July 1, 1978

Recall the construction of the unitary dilation $(\tilde{\mathcal{H}}, U)$ of a contraction T on $\mathcal{H}$. One forms the space $\tilde{\mathcal{H}}$ by completing the space of finite sums $\sum U^n a_n$, $a_n \in \mathcal{H}$ with inner product

$$(U^n a, U^m b) = (T_{n-m} a, b) \qquad T_p = \begin{cases} T^p & p \geq 0 \\ (T^*)^{-p} & p \leq 0 \end{cases}$$

$$= \int \Big(\sum_{p \in \mathbb{Z}} z^{-p} T_p \cdot z^n a, z^m b \Big)_{\mathcal{H}} \frac{d\theta}{2\pi}$$

where

$$\sum z^{-p} T_p = (1 - z^{-1}T)^{-1} (1 - TT^*)(1 - zT^*)^{-1}$$

and we are assuming $\|T\| < 1$ to make sense out of $(1 - zT^*)^{-1}$.

Another method to get $\tilde{\mathcal{H}}$ is to equip $\mathcal{H}$ with the norm $((1-TT^*)u, u)$ and complete to get $\rho : \mathcal{H} \to \mathcal{N}$. Then we can define

$$j : \mathcal{H} \longrightarrow L^2(S^1, \mathcal{N})$$

$$h \longmapsto \rho(1 - zT^*)^{-1} h = \sum_{n \geq 0} z^n \rho(T^{*n} h)$$

Then

$$(j^* U^m j h, h') = [redacted] \Big(z^m \sum_{n \geq 0} z^n \rho(T^{*n} h), \sum_{n \geq 0} z^n \rho T^{*n} h' \Big)$$

$(U^m jh, jh')$

$$= \sum_{n \geq 0, -m} ((1 - TT^*) T^{*n} h, T^{*n+m} h')$$

say $m \geq 0$

$$= (h, T^{*m} h') - \lim_{n \to \infty} (T^{*n} h, T^{*m+n} h')$$

This last formula shows that for any contraction T the map j is defined, i.e. the series $\sum z^n \rho T^{*n} h$ converges in $L^2(S^1, \mathcal{N})$, and $\|jh\|^2 = \|h\|^2 - \lim_{n \to \infty} \|T^{*n} h\|^2$. When $T^{*n} h \to 0$ for all $h \in \mathcal{H}$, then we know j is an embedding and hence we get the unitary dilation of T.

Next we want the continuous version of the preceding. Let $T(t)$ $t \geq 0$ be a contraction semi-group on $\mathcal{H}$ such that $T(t)h$ is continuous in t for each $h \in \mathcal{H}$, whence $T(t)$ has an infinitesimal generator B by Hille theory. We can construct $\tilde{\mathcal{H}}$ by completing finite sums of formal expressions $U(t)h$ in the norm

$$(U(s)h,\ U(t)h') = (T_{s-t}\,h,\ h') \qquad T_x = \begin{cases} T(x) & x \geq 0 \\ T(-x)^* & x \leq 0 \end{cases}$$

$$= \int \frac{dk}{2\pi}\left(\int e^{-ikx}T_x\,dx \cdot e^{iks}h,\ e^{ikt}h'\right)$$

where
$$\int e^{-ikx}T_x\,dx = \int_0^\infty e^{-ikx}T(x)\,dx + \int_{-\infty}^0 e^{-ikx}T(-x)^*\,dx$$

$$= (ik-B)^{-1} + (-ik-B^*)^{-1}$$

$$= (-ik-B^*)^{-1}\left\{-ik-B^* + ik - B\right\}(ik-B)^{-1}$$

$$= (-ik-B^*)^{-1}(-B-B^*)(ik-B)^{-1}$$

Second approach to $\tilde{\mathcal{H}}$ is to form the completion $\rho : \mathcal{H} \to \mathcal{N}$ of $\mathcal{H}$ wrt the norm $((-B-B^*)u, u)$ and to define

$$j : \mathcal{H} \longrightarrow L^2\left(\mathbb{R}, \frac{dk}{2\pi}; \mathcal{N}\right)$$

$$jh = \int_0^\infty e^{+ikx}\rho(T(x)^*h)\,dx = \rho\,(-ik-B^*)^{-1}h$$

provided this has a sense. One has

$$\|jh\|^2 = \int \frac{dk}{2\pi}\left\|\int_0^\infty e^{ikx}\rho(T(x)^*h)\,dx\right\|^2$$

$$= \int_0^\infty \|\rho\,T(x)^*h\|^2\,dx = \int_0^\infty \left(-(B+B^*)T(x)^*h,\ T(x)^*h\right)dx$$

$$= \int_0^\infty -\frac{d}{dx}\|T(x)^*h\|^2\,dx = \|h\|^2 - \lim_{x\to+\infty}\|T(x)^*h\|^2$$

This equation shows that for any contraction semi-group the map j is defined because $x \mapsto \rho T(x)^* h$ is in $L^2([0,\infty);\mathcal{N})$. If we have $T(x)^*h \to 0$ as $x \to \infty$ then j is an isometric embedding.

Problem: To understand well the relation between contraction semi-groups and their infinitesimal generators. The theorem seems to be that the map $T(t) \longmapsto B = \lim_{\varepsilon\to 0}\frac{T(\varepsilon)-I}{\varepsilon}$ is a one-one correspondence of contraction semi-groups with closed densely-defined operators B such that $-(Bu,u)-(u,Bu) \geq 0$ for all $u \in \mathcal{D}_B$. Most of the proof is in Riesz-Nagy § 143. Starting with B one constructs $T(t)$ via the Thm. on page 143. One has to show that $(I-\varepsilon B)^{-1}$ exists for every $\varepsilon > 0$ and that its norm is ≤ 1. But for $u \in \mathcal{D}_B$ and $\varepsilon > 0$

$$\|(1-\varepsilon B)u\|^2 = \|u\|^2 - \varepsilon(Bu,u) - \varepsilon(u,Bu) + \varepsilon^2\|Bu\|^2$$

$$\geq \|u\|^2$$

so $(1-\varepsilon B)^{-1}$ exists and has norm ≤ 1 provided $(1-\varepsilon B)\mathcal{D}_B$ is dense in $\mathcal{H}$. Now the orthogonal complement of $(1-\varepsilon B)\mathcal{D}_B$ consists of v with

$$((1-\varepsilon B)u, v) = (u,v) - \varepsilon(Bu,v) = 0 \qquad \text{all } u \in \mathcal{D}_B$$

and hence $v \in \mathcal{D}_{B^*}$ and $(1-\varepsilon B^*)v = 0$. Thus I seem also to want

$$-(B^*v,v) - (v,B^*v) \geq 0 \qquad \forall\, v \in \mathcal{D}_{B^*}$$

July 2, 1978

Let B be a closed densely-defined operator on $\mathcal{H}$ such that $(Bu,u)+(u,Bu)\le 0$ for all $u\in\mathcal{D}_B$. If $k=\alpha+i\beta$ $\alpha\ge 0$, β real, then

$$\|(B-k)u\|^2 = \|(B-i\beta)u\|^2 \underbrace{-2\,\mathrm{Re}\,((B-i\beta)u,u)\alpha}_{-2\,\mathrm{Re}\,(Bu,u)\alpha \;\leftarrow\; \ge 0} + \|\alpha u\|^2$$

$$\ge \|(B-i\beta)u\|^2 + \|\alpha u\|^2$$

$$\ge \|(B-i\beta)u\|^2 + 2\,\mathrm{Re}\,((B-i\beta)u,u)\alpha + \|\alpha u\|^2$$

$$= \|(B-i\beta+\alpha)u\|^2 = \|(B+\bar{k})u\|^2$$

Hence for $\alpha>0$ it follows that $(B-k)\mathcal{D}_B$ is closed and that

$$T=(B+\bar{k})(B-k)^{-1}$$

is a contraction operator on $(B-k)\mathcal{D}_B$.

Claim that $(B-k)\mathcal{D}_B$ has the same codimension as k ranges over $\mathrm{Re}\,k>0$. To see this consider

$$\mathcal{D}_B \oplus (B-1)\mathcal{D}_B^{\perp} \longrightarrow \mathcal{H}$$

$$\begin{pmatrix} x \\ y\end{pmatrix} \longmapsto (B-1)x+y$$

This map is unitary if $\mathcal{D}_B$ is equipped with the norm $\|(B-1)u\|^2$, which is $\ge \|Bu\|^2+\|u\|^2$ and hence is equivalent to the usual norm on $\mathcal{D}_B$. Then $\begin{pmatrix} x \\ y\end{pmatrix}\mapsto (B+1)x$ is of norm ≤ 1 so the map

$$\begin{pmatrix} x \\ y\end{pmatrix} \longmapsto (B-1)x+y+u(B+1)x$$

is an isomorphism for $|u|<1$. This means

$$\big((1+u)B-(1-u)\big)\mathcal{D}_B = \left(B-\tfrac{1-u}{1+u}\right)\mathcal{D}_B$$

is a closed subspace of $\mathcal{H}$ complementary to $(B-1)\mathcal{D}_B{}^{\perp}$.
Finally $k=\frac{1-u}{1+u}$ runs over $\operatorname{Re}(k)>0$ as u runs over $|u|<1$.

Take $k=1$, so $T=(B+1)(B-1)^{-1}$ and

$$T-1 = (B+1)(B-1)^{-1}-(B-1)(B-1)^{-1}$$
$$= 2(B-1)^{-1}$$

$$\mathcal{D}_B \xrightarrow[\sim]{B-1} \mathcal{D}_T \quad \text{has inverse map} \quad \tfrac{1}{2}(T-1)$$

hence $T-1$ is injective with image $\mathcal{D}_B$. So we have associated to any closed operator B with $\operatorname{Re}(Bu,u)\le 0$ on $\mathcal{D}_B$ a partially-defined contraction T with $\mathcal{D}_T=(B-1)\mathcal{D}_B$ such that $T-1$ is injective.

Conversely given a partially-defined contraction $T:\mathcal{D}_T\to\mathcal{H}$ with $T-1$ injective, we can define a closed operator B with $\mathcal{D}_B=(T-1)\mathcal{D}_T$ and

$$B-1 = 2(T-1)^{-1}$$

Then $$B+1 = 2(T-1)^{-1}+2(T-1)(T-1)^{-1} = 2T(T-1)^{-1}$$

so $$\|(B+1)u\| = \|T\,2(T-1)^{-1}u\| \le \|2(T-1)^{-1}u\| = \|(B-1)u\|$$

for all $u\in\mathcal{D}_B$. Squaring and subtracting we get $4\operatorname{Re}(Bu,u)\le 0$.

(terminology not exactly same as Nagy.)

Let's call B dissipative when $\operatorname{Re}(Bu,u)\le 0$. We have established a 1-1 correspondence between dissipative operators and partially-defined contractions T without the eigenvalue $+1$. Maximal dissipative operators correspond to contractions T defined on all of $\mathcal{H}$.

Suppose T is a contraction on $\mathcal{H}$ with $T-1$ injective. I claim T^*-1 is also injective. In effect because $1-TT^*$ is self-adjoint and ≥ 0 we have,

$$\|T^*f\| = \|f\| \Rightarrow \left((1-TT^*)f, f\right) = 0 \Rightarrow (1-TT^*)f = 0$$

so $T^*f = f \Rightarrow f = TT^*f = Tf$ and so f has to be zero.

Now $\left((T-1)\mathcal{H}\right)^\perp = \operatorname{Ker}(T^*-1) = 0$, hence $\mathcal{D}_B = (T-1)\mathcal{H}$ is dense. Thus any maximal dissipative B is automatically densely-defined. Finally from

$$B-1 = 2(T-1)^{-1} \qquad \text{we get}$$

$$B^*-1 = 2(T^*-1)^{-1}$$

and so B^* is also maximal dissipative.

If instead of $k=1$ we take a general k with $\operatorname{Re}(k)>0$ and define

$$T = \frac{B+\bar{k}}{B-k} = \begin{pmatrix} 1 & \bar{k} \\ 1 & -k \end{pmatrix}(B)$$

then

$$B = \begin{pmatrix} -k & -\bar{k} \\ -1 & 1 \end{pmatrix}(T) = k\,\frac{T+(\bar{k}/k)}{T-1}$$

Better to write $T-1 = 2\operatorname{Re}(k)(B-k)^{-1}$. This shows that the eigenvalue 1 is still the critical one for T. Moreover if k is real then

$$\frac{T+1}{T-1} = \frac{1}{k}B$$

so that all we are doing is scaling on the B side.

Suppose we start with a maximal dissipative operator B on $\mathcal{H}$ and form the associated contraction $T = (B+1)(B-1)^{-1}$. Let $(\tilde{\mathcal{H}}, U)$ be the unitary dilation associated to T. One knows $\tilde{\mathcal{H}}$ admits the decomposition (orthogonal)

~~$\tilde{\mathcal{H}} : \cdots \oplus U^{-1}\mathcal{L}_i \oplus \mathcal{H} \oplus U\mathcal{L}_i \oplus U^2\mathcal{L}_i \oplus \cdots$~~

~~defined as follows. $\mathcal{L}_i = \overline{(1-T^*T)^{1/2}\mathcal{H}}$, $\mathcal{L}_i = (1-TT^*)^{1/2}\mathcal{H}$. In fact recall that~~

~~$(1-T^*T)^{1/2} : \mathcal{H} \longrightarrow \mathcal{L}_i$~~

~~is the completion of $\mathcal{H}$ for the norm $((1-T^*T)h, h)$.~~

$$\tilde{\mathcal{H}} : \cdots \oplus U^{-1}\mathcal{L}^* \oplus \mathcal{L}^* \oplus \mathcal{H} \oplus \mathcal{L} \oplus U\mathcal{L} \oplus \cdots$$

where $\mathcal{L} = \overline{(U-T)\mathcal{H}}$, $\mathcal{L}^* = \overline{(U^{-1}-T^*)\mathcal{H}}$ in $\tilde{\mathcal{H}}$.
In effect clearly we have

$$((U-T)h, h') = (Uh, h') - (Th, h') = 0$$

so that $Uh = (U-T)h + Th$ decomposes Uh orthogonally wrt $\mathcal{H}$. Hence we have the orth. decomp.

$$U\mathcal{H} + \mathcal{H} = (U-T)\mathcal{H} \oplus \mathcal{H}$$

and so $\overline{U\mathcal{H} + \mathcal{H}} = \mathcal{L} \oplus \mathcal{H}$. Similarly

$$(U^n(U-T)h, h') = (U^{n+1}h, h') - (U^nTh, h') = 0$$

$$\left(U^{-m}(U^{-1}-T^*)h,\ U^n(U-T)h'\right) = \left(h,\ U^{m+1+n}(U-T)h'\right) - \left(T^*h,\ U^{m+n}(U-T)h'\right)$$

$$= 0$$

hence we conclude the subspaces $\mathcal{H}$, $U^n\mathcal{L}$, $U^{-n}\mathcal{L}^*$ have to be orthogonal, and then they exhaust $\tilde{\mathcal{H}}$.

Note that because

$$\left((U-T)h, (U-T)h\right) = \|h\|^2 - \|Th\|^2$$

the map $h \longmapsto (U-T)h$ identifies $\mathcal{L}$ with the completion of $\mathcal{H}$ wrt the norm $\left((1-T^*T)h, h\right)$.

Notice also that we get biinvariant subspaces

$$\oplus\, U^n\mathcal{L}^* \subset \tilde{\mathcal{H}} \supset \oplus\, U^n\mathcal{L}$$

leading to "in" and "out" representations. The orthogonal complement of their sum is a closed subspace biinvariant under U and contained in $\mathcal{H}$, hence it coincides with the unitary part of T.

One sees also that discrete eigenvalues of modulus 1 for U can't appear in the "in" and "out" representations, hence they are the same for T, U.

Recall the isom

$$\mathcal{D}_B \underset{T-1}{\overset{(B-1)/2}{\rightleftarrows}} \mathcal{H}$$

hence if $h = \frac{1}{2}(B-1)u$, then

$$\|h\|^2 - \|Th\|^2 = \frac{1}{4}\left\{\|(B-1)u\|^2 - \|(B+1)u\|^2\right\}$$

$$= -\,\mathrm{Re}\,(Bu, u)$$

hence completing $\mathcal{H}$ wrt $\|h\|^2 - \|Th\|^2$ is isomorphic to completing $\mathcal{D}_B$ wrt $-\mathrm{Re}(Bu,u)$.

July 3, 1978:

Let B be a maximal dissipative operator on $\mathcal{H}$. To construct the semi-group e^{tB}, $t \geq 0$ one uses the following indirect method (from Sz.-Nagy, Foias). Form the contraction $T = (B+1)(B-1)^{-1}$, whence $B = (T+1)(T-1)^{-1}$ and we want to use

$$e^{tB} = e_t(T)$$

where $e_t(z) = \exp\left(t\frac{z+1}{z-1}\right)$. So we want to substitute the contraction T into the bounded analytic function e_t on $|z|<1$. In general given $f(z)$ bounded analytic one can try to define

$$f(T) = \lim_{r \uparrow 1} f(rT)$$

(note $f(rT) = \sum a_n r^n T^n$ converges in norm.). To see whether this limit exists one can use the unitary dilation U of T. One has $f(rT) = j^* f(rU) j$. In the particular case $T = (B+1)(B-1)^{-1}$ we know that T doesn't have the eigenvalue 1, hence U doesn't either. Moreover $e_t(rz) \longrightarrow e_t(z)$ boundedly for all $z \in S^1$ except $z=1$, hence boundedly for almost all z wrt the spectral measure of U. Hence by dominated convergence $e_t(rU) \longrightarrow e_t(U)$ (strong sense).

Unfortunately it does not seem to be possible to define e^{tB} via inverse Laplace transform

$$e^{tB} = \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{kt} (k-B)^{-1} \frac{dk}{2\pi i}$$

Let A be a symmetric operator on $\mathcal{H}$ with indices $(1,1)$ and put $V = \frac{iA+1}{iA-1} = \frac{A-i}{A+i}$, and let $u_i \perp \mathcal{D}_V$, $u_{-i} \perp V\mathcal{D}_V$ be unit vectors. We can extend V to a contraction T sending u_i to zero and this gives us a maximal dissipative operator B extending iA. We can form the unitary dilation $(\tilde{\mathcal{H}}, U)$ of $(\mathcal{H}, T)$ and this gives us a unitary group $U(t)$ dilation of e^{tB}. Assuming A has no self-adjoint component, we know that T is completely non-unitary and hence that $\tilde{\mathcal{H}}$ is completely described by an analytic function $S(z)$ in the disk of modulus ≤ 1. I want to describe the whole picture completely in terms of the original symmetric operator A.

July 4, 1978

Let's review how we get the S function for a discrete 1-port $(\mathcal{H}, V, u_i, u_{-i})$. We extend V to T and construct the unitary dilation $(\tilde{\mathcal{H}}, U)$ of T. $\tilde{\mathcal{H}}$ is constructed by completing finite sums $\sum U^n h_n$ with respect to the inner product

$$(U^n h, U^m h') = (T_{n-m} h, h') = \int \left(\sum z^{-p} T_p \, z^n h, z^m h' \right)_{\mathcal{H}} \frac{d\theta}{2\pi}$$

where

$$\sum z^{-p} T_p = (1 - z^{-1}T)^{-1}(1 - TT^*)(1 - zT^*)^{-1}.$$

[redacted] The completion of $\mathcal{H}$ wrt $\|h\|^2 - \|T^*h\|^2$ can be identified with $\rho(h) = (h, u_{-i})$, hence we can define

$$\begin{aligned} j : \mathcal{H} &\longrightarrow H^2_+(S^1) \subset L^2(S^1) \\ h &\longmapsto \rho\,(1 - zT^*)^{-1} h = ((1 - zT^*)^{-1} h, u_{-i}) \end{aligned}$$

When $T^{*n} \to 0$ this map j satisfies $j^* U^n j = T^n$ for $n \geq 0$, so that it gives rise to an isomorphism of $\tilde{\mathcal{H}}$ with $L^2(S^1)$ sending u_{-i} to 1. This is the "out" representation. We have (assuming also $T^n \to 0$)

$$\begin{array}{ccccc} L^2(S^1) & \xleftarrow[\sim]{\text{in}} & \tilde{\mathcal{H}} & \xrightarrow[\sim]{\text{out}} & L^2(S^1) \\ ((1 - z^{-1}T)^{-1} h, u_i) & \longleftarrow\!\shortmid & h & \longmapsto & ((1 - zT^*)^{-1} h, u_{-i}) \end{array}$$

and the S-function gives the composite $\text{out} \circ \text{in}^{-1}$. Take $h = u_i$.

$$S(z) = ((1 - zT^*)^{-1} u_i, u_{-i})$$

Now we do the same in the continuous case. Let's begin with a symmetric operator A of type $(1,1)$ on $\mathcal{H}$. Form $V = \frac{iA+1}{iA-1} = \frac{A-i}{A+i}$ and let u_i, u_{-i} be unit vectors $\perp$ to $\mathcal{D}_V = (A+i)\mathcal{D}_A$, $\mathcal{R}_V = (A-i)\mathcal{D}_A$ resp. Extend V to the contraction T with $T(u_i) = 0$ and let $B = \frac{T+1}{T-1}$. Then B is maximal dissipative extending iA. $\mathcal{D}_B = (T-1)\mathcal{H} = \langle (T-1)u_i \rangle + (V-1)\mathcal{D}_V = \langle -u_i \rangle + \mathcal{D}_A$. Hence $\mathcal{D}_B = \mathcal{D}_A + \langle u_i \rangle$ and

$$Bu_i = (T+1)(T-1)^{-1}u_i = -u_i$$

Similarly $\mathcal{D}_{B^*} = \mathcal{D}_A + \langle u_{-i} \rangle$ and $B^* u_{-i} = -u_{-i}$.

Let $T(t) = e^{tB}$ be the semi-group of contractions belonging to B, and let $(\tilde{\mathcal{H}}, U(t))$ be the associated unitary dilation. $\tilde{\mathcal{H}}$ is generated by $U(t)h$ with the norm

$$\big(U(t)h, U(t')h'\big) = \big(T_{t-t'}h, h'\big) = \int \frac{dk}{2\pi}\left(\int_{-\infty}^{\infty} e^{-ikp}T_p\,dp \cdot e^{ikt}h, e^{ikt'}h'\right)$$

and

$$\int e^{-ikp}T_p\,dp = \int_0^\infty e^{-ikp+pB}\,dp + \int_{-\infty}^0 e^{-ikp-pB^*}\,dp$$

$$= (ik-B)^{-1} + (-ik-B^*)^{-1}$$

$$= (ik-B)^{-1}(-B-B^*)(-ik-B^*)^{-1}$$

Proceeding formally, we should interpret $-B-B^*$ as the form $-2\operatorname{Re}(B^*u, u)$ on $\mathcal{D}_{B^*}$. Notice that this form vanishes on $\mathcal{D}_A$ and has the value 2 on u_{-i}, hence the completion of $\mathcal{D}_{B^*}$ for this form can be identified with the map $\rho : \mathcal{D}_{B^*} \longrightarrow \mathbb{C}$ given by

$$u \longmapsto \{-(B^*u, u_{-i}) - (u, B^*u_{-i})\}/\sqrt{2}$$
$$= \{(-B^*u, u_{-i}) - (u, -u_{-i})\}/\sqrt{2}$$

$$\rho(u) = \left(\frac{1-B^*}{\sqrt{2}} u, u_{-i}\right)$$

so define

$$j : \mathcal{H} \longrightarrow L^2\left(\frac{dk}{2\pi}\right)$$
$$h \longmapsto \rho\left((-ik-B^*)^{-1}h\right) = \int_0^\infty e^{ikt} \rho(T(t)^*h)\,dt$$
$$= \left(\frac{1-B^*}{\sqrt{2}}(-ik-B^*)^{-1}h, u_{-i}\right)$$

Assuming $T(t)^* \to 0$ it should follow that j induces an isomorphism of $\tilde{\mathcal{H}}$ with $L^2(\frac{dk}{2\pi})$ giving the "out" representation. If $T(t) \to 0$ also, then we should have

$$L^2\left(\frac{dk}{2\pi}\right) \xleftarrow{\text{in}} \tilde{\mathcal{H}} \xrightarrow{\text{out}} L^2\left(\frac{dk}{2\pi}\right)$$

$$\left(\frac{1-B}{\sqrt{2}}(ik-B)^{-1}h, u_i\right) \longleftarrow\!\shortmid\; h \longmapsto \left(\frac{1-B^*}{\sqrt{2}}(-ik-B^*)^{-1}h, u_{-i}\right)$$

The S-function is then obtained by taking $h = u_i$ and using $Bu_i = -u_i$

$$S(k)\frac{2}{\sqrt{2}}(ik+1)^{-1} = \left(\frac{1-B^*}{\sqrt{2}}(-ik-B^*)^{-1}u_i, u_{-i}\right)$$

$$\boxed{S(k) = (ik+1)\left(\frac{1-B^*}{2}(-ik-B^*)^{-1}u_i, u_{-i}\right)}$$

Notice that
$$\frac{1}{1-zT^*} = \frac{1}{1-\frac{ik+1}{ik-1}\,\frac{B^*+1}{B^*-1}} = \frac{(B^*-1)(ik-1)}{-ik-B^*-ik-B^*}$$
$$= (ik-1)\frac{B^*-1}{2}(-ik-B^*)^{-1}$$

Hence we see that

$$S(k) = -z\,S(z) \qquad z = \frac{ik+1}{ik-1} \qquad k = \frac{1}{i}\frac{z+1}{z-1}$$

Next we recall that $(1-\bar{z}T^*)^{-1}u_i$ is the unique element perpendicular to $(1-\bar{z}V)\mathcal{D}_V$ whose inner product with u_i is 1. Now

$$(1-\bar{z}V)\mathcal{D}_V = \left(1-\bar{z}\frac{A-i}{A+i}\right)(A+i)\mathcal{D}_A = \left[(1-\bar{z})A + i(1+\bar{z})\right]\mathcal{D}_A$$

$$= \left[A - i\,\frac{\bar{z}+1}{\bar{z}-1}\right]\mathcal{D}_A = \left[A-\bar{k}\right]\mathcal{D}_A$$

Hence we see that

$$(B^*-1)(-ik-B^*)^{-1}u_i$$

is some element of $(A-\bar{k})\mathcal{D}_A{}^{\perp} = \operatorname{Ker}(A^*-k)$.

Example: Suppose we are given a Dirac system on $[-b,\infty)$ with $-b<0$, with a self-adjoint boundary condition at $-b$, and suppose $p(x)=0$ for $x \geq -\varepsilon$ so that

$$\phi(x,k) = \begin{pmatrix} e^{ikx}A(k) \\ e^{-ikx}B(k) \end{pmatrix} \qquad x \geq 0$$

The Dirac system provides a symmetric operator A on $L^2(-b,0)^{\oplus 2}$ of type $(1,1)$. In what sense can we view the space $L^2(-b,\infty)^{\oplus 2}$, with the self-adjoint operator defined by the Dirac system, as the unitary dilation of A?

The basic problem here seems to be the fact that we have a natural identification of $\mathcal{D}_{A^*}/\mathcal{D}_A$ given by boundary values at $x=0$, but that this identification is not compatible with the u_i, u_{-i} basis unless $B(i) = 0$.

<u>July 5, 1978</u>:

Let V be a partial isometry of type $(1,1)$ on $\mathcal{H}$. Describe all contractions T extending V. Let u_i, u_{-i} be as usual. If w is an [redacted] arbitrary element of $\mathcal{D}_V$ we must have

[redacted]
$$\|T(w+u_i)\|^2 \leq \|w+u_i\|^2 = \|w\|^2 + 1$$
$$\|T(w+u_i)\|^2 = \|w\|^2 + 2\,\mathrm{Re}\,(Tw, Tu_i) + \|Tu_i\|^2 .$$

[redacted] Cancelling $\|w\|^2$, w appears linearly on one side of this inequality, hence we conclude

$$(Tw, Tu_i) = 0 \qquad \forall w \in \mathcal{D}_V$$

and hence $Tu_i = a\,u_{-i}$ where $|a| \leq 1$.

July 7, 1978

Suppose $(\mathcal{H}, V)$ is a port, u_i, u_{-i} are chosen, and T is the contraction extending V with $T(u_i)=0$, and $\tilde{\mathcal{H}}$ is the unitary dilation of T. The "in" representation is obtained from the ortho[redacted]normal system $\{U^n u_i\}$:

$$in(f) = \sum_{n\in\mathbb{Z}} (f, U^n u_i) z^n$$

Hence for $h \in \mathcal{H}$

$$in(h) = \sum_{n \in \mathbb{Z}} (h, U^n u_i) z^n \qquad 0 \text{ for } n>0$$

$$= \sum_{j\geq 0} (h, U^{-j} u_i) z^{-j}$$

$$= \left(h, (1-\bar{z}^{-1}T^*)^{-1} u_i\right)$$

Similarly

$$out(h) = \sum_{n \geq 0} (h, U^n u_{-i}) z^n$$

$$= \left(h, (1-\bar{z}T)^{-1} u_{-i}\right)$$

But recall that $(1-zT^*)^{-1}u_i$ is the unique element of $\mathcal{H}$ perp to $(1-\bar{z}V)\mathcal{D}_V$ whose inner product with u_i is 1. Hence $(1-\bar{z}^{-1}T^*)^{-1}u_i$ is the unique element [redacted] perp. to $(1-z^{-1}V)\mathcal{D}_V = (V-z)\mathcal{D}_V$ whose inner product with u_i is 1. Similarly $(1-\bar{z}T)^{-1}u_{-i}$ is the unique element [redacted] $\perp (1-zV^{-1})V\mathcal{D}_V = (V-z)\mathcal{D}_V$ whose inner product with u_{-i} is 1. [redacted]

Now recall that over the set of z for which $(V-z)\mathcal{D}_V = (1-z^{-1}V)\mathcal{D}_V$ is closed we get a holomorphic

line bundle $\mathcal{L}$ whose fibre at z is $\mathcal{H}/(V-z)\mathcal{D}_V$.
Moreover u_{-i} is a section of $\mathcal{L}$ over $|z|<1$, because we have $\mathcal{H} = \langle u_{-i}\rangle + (V-z)\mathcal{D}_V$. Hence to $h \in \mathcal{H}$ we can define a holomorphic function $\hat{h}$ in $|z|<1$ by

$$h \equiv \hat{h}(z)u_{-i} \quad \text{mod}\,(V-z)\mathcal{D}_V$$

But because $(1-\bar{z}T)^{-1}u_{-i}$ is $\perp$ to $(V-z)\mathcal{D}_V$ we have

$$\begin{aligned}\left(h, (1-\bar{z}T)^{-1}u_{-i}\right) &= \hat{h}(z)\left(u_{-i}, (1-\bar{z}T)^{-1}u_{-i}\right) \\ &= \hat{h}(z)\end{aligned}$$

Thus $\quad \text{out}(h)(z) = h \mod (V-z)\mathcal{D}_V$ relative to the basis u_{-i} for $\mathcal{L}_z$

Similarly $\text{in}(h)(z) = h \mod (V-z)\mathcal{D}_V$ relative to the basis u_i for $\mathcal{L}_z$.

July 8, 1978

Let V be a partial isometry of type $(1,1)$ on $\mathcal{H}$ without unitary component. Associated to $(\mathcal{H}, V)$ is a 2-diml vector space

$$W = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Gamma_V\right)^{\perp} \Big/ \Gamma_V$$

with a hermitian form of signature $+,-$. Here

$$\Gamma_V = \left\{ \begin{pmatrix} x \\ Vx \end{pmatrix} \in \mathcal{H}^{\oplus 2} \,\middle|\, x \in \mathcal{D}_V \right\}$$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Gamma_V\right)^{\perp} = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, (y_1, x) = (y_2, Vx) \;,\; \forall x \in \mathcal{D}_V \right\}$$

Clearly the latter contains $\begin{pmatrix} u_i \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ u_{-i} \end{pmatrix}$ and Γ_V. On the other hand

$$\Gamma_V^{\perp} \cap \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Gamma_V\right)^{\perp} = \Gamma_V^{\perp} \cap \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\Gamma_V^{\perp})$$

is stable under $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, hence one sees easily it is spanned by $\begin{pmatrix} y \\ 0 \end{pmatrix}$ with $y \in \mathcal{D}_V^{\perp}$ and $\begin{pmatrix} 0 \\ y \end{pmatrix}$, $y \in R_V^{\perp}$. Thus

$\begin{pmatrix} u_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_{-i} \end{pmatrix}$ is an orthonormal basis for the above intersection.

I have seen that extensions of V to a contraction T (p. 89) are determined by

$$T u_i = a u_{-i}$$

for some a with $|a| \leq 1$. Then Γ_T corresponds to the line in W generated by $\begin{pmatrix} u_i \\ a u_{-i} \end{pmatrix}$.

So if we equip W with the hermitian form

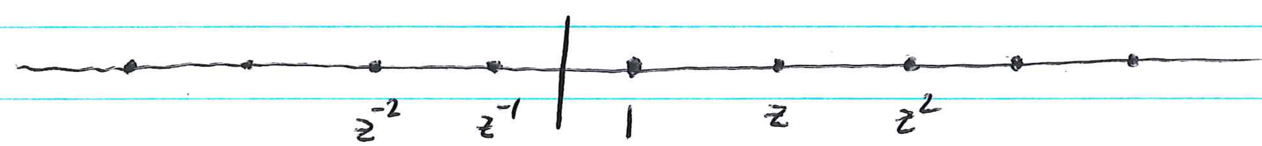
$$P\left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = |y_1|^2 - |y_2|^2$$

then contractions T extending V are in one-one correspondence with lines in W on which $P \geq 0$.

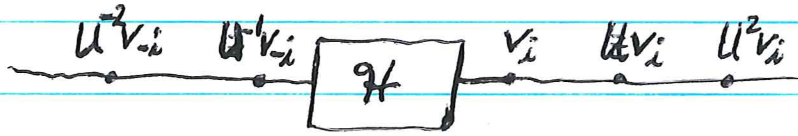
The program now will be to make correspond such contractions T with unitary bases for the form P. My idea is that a unitary basis for P amounts to a way of connecting the port $(\mathcal{H}, V)$ to a transmission line and that T is the contraction associated the scattering on the line.

July 9, 1978.

Consider $L^2(S^1)$ with $U =$ mult. by z. I can picture this as a line



If I cut the line, then U is not defined on z^{-1} and U^{-1} is not defined on 1. This is my model for a transmission line. Now I want to connect this line to the port $(\mathcal{H}, V)$, i.e. to obtain a U on an $\tilde{\mathcal{H}}$ with the picture



Here $v_{-i} \in \mathcal{H} + \langle v_i \rangle$. In fact we have

$$v_i - (v_{-i}, v_i) v_i \in \mathcal{H}$$

Let's put
$$\gamma = (v_i, v_{-i}) \qquad \text{so } |\gamma| < 1$$

and
$$v_{-i} - \bar{\gamma} v_i = h_1 . \qquad |h_1| = \sqrt{1 - |\gamma|^2}$$

Similarly $u^{-1} v_i \in \langle u^{-1} v_{-i} \rangle + \mathcal{H}$ so we have
$$v_i - \gamma v_{-i} = u h_2 \qquad |h_2| = \sqrt{1-|\gamma|^2}$$

So if $j: \mathcal{H} \to \tilde{\mathcal{H}}$ is the embedding we have

$$j^* v_{-i} = h_1$$
$$-\gamma j^* v_{-i} = j^* u h_2 = T h_2$$
$$\boxed{T h_2 = -\gamma h_1}$$

$$T^* h_1 = j^*(u^{-1} h_1) = j^*(u^{-1}(v_{-i} - \bar{\gamma} v_i)) = -\bar{\gamma} j^* u^{-1} v_i$$
$$h_2 = j^*(u^{-1}(v_i - \gamma v_{-i})) = j^* u^{-1} v_i$$
$$\therefore \boxed{T^* h_1 = -\bar{\gamma} h_2}$$

So we have $\quad T^* T h_2 = |\gamma|^2 h_2 \qquad T T^* h_1 = |\gamma|^2 h_1$

and hence $h_2 \in \langle u_i \rangle$, $\quad h_1 \in \langle u_{-i} \rangle$.

Choose u_i and u_{-i} so that
$$h_2 = \sqrt{1-|\gamma|^2}\, u_i \qquad h_1 = \sqrt{1-|\gamma|^2}\, u_{-i}$$

The problem now is to find a symplectic basis for
$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_V^{\perp} / \Gamma_V$$

which represents the connection of $\mathcal{H}, V$ to the transmission line.

Work in $\tilde{\mathcal{H}}^{\oplus 2}$. We want to determine the subspace $\Gamma_{\mathcal{U}}$. Put $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}'$ where

$$\mathcal{H}' = \langle v_i, \mathcal{U} v_i, \dots \rangle \oplus \langle \mathcal{U}^{-1} v_{-i}, \mathcal{U}^{-2} v_{-i}, \dots \rangle$$

In $\mathcal{H}'^{\oplus 2}$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_{V'}^{\perp} = \Gamma_{V'} \oplus \left\langle \begin{pmatrix} \mathcal{U}^{-1} v_{-i} \\ 0 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 0 \\ v_i \end{pmatrix} \right\rangle$$

The connection we are after will give us $\Gamma_{\mathcal{U}}$ as a subspace of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_V^{\perp} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_{V'}^{\perp}$$

of codim 2, containing $\Gamma_V \oplus \Gamma_{V'}$ as a subspace of codim 2. The connection will appear as essentially the graph of a symplectic isom. $W \simeq W'$. So what I want is an expression

$$\begin{pmatrix} u_i \\ 0 \end{pmatrix} \equiv \alpha \begin{pmatrix} \mathcal{U}^{-1} v_{-i} \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ v_i \end{pmatrix} \qquad \mathrm{mod}\ (\Gamma_{\mathcal{U}})$$

But

$$\begin{pmatrix} u_i \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1-|\gamma|^2}} \begin{pmatrix} \mathcal{U}^{-1} v_i - \gamma\, \mathcal{U}^{-1} v_{-i} \\ 0 \end{pmatrix}$$

$$\equiv \frac{-\gamma}{\sqrt{1-|\gamma|^2}} \begin{pmatrix} \mathcal{U}^{-1} v_{-i} \\ 0 \end{pmatrix} + \frac{-1}{\sqrt{1-|\gamma|^2}} \begin{pmatrix} 0 \\ v_i \end{pmatrix} \qquad \mathrm{mod}\ (\Gamma_{\mathcal{U}})$$

and $$\begin{pmatrix} 0 \\ u_{-i} \end{pmatrix} = \frac{1}{\sqrt{1-|\gamma|^2}} \begin{pmatrix} 0 \\ v_{-i} - \bar{\gamma} v_i \end{pmatrix}$$

$$\equiv \frac{-1}{\sqrt{1-|\gamma|^2}} \begin{pmatrix} U^{-1} v_{-i} \\ 0 \end{pmatrix} + \frac{-\bar{\gamma}}{\sqrt{1-|\gamma|^2}} \begin{pmatrix} 0 \\ v_i \end{pmatrix} \quad \bmod \Gamma_U$$

More generally given two ports $(\mathcal{H}, V)$ $(\mathcal{H}', V')$ one can connect them together. One forms $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}'$ and looks for a unitary extension $\mathcal{U}$ of $V \oplus V'$. Then

$$\Gamma_V \oplus \Gamma_{V'} \subset \Gamma_U \subset \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_V^\perp \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_{V'}^\perp$$

and for U to be unitary ~~[illegible]~~ means that the subspace

$$\Gamma_U / \Gamma_V \oplus \Gamma_{V'} \subset W \oplus W'$$

is maximal isotropic. In fact in order that there be a definite transfer between the two ports we want this subspace to be the graph of a symplectic isom. of W and W'. Thus we give ~~[illegible]~~ coefficients such that

$$\begin{pmatrix} u_i \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} u_i' \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ u_{-i}' \end{pmatrix} \in \Gamma_U$$

$$\begin{pmatrix} 0 \\ u_{-i} \end{pmatrix} + \gamma \begin{pmatrix} u_i' \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ u_{-i}' \end{pmatrix} \in \Gamma_U$$

This gives the equations

$$u_i + \alpha u_i' - \beta U^{-1} u_{-i}' = 0$$

$$u_{-i} - \gamma U u_i' + \delta u_{-i}' = 0.$$

where $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in U(1,1)$.

Consider a standard situation $T(u_i) = 0$.

$$\tilde{\mathcal{H}} : \cdots \oplus \langle U^{-1} u_i \rangle \oplus \mathcal{H} \oplus \langle U u_i \rangle \oplus \cdots$$

and look for eigenvectors for U (formally). An example is

$$\psi_\zeta = \sum_{n \in \mathbb{Z}} \zeta^{-n} U^n u_i \qquad U\psi_\zeta = \zeta \psi_\zeta$$

Projecting ψ_ζ onto $\mathcal{H}$ gives

$$j^* \psi_\zeta = \sum_{n \leq 0} \zeta^{-n} T^{*-n} u_i = (1 - \zeta T^*)^{-1} u_i$$

Thus $j^*\psi_\zeta$ is the unique element of $\mathcal{H}$ perpendicular to

$$(1 - \bar{\zeta} T) \mathcal{D}_V = (1 - \bar{\zeta} V) \mathcal{D}_V$$

Clearly

$$\text{in } [redacted] (\psi_\zeta) = \sum_n \zeta^{-n} z^n$$

$$\text{out } (\psi_\zeta) = \sum_n (\psi_\zeta, U^n u_{-i}) z^n$$

$$= \sum_n \zeta^{-n} \underbrace{(\psi_\zeta, u_{-i})}_{\substack{\parallel \\ (j^*\psi_\zeta,\, u_{-i}) \,=\, S(\zeta)}} z^n$$

$$\text{out } (\psi_\zeta) = S(\zeta) \sum_n \zeta^{-n} z^n$$

Continuous analogue. First look at a continuous transmission line.

Let $\mathcal{H} = L^2(\mathbb{R}, dx) \cong L^2(\mathbb{R}, \frac{dk}{2\pi})$ the isomorphism being given by Fourier transform

$$f(x) = \int e^{-ikx} \hat{f}(k) \frac{dk}{2\pi}$$

Then $\hat{f} \mapsto e^{ikt} \hat{f}$ corresponds to $f \longmapsto f(\cdot - t)$ and so multiplication by k corresponds to $i\frac{d}{dx}$. Let A be the symmetric operator which is the restriction of $\hat{A}$ = mult. by k to

$$\mathcal{D}_A = \left\{ \hat{f} \;\middle|\; \hat{f}, k\hat{f} \in L^2 \text{ and } \int \hat{f} \frac{dk}{2\pi} = 0 \right\}$$

Equivalently $\mathcal{D}_A$ consists of abs. cont f with $f' \in L^2$ such that $f(0) = 0$. $\mathcal{D}_{A^*}$ consists of f which are absolutely continuous except at $x=0$ with $f' \in L^2$, hence $\mathcal{D}_{A^*}/\mathcal{D}_A$ can be identified with the pair of boundary values $\begin{pmatrix} f(0^-) \\ f(0^+) \end{pmatrix}$. $\mathrm{Ker}(A^* - \lambda)$ for $\lambda \in UHP$ ~~consists~~ is generated by an L^2 solution of

$$i\frac{d}{dx} u = \lambda u \qquad u = e^{-i\lambda x}$$

hence by the element

$$\psi_\lambda = \begin{cases} ie^{-i\lambda x} & x < 0 \\ 0 & x > 0 \end{cases}$$

whose transform is

$$\hat{\psi}_\lambda(k) = i\int_{-\infty}^{0} e^{ikx - i\lambda x} \, dx = \frac{1}{(k-\lambda)}$$

Similarly for $Re(\lambda) < 0$

$$\psi_\lambda = \begin{cases} 0 & x < 0 \\ \frac{1}{i} e^{-i\lambda x} & x > 0 \end{cases}$$

and
$$\hat{\psi}_\lambda(k) = \frac{1}{i}\int_0^\infty e^{-i\lambda x} e^{ikx}\, dx = \frac{1}{k-\lambda}$$

In the k-picture $\mathcal{D}_{A^*}$ is generated by $\mathcal{D}_A$ and the elements $\hat{\psi}_\lambda = \frac{1}{k-\lambda}$ for $\lambda = \pm i$. One has

$$(\psi_\lambda, \psi_\mu) = \int_0^\infty e^{-i\lambda x + i\bar{\mu} x}\, dx = \frac{i}{\lambda - \bar{\mu}} \qquad Re(\lambda), Re(\mu) > 0$$

$$= 0 \qquad \text{if} \quad Re(\lambda)\cdot Re(\mu) < 0$$

$$= \frac{-i}{\lambda - \bar{\mu}} \qquad \text{if} \quad Re(\lambda), Re(\mu) < 0.$$

So $(\psi_i, \psi_i) = \frac{1}{2}$, hence $\underbrace{\text{put}}_{\text{we can}}$ $u_i = \frac{\sqrt{2}}{k-i}$, $u_{-i} = \frac{\sqrt{2}}{k+i}$.

If $f \in \mathcal{D}_{A^*}$ we have

$$(A^*f, f) - (f, A^*f) = \int i\left[\frac{d}{dx}(f)\bar{f} + f\frac{d\bar{f}}{dx}\right] dx$$

$$= i\left\{|f(0^-)|^2 - |f(0^+)|^2\right\}$$

Finally we compute the scattering function. For $\lambda \in UHP$

$$(\psi_\lambda, u_i) = \frac{\sqrt{2}}{\lambda + i} \qquad (\psi_\lambda, u_{-i}) = 0$$

and hence $S(\lambda) = 0$ in the UHP
$S(\lambda) = \infty$ " " LHP.