gamma_x^*\gamma, 1\right)$$

We run into the following problem: $\Gamma_x^* = P_- R_x$, how do we evaluate

$$(\Gamma_x^*\gamma, 1) = (P_- R_x\gamma, 1)$$

or for that manner what is $(P_- f, 1)$ for an element f of L^2. Here P_- on the Fourier transform level is multiplication by the characteristic function of $(-\infty, 0]$, so $(P_- f, 1) = \check{P_- f}(0)$ is the value at a discontinuity.

July 24, 1978

Consider a Schroedinger DE on $\mathbb{R}$

$$Lu = -u'' + qu = k^2 u$$

with $q \in C_0^\infty(\mathbb{R})$. I want to construct $(\mathcal{H}, U(t), e_{out}, e_{in})$ which belongs to this DE. $\mathcal{H}$ should be built up out of solutions ψ of the wave equation

$$L\psi = -\frac{\partial^2 \psi}{\partial t^2}$$

Such solutions correspond under Fourier transform:

$$\psi(x,t) = \int e^{ikt} u(x,k)\, dk/2\pi$$

to solutions $u(x,k)$ of $Lu = k^2 u$ in the space of functions of k.

(Check time dependence: After b seconds the wave ψ evolves into the wave ψ_b given by

$$\psi_b(x,t) = \psi(x, b+t) = \int e^{ikt} e^{ikb} u(x,k)\, dk/2\pi$$

and hence time evolution for solutions of the wave equation corresponds to multiplication by e^{ikt}.)

The next thing is to identify e_{in} and e_{out}. e_{in} should correspond to a solution of the wave equation consisting of a δ impulse coming in from the right giving rise to a transmitted and a reflected wave. Hence if we take the solution with the asymptotic behavior

$$(*) \qquad e^{+ikx} \longleftrightarrow Ae^{ikx} + Be^{-ikx}$$

[redacted] outgoing to left; incoming from right; outgoing to right

and divide by $A(k)$ to get

[redacted] $$\frac{1}{A}e^{+ikx} \longleftrightarrow e^{ikx} + \frac{B}{A}e^{-ikx}$$

we get what should be e_{in}.

If $u(x,k)$ is a solution of $Lu = k^2 u$, then we have

$$u(x,k) = \alpha(k)e^{ikx} + \beta(k)e^{-ikx} \qquad x >> 0$$

and clearly we want to define

$$in(u) = \alpha$$
$$out(u) = \beta$$

Hence
$$in(e_{in}) = 1$$
$$out(e_{in}) = \frac{B}{A}$$

so the reflection coefficient is $R = \frac{\bar{B}}{\bar{A}}$. As a check notice that for $k \in LHP$ the function e^{+ikx} decays as $x \to -\infty$, and grows as $x \to +\infty$; hence A is "well-defined" in the LHP and doesn't vanish there assuming no bound states. Thus R is analytic in the UHP (strictly speaking admits an analytic extension to the UHP).

Similarly one has

$$e^{-ikx} \longleftrightarrow \bar{B}e^{ikx} + \bar{A}e^{-ikx}$$

incoming from left; outgoing to right

so e_{out} is the solution with asymptotic behavior

$$\frac{1}{\bar{A}} e^{-ikx} \xleftrightarrow{e_{out}} \frac{\bar{B}}{\bar{A}} e^{ikx} + e^{-ikx}$$

Next I want to relate the Schroedinger DE to a Dirac DE. The point is that for a Dirac D.E. the inner product on solutions of the wave equation is the obvious one. So begin with

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & h \\ h & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad h \text{ real} .$$

then

$$\frac{d}{dx}(u_1+u_2) = ik(u_1-u_2) + h(u_1+u_2)$$

$$\frac{d}{dx}(u_1-u_2) = ik(u_1+u_2) - h(u_1-u_2)$$

$$\left(\frac{d}{dx}-h\right)(u_1+u_2) = ik(u_1-u_2)$$

$$\left(\frac{d}{dx}+h\right)(u_1-u_2) = ik(u_1+u_2)$$

$$\left(\frac{d^2}{dx^2} - h' - h^2\right)(u_1+u_2) = \left(\frac{d}{dx}+h\right)\left(\frac{d}{dx}-h\right)(u_1+u_2) = -k^2 u$$

Hence given the potential q I choose h to satisfy

$$h' + h^2 = q.$$

and then a solution $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of the Dirac equation gives rise to a solution $u = u_1 + u_2$ of the Schroedinger DE. Solutions of the above Riccati equation are given by

$$h = \frac{u'}{u} \qquad \text{where} \qquad -u'' + qu = 0$$

e.g. $\left(\frac{u'}{u}\right)' = \frac{u''}{u} - \left(\frac{u'}{u}\right)^2$.

and since we don't want h to have poles we need $u \neq 0$ for all x. For example $u = f(x,0)$ works and gives an h with $h(x) = 0$ for $x \gg 0$. In general one expects $f(x,0)$ to be independent from the solution ~ 1 as $x \to -\infty$, hence there appear to be many possibilities for h.

However we want a solution u of Schroedinger to be given by $u = u_1 + u_2$, where $\binom{u_1}{u_2}$ is a solution of Dirac with the same in and out functions.

The point maybe is that there ought to be some sort of mathematics connected with the fact that h cannot be chosen to be zero for $x \gg 0$ and $x \ll 0$ simultaneously, in general.

Point: Yesterday I saw that there were in general many Dirac equations belonging to a Schroedinger equation with potentials of compact support. However since the reflection coefficient R is analytic in the UHP (and bounded by 1 if q supported in $(-\infty, 0]$) the Dirac system I am after has $h = 0$ for $x \gg 0$. This means that the Dirac systems associated to the left and right scattering will be different.

For a Dirac system

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & h \\ \bar{h} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

one gets a Hilbert space $\mathcal{H}$ out of solutions $\vec{\psi}(x,t)$ of the associated wave equation using the norm

$$\int_{-\infty}^{\infty} |\vec{\psi}(x,t)|^2 dx$$

which is independent of t. Assuming $h = 0$ for $x \geq 0$ one has "incoming" solutions to the wave equation of the form

$$\vec{\psi}(x,t) = \begin{pmatrix} f(x+t) \\ 0 \end{pmatrix} \qquad x \geq 0$$

with $f \in C_0^\infty(\mathbb{R})$, and $\vec{\psi}$ is to be zero for $x \leq 0$ and $t \ll 0$. This is just the solution αe_{in} where $f = \underbrace{\int e^{ikx} \alpha(k)\, dk/2\pi}_{\tilde{\alpha}(x)}$. In fact

$$e_{in}(x,k) = \begin{pmatrix} e^{ikx} \\ \tilde{R} e^{-ikx} \end{pmatrix} \qquad x \geq 0$$

so

$$\vec{\psi}(x,t) = \int e^{ikt} e_{in}(x,k)\, \alpha(k)\, dk/2\pi$$

$$= \begin{pmatrix} f(x+t) \\ \widetilde{\tilde{R}\alpha}(t-x) \end{pmatrix} \qquad x \geq 0$$

It's clear more or less that we actually do get the formulas

$$\|\alpha e_{in}\|^2 = \|\alpha\|^2, \quad \|\beta e_{out}\|^2 = \|\beta\|^2, \quad (\beta e_{out}, \alpha e_{in}) = (R\beta, \alpha)$$

this way.

<u>Question</u>: Is it possible that of the different possible h satisfying $g = h' + h^2$, that only the h with $h = 0$ for $x \gg 0$ has the correct R function? Otherwise you will have different Dirac systems with the same R.

So suppose $g = h' + h^2$ where g has compact support. Solutions of $u'' = gu$ are linear outside the support of g, say $u = ax + b$, hence

$$h = \frac{u'}{u} = \frac{a}{ax+b} = O\left(\frac{1}{x}\right) \quad \text{as } |x| \to \infty$$

$$= 0 \qquad \text{if } a = 0$$

Consider now the solution e_{in} of Schroedinger with asymptotic behavior

$$T e^{+ikx} \longleftrightarrow e^{ikx} + \bar{R} e^{-ikx}$$

Now if u is a solution of Schroed., then the corresponding Dirac solution $\binom{u_1}{u_2}$ is found by solving

$$u_1 + u_2 = u$$
$$u_1 - u_2 = \frac{1}{ik}\left(\frac{d}{dx} - h\right) u$$

If $u = e^{ikx}$ for x outside the support of q, then we find

$$2u_1 = e^{ikx}\left(1 + 1 - \frac{h}{ik}\right) \qquad \text{or}$$

$$u_1 = e^{ikx}\left(1 - \frac{h}{2ik}\right) \sim e^{ikx} \qquad \text{as } |x| \to \infty$$

$$u_2 = \frac{h}{2ik} e^{ikx} \sim 0 \qquad \text{as } |x| \to \infty.$$

Similarly for $u = e^{-ikx}$ we have

$$u_1 = -\frac{h}{2ik} e^{-ikx} \sim 0$$

$$u_2 = \left(1 + \frac{h}{2ik}\right) e^{-ikx} \sim e^{-ikx} \qquad \text{as } |x| \to \infty$$

and so therefore no matter what h is, $u = e^{ikx}$ for $x \ll 0$ corresponds to the Dirac solution with asymptotic behavior $\binom{e^{ikx}}{0}$, etc. Hence we have

$$T\binom{e^{ikx}}{0} \longleftrightarrow \binom{e^{ikx}}{\bar{R} e^{-ikx}}$$

and so we get the same scattering function R.