June 13, 1978

Easy derivation of the recursion relation for p_n, q_n.

Recall $\quad p_n = pr_{F_n}(z^n u_i)/\text{norm}$ is a unit vector in $F_n \ominus F_{n-1}$

$\quad q_n = pr_{F_n}(u_{-i})/\text{norm}$ " " " " in $F_n \ominus zF_{n-1}$

Then automatically we have

$$q_n = k' q_{n-1} + h' p_n \qquad k' > 0$$

by decomposing q_n into its projection on F_{n-1} and its projection onto the orthogonal complement. We have

$$1 = k'^2 + |h'|^2 \qquad h' = (q_n, p_n) = \bar{h}_n$$

$$k' = \sqrt{1-|h'|^2} = k_n$$

Similarly we have

$$p_n = k'' z p_{n-1} + h'' q_n \qquad k'' > 0$$

and $h'' = (p_n, q_n) = h_n$, $k'' = \sqrt{1-|h''|^2} = k_n$. Thus we get

$$k\begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} p_n - h_n q_n \\ -\bar{h}_n p_n + q_n \end{pmatrix}$$

or

$$\begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix} = R(-h_n)\begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

whence the desired recursion relation follows by inversion.

Continuous case: Consider on the line

$$Lu = -u'' + qu = k^2 u$$

where q decays fast. We propose to form a Hilbert space out of solutions of the wave equation

$$L\psi = -\psi_{tt}$$

If $\psi(x,t)$ is a solution, then by Fourier transforming

$$\psi(x,t) = \int e^{-ikt} u(x,k) \frac{dk}{2\pi}$$

we get a family of solutions of the Schroedinger equation.

Denote by $f(x,k) = f_k(x)$ the solution asymptotic to e^{ikx} as $x \to +\infty$. Then we can express $u(x,k)$ in terms of $f_{\pm k}$:

$$u(x,k) = a(k) f_k(x) + b(k) f_{-k}(x)$$

(modulo technical problems at $k=0$), and we have as $x \to \infty$

$$\psi(x,t) \sim \hat{a}(x-t) + \hat{b}(-x-t).$$

Thus for large x, t we see the outgoing wave $\hat{a}(x-t)$ and for large $x, -t$ we see the incoming wave $\hat{b}(-x-t)$. Therefore the out representation for ψ is $a(k)$ and the in representation is $b(k)$. Hence it is clear that $f_k(x)$ is not the desired u_i or u_{-i}.

We should find u_i by looking for something without a left-incoming component. Try

$$\underbrace{T(k) f^{-}_{-k}(x)}_{\text{its transmitted wave}} = \underbrace{S(k) f_k(x)}_{\text{its reflection}} + \underbrace{f_{-k}(x)}_{\text{simple incoming wave}}$$

$$T(-k) f^-_{+k}(x) = \underbrace{f_{+k}(x)}_{\substack{\text{simple}\\ \text{outgoing}\\ \text{wave}}} + S(-k) f_{-k}(x)$$

Then if we express u_k in terms of these

$$u_k = a(k)\left(T(k) f^-_{-k}\right) + b(k)\left(T(-k) f^-_k\right)$$

$$= a(f_{-k} + S f_k) + b(f_k + S(-k) f_{-k})$$

$$= (Sa + b) f_k + (a + \bar{S} b) f_{-k}$$

and hence

$$in(u_k) = a + \bar{S} b$$
$$out(u_k) = Sa + b$$

Curiosity: Suppose $S(z)$ analytic on $|z| = 1$ and of modulus 1, and of degree 0. Then the space

$$F_0 \xrightarrow[out]{\sim} H_+ \cap S H_-$$

is one-dimensional and it has 2 generators $p_0 = pr_{F_0}(u_i)/norm$, $q_0 = pr_{F_0}(u_{-i})/norm$, and hence there is a scalar p_0/q_0 associated to S. To compute it we use the isometry $out: \tilde{\mathcal{H}} \to L_2$ which takes $u_{-i} \mapsto 1$, $u_i \mapsto S$, F_0 to $H_+ \cap S H_-$ which is generated by an outer function t with $S = t/\bar{t}$ and $|t| = 1$. Then projecting u_i onto F_0 is identified with

$$\tilde{q}_0 \mapsto (1, t)t = \left(\int \bar{t} \frac{d\theta}{2\pi}\right) t = \bar{t}(0) t$$

Similarly $\tilde{p}_0 \mapsto (S, t)t = \left(\int \frac{t}{\bar{t}} \bar{t} \frac{d\theta}{2\pi}\right) t = t(0) t$

So $\dfrac{p_0}{q_0} = \dfrac{t(0)}{\bar{t}(0)}$. Recall how we get t

$$\log S = \sum_{n\in\mathbb{Z}} c_n z^n \qquad -\bar{c}_n = c_{-n}$$

$$t = \exp\left\{\frac{c_0}{2} + \sum_{n\geq 1} c_n z^n\right\}$$

So $t(0) = e^{c_0/2}$ and hence

$$\frac{p_0}{q_0} = e^{c_0/2 - \bar{c}_0/2} = e^{c_0} = \boxed{\exp\int_0^{2\pi} \log S \,\frac{d\theta}{2\pi}}$$

~~Note that although we've used $\deg S = 0$ in order to defined $\log S$ as an analytic function on S^1 the formula in the box makes sense quite generally. The point is that if one has a prob. measure $d\mu$ on X and a map $f : X \to S^1$ then~~

~~$$\exp\int \log f \, d\mu \qquad \log : S^1 \to i\mathbb{R} \text{ jumps at } z = 1$$~~

~~is a way of making sense! NO NO!!~~

We want to understand the continuous case. So lets consider a Dirac system on $0 \leq x < \infty$

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & \bar{p} \\ p & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and define the [redacted] solution $\phi(x,k)$ by the condition $\phi(0,k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. One knows that

$$\phi(x,k) = \begin{pmatrix} e^{-ikx} \\ e^{-ikx} \end{pmatrix} + \int_{-x}^{x} e^{+ikt}\, v(x,t)\,dt$$

where $\psi = v(x,t) + \begin{pmatrix} \delta(x-t) \\ \delta(-x-t) \end{pmatrix}$ is the solution of the wave equation

$$i\frac{\partial}{\partial t}\psi = L\psi = \frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ p & -\frac{d}{dx} \end{pmatrix}\psi$$

with the initial data $\psi(0,t) = \begin{pmatrix} \delta(t) \\ \delta(t) \end{pmatrix}$. The singularities of ψ can be detected from the behavior of $\phi(x,k)$ at $k \to \infty$. Recall that for φ smooth on $\mathbb{R}_{\geq 0}$ and decaying fast, we have

$$\int_{-\infty}^{0} e^{iku}\varphi(u)\,du = \left[\frac{e^{iku}}{ik}\varphi(u)\right]_{-\infty}^{0} - \int_{-\infty}^{0} \frac{e^{iku}}{ik}\varphi'(u)\,du$$

$$= \frac{1}{ik}\varphi(0) - \frac{1}{ik}\cdot O\left(\frac{1}{k}\right)$$

and that singularities of φ for $u<0$ wouldnt matter if e^{iku} decays as $u \to -\infty$, i.e. $\operatorname{Im} k < 0$. Consequently assuming $v(x,t)$ smooth for $|t| \leq x$ we expect

$$\int_{-x}^{x} e^{ikt} v(x,t)\,dt = e^{ikx}\int_{-2x}^{0} e^{iku} v(x, x+u)\,du$$

$$\sim \frac{e^{ikx}}{ik} v(x,x) \qquad \operatorname{Im} k < 0 \quad |k| \to \infty$$

and if $|k| \to \infty$ and $k \in \mathbb{R}$ we expect

$$\int_{-x}^{x} e^{ikt} v(x,t)\,dt \sim \frac{e^{ikx}}{ik} v(x,x) - \frac{e^{-ikx}}{ik} v(x,-x)$$

So we should review finding asymptotic solutions to the Dirac equation. Try

$$u = \begin{pmatrix} a \\ b \end{pmatrix} e^{ikx} \qquad \begin{aligned} a &= a_0 + a_1 k^{-1} + a_2 k^{-2} + \cdots \\ b &= b_0 + b_1 k^{-1} + \cdots \end{aligned}$$

$$\begin{pmatrix} a' + \cancel{ika} \\ b' + ikb \end{pmatrix} = \begin{pmatrix} \cancel{ika} + \bar{p} b \\ pa - ikb \end{pmatrix} \qquad b_0 = 0, \quad a_0' = 0$$

so we can put $a_0 = 1$

$$\cancel{b_0'} + 2i\cancel{k}\, b_1/\cancel{k} = pa_0 \qquad \therefore\ b_1 = \frac{p}{2i}$$

$$a_1' = \frac{|p|^2}{2i}$$

So we get the asymptotic solutions to $O(\frac{1}{k^2})$

$$\begin{pmatrix} 1 + \int^x \frac{|p|^2}{2ik} \\ \frac{p(x)}{2ik} \end{pmatrix} e^{ikx} \qquad \begin{pmatrix} 0 & -\frac{\bar{p}(x)}{2ik} \\ 1 - \int^x \frac{|p|^2}{2ik} \end{pmatrix} e^{-ikx}$$

To fit these together to get $\phi(x,k)$ we must combine them to get $\binom{1}{1}$ at $x=0$, and choose the integration constant right. Clearly we want

$$a_1 = \int_0^x \frac{|p|^2}{2i} + \frac{\bar{p}(0)}{2i} \qquad b_1 = -\int_0^x \frac{|p|^2}{2i} - \frac{p(0)}{2i}$$

and to add the above. Then we conclude

$$V(x,x) = \begin{pmatrix} \int_0^x \frac{|p|^2}{2}dx + \frac{\bar{p}(0)}{2} \\ \frac{p(x)}{2} \end{pmatrix} \qquad V(x,-x) = \begin{pmatrix} +\frac{\bar{p}(x)}{2} \\ \int \frac{|p|^2}{2}dx + \frac{p(0)}{2} \end{pmatrix}$$

I can check this as follows

$$\phi(x,k) = \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} + \int_{-x}^{x} e^{ikt} V(x,t)\,dt$$

$$\frac{d\phi_1}{dx} = ike^{ikx} + \int_{-x}^{x} e^{ikt}\frac{\partial v_1}{\partial x}(x,t)\,dt + e^{ikx}v_1(x,x) - \underbrace{(-1)}_{\frac{d}{dx}(-x)} e^{-ikx} v_1(x,-x)$$

$$(-)\quad ik\phi_1 = ike^{ikx} + \underbrace{\int_{-x}^{x} ike^{ikt}v_1(x,t)\,dt}_{\left[e^{ikt}v_1(x,t)\right]_{-x}^{x} - \int_{-x}^{x} e^{ikt}\frac{\partial v_1}{\partial t}dt}$$

$$(-)\quad \bar{p}\phi_2 = \bar{p}e^{-ikx} + \int_{-x}^{x} e^{ikt}\bar{p}(x)v_2(x,t)\,dt$$

$$0 = \int_{-x}^{x} e^{ikt}\left\{\frac{\partial v_1}{\partial x} + \frac{\partial v_1}{\partial t} - \bar{p}v_2\right\}dt + e^{-ikx}\left\{2v_1(x,-x) - \bar{p}\right\}$$

So we conclude $v_1(x,-x) = \frac{\bar{p}(x)}{2}$. Similarly $v_2(x,x) = \frac{p(x)}{2}$

also

$$\frac{d}{dt} v_1(t,t) = \frac{\partial v_1}{\partial x}(t,t) + \frac{\partial v}{\partial t}(t,t) = \bar{p}(t)\, v_2(t,t)$$

$$= \bar{p}(t)\,\frac{p(t)^*}{2} = \frac{|p|^2}{2}$$

hence $v_1(x,x) = \int_0^x \frac{|p|^2}{2} + v_1(0,0) = \int_0^x \frac{|p|^2}{2}dx + \frac{\bar{p}(0)}{2}$.

Thus it checks.

June 14, 1978:

Let's take the discrete case with $|S(z)| = 1$ on S^1, and let's compute p_n, q_n. Suppose $\deg S = 0$ and choose $t \in H_+ \cap SH_-$ with $t = S\bar{t}$ and $\|t\| = 1$. Notice that are elements of the abstract space $\tilde{\mathcal{H}}$; they do not become functions of z until we choose a representation.

In order to get orthogonal polys we choose the repn.

$$\tilde{\mathcal{H}} \xrightarrow{\sim} L^2\left(|t|^2 \frac{d\theta}{2\pi}\right)$$

$$u_1 \longmapsto \frac{1}{\bar{t}}$$

$$u_{-1} \longmapsto \frac{1}{t}$$

and then we have

$$\begin{tikzcd} L^2(S^1) & L^2\left(|t|^2 \frac{d\theta}{2\pi}\right) \arrow[l, "\cdot \bar{t}"'] \arrow[r, "\cdot t"] & L^2(S^1) \\ & \tilde{\mathcal{H}} \arrow[ul, "in"] \arrow[u, "\wr"] \arrow[ur, "out"'] & \end{tikzcd}$$

so what ends up under out in H_+ is $\frac{1}{t}H_+(d\mu) \simeq H_+^{(d\mu)}$

and ends up under "in" in H_- is $\frac{1}{\bar{t}} H_- = H_-(d\mu)$. Then F_n in this (call it central) representation is

$$H_+(d\mu) \cap z^n H_-(d\mu) = \text{span of } 1, \dots, z^n \text{ in } L^2(d\mu).$$

Next ask what p_n, q_n are. They differ by a scalar of modulus 1 from the orthogonal polys, the scalar being related to $\exp \int (\log g) \frac{d\theta}{2\pi}$.

$q_n = pr_{F_n}(u_{-i})$ In the central representation suppose q_n is given by $\sum_{k=0}^{n} a_{nk} z^k$. Then for $j = 0, \dots, n$

$$\int \frac{1}{\bar{t}} z^{-j} |t|^2 \frac{d\theta}{2\pi} = \sum_{k=0}^{n} a_{nk} \int z^{k-j} |t|^2 \frac{d\theta}{2\pi}$$

or

$$\int \bar{t} z^{-j} \frac{d\theta}{2\pi} = \sum_{k=0}^{n} a_{nk}\, c(k-j) \qquad j = 0, \dots, n$$

$$\| \qquad\qquad \text{where } c_n = \int z^n d\mu$$

$$\begin{cases} 0 & j > 0 \\ \bar{t}(0) & j = 0 \end{cases}$$

On the other hand if $1 + \sum_{k=1}^{n} b_{nk} z^k$ is orthogonal to $z, \dots, z^n$ we get the equations

$$c(-j) + \sum_{k=1}^{n} b_{nk}\, c(k-j) = 0 \qquad j = 1, \dots, n$$

These are the same as the equation

$$\sum_{k=0}^{n} a_{nk}\, c(k-j) = 0 \qquad j = 1, \dots, n$$

with the normalization $a_{n0} = 1$, so one gets essentially the same equation with the normalization furnished by

$$\sum_{k=0}^{n} a_{nk}\, c(k) = \bar{t}(0).$$

Return to the continuous case. Suppose $S(k) = \frac{A(k)}{B(k)}$ given with $A(k) = \overline{B(\bar{k})}$ and $B(k)$ holom. non-vanishing for $\mathrm{Im}(k) \geq 0$. We probably also want $B(k) \sim 1 + b_1/k + b_2/k^2 + \cdots$ as $k \to \pm\infty$. Put $d\mu(k) = \frac{dk}{2\pi |B|^2}$ and put

$$c(x) = \int e^{ikx} d\mu(k) = \delta(x) + \tilde{c}(x)$$

where $\tilde{c}$ should be smooth away from 0, in fact smooth on $[0,\infty)$ and on $(-\infty, 0]$ separately.

We propose to define

$$\phi_1(x,k) = e^{ikx} + \int_{-x}^{x} v_1(x,t) e^{-ikt} dt$$

so as to be orthogonal to e^{-iky} for $-x < y < x$. This leads to the Gelfand-Levitan equation

$$\tilde{c}(x-y) + \int_{-x}^{x} v_1(x,t)\,\tilde{c}(t-y)\,dt + v_1(x,y) = 0$$

for the function $y \mapsto v_1(x,y)$ defined on the interval $-x \leq y \leq x$. It should be the case that this is a pseudo-differential operator equation by virtue of the asymptotic expansion of B, and solutions exist because of the positivity, i.e. positive definiteness of the kernel.

Still proceeding heuristically we have

$$\frac{d}{dx}\phi_1 = ike^{ikx} + \int_{-x}^{x} \frac{\partial v_1}{\partial x} e^{ikt} dt + v_1(x,x)e^{ikx} + v_1(x,-x)e^{-ikx}$$

$$ik\phi_1 = ike^{ikx} + \underbrace{\int_{-x}^{x} v_1 \frac{\partial}{\partial t} e^{ikt} dt}_{\left[v_1(x,t)e^{ikt}\right]_{-x}^{x} - \int_{-x}^{x} \frac{\partial v_1}{\partial t} e^{ikt} dt}$$

So
$$\left(\frac{d}{dx}-ik\right)\phi_1 = \int_{-x}^{x}\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_1}{\partial t}\right)(x,t)\,e^{ikt}\,dt + 2v_1(x,-x)\,e^{-ikx}$$

Now if $y \in (-x,x)$, then because $\phi_1(x,k)$, $\phi_1(x+\varepsilon,k)$ are orthogonal to e^{iky} the same will be true for the above. Hence if
$$\phi_2(x,k) = e^{-ikx} + \int_{-x}^{x} v_2(x,t)\,e^{-ikt}\,dt$$
is defined analogously to ϕ_1, we get the equations
$$\left(\frac{d}{dx}-ik\right)\phi_1 = \bar{p}\,\phi_2 \qquad \bar{p}(x) = 2v_1(x,-x)$$
$$\frac{\partial v_1}{\partial x}+\frac{\partial v_1}{\partial t} = \bar{p}\,v_2$$
Similarly we get the other equation
$$\left(\frac{d}{dx}+ik\right)\phi_2 = p\,\phi_1$$
or more simply by conjugating.

Next we derive the Deift-Trubowitz formula. First we have
$$\int \phi_1(x,k)\,e^{-ikt}\,d\mu(k) = \int \phi_1(x,k)\,e^{-ikt}\left\{\frac{dk}{2\pi}+\left(d\mu-\frac{dk}{2\pi}\right)\right\}$$
$$= \underbrace{\delta(x-t)+v_1(x,t)}_{\text{by Fourier inversion}} + \tilde{c}(x-t) + \int_{-x}^{x} v_1(x,u)\,\tilde{c}(u-t)\,du$$
$$= \delta(x-t) + \text{function of } t \text{ which vanishes for } -x \le t \le x$$
by the Gelfand-Levitan equation

(to be understood as distributions ultimately?)

Hence

$$\int \phi_1(x,k)^2 \, d\mu(k) = \int \phi_1(x,k)\, \phi_2(x,k)^* \, d\mu(k)$$

$$= \int \phi_1(x,k) \left\{ e^{ikx} + \int_{-x}^{x} v_1(x,t)\, e^{ikt} \, dt \right\} d\mu(k)$$

$$= \int_{-x}^{x} v_1(x,t) \left(\delta(x+t)\right) dt = v_1(x,-x) = \frac{\bar{p}(x)}{2}$$

which is the desired formula, ?? except that the DT formula is for the Schroedinger equation.

June 15, 1978

Let's consider [redacted] a Schroedinger equation $-u''+qu=k^2$ on $0 \le x < \infty$ and define $\phi(x,k^2)$ to be the solution with

$$\phi(0)=1 \qquad \phi'(0)=\gamma$$

Suppose q vanishes for large x whence we have

$$\phi(x,k^2)= Ae^{ikx}+Be^{-ikx} \qquad x>>0$$

We consider solutions of the wave equation

$$-\frac{\partial^2 u}{\partial t^2} = Lu \qquad \text{where } Lu=-u''+qu$$

with bdry condition $\frac{\partial u}{\partial x}=\gamma u$ at $x=0$. An example is

$$u_\alpha(x,t) = \int e^{-ikt}\phi(x,k^2)\alpha(k)\frac{dk}{2\pi}$$

which for large x has the form

$$u_\alpha(x,t) = \widehat{A\alpha}(x-t) + \widehat{B\alpha}(-x-t) \qquad x>>0$$

By Riemann-Lebesgue $u_\alpha(x,t)$ decays as $t\to\pm\infty$ for x in a compact set, so one sees that solutions u_α [redacted] form a space with incoming and outgoing representations

$$L^2\left(\frac{dk}{2\pi}\right) \xleftarrow{\text{in}} \{u_\alpha\} \xrightarrow{\text{out}} L^2\left(\frac{dk}{2\pi}\right)$$

$$B\alpha \longleftarrow\!\shortmid\; u_\alpha \;\longmapsto A\alpha$$

The scattering function is $S=\frac{A}{B}$.

But I already have a machine which associates to S a measure $d\mu$ and a Dirac system, so the question arises [redacted] what is the relation between the Dirac system and the Schroedinger equation.

Interlude: Calculate the asymptotic expansion of $\phi(x,k^2)$.

$$u = \left(a_0 + \frac{a_1}{k} + \cdots\right) e^{ikx}$$

$$-u'' + qu = \left(-\left(A'' + 2A'ik + A(ik)^2\right) + qA\right)e^{-ikx} = k^2 A e^{-ikx}$$

$$(2ik)A' = qA - A''$$

$$a_0' = 0 \qquad \text{so} \quad a_0 \text{ is constant ,}$$

$$2i\, a_1' = q a_0 \qquad\qquad a_1 = a_0 \int^x \frac{q}{2i}\, dx + \text{const}$$

$$2i\, a_2' = q a_1 - a_1'$$

$$a_2' = \frac{a_1' a_1}{a_0} - a_1' \qquad\qquad a_2 = \frac{a_1^2}{2a_0} - \frac{a_0 q}{2i} + \text{const}$$

To determine the constant we use initial data at 0.

$$\phi(x,k^2) \approx A e^{ikx} + B e^{-ikx}$$

$$1 = \phi(0,k^2) \approx A(0,k) + B(0,k) \qquad \therefore \quad a_0(0) + b_0(0) = 1$$

$$a_1(0) + b_1(0) = 0 \qquad \therefore a_0(0) = \tfrac{1}{2}$$

$$\gamma \sim ikA(0,k) - ik B(0,k) + A'(0,k) + B'(0,k)$$

$$a_0(0) - b_0(0) = 0$$

$$i a_1(0) - i b_1(0) = \gamma$$

$$\therefore a_1(0) = \frac{\gamma}{2i}$$

$$\text{so} \quad a_0 = \frac{1}{2} \qquad a_1 = \frac{1}{2}\int_0^x \frac{q}{2i} + \frac{\gamma}{2i}$$

and

$$\phi(x,k^2) = \left(\frac{1}{2} + \frac{\gamma}{2ik} + \frac{1}{2ik}\left(\frac{1}{2}\int_0^x q\right)\right)e^{ikx} + \text{conjugate}$$

$$\boxed{\phi(x,k^2) = \cos kx + \frac{\sin kx}{k}\left(\gamma + \frac{1}{2}\int_0^x q\right) + O\left(\frac{1}{k^2}\right)}$$

The same result can be obtained from the integral equation

$$\phi(x,k^2) = \cos kx + \frac{\sin kx}{k} + \int_0^x \frac{\sin k(x-\hat{x})}{k} q(\hat{x})\phi(\hat{x},k^2)\, d\hat{x}$$

There is a difficulty with the program.

Example: Take $-u'' = k^2 u$ with $\phi(0) = 1$, $\phi'(0) = \gamma > 0$ so that

$$\phi(x, k^2) = \cos kx + \gamma \frac{\sin kx}{k}$$

hence
$$A(k) = \frac{1}{2} + \frac{\gamma}{2ik} \quad \text{and}$$

$$S(k) = \frac{\frac{1}{2} + \frac{\gamma}{2ik}}{\frac{1}{2} - \frac{\gamma}{2ik}} = \frac{k - i\gamma}{k + i\gamma}$$

Now notice that as a map from $\mathbb{R}$ to S^1 sending ∞ to 1 $S(k)$ has degree 1, and consequently it is not possible to define $\log S(k)$ so as to vanish at $\pm\infty$ without putting in a jump discontinuity. Note that $1 - \frac{\gamma}{ik}$ is non-vanishing in the upper half plane. Its argument as k goes from $-\infty$ to ∞ goes from 0 to $-\pi/2$ as $k \to 0^-$, then it jumps $+\pi/2$ on the other side of 0, and then goes to 0. Hence if we define

$$\arg S(k) = \arg\left(1 + \frac{\gamma}{ik}\right) - \arg\left(1 - \frac{\gamma}{ik}\right) = -2\arg\left(1 - \frac{\gamma}{2i}\right)$$

this argument goes from 0 to $+\pi$ as $k \to 0-$, then it jumps to $-\pi$ and goes to zero as $k \to +\infty$

Let go back to some discrete examples. Take the Jacobi system $(Ly)_n = \frac{1}{2}(y_{n+1} + y_{n-1}) = \lambda y_n$ and define $\phi(n, \lambda) = \phi_\lambda(n)$ so that $\phi(0), \phi(1)$ are constant and hence $\phi_\lambda(n)$ is of degree $n-1$ in λ for $n \geq 1$. Example:

$$\frac{z^{-1/2} z^n - z^{1/2} z^{-n}}{z^{1/2} - z^{-1/2}} \quad \text{has initial values} \quad \begin{array}{ll} -1 & n=0 \\ 1 & n=1 \end{array}$$

Here $A = \dfrac{z^{-1/2}}{z^{1/2} - z^{-1/2}} = \dfrac{1}{z-1}$ $B = \dfrac{-z^{1/2}}{z^{1/2} - z^{-1/2}} = -\dfrac{z}{z-1}$

and so $S = \dfrac{A}{B} = -z^{-1}$ is not of degree 0

as I once thought.

Another example

$$\frac{z^{-1/2}z^{n} + z^{1/2}z^{-n}}{z^{1/2} + z^{-1/2}}$$ initial values 1 $n=0$; 1 $n=1$

$A = \dfrac{1}{z+1}$ $B = \dfrac{z}{z+1}$ $S = z^{-1}$

June 16, 1978

Return to the example $\phi(x, k^2) = \cos kx + \gamma \dfrac{\sin kx}{k}$

$$S = \frac{\frac{1}{2} + \frac{\gamma}{2ik}}{\frac{1}{2} - \frac{\gamma}{2ik}} = \frac{k - i\gamma}{k + i\gamma}$$

$\gamma > 0$, so there are no bound states. Notice that

$$h(k) = \frac{1}{k + i\gamma}$$

is analytic non-vanishing in the closed UHP and it decays as $|k| \to \infty$. Also

$$\log|h| = -\log|k + i\gamma| \sim -\log|k| \quad \text{as } |k| \to \infty$$

is integrable with respect to $\dfrac{dk}{(1+k^2)}$, hence h ought to be an outer function.

Clearly $h \in H_+$ since it's L^2 and extends analytically to the UHP. Thus $\bar{h} \in H_-$ and so

$$h \in H_+ \cap SH_- .$$

Thus the associated measure $d\mu = |h|^2 \frac{dk}{2\pi} = \frac{dk}{(k^2+\gamma^2)2\pi}$ has $\int d\mu < \infty$.

What this probably means is that the Dirac system associated to S has a little Schur piece at the beginning in order to get started.

June 17, 1978

We continue with $S = h/\bar{h}$ where $h(k) = \frac{1}{k+i\gamma}$, $\gamma > 0$. Put $d\mu = |h|^2 \frac{dk}{2\pi} = \frac{dk}{(k^2+\gamma^2)2\pi}$ and consider the central representation for $\tilde{\mathcal{H}}$

$$\tilde{\mathcal{H}} \simeq L^2(d\mu) \qquad u_i \mapsto \frac{1}{h} \qquad u_{-i} = \frac{1}{\bar{h}}$$

Recall that in some sense q_x is obtained by projecting u_{-i} onto

$$F_{[0,x]}\tilde{\mathcal{H}} \xrightarrow{\sim} H_+(d\mu) \cap e^{ikx} H_-(d\mu)$$

The latter is spanned by e^{ikt} for $0 \le t \le x$, so in some sense I am looking for a thing

$$q_x = \int v(x,t)\, e^{ikt}\, dt$$

where v is a distribution supported in $[0,x]$ and which is orthogonal to e^{iky} for $0 < y \le x$.

Now one knows in general that $u_{-i} = \frac{1}{\bar{h}}$ is orthogonal to $e^{iky} H_-$ for $y > 0$ for

$$\int \frac{1}{h} e^{-iky}\, d\mu(k) = \int \bar{h}(k) e^{-iky} \frac{dk}{2\pi} = 0$$

because $\bar{h}$ is analytic in the LHP and $|e^{-iky}| = e^{y \operatorname{Im} k}$ decays as $\operatorname{Im} k \to -\infty$ for $y > 0$.

So we should have $g_x = \frac{1}{h}$ when the latter is expressible in terms of e^{ikt} for $t \in [0,x]$, i.e. when $\frac{1}{h}$ is the Fourier transform of a distribution supported in $[0,x]$. But

$$\frac{1}{h} = k + i\gamma = \int e^{ikt} \left(i\delta'(t) + i\gamma\delta(0) \right) dt$$

Hence we should have $g_x = k + i\gamma$ for all $x > 0$ and hence for $x > 0$ the Dirac system should be

$$\begin{pmatrix} \phi_1(x,k) \\ \phi_2(x,k) \end{pmatrix} = \begin{pmatrix} (k - i\gamma) e^{ikx} \\ (k + i\gamma) e^{-ikx} \end{pmatrix}$$

Now we want to understand what happens as we pass from $x = 0$ to $x > 0$.

June 18, 1978

The problem: Given a Schroedinger equation

$$-u'' + qu = k^2 u \qquad \text{on} \quad 0 \leq x < \infty$$

with $q = 0$ for $x >> 0$ and some boundary condition at $x = 0$, let

$$\phi(x, k^2) = Ae^{ikx} + Be^{-ikx} \qquad S = \frac{A}{B}$$

be the scattering function associated to the wave equation belonging to the Schroedinger equation. There is a Dirac system belonging to S, in particular a function $p(x)$ and the idea is to relate this p and the solution $\begin{pmatrix} \phi_1(x,k) \\ \phi_2(x,k) \end{pmatrix}$ to the Schroed. equation.

The idea which comes from the discrete case goes as follows. We have ~~from~~ a measure $d\mu$ giving the central representation of $\tilde{\mathcal{H}}$. Note we have

$$\phi'(x, k^2) + ik\phi(x, k^2) = 2ik\,Ae^{ikx} \qquad x >> 0$$

$$\phi'(x, k^2) - ik\phi(x, k^2) = -2ik\,Be^{-ikx}$$

so that $2ikB$ is an entire function of k. Assuming no bound states we know B doesn't vanish for $\operatorname{Im} k > 0$ and we know it doesn't vanish for k real $\neq 0$. Ruling out the case where B doesn't have a pole at $k = 0$, or equivalently that $\phi(x, 0)$ is not constant for $x >> 0$, we see that

$$S = \frac{h}{\bar{h}} \qquad h = \frac{1}{2kB}$$

is the factorization needed for the central representation, and that hence

$$d\mu = \frac{1}{|2kB|^2} \frac{dk}{2\pi}$$

On the other hand we know that the spectral measure for the Schr. problem is $\dfrac{dk}{|B|^2 2\pi}$

and hence we can embed the Hilbert space for the Schroedinger problem into $\tilde{\mathcal{H}} \simeq L^2(d\mu)$ by means of the map

$$\alpha(k) \longmapsto 2ik\,\alpha(k), \quad L^2_{even}\left(\frac{dk}{|B|^2 2\pi}\right) \hookrightarrow L^2\left(\frac{dk}{|2ikB|^2 2\pi}\right)$$

Here α is an even function of k and we are confused.

June 19, 1978:

Interesting point: A scattering function $S(k)$ $k \in \mathbb{R}$ can be viewed as a function on the circle $S^1 - \{1\}$ via Cayley $z = \dfrac{k-i}{k+i}$ so that to have $S = h/\bar{h}$ with h non-vanishing and analytic for $\operatorname{Im} k \geq 0$ means one has such a representation for $S(z)$ with h analytic non-van. on $|z| \leq 1$ except for $z=1$.

Suppose S analytic for $\operatorname{Im} k \geq 0$ and $|S(k)| < 1$ in the UHP. Then S is an inner function and so it has a standard factorization into a Blaschke product, e^{ika}, $a \geq 0$, singular factor. Since S is analytic on the line ~~the~~ the singular factor is trivial so that S is e^{ika} times a Blaschke product.

So consider $-u'' + qu = k^2 u$ on $0 \leq x < \infty$ with a boundary condition at 0 and $q = 0$ for $x >> 0$. Put $\phi(x, k^2) = Ae^{ikx} + Be^{-ikx}$ $S = \frac{A}{B}$ as usual.

Assume no bound states and that $\phi(x, 0)$ is not constant for $x >> 0$. Then kB is entire and non-vanishing for $\operatorname{Im} k \geq 0$. Also since $B = \text{const}\left(1 + O\left(\frac{1}{k}\right)\right)$ as $k \to$ [redacted] ∞ in the UHP we have $kB \sim (\text{const})$ [redacted] k [redacted]. [redacted]

$$S = \frac{h}{\bar{h}} \qquad \text{where} \qquad h = \frac{1}{kB}$$

Example: Take $S(k) = \frac{k - i}{k + i} = z$ and consider h analytic in the closed disk with $S\bar{h} = h$. This consists of $h(z) = az + b$ with $\bar{a} = b$. Such an element is determined up to a non-zero real scalar by where on S^1 it vanishes. For $h(k)$ to be L^2 the root must be $z = 1$.

Suppose $S(z) = z^2$ whence h is of the form $az^2 + bz + c$ where

$$z^2\left(\bar{a}z^{-2} + \bar{b}z^{-1} + \bar{c}\right) = \bar{a} + \bar{b}z + \bar{c}z^2 = az^2 + bz + c$$

$$\bar{a} = c \qquad \bar{b} = b.$$

If [redacted] $a = 0$, then $h = bz$ vanishes at 0. If $a \neq 0$, then also $c \neq 0$ so one has two [redacted] roots $\neq 0, \infty$ symmetrically placed wrt S^1. If h is not to vanish for $|z| < 1$, then both roots have to be on S^1. If one of the roots is $z = 1$, then the corresponding $h(k)$ is in $L^2(\mathbb{R})$ so in this case there are many possible h.

Let's consider now the general question of solving

$$S\bar{h} = h.$$

First consider the circular case: S is analytic and of modulus 1 on S^1; h is to be analytic on $|z|<1$ at least and eventually non-vanishing.

Because of the assumption on S, we get a holomorphic line bundle over P^1 whose global sections are $H_+ \cap SH_-$. If we give f analytic for $|z|<1$ and g analytic for $|z|>1$ such that $f = Sg$ we don't get a holom. section of the line bundle because this section might have singularities along $|z|=1$. One can prevent these singularities by requiring f, g to be square-integrable (perhaps even integrable works).

So it is clear that solutions of

$$S\bar{h} = h \qquad \text{with } h \in H_+$$

form the "real" elements of the space of holom. sections of the line bundle. Any such h is analytic for $|z| \leq 1$.

Suppose S has degree n (≥ 0) and let q be the unique non-vanishing analytic function on $|z| \leq 1$ such that $z^{-n}S = q/\bar{q}$. Then

$$H_+ \cap SH_- \simeq \frac{1}{q}H_+ \cap z^n \frac{1}{\bar{q}} H_- = H_+ \cap z^n H_-$$
$$= \operatorname{span}\langle 1, \dots, z^n \rangle.$$

If p is a poly of degree n such that $z^n \bar{p} = p$, then $h = (p/q) \in H_+$ and

$$S\,\overline{pq} = \quad = z^n \frac{q}{\bar{q}}\, \bar{p}\bar{q} = pq$$

So in this way in order to study $S\bar{h} = h$

we can suppose $S = z^n$.

The next point is to look for polys h satisfying $z^n \bar{h} = h$ such that h doesn't vanish for $|z|<1$. Since

$$z^n \overline{h\left(\frac{1}{\bar{z}}\right)} = h(z)$$

we see that h doesn't vanish for $|z|>1$, hence the roots of h are all on S^1. Conversely any poly

$$(*) \qquad p(z) = \prod_{i=1}^{n} \left(\zeta_i^{-1} z + \zeta_i\right) \qquad |\zeta_i| = 1$$

satisfies

$$z^n \bar{p} = \prod_{i=1}^{n} \left(\zeta_i z^{-1} + \zeta_i^{-1}\right) z = p$$

Consequently we see that all solutions of $S\bar{h} = h$ with $h \in H_+$ non-vanishing for $|z|<1$ are of the form

$$h = pg$$

where p is a polynomial of degree n in the form $(*)$.

June 20, 1978:

Yesterday I saw that given $S : S^1 \to S^1$, analytic of degree $n \geq 1$ there were many ways of expressing it in the form

$$S = \frac{h}{\bar{h}}$$

with $h \in H_+$ an outer function. In fact such h's turn out to be analytic on $|z| \leq 1$ and non-vanishing on $|z| < 1$. Up to real scalars they are described by their zeroes on S^1 which can be any n points of S^1. So one gets an n-parameter family of such h, all giving the same S.

Recall what it means for a measure $d\mu$ on S^1 to give rise to a scattering function S. In $L^2(d\mu)$ consider the subspace $H_+(d\mu)$ spanned by $1, z, z^2, \ldots$. In the scattering situation this subspace is incoming, i.e.

$$\bigcap_n z^n H_+(d\mu) = 0$$

and hence if g is a unit vector in $H_+(d\mu) \ominus z H_+(d\mu)$ we have

$$\begin{aligned} L^2(S^1) &\xrightarrow{\sim} L^2(d\mu) \\ f &\longmapsto fg \\ gh &\longleftarrow g \end{aligned}$$

where $h = \frac{1}{g}$. It follows that

$$d\mu = |h|^2 \frac{d\theta}{2\pi}$$

and that $h H_+(d\mu) = H_+$, so hz^n, $n \geq 0$ span H_+, and so h is an outer function by Beurling's thm.

So one gets the Szego criterion for $d\mu$ to have scattering: $d\mu$ is abs. continuous wrt Lebesgue measure and $\log\left(\frac{d\mu}{d\theta}\right) \in L^1$. (This uses the fact that $h \in H_+$, $h \neq 0 \Rightarrow \log|h| \in L^1$.)

Next point is that if p_n [redacted] is the sequence of ortho[redacted]normal polys. belonging to $d\mu$ then

$$q_n = z^n p_n^* \longrightarrow q \qquad \bar{z}^n p_n \longrightarrow \bar{q}$$

and so the scattering function for the [redacted] Schur system belonging to $d\mu$ is

$$S = \frac{\bar{q}}{q} = \frac{h}{\bar{h}} .$$

So what you find is a whole d-parameter family of Schur systems belonging to an S of degree d, but they start not at $n=-d$, but at $n=0$.

June 21, 1978

Suppose $S: S^1 \to S^1$ is analytic and of degree d. We've seen that there is a d-parameter family of probability measures $d\mu = |h|^2 \frac{d\theta}{2\pi}$ with h analytic for $|z| \leq 1$ and non-zero for $|z| < 1$ such that $S = h/\bar{h}$. The canonical Schur system for S obtained from the filtration $H_+ \cap z^n S H_-$ begins in degree $n = -d$, unlike the ones for such a $d\mu$.

Note that for such a $d\mu$, we have that

$$H_+(d\mu) \cap H_-(d\mu) \xrightarrow[\cong]{\cdot h} H_+ \cap h H_-(d\mu) = H_+ \cap h(\bar{h})^{-1} H_-$$

$$= H_+ \cap S H_-$$

is $(d+1)$-dimensional

Return to the example with $(Ly)_n = \frac{1}{2}(y_{n+1} + y_{n-1})$ and define $\phi(0) = \gamma$, $\phi(1) = 1$. ~~If~~ $\phi_\lambda(n) = Az^n + Bz^{-n}$ then

$$A + B = \gamma \qquad\qquad A = \frac{\begin{vmatrix} \gamma & 1 \\ 1 & z^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ z & z^{-1} \end{vmatrix}} = \frac{\gamma z^{-1} - 1}{z^{-1} - z}$$

$$Az + Bz^{-1} = 1$$

$$B = \frac{\gamma z - 1}{z - z^{-1}} \qquad\qquad S = \frac{\gamma z^{-1} - 1}{1 - \gamma z} = -z^{-1}\left(\frac{z - \gamma}{1 - \gamma z}\right)$$

No bound states means that $-1 \leq \gamma \leq 1$. If $|\gamma| < 1$, then S has degree 0. We have

$$|\gamma| < 1 \implies \deg S = 0$$

$$\left.\begin{aligned} \gamma = +1 &\implies S = z^{-1} \\ \gamma = -1 &\implies S = -z^{-1} \end{aligned}\right\} \text{ so } \deg S = -1$$

$$|\gamma| > 1 \implies \deg S = -2$$

June 24, 1978

Consider a Dirac system with p real

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ p & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad 0 \le x < \infty$$

and let $\vec{\phi}(x,k) = \begin{pmatrix} \phi_1(x,k) \\ \phi_2(x,k) \end{pmatrix}$ be the solution with $\vec{\phi}(0,k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

There are two possible Schroedinger DE's on $0 \le x < \infty$ we can obtain from the Dirac system. Note the Dirac system can be put in the form

$$\frac{d}{dx}(u_1+u_2) = ik(u_1-u_2) + p(u_1+u_2)$$

$$\frac{d}{dx}(u_1-u_2) = ik(u_1+u_2) - p(u_1-u_2)$$

or

$$\left(\frac{d}{dx} - p\right)\left(\frac{u_1+u_2}{2}\right) = -k\left(\frac{u_1-u_2}{2i}\right)$$

$$\left(\frac{d}{dx} + p\right)\left(\frac{u_1-u_2}{2i}\right) = k\left(\frac{u_1+u_2}{2}\right)$$

Hence if I put

1) $$\phi(x,k^2) = \frac{\phi_1(x,k) + \phi_2(x,k)}{2}$$

then this is the solution of $Lu = k^2 u$ with

$$L = -\left(\frac{d}{dx}+p\right)\left(\frac{d}{dx}-p\right) = -\frac{d^2}{dx^2} + \underbrace{p^2+p'}_{q}$$

and the initial condition

$$\phi(0,k^2) = 1 \qquad \phi'(0,k^2) = p(0)\,\phi(0,k^2) = p(0)$$

Now suppose I start with a Schroedinger equation

$$-u'' + qu = k^2 u$$

$$\phi(0,k^2) = 1, \quad \phi'(0,k^2) = \gamma$$

and we wish to obtain it from a Dirac system in the above way. Then clearly we have

$$p(x) = \frac{\phi'(x,0)}{\phi(x,0)}$$

and $p(x)$ will be defined for all $x \geq 0$ provided $\phi(x,0) \neq 0$ which is the case iff the Schroedinger problem has no bound states. But we want the Dirac potential p to have the property that it decays as $x \to \infty$, assuming that q does. Assume q vanishes for large x. Then we have that $\phi(x,0)$ is a linear function of x for x large, say $\phi(x,0) = ax + b$. Then

$$p(x) = \frac{a}{ax+b} = O\left(\frac{1}{x}\right) \quad \text{as} \quad x \to \infty$$

which decays but not very fast, unless $a = 0$, i.e. $\phi(x,0)$ is constant for large x.

The other possibility is to put

2) $$\phi(x,k^2) = \frac{\phi_1(x,k) - \phi_2(x,k)}{2ik}$$

This is the solution of $Lu = \left(-\frac{d}{dx} + p\right)\left(\frac{d}{dx} + p\right)u = k^2 u$, or

$$q = p^2 - p'$$

with Dirichlet initial data

$$\phi(0,k^2) = 0, \quad \phi'(0,k^2) = 1.$$

So [redacted] next suppose that q is given vanishing for large x. In this [redacted] possibility there is no restriction on $p(0)$, hence we can take p to be the solution of $p^2 - p' = q$ with $p = 0$ for large x. This solution is clearly

$$p = -\frac{f'(x,0)}{f(x,0)}$$

where $f(x,k)$ is the solution of $Lu = k^2 u$ with asymptotic e^{ikx} as $x \to +\infty$. We need $f(x,0) \neq 0$ for $x \geq 0$, and for $x > 0$ this follows from the hypothesis there are no bound states (I think). But for $x = 0$ there will be a problem, i.e. when $\phi(x,0)$ is constant for large x.

June 25, 1978

Let's suppose given $p(x)$ smooth on $0 \leq x < \infty$ vanishing for large x and let $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}(x,k)$ be the solution of the Dirac system

$$\frac{d}{dx}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix} ik & \bar{p} \\ p & -ik\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$

with initial data $\begin{pmatrix}1\\ 1\end{pmatrix}$. Then I know that

$$\phi_1(x,k) = e^{ikx} + \int_{-x}^{x} v_1(x,t)\, e^{ikt}\, dt$$

where $v_1(x,t)$ is smooth for $|x| \leq t$ and moreover that

$$v_1(x,-x) = \frac{\overline{p(x)}}{2}.$$

June 26, 1978

Problem: Understand the continuous analogue of a port. In the discrete case we described ports in 3 ways: 1) using $\tilde{\mathcal{H}}$ with unitary operator z and with incoming and outgoing reps., 2) using S, 3) using partial isometries.

Let's begin with the continuous version of 1). z^n is to be replaced by e^{-ikx}. $\tilde{\mathcal{H}}$ is a Hilbert space with a 1-parameter unitary group $U(t) =$ mult. by e^{-ikt} (we think of elements of $\tilde{\mathcal{H}}$ as functions of k.) $\tilde{\mathcal{H}}$ comes with outgoing & incoming subspaces.

$$\tilde{\mathcal{H}} = \underbrace{H_- u_{-i}}_{\text{outgoing}} \oplus \mathcal{H} \oplus \underbrace{H_+ u_i}_{\text{incoming}} \quad \text{(means stable under } e^{ikt}\text{)}$$

Think of u_i as being an embedding of $L^2(\frac{dk}{2\pi})$ into $\tilde{\mathcal{H}}$. One has

$$\begin{array}{ccccc} L^2(\frac{dk}{2\pi}) & \overset{\text{out}}{\underset{}{\rightleftarrows}} & \tilde{\mathcal{H}} & \overset{\text{in}}{\rightleftarrows} & L^2(\frac{dk}{2\pi}) \\ g & \longmapsto & g u_i & & \\ & & f u_i & \longleftarrow & f \end{array}$$

and the scattering or reflection function $S(k)$ gives the scattering:

$$f \longmapsto f u_i \overset{\text{out}}{\longmapsto} fS$$

Assuming that $H_- u_{-i} \perp H_+ u_i$ are perpendicular we should be able to deduce that S is analytic of modulus ≤ 1 in the UHP. If $f \in H_-$

$$e^{-ikx} f u_i = pr_{\mathcal{H}}(e^{-ikx} f u_i) + pr_{H_- u_{-i}}(e^{-ikx} f u_i) \quad ?$$

The point is that if $f \in H_+$, then $f u_i \in H_+ u_i$ so that $\text{out}(fu_i) = fS \in H_+$. So the map $f \mapsto fS$ from $L^2(\frac{dk}{2\pi})$ to itself decreases norm (whence $|S| \leq 1$ for $k \in \mathbb{R}$) and carries H_+ into itself. These facts ought to imply $S(k)$ analytic of modulus ≤ 1 in the UHP.

Let $j : \mathcal{H} \hookrightarrow \tilde{\mathcal{H}}$ be the embedding and define a contraction $T(t)$ on $\mathcal{H}$ by

$$T(t) = j^* U(t) j.$$

Then $T(t)^* = T(-t)$, $T(0) = \text{id}$. If $t \geq 0$, then because $H_+ u_i$ is stable under $U(t)$ and killed by j^*, given any element $z = x \oplus y \in \mathcal{H} \oplus H_+ u_i$ with $j^*(z) = x$, we have

$$T(t)x = j^* U(t) x = j^* U(t) z$$

Consequently if we take $t' \geq 0$ and $z = U(t')\alpha$ with $\alpha \in \mathcal{H}$, then $x = j^* z = T(t')\alpha$, so

$$T(t)T(t')\alpha = j^* U(t) U(t')\alpha = j^* U(t+t')\alpha = T(t+t')\alpha$$

It follows that $T(t)$ is a 1-parameter semigroup of contractions for $t \geq 0$, and also for $t \leq 0$.

The next step is to work in the infinitesimal generator R of the semi-group $T(t)$ $t \geq 0$. Formally I want to proceed as follows

$$S = \text{out}(u_i) = \int (u_i, \underbrace{e^{ikt}}_{U(t)} u_{-i}) e^{ikt} \, dt \qquad ?$$

I am trying to use the analogue of the orthonormal basis $z^n u_{-i}$ for the out representation, which leads to the formula

$$\mathrm{out}(f) = \sum_{n \in \mathbb{Z}}' \left(f, z^n u_{-i}\right) z^n$$

The analogous formula should be

$$\mathrm{out}(f) = \int \Big(f, \underbrace{e^{ikx}}_{U(x)} u_{-i}\Big) e^{ikx}\, dx$$

Hence

$$S = \mathrm{out}(u_i) = \int \left(u_i, U(x) u_{-i}\right) e^{ikx}\, dx$$

Since $H_+ u_i \perp H_- u_{-i}$ the integrand vanishes for $x < 0$ so we get

$$S(k) = \int_0^\infty \left(U(-x) u_i, u_{-i}\right) e^{ikx}\, dx$$

If $U(t) = e^{itA}$ this becomes

$$S(k) = \left(\int_0^\infty e^{i(k-A)x}\, dx \; u_i, u_{-i}\right)$$

$$= -i\left((k-A)^{-1} u_i, u_{-i}\right)$$

June 27, 1978

Example: Let $\tilde{\mathcal{H}} = L^2(dx) = L^2(-\infty, 0) \oplus L^2(0,b) \oplus L^2(b,\infty)$ with $(U(t)f)(x) = f(x-t)$. By the Fourier transform

$$\hat{f}(k) = \int e^{ikx} f(x)\,dx \qquad f(x) = \int e^{-ikx} \hat{f}(k) \frac{dk}{2\pi}$$

we can identify $\tilde{\mathcal{H}}$ with $L^2\left(\frac{dk}{2\pi}\right)$ and

$$\widehat{U(t)f}(k) = \int e^{ikx} f(x-t)\,dt = e^{ikt}\hat{f}(k)$$

Thus

$$\tilde{\mathcal{H}} = L^2\left(\frac{dk}{2\pi}\right) = H_- \oplus \mathcal{H} \oplus e^{ikb} H_+$$

where

$$u_i = \delta(x-b) \quad \text{or} \quad e^{ikb}$$
$$u_{-i} = \delta(x) \quad \text{or} \quad 1$$

$$S(k) = e^{ikb}$$

Now I want compute the semi-group $T(t)$ $t>0$ and its infinitesimal generator. $T(t)$ takes $f \in L^2(0,b)$ shifts it a distance t and then projects onto $(0,b)$. The infinitesimal generator R is defined by

$$Rf = \lim_{t\to 0^+} \frac{T(t)-I}{t} f.$$

It is a closed densely-defined operator whose domain consists of all $f \in L^2(0,b)$ for which the above limit exists. One knows by the Fourier transform that the infinitesimal generator of $U(t)$ is $\frac{d}{dt}$ defined on the subspace of $L^2(\mathbb{R}, dx)$ consisting of absolutely continuous functions with L^2 derivatives.

Notice that R must extend the closure of $\frac{d}{dx}$ given on $C_0^\infty((0,b))$

June 29, 1978

Let $T(t)$, $t \geq 0$ be a semi-group of bounded operators on a Hilbert space $\mathcal{H}$, such that $t \mapsto T(t)u$ is continuous for each $u \in \mathcal{H}$. Let $\mathcal{D}_B$ denote the subspace of u in $\mathcal{H}$ such that $T(t)u$ is differentiable at $t=0$, and let Bu be the derivative at $t=0$. Because

$$\lim_{\varepsilon \searrow 0} \frac{T(t+\varepsilon)-T(t)}{\varepsilon} u = \lim T(t)\left\{\frac{T(\varepsilon)-I}{\varepsilon}\right\} u .$$

$$= \lim \left\{\frac{T(\varepsilon)-I}{\varepsilon}\right\} T(t)u$$

it follows that $T(t)\mathcal{D}_B \subset \mathcal{D}_B$ and that

$$\lim_{\varepsilon \searrow 0} \frac{T(t+\varepsilon)-T(t)}{\varepsilon} u = T(t)Bu = BT(t)u.$$

Assume there exists an a [redacted] $\in \mathbb{R}$ such that

$$\|T(t)\| \leq \text{[redacted]}\, e^{at}$$

e.g. $a=0$ if $T(t)$ is a contraction semi-group. Then we can form the Laplace transform

$$\Psi(k)f = \int_0^\infty e^{-kt}\, T(t)f\,dt$$

which is an analytic [redacted] function of k in the half-plane $\mathrm{Re}(k) >$ [redacted] a.

[redacted]

$$T(s)\Psi(k)f = \int_0^\infty e^{-kt}\, T(s+t)f\,dt = \int_s^\infty e^{-k(t-s)}T(t)f\,dt$$

$$= e^{ks}\left\{\Psi(k) - \int_0^s e^{-kt}\,T(t)\,dt\right\} f$$

$$T(s)\Phi(k)f = T(s)\int_0^\infty e^{-kt}T(t)f\,dt \overset{\substack{T(s)\text{ continuous}\\ \downarrow}}{=} \int_0^\infty e^{-kt}T(s+t)f\,dt = \Phi(k)T(s)f$$

[redacted] $\therefore\ T(s)\Phi(k)f = \Phi(k)T(s)f = \int_0^\infty e^{-kt}T(s+t)f\,dt$

$$= \int_s^\infty e^{-k(t-s)}T(t)f\,dt = e^{ks}\left\{\Phi(k)f - \int_0^s e^{-kt}T(t)f\,dt\right\}$$

This expression is C^1 in s for all f in $\mathcal{H}$, hence $\operatorname{Im}\Phi(k) \subset \mathcal{D}_B$. Differentiating at $s=0$ gives for any f

$$B\,\Phi(k)f = \lim_{\varepsilon\downarrow 0} \Phi(k)\,\frac{T(\varepsilon)-I}{\varepsilon}f = k\Phi(k)f - f$$

Thus $(k-B)\Phi(k) = I$ on $\mathcal{H}$. If $f\in\mathcal{D}_B$ then because $\Phi(k)$ is bounded [redacted] the middle limit is $\Phi(k)Bf$, so we get

$$\Phi(k)(k-B)\,[redacted]\,f = f$$

for all $f\in\mathcal{D}_B$. Hence $\operatorname{Im}\Phi(k)\supset\mathcal{D}_B$, so they coincide, ~~so~~ $k-B$ is the inverse of $\Phi(k)$ showing that B is a closed operator. Also one sees that $T(t)f$ is C^1 for any $f\in\mathcal{D}_B = \operatorname{Im}\Phi(k)$.

Summarizing: Given a semi-group of operators $T(t)$, $t\geq 0$ on a Banach space $\mathcal{H}$ with $T(t)$~~f~~ continuous for all $f\in\mathcal{H}$ and $\|T(t)\|\leq e^{at}$ for some real number a, we have [redacted] that

$$Bf = \lim_{\varepsilon\downarrow 0}\frac{T(\varepsilon)-I}{\varepsilon}f$$

is a closed densely-defined operator on $\mathcal{H}$ such that

$$\frac{d}{dt} T(t) = T(t)B = BT(t)$$

and such that $k - B$ is invertible for $Re(k) > a$. (The reason $\mathcal{D}_B$ is dense in $\mathcal{H}$ is because

$$\lim_{k \to +\infty} \Phi(k) f = \lim_{k \to \infty} \int_0^\infty e^{-kt} T(t) f \, dt \overset{?}{=} T(0) f = f.) \Leftarrow \text{wrong}$$

Remaining questions: In what sense is $T(t) = e^{tB}$? Suppose given B closed densely-defined with $k-B$ invertible for $Re(k) > a$, does it come from a unique $T(t)$?

~~Correction to the above error. Let $g(t)$ be continuous at $t = 0$ and [illegible] $\|g(t)\| \leq Ce^{\varepsilon t}$. If~~

Correction to the above error: If $\lim_{\varepsilon \downarrow 0} g(\varepsilon) = 0$, and $\|g(t)\| \leq e^{at}$, then

$$\int_0^\infty ke^{-kt} g(t)\, dt = \int_0^\varepsilon ke^{-kt} g(t)\, dt + \int_\varepsilon^\infty ke^{-kt} g(t)\, dt$$

The second integral goes to zero because $ke^{-(k-2a)t}$ goes to zero uniformly (or just use dominated convergence). The first integral is bounded

$$\left\| \int_0^\varepsilon ke^{-kt} g(t)\, dt \right\| \leq \int_0^\varepsilon ke^{-kt} M(\varepsilon)\, dt \leq M(\varepsilon)$$

where $M(\varepsilon)$ is the sup [redacted] of $\|g(t)\|$ on $0 \leq t \leq \varepsilon$ and this goes to zero as $\varepsilon \to 0$. Thus

$$\lim_{k \to +\infty} k \int_0^\infty e^{-kt} g(t)\, dt = 0 \quad \text{whence} \quad \lim_{\varepsilon \downarrow 0} g(\varepsilon) = 0$$

and from this one gets easily that $\lim_{k \to \infty} k\Phi(k) f = f$.