April 11, 1978:

Let T be a contraction operator on $\mathcal{H}$, and denote by $\rho : \mathcal{H} \longrightarrow \mathcal{H}_{1-TT^*}$ the completion of $\mathcal{H}$ w.r.t. the norm $((1-TT^*)x, x) = \|x\|^2 - \|T^*x\|^2$. Define

$$i : \mathcal{H} \longrightarrow \mathcal{H}^2(S^1, \mathcal{H}_{1-TT^*})$$

$$i(h) = \rho\,(1-zT^*)^{-1}h.$$

To be precise: The series

$$(1-zT^*)^{-1} = \sum_{n\geq 0} z^n (T^*)^n$$

converges for $|z|<1$, so $\rho(1-zT^*)^{-1}h$ is analytic in $|z|<1$. Moreover the series

$$\sum_{n\geq 0} \rho\, z^n T^{*n} h = \sum_{n\geq 0} (\rho T^{*n} h) z^n$$

represents an element in $L^2(S^1, \mathcal{H}_{1-TT^*})$ because

$$\sum_{n\geq 0} \|\rho T^{*n} h\|_{\mathcal{H}_{1-TT^*}} = \sum_{n\geq 0} \left((1-TT^*)(T^{*n}h), (T^{*n}h)\right)_{\mathcal{H}}$$

assumes $\|T^{*n}h\| \to 0$

$$= \sum_{n\geq 0} \left(\|T^{*n}h\|^2 - \|T^{*(n+1)}h\|^2\right) = \|h\|^2 < \infty.$$

Therefore i is a well-defined isometric embedding. Furthermore

$$(i^* U^m i h, h')_{\mathcal{H}} = \left(z^m \sum_{n\geq 0} \rho z^n T^{*n} h, \sum_{n\geq 0} \rho z^n T^{*n} h'\right)_{L^2(S^1, \mathcal{H}_{1-TT^*})}$$

$m \geq 0$

$$= \sum_{n\geq 0} \left(\rho T^{*n} h, \rho T^{*(m+n)} h'\right)_{\mathcal{H}_{1-TT^*}}$$

$$= \sum_{n \geq 0} \left((1-TT^*) T^{*n} h,\ T^{*m+n} h' \right)$$

telescopes to

$$= (h, T^{*m} h') = (T^m h, h').$$

So therefore one will obtain at least an embedding

$$\tilde{\mathcal{H}} \hookrightarrow L^2(S^1, \mathcal{H}_{1-TT^*})$$

only assuming $\|T\| \leq 1$. NO must assume $T^{*n} \to 0$ weakly.

So suppose now that T is the backwards shift on $\ell^2(\mathbb{N})$ i.e. $T(e_i) = \begin{cases} e_{i-1} & i \geq 1 \\ 0 & i = 0 \end{cases}$. Then T^* is the forward shift, so that

$$TT^* = 1$$

and hence there is a mistake in the above. The mistake consists in assuming that

$$T^{*n} h \longrightarrow 0 \qquad \text{as } n \to \infty$$

for all $h \in \mathcal{H}$.

Note that

$$i(T^* h) = \sum_{n \geq 0} (\rho\, T^{*n+1} h) z^n$$

so

$$i(h) - z\, i(T^* h) = \rho(h)$$

Elements of the form $\rho(h)$ are dense in $\mathcal{H}$, hence we see that

$$\tilde{\mathcal{H}} = L^2(S^1, \mathcal{H}_{1-TT^*})$$

with $\overline{\sum_{n\geq 0} z^n i \mathcal{H}} = H^2(S^1, \mathcal{H}_{1-TT^*})$.

Hence this isomorphism gives the outgoing representation for $\mathcal{H}$. Similarly assuming $T^n \longrightarrow 0$ strongly we get the incoming representation

$$\tilde{\mathcal{H}} \simeq L^2(S^1, \mathcal{H}_{1-T^*T})$$

induced by $\quad i(h) = \rho(1-z^{-1}T)^{-1}h = \sum_{n\geq 0} (\rho T^n h)\, z^{-n}$

April 13, 1978

Suppose V partial isometry on $\mathcal{H}$ with indices $(1,1)$ without unitary component and T = its canonical extension as a contraction. I'd like to understand what it means for $T^n, T^{*n} \to 0$ strongly.

We can identify $\mathcal{H} \to \mathcal{H}_{1-TT^*}$ with $h \mapsto (h, u_{-i})$ and $\mathcal{H} \to \mathcal{H}_{1-T^*T}$ with $h \mapsto (h, u_i)$. Under the assumption $T^n, T^{*n} \to 0$ we computed the scattering operator

$$L^2(S^1) = L^2(S^1, \mathcal{H}_{1-T^*T}) \xleftarrow{\sim} \tilde{H} \xrightarrow{\sim} L^2(S^1, \mathcal{H}_{1-TT^*}) = L^2(S^1)$$

and found it to be multiplication by the function

$$(1) \qquad S(z) = \left((1-zT^*)^{-1}u_i, u_{-i}\right)$$

In effect we have

$$\left((1-z^{-1}T)^{-1}h, u_i\right) \longleftarrow h \longmapsto \left((1-zT^*)^{-1}h, u_{-i}\right)$$

$$(h, u_i) \longleftarrow h - z^{-1}Th \longmapsto \left((1-zT^*)^{-1}(1-z^{-1}T)h, u_{-i}\right)$$

Take $h = u_i$

$$1 \longleftarrow \underset{\substack{\parallel \\ u_i}}{(1-z^{-1}T)u_i} \longmapsto \left((1-zT^*)^{-1}u_i, u_{-i}\right)$$

because $Tu_i = 0$.

In general the function $S(z)$ defined by (1) is defined for $|z| < 1$. Under the assumption $T^n, T^{*n} \to 0$ we know that it is of modulus 1 a.e. for $z \in S^1$.

More precisely even when $T^{*n} \not\to 0$ we have a well-defined map $\mathcal{H} \to L^2(S^1, \mathcal{H}_{1-TT^*})$ given by

$$h \longmapsto \rho(1-zT^*)^{-1}h$$

Hence

$$\rho(1-zT^*)^{-1}h = \sum z^n(\rho T^{*n}h) \quad \in H^2(S^1; \mathcal{H}_{1-TT^*})$$

so

$$S(z) = ((1-zT^*)^{-1}u_i, u_{-i}) \quad \in H^2(S^1).$$

In particular $S(z)$ for $z \in S^1$ is a well-defined measurable function up to null-set equivalence. (It is also known maybe that

$$S(e^{i\theta}) = \lim_{r \uparrow 1} S(re^{i\theta}) \qquad \text{a.e. } \theta.)$$

So the problem is whether S has modulus 1 on S^1!

Let's compute S by choosing the unitary extension U of V with $U(u_i) = u_{-i}$, whence we can identify $\mathcal{H}$ with $L^2(S^1 d\nu)$ and $u_i = \zeta^{-1}$, $u_{-i} = 1$. Here we use ζ to note the S^1 variable associated with $d\nu$.

Recall that $(1-zT^*)^{-1}u_i$ for $|z|<1$ is determined by the properties

$$\begin{cases} ((1-zT^*)^{-1}u_i, (1-\bar{z}V)\mathcal{D}_V) = ((1-zT^*)^{-1}u_i, (1-\bar{z}T)\mathcal{D}_V) = (u_i, \mathcal{D}_V) = 0 \\ ((1-zT^*)^{-1}u_i, u_i) = (u_i, (1-\bar{z}T)^{-1}u_i) = (u_i, u_i) = 1 \end{cases}$$

On the other hand for $|z| < 1$

$$((1-zU^{-1})^{-1}u_i, (1-\bar{z}V)\mathcal{D}_V) = 0$$

so $(1-zU^{-1})^{-1}u_i$ is proportional to $(1-zT^*)^{-1}u_i$. But

$$((1-zU^{-1})^{-1}u_i, u_i) = \left(\frac{\zeta^{-1}}{1-z\zeta^{-1}}, \zeta^{-1}\right) = \int \frac{d\nu(\zeta)}{1-z\zeta^{-1}}$$

so $$(1-zT^*)^{-1}u_i = \frac{1}{\int \frac{d\nu}{1-z\zeta^{-1}}} \cdot \frac{\zeta^{-1}}{1-z\zeta^{-1}}$$

hence
$$S(z) = \frac{\displaystyle\int \frac{\rho^{-1}}{1 - z\rho^{-1}}\, d\nu}{\displaystyle\int \frac{1}{1 - z\rho^{-1}}\, d\nu}$$

$$\frac{1 + zS(z)}{1 - zS(z)} = \int \frac{1 + z\rho^{-1}}{1 - z\rho^{-1}}\, d\nu \qquad \text{since } \int d\nu = 1.$$

Note that if $\quad g(z) = \displaystyle\int \frac{d\nu}{1 - z\rho^{-1}}$, then

$$\frac{1}{z}\,\overline{g(z^*)} = \frac{1}{z}\int \frac{d\nu}{1 - z^{-1}\rho} = -\int \frac{d\nu}{\rho - z} = -\int \frac{\rho^{-1}\,d\nu}{1 - z\rho^{-1}}$$

so that
$$-zS(z) = \frac{\overline{g(z^*)}}{g(z)}$$

Recall that
$$\operatorname{Re} \frac{1 + z\rho^{-1}}{1 - z\rho^{-1}} = \operatorname{Re} \frac{(1 + z\rho^{-1})(1 - \bar{z}\rho)}{|1 - z\rho^{-1}|^2} = \frac{1 - |z|^2}{|1 - z\rho^{-1}|^2} > 0$$

It follows that
$$\operatorname{Re} \frac{1 + zS(z)}{1 - zS(z)} > 0 \qquad \text{for } |z| < 1$$

which implies that $\quad |zS(z)| < 1 \quad$ for $|z| < 1$ and hence $\quad |S(z)| \leq 1 \quad$ by maximum modulus.

April 17, 1978:

Let's go over Szegö's thm. Let $d\nu$ be a probability measure on S^1 and let $H^2(d\nu)$ be the [redacted] closure of $\mathbb{C}[z]$ in $L^2(d\nu)$. [redacted] The problem is to compute the length of the projection of $1 \in H^2(d\nu)$ perpendicular to $z H^2(d\nu)$.

We can form the sequence of orthonormal polys $1 = p_0, p_1, \ldots$ with respect to $d\nu$, and define numbers h_n, $n = 1, 2, \ldots$ by

$$z p_n = k_{n+1} p_{n+1} - h_{n+1} z^n p_n^* \qquad k_{n+1} = \sqrt{1 - |h_{n+1}|^2}$$

whence $|h_n| < 1$ for all n. We assume $d\nu$ has infinite support so that p_n is defined for all n. Recursion relation:

$$\begin{pmatrix} p_{n+1} \\ z^{n+1} p_{n+1}^* \end{pmatrix} = \frac{1}{k_{n+1}} \begin{pmatrix} 1 & h_{n+1} \\ \bar{h}_{n+1} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_n \\ z^n p_n^* \end{pmatrix}$$

This shows that for $|z| < 1$ one has

$$\left| \frac{p_n}{z^n p_n^*} \right| < 1 \qquad n \geq 1.$$

hence $z^n p_n^*$ has its roots outside S^1 and p_n has its roots inside S^1.

Suppose that T arises from V as before and that U is a unitary extension of V so that $\mathcal{H}$ can be identified with $L^2(S^1, d\nu)$, $u_i = \zeta^{-1}$, $u_{-i} = 1$. For $|z|<1$ we have the analytic function

$$S(z) = \left((1-zT^*)^{-1}u_i, u_{-i}\right)$$

which we have computed to be given by

$$S(z) = \frac{\displaystyle\int \frac{\zeta^{-1}d\nu}{1-\zeta^{-1}z}}{\displaystyle\int \frac{d\nu}{1-\zeta^{-1}z}}$$

Thus

$$\frac{1+zS(z)}{1-zS(z)} = \int \frac{\zeta+z}{\zeta-z}\,d\nu$$

maps $|z|<1$ into $\mathrm{Re}>0$. It follows that $|zS(z)|<1$ for $|z|<1$, hence by maximum modulus that

$$|S(z)| \leq 1$$

Equality can occur only when $S(z)$ is a constant of modulus 1, whence $d\nu$ is the Dirac measure at some point of S^1.

It follows that $S(z) \in H^2(S^1, \frac{d\theta}{2\pi})$ and that $\|S\| = 1 \iff |S| = 1$ a.e. on $S^1 \iff \mathrm{Re}\int \frac{\zeta+z}{\zeta-z}d\nu = 0$ for a.e. $z \in S^1$. Here we define values for $z \in S^1$ by radial limits. But Fatou's thm. says

$$\mathrm{Re}\int \frac{\zeta+z}{\zeta-z}\,d\nu = 2\pi\frac{d\nu}{d\theta} \qquad \text{a.e. } z \in S^1.$$

Hence it follows that $\|S\| = 1 \iff d\nu$ is singular wrt Lebesgue measure. But

$$\|S\|^2 = \Big\| \sum z^n (T^{*n} u_i, u_{-i}) \Big\|^2$$

$$= \sum_{n \geq 0} \|\rho T^{*n} u_i\|^2 = 1 - \lim_{n\to\infty} \|T^{*n} u_i\|^2$$

So

$$\begin{array}{ccc} \|S\| = 1 & \iff & T^{*n} u_i \longrightarrow 0 \\ \Updownarrow & & \\ |S| = 1 \text{ a.e. on } S^1 & \iff & \dfrac{d\nu}{d\theta} = 0. \end{array}$$

The last condition should be symmetrical under interchanging T and T^* hence we also get $T^n u_{-i} \longrightarrow 0$.

April 20, 1978

Let $V: \mathcal{D}_V \to \mathcal{H}$, T be as above, and let $(\tilde{\mathcal{H}}, U)$ be the unitary dilation of T. I recall proving that $\tilde{\mathcal{H}} \ominus i\mathcal{H}$ has the orthonormal basis $U^n(u_i)$ $n \geq 1$, $U^{-n}(u_{-i})$ $n \geq 1$. Put

$$\begin{cases} e_n = U^n(u_i) & n \geq 1 \\ e_n = U^{-n}(u_{-i}) & n \leq -1. \end{cases}$$

What can we say about the structure of $(\tilde{\mathcal{H}}, U)$? [redacted] When is $(\tilde{\mathcal{H}}, U)$ a scattering situation?

$$\begin{matrix} \langle \ldots\ldots, & U^{-2}u_{-i}, & U^{-1}u_{-i} \rangle & \oplus \ \mathcal{H} \ \oplus & \langle U u_i, & U^2 u_i, & \ldots\ldots \rangle \\ & \| & \| & & \| & \| & \\ & e_{-2}, & e_{-1} & & e_1, & e_2, \ldots & \end{matrix}$$

Let's follow the trajectory $U^n u_i$ as $n \to -\infty$. $u_i \in \mathcal{H}$. [redacted] Clearly $\langle e_2, e_{-1} \rangle \oplus \mathcal{H}$ is stable under U^{-1}. If $i: \mathcal{H} \to \tilde{\mathcal{H}}$ is the embedding, we have

$$i^* U^{-n}(u_i) = T^{*n} u_i$$

hence for this to tend to zero means that we pick up all of u_i by taking $U^{-n} u_i$ $n \to \infty$ and projecting onto $\langle \cdot, e_{-2}, e_{-1} \rangle$. What do we actually get. Suppose

$$U^{-n} u_i = a_0 e_{-n} + a_1 e_{-n+1} + \cdots + a_{n-1} e_1 \quad \text{mod } \mathcal{H}.$$

Do in stages: $n=0$ $\quad u_i = u_i = (u_i, u_{-i}) u_{-i} + \text{something in } R_V$

$$U^{-1} u_i = (u_i, u_{-i}) U^{-1} u_{-i} + h$$

$$T^* u_i = h$$

$$U^{-1}u_i = (u_i, u_{-i})e_{-1} + T^* u_i$$
$$+ (T^* u_i, u_{-i})u_{-i} + \text{something in } R_V$$

$$U^{-2}u_i = (u_i, u_{-i})e_{-2} + (T^* u_i, u_{-i})e_{-1} + T^{*2}u_i$$

Hence we get for the scattering operator

$$S(z) = \sum_{n \geq 0} (T^{*n}u_i, u_{-i})z^n = ((1 - zT^*)^{-1}u_i, u_{-i})$$
$$(u_i = S(z)u_{-i})$$

The good case is when $T^{*n}u_i \longrightarrow 0$ for then S is a unitary operator, i.e. $|S(z)| = 1$ for $z \in S^1$. When this case holds it seems clear that we can invert the scattering operator, so that the trajectory $U^n(u_{-i})$ $n \geq 0$ must be captured in the limit by projection onto $\langle e_1, e_2, \ldots \rangle$. Thus we obtain:

<u>Proposition</u>: $T^{*n}u_i \longrightarrow 0 \iff T^n u_{-i} \longrightarrow 0$

If this holds, then the U, U^{-1} invariant subspaces generated by u_i and u_{-i} coincide and hence must equal $\tilde{\mathcal{H}}$, for otherwise the orthogonal complement would be a subspace of $\mathcal{H}$ stable under U, U^{-1} perpendicular to u_i, u_{-i} which contradicts the assumption that V has no unitary component. Hence we see that every element of $\mathcal{H}$ has to be "seen" in both scattering representations, so

$$T^n h \longrightarrow 0 \quad , \quad T^{*n}h \longrightarrow 0$$

for all $h \in \mathcal{H}$.

Summary: Let T be the contraction on $\mathcal{H}$ obtained from a partial isometry V with indices $(1,1)$ and no unitary component. Choose u_i, u_{-i} and let $d\nu$ be the spectral measure of the unitary extension U of V with $U(u_i) = u_{-i}$. Define
$$S(z) = \left((1 - zT^*)^{-1} u_i, u_{-i} \right)$$

Then we have found that the following conditions are equivalent

1) S inner, i.e. $|S| = 1$ on boundary
2) $T^{*n} \to 0$ strongly (enough that $T^{*n} u_i \to 0$)
3) $T^n \to 0$ strongly (" " $T^n u_{-i} \to 0$)
4) $d\nu$ singular with respect to Lebesgue measure.

Suppose given an inner function S take $\tilde{\mathcal{H}} = L^2(S^1)$ and $\mathcal{H} = H^2(S^1) \ominus zS\,H^2(S^1)$. Put

$$e_1 = zS \qquad e_{-1} = z^{-1}$$
$$e_2 = z^2 S \qquad e_{-2} = z^{-2}$$
$$\text{etc.} \qquad \text{etc.}$$

so that $u_i = S$, $u_{-i} = 1$. Does this work? We know multiplication by z induces a contraction T on $\mathcal{H}$ with $T^n = i^* U^n i$. If $h \perp S = u_i$, then $h \perp S H^2(S^1)$ so $zh \perp zS H^2(S^1)$ so $Th = zh$; on the other hand $TS = i^* zS = 0$. Similarly if $h \perp 1 = u_{-i}$, then $h \perp H^2(S^-)^-$ so $z^{-1} h \perp z^{-1} H^2(S^-)^-$ so $z^{-1} h \in h$ and so $T^* h = h$, etc.

Conclude: V's belonging to measures $d\nu$ singular wrt $\frac{d\theta}{2\pi}$ are in 1-1 correspondence with inner functions modulo multiplication by scalars.

The problem is now to understand invariant subspaces for a given V. Begin with a finite-dimensional situation.

Suppose $\mathcal{H}$ is finite-dimensional. I can choose u_i, u_{-i} so that the measure $d\nu$ is not supported at $\zeta = 1$, which means that V is the Cayley transform of a symmetric operator A. Specifically, I've seen that $\mathcal{H} \simeq L^2(d\mu)$ where $d\mu$ is a measure with finite support on $\mathbb{R}$ with $\int \frac{d\mu}{x^2+1} = 1$ and where under this isom one has

$$u_i = \frac{1}{x-i} \qquad u_{-i} = \frac{1}{x+i}$$

$$\mathcal{D}_A = \left\{ f \in L^2(d\mu) \,\middle|\, \int f\, d\mu = 0 \right\}$$

$$\tilde{A} = \text{mult. by } x. \qquad U = \frac{\tilde{A}-i}{\tilde{A}+i} \qquad V = \frac{A-i}{A+i}$$

Let $\mathcal{K}$ be a subspace of $\mathcal{H}$. Then $\mathcal{K} \cap \mathcal{D}_A$ is of codim 1 or 0 in $\mathcal{K}$. Assume that

$$A(\mathcal{K} \cap \mathcal{D}_A) \subset \mathcal{K}$$

so that A induces an operator in $\mathcal{K}$ with domain of codim 1. Note that $\mathcal{K} \cap \mathcal{D}_A = \mathcal{K}$ is impossible for $\mathcal{K} \neq 0$ since A has no self-adjoint component by hypothesis.

But note for $\nu \geq 1$

$$A(\mathcal{K} \cap \mathcal{D}_{A^\nu}) \subset \mathcal{K} \cap \mathcal{D}_{A^{\nu-1}}$$

Now $\mathcal{H} > \mathcal{D}_A > \mathcal{D}_{A^2} > \cdots\cdots > \mathcal{D}_{A^d} = 0$ is a complete flag, hence $\mathcal{K} \cap \mathcal{D}_{A^\nu}$ is of codim 1 or or in $\mathcal{K} \cap \mathcal{D}_{A^{\nu-1}}$. But $\mathcal{K} \cap \mathcal{D}_{A^\nu} = \mathcal{K} \cap \mathcal{D}_{A^{\nu-1}}$ would imply $\mathcal{K} \cap \mathcal{D}_{A^{\nu-1}}$ is A-invariant which is impossible unless it is zero.

Let $m(\lambda)$ be a rational function of λ such that $\operatorname{Im}(\lambda)$ has the same sign as $\operatorname{Im}(\lambda)$, so that we have the Riesz-Herglotz representation

$$m(\lambda) = p\lambda + \sum \frac{r_i}{x_i - \lambda} + c$$

$p \geq 0$, c real
$r_i > 0$
x_i real distinct

One has a continued fraction development

$$m(\lambda) = p_0\lambda + q_0 - \frac{1}{p_1\lambda + q_1 -} \frac{1}{p_2\lambda + q_2 -} \cdots\cdots .$$

Let $S(z)$ be an inner function which is rational. This means that S has a Blaschke product representation

$$S(z) = c \prod_{i=1}^{d} \frac{z - a_i}{1 - \bar{a}_i z}$$

$|a_i| < 1$
$|c| = 1$

Also S has a Schur development

$$S(z) = \begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & h_d \\ \bar{h}_d & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} (\alpha)$$

where $|h_i| < 1$ and $|\alpha| = 1$.

On the other hand we can pass between $m(\lambda)$ and $S(z)$ via the substitution

$$S(z) = \frac{m(\lambda) - i}{m(\lambda) + i} \qquad z = \frac{\lambda - i}{\lambda + i}$$

April 28, 1977:

Program: Let A on $\mathcal{H} = \ell^2(1,\infty)$ be given by a J-matrix. I want to describe the filtration $(\mathcal{D}_{A^n}^{\circ})^{\perp}$ in another way, somehow related to decomposing a 1-port into a 2-port connected to a 1-port. To get insight let's start with the analogue of this filtration in the circular case:

Let V be a partial isometry on $\mathcal{H}$ with no unitary component and indices $(1,1)$ and put

$$(*) \qquad S(z) = \left(\left(1 - zT^*\right)^{-1} u_i, u_{-i}\right) \qquad |z|<1$$

as usual. We know $|S(z)| \leq 1$ for $|z|<1$, hence if S is not a constant of modulus 1 we can Schur develop S one-step

$$\frac{S-h}{1-\bar{h}S} = zS_1 \qquad h = S(0) = (u_i, u_{-i})$$

S_1 ought to belong to a partial isometry $(\mathcal{H}_1, V_1)$ which I now want to find.

The obvious candidate is to take $\mathcal{H}_1 = \mathcal{D}_V$ with V_1 induced by V, i.e.

$$\mathcal{D}_{V_1} = \{x \in \mathcal{D}_V \mid Vx \in \mathcal{D}_V\} = \mathcal{D}_{V^2}.$$

The problem is to show this leads to S_1.

Recall that in $(*)$ T denote the contraction obtained by extending V by zero, however more generally if $\tilde{T}$ is any contraction extending V to $\mathcal{H}$ we have a similar formula obtained as follows. Recall that the element $(1-zT^*)^{-1}u_i$ is characterized by being $\perp$ to $(1-\bar{z}V)\mathcal{D}_V$ and having inner product 1 with u_i. Hence the general formula is

$$S(z) = \frac{\left((1-z\tilde{T}^*)^{-1}u_i, u_{-i}\right)}{\left((1-z\tilde{T}^*)^{-1}u_i, u_i\right)}$$

which we used already when $\tilde{T}=$ the unitary extension with $\tilde{T}u_i = u_{-i}$.

~~Let $j: \mathcal{D}_V \to \mathcal{H}$ be the inclusion and put~~

~~$$T_1 = j^*V.$$~~

~~If $x \in \mathcal{D}_{V^2}$, then $T_1x = j^*Vx = V_1x$ where $V_1: \mathcal{D}_{V^2} \to \mathcal{D}_V$ is the restriction of V. Thus T_1 is a contraction on $\mathcal{D}_V$ extending V_1.~~

Put $\hat{u}_z = (1-zT^*)^{-1}u_i$. It is the unique element of $\mathcal{H}$ with $\hat{u}_z \perp (1-\bar{z}V)\mathcal{D}_V$ and $(\hat{u}_z, u_i) = 1$. So look at

$$\begin{tikzcd}[column sep=small] & \mathcal{H} & \\ \mathcal{D}_V \arrow[ur, hook] & & (1-\bar{z}V)\mathcal{D}_V \arrow[ul, hook'] \\ & (1-\bar{z}V_1)\mathcal{D}_{V^2} \arrow[ul, hook] \arrow[ur, hook'] & \end{tikzcd}$$

It follows that

$$pr_{\mathcal{D}_V}(\hat{u}_z) \perp (1-\bar{z}V_1)\mathcal{D}_{V^2}.$$

Now

$$pr_{\mathcal{D}_V}(\hat{u}_z) = \hat{u}_z - (\hat{u}_z, u_i)u_i = \hat{u}_z - u_i$$

I need in $\mathcal{D}_V$ unit vectors $\perp$ $\mathcal{D}_{V^2}$ and $V\mathcal{D}_{V^2}$

$$\begin{tikzcd}[column sep=small] & & u_{-i} \arrow[dr] & \\ & \mathcal{H} \arrow[dl, phantom, "\subset"] \arrow[dr, phantom, "\supset"] & & \circ \\ \mathcal{D}_V \arrow[d, phantom, "\subset"] \arrow[dr, phantom, "\supset"] & & V\mathcal{D}_V \arrow[dl, phantom, "\subset"] & \\ \mathcal{D}_{V^2} & V\mathcal{D}_{V^2} & & \end{tikzcd}$$

$$pr_{\mathcal{D}_V}(u_{-i}) \perp V\mathcal{D}_{V^2}$$
$$\parallel$$
$$u_{-i} - (u_{-i}, u_i)u_i = u_{-i} - \bar{h}u_i$$

has norm $\sqrt{1-|h|^2}$

so $\quad u'_{-i} = \dfrac{u_{-i} - \bar{h}u_i}{\sqrt{1-|h|^2}}$ is a unit vector $\perp V\mathcal{D}_{V^2} = R_{V_1}$

Also $\quad pr_{V\mathcal{D}_V}(u_i) \perp V\mathcal{D}_{V^2}$

$$\parallel$$
$$u_i - (u_i, u_{-i})u_i = u_i - hu_{-i}$$

so $\quad V^{-1}(u_i - hu_{-i}) \perp \mathcal{D}_{V^2}$ and $\in \mathcal{D}_V$

hence

$$u'_i = V^{-1}\left(\frac{u_i - hu_{-i}}{\sqrt{1-|h|^2}}\right)$$ is a unit vector in $\mathcal{D}_V$ $\perp$ to $\mathcal{D}_{V^2}$

So now I can calculate $\quad S_1(z) = \dfrac{(\hat{u}_z - u_i, u'_{-i})}{(\hat{u}_z - u_i, u'_i)}$

Take numerator $\quad (\hat{u}_z - u_i, u'_{-i}) = (\underbrace{pr_{\mathcal{D}_V}}_{\text{drop}}(\hat{u}_z), u_{-i} - \bar{h}u_i)/\sqrt{1-|h|^2}$

$$= (S(z) - h)/\sqrt{1-|h|^2}$$

Take denominator

$$(\underbrace{pr_{\mathcal{D}_V}}_{\text{drop}}(\hat{u}_z), V^{-1}\underbrace{(u_i - hu_{-i})}_{\in R_V \text{ on which } V^{-1} = T^*})/\sqrt{1-|h|^2}$$

$$= (\hat{u}_z, T^*(u_i - h\underset{\substack{\uparrow \\ \text{killed by } T^*}}{u_{-i}}))/\sqrt{} = (T(1-zT^*)^{-1}u_i, u_i - \cancel{hu_{-i}})$$

<u>April 29, 1978:</u>

$$\begin{tikzcd}[column sep=small] & & \mathcal{H} \arrow[dl, dash] \arrow[dr, dash] & \\ & \mathcal{D}_V \arrow[dl, dash] \arrow[dr, dash] & & V\mathcal{D}_V \arrow[dl, dash] \\ \mathcal{D}_{V^2} & & V\mathcal{D}_{V^2} & \end{tikzcd}$$

It is important to concentrate on the line L_z perpendicular to $(1-\bar{z}V)\mathcal{D}_V$:

$$L_z = \mathcal{H} \ominus (1-\bar{z}V)\mathcal{D}_V$$

$$\begin{tikzcd}[column sep=small] & \mathcal{H} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{D}_V \arrow[dr, dash] & & (1-\bar{z}V)\mathcal{D}_V \arrow[dl, dash] \\ & (1-\bar{z}V)\mathcal{D}_{V^2} & \end{tikzcd}$$

If V_1 is the partial isom. on $\mathcal{D}_V$ induced by V, then clearly $L'_z = pr_{\mathcal{D}_V}(L_z)$ when the latter is a line. This is allright except when $z = 0$. L_z is spanned by $(1-\dot{z}T^*)^{-1}u_i$ whose projection on $\mathcal{D}_V$ is $(1-zT^*)^{-1}u_i - u_i$. Thus L'_z is spanned by

$$\frac{(1-zT^*)^{-1}u_i - u_i}{z} = T^*(1-zT^*)^{-1}u_i$$

To find the function $S_1(z)$ belonging to V_1 we need unit vectors in $\mathcal{D}_V$ $\perp$ to $\mathcal{D}_{V^2}$, $V\mathcal{D}_{V^2}$ resp.

$$pr_{\mathcal{D}_V}(u_{-i}) = u_{-i} - (u_{-i}, u_i)u_i = u_{-i} - \bar{h}u_i \qquad \perp V\mathcal{D}_{V^2}$$

$$u'_{-i} = \frac{u_{-i} - \bar{h}u_i}{\sqrt{1-|h|^2}} \qquad\qquad h = (u_i, u_{-i}) = S(0)$$

$$pr_{V\mathcal{D}_V}(u_i) = u_i - hu_{-i} \quad \perp V\mathcal{D}_{V^2}$$

so
$$u'_i = \frac{V^{-1}(u_i - hu_{-i})}{\sqrt{1-|h|^2}}$$

$$\left(T^*(1-zT^*)^{-1}u_i, V^{-1}(u_i - hu_{-i})\right) = \left((1-zT^*)^{-1}u_i, \underset{\substack{\uparrow \\ \text{in } \mathcal{D}_V \text{ on which } T=V}}{T}V^{-1}(u_i - hu_{-i})\right)$$

$$= \left((1 - zT^*)^{-1} u_i , u_i - h u_{-i} \right)$$
$$= 1 - \bar{h} S(z).$$

$$\left(T^*(1-zT^*)^{-1} u_i , \underset{-}{u_i} - \bar{h} u_{+i} \right) = \frac{1}{z}\left((1-zT^*)^{-1} u_i - u_i , \underset{-}{u_i} - \bar{h} u_{+i} \right)$$
$$= \frac{1}{z}\left(S(z) - h \right)$$

Therefore we have

$$S_1(z) = \frac{\left(T^*(1-zT^*)^{-1} u_i , \frac{u_{-i} - \bar{h} u_i}{\sqrt{\ }} \right)}{\left(T^*(1-zT^*)^{-1} u_i , \frac{u_i - h u_{-i}}{\sqrt{\ }} \right)} = \frac{1}{z} \frac{S(z) - h}{1 - \bar{h} S(z)}$$

as I conjectured.

Summarizing I get:

Thm: Let V be a partial isometry on $\mathcal{H}$ with indices $(1,1)$, and [redacted] let $S(z)$ be the analytic function in the disk associated to V and a choose of unit vectors $u_i \in \mathcal{H} \ominus \mathcal{D}_V$, $u_{-i} \in \mathcal{H} \ominus V\mathcal{D}_V$. (Thus $S(z) = (\psi_z, u_{-i})$ where ψ_z is the unique vector with $\psi_z \perp (1 - \bar{z}V)\mathcal{D}_V$ and $(\psi_z, u_i) = 1$). Let V_1 be the isometry induced on $\mathcal{D}_V$ by V and let $S_1(z)$ be the analytic function belonging to V_1 and the unit vectors

$$u_i^1 = \frac{V^{-1}(u_i - h u_{-i})}{\sqrt{1 - |h|^2}} \in \mathcal{D}_V \ominus \mathcal{D}_{V^2}$$

$$u_{-i}^1 = \frac{u_{-i} - h u_i}{\sqrt{1 - |h|^2}} \in \mathcal{D}_V \ominus V\mathcal{D}_{V^2}$$

where $h = (u_i, u_{-i}) = S(0)$. Then $S_1(z)$ is the first function in the Schur development of $S(z)$, i.e.

$$S_1(z) = \frac{1}{z} \cdot \frac{S(z) - h}{1 - \bar{h} S(z)}$$

or

$$S(z) = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} (z S_1(z))$$

Let V be a partial isometry on $\mathcal{H}$ without unitary component. For each z such that $(1-\bar{z}V)\mathcal{D}_V$ is closed let N_z denote its orthogonal complement. Note for $z \neq 0$ one has $(1-\bar{z}V)\mathcal{D}_V = (V-z^*)\mathcal{D}_V$ so we interpret this space to be $V\mathcal{D}_V$ at $z=\infty$. I know that $(1-\bar{z}V)\mathcal{D}_V$ is closed for all z not on S^1. Recall the proof: We have an isometry

$$\mathcal{D}_V \oplus \mathcal{D}_V^\perp \longrightarrow \mathcal{H} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x+y$$

For $|z|<1$ the map $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto -\bar{z}Vx$ is of norm less than 1 so on adding it to the above isometry we get an isomorphism

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto (1-\bar{z}V)x + y$$

which shows for $|z|<1$, that $\mathcal{D}_V^\perp$ is complementary to $(1-\bar{z}V)\mathcal{D}_V$. (Actually the above map is $1-\bar{z}T$ which is invertible for all z such that z^* is not in the spectrum of T.) So $(1-\bar{z}V)\mathcal{D}_V$ is closed for $|z|<1$. Case $|z|>1$ is similar.

Claim that N_z injects into $\mathcal{D}_V^\perp \oplus (V\mathcal{D}_V)^\perp$ for any z for which it is defined. In effect let $\alpha \in N_z$ be orthogonal to $\mathcal{D}_V^\perp$ and $R_V^\perp$ i.e. $\alpha \in \mathcal{D}_V$ and $\alpha \in R_V$. Then by definition of N_z one has

$$(\alpha, x - \bar{z}Vx) = 0 \qquad \forall x \in \mathcal{D}_V$$

Since $V^{-1}\alpha$ is defined this says $(\alpha, x) = z(\alpha, Vx) = z(V^{-1}\alpha, x)$ or $((1-zV^{-1})\alpha, x) = 0$. But $(1-zV^{-1})\alpha \in \mathcal{D}_V$ hence we have $\alpha = zV^{-1}\alpha$ or $V\alpha = z\alpha$ contradicting the assumption that V has no unitary component. (This proof makes sense even if $(1-\bar{z}V)\mathcal{D}_V$ isn't closed).

Important case: $\mathcal{H}$ finite-dimensional in which case $(1-\bar{z}V)\mathcal{D}_V$ is closed for all z. Then we can define a scattering operator

$$S(z) : \mathcal{D}_V^{\perp} \longrightarrow \mathcal{R}_V^{\perp}$$

by

$$\big(S(z)u, v\big) = \big((1-zT^*)^{-1}u, v\big) \qquad u \in \mathcal{D}_V^{\perp},\ v \in \mathcal{R}_V^{\perp}$$

This is defined as long as $(1-zT^*)$ is invertible in particular for $|z| \leq 1$. If I recall that $(1-zT^*)^{-1}u$ is the unique element of N_z projecting onto u, then it is clear that

$$\text{Image of } N_z \text{ in } \mathcal{D}_V^{\perp} \times \mathcal{R}_V^{\perp} = \text{graph of } S(z)$$

In other words we a map $z \mapsto N_z$ into the Grassman manifold of subspaces of $\mathcal{D}_V^{\perp} \times \mathcal{R}_V^{+}$ of $\dim = \dim \mathcal{D}_V$, and to get S we intersect with the open sell of those subspaces projecting non-trivially on $\mathcal{D}_V^{\perp}$.

Comments: 1) One gets via the Grassman interpretation an interpretation of the scattering operator for all z

2) In $\mathrm{Grass}_d(\mathcal{D} \times \mathcal{D})$, $d = \dim \mathcal{D}$, the maximal isotropic subspaces for the hermitian form

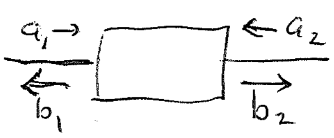
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto |x_1|^2 - |x_2|^2$$

are graphs of unitary operators on $\mathcal{D}$.

I want to discuss 2 ports. Here one thinks of $\mathcal{D}_V^\perp$ as consisting of incoming waves with amplitudes a_1, a_2 and $\mathcal{R}_V^\perp$ as consisting of outgoing waves with amplitudes b_1, b_2. The scattering matrix gives the relation between the two:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = S(z)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Power ~~into~~ the 2-port is $|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2$. But now I want to describe things interms of a transfer matrix



$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = T\begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

and the power form:

$$\underbrace{\left(|a_1|^2 - |b_1|^2\right)}_{\text{power in at } 1} - \underbrace{\left(|b_2|^2 - |a_2|^2\right)}_{\text{power out at } 2}$$

Let's fix $z \in S^1$. The 2 dimensional subspaces of $\mathcal{D}_V^\perp \times \mathcal{R}_V^\perp$ which are isotropic for the power form are the graphs of unitary transformations. So we see that the possible T i.e. $U(1,1)$ should correspond bijectively to a subset of $U(2)$.

Formulas

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{S}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{T}\begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

$$b_2 = \gamma a_1 + \delta a_2 \qquad\qquad b_1 = \alpha a_1 + \beta a_2$$

$$a_1 = \frac{1}{\gamma} b_2 - \frac{\delta}{\gamma} a_2 \qquad\qquad = \alpha\left(\frac{1}{\gamma} b_2 - \frac{\delta}{\gamma} a_2\right) + \beta a_2$$

$$= \frac{\alpha}{\gamma} b_2 + \frac{\beta\gamma - \alpha\delta}{\gamma} a_2$$

So

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \Longrightarrow T = \begin{pmatrix} \frac{1}{\gamma} & -\frac{\delta}{\gamma} \\ \frac{\alpha}{\gamma} & \frac{\beta\gamma - \alpha\delta}{\gamma} \end{pmatrix}$$

Similarly

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Longrightarrow S = \begin{pmatrix} \frac{C}{A} & \frac{AD - BC}{A} \\ \frac{1}{A} & -\frac{B}{A} \end{pmatrix}$$

Recall that any element T in $U(1,1)$ can be expressed in the form

$$\zeta \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \qquad |\zeta| = 1, \quad |a|^2 - |b|^2 = 1$$

from which we see that $|A| \geq 1$. Consequently the above formulas give an embedding of $U(1,1)$ into $U(2)$ with image those S with γ (hence β) $\neq 0$, i.e. the complement of the diagonal matrices.