March 29, 1978

The intelligent question: Consider the category of symmetric (1,1) operators without self-adjoint component. What are the automorphisms of such a gadget?

Fix $\mathcal{H}$ and a isometric relation [redacted] V on $\mathcal{H}$ of type (1,1), that is, $V: R_V \to D_V$ is unitary, where R_V and D_V are closed subspaces of codim. 1. Assume V has no unitary component, i.e. there is no non-trivial closed subspace of $\mathcal{H}$ on which V induces a unitary [redacted] automorphism.

Q: What are the autos. of such a thing?

Other questions: Let T be a contraction operator extending V, [redacted] such that T^* extends V^{-1}. I saw before this means that $T(u_i) = a(u_{-i})$ for some $|a|<1$. A canonical choice would be for $T(u_i) = 0$. What does the associated unitary operator: $\tilde{\mathcal{H}}, U$ look like in this case.

Let θ be an automorphism of $(\mathcal{H}, V)$. By multiplying θ by a scalar we can suppose $\theta(u_i) = u_i$. Since θ is an auto of the setup

$$D_V \underset{V}{\overset{i}{\rightrightarrows}} \mathcal{H}$$

we can decompose $\mathcal{H}$ into $\mathcal{H}' = \mathrm{Ker}(\theta - 1)$ and its orthogonal complement. [redacted] Denoting by a prime fixpts under θ we have

$$D_V' \oplus \mathbb{C} u_i = \mathcal{H}'$$
$$R_V' \oplus (\mathbb{C} u_{-i})' = \mathcal{H}'$$

Case 1: $\Theta u_{-i} = u_{-i}$. In this case V would induce a unitary transf. on $\mathcal{H} \ominus \mathcal{H}'$, hence because V has no unitary component we have $\mathcal{H}' = \mathcal{H}$, so $\Theta =$ [redacted] 1.

Case 2: $\Theta u_{-i} = \zeta u_{-i}$ [redacted] with $\zeta \neq 1$. Then on $\mathcal{H}'$, V induces V' which is the inverse of an isometric embedding $(V')^{-1} : \mathcal{H}' \hookrightarrow \mathcal{H}'$ with range of codimension 1. We have $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$ where $\mathcal{H}''$ is the ζ-eigenspace for Θ. [redacted] On $\mathcal{H}''$, V induces an isometric embedding V'' with range of codim 1. Thus $(\mathcal{H}, V)$ is the direct sum of the shift: multiplication by z on H^2 and the anti-shift: multiplication by z^{-1} on $\bar{z}H^2_-$ followed by projection.

In this example, $\tilde{\mathcal{H}}$ is two copies of $L^2(S^1)$ for the choice $Tu_i = au_{-i}$, $a = 0$. It seems one gets the same $\tilde{\mathcal{H}}$ for any $|a| < 1$.

How can we characterize the particular V that occurs in Case 2. First we have that $(u_i, u_{-i}) = 0$. So [redacted] is defined $(V^{-1}u_i)$. Necessarily $(V^{-1}u_i, u_i) = 0$, because we can always extend V to a unitary map U with $U(u_i) = u_{-i}$. 2ndly we have $(V^{-1}u_i, u_{-i}) = 0$ so $V^{-2}u_i$ is defined, and so forth. Similarly $V^n u_{-i}$ is defined for all $n \geq 0$ and [redacted] $\{V^{-n}u_i, V^n u_{-i}, n \geq 0\}$ is an orthonormal basis for $\mathcal{H}$. So we can characterize this case as the one such that any unitary extension [redacted] U yields Lebesgue measure $\frac{d\theta}{2\pi}$ on S^1. [redacted]

The A-description involves [redacted] Lebesgue measure on $\mathbb{R}$.

[redacted] What are the possible Θ?

There is an interesting action of S^1 on the set of probability measures on S^1. Given $d\nu$ we can associate $\mathcal{H} = L^2(S^1, d\nu)$ and the isometric relation V given by mult. by z on $D_V = \{ f \mid (f, z^{-1}) = 0 \}$. The [redacted] possible extensions of V to a unitary operator U on $\mathcal{H}$ form a torsor under S^1 and to each U belongs a probality measure.

This action should be related to the following. To any probability measure $d\nu$ on S^1 belongs an analytic function $\varphi(z)$ on $|z| < 1$ with $\varphi(0) = i$ and values in the UHP given by

$$\varphi(z) = \frac{1}{i} \int_{S^1} \frac{z + \zeta}{z - \zeta} \, d\nu(\zeta)$$

and $d\nu$ is uniquely determined by $\varphi(z)$.

$$\operatorname{Im} \varphi(z) = \int_{S^1} \frac{1 - |z|^2}{|z - \zeta|^2} \, d\nu(\zeta)$$

However S^1 acts as autos. of UHP preserving i, so it acts on these probability measures.

Another version: Let A be densely-defined symm. (1,1) operator with no non-trivial self-adjoint component. [redacted] $S^1 \times S^1$ acts on possible choices for u_i, u_{-i} & hence on the self-adjoint extensions $\tilde{A}$ of A. [redacted] ΔS^1 acts trivially so we get an action of $S^1 \times S^1 / \Delta S^1 = S^1$ on the possible $\tilde{A}$. [redacted] To each $\tilde{A}$ belongs a unique measure $d\mu$ on $\mathbb{R}$ with $\int \frac{d\mu}{x^2+1} = 1$ hence we get an action of S^1 on the[redacted] measures. It seems that we get any measure on $\mathbb{R}$ with $\int d\mu = \infty$ [redacted] (so that

$\mathcal{D}_A$ is densely-defined. Somehow this seems to imply that $d\nu$ is a prob. measure on S^1 without point spectrum at $z=1$, then [redacted] none of the other measures in its orbit have point spectrums at $z=1$ [redacted] (assume 1 is in the spectrum).

Suppose $\mathcal{H}$, V given, V type $(1,1)$ isometric relation without unitary part.

Prop: V is the Cayley transform of a partial isometry...

<u>Prop:</u> V is the Cayley transform of a densely-defined symmetric op. [redacted] A $\Longleftrightarrow$ $(1-V)\mathcal{D}_V$ is dense in $\mathcal{H}$.

<u>Proof.</u> Recall that when $V=(A-i)(A+i)^{-1}$ we have

$$x=(A+i)(A+i)^{-1}x \qquad x\in\mathcal{D}_V=(A+i)\mathcal{D}_A$$

$$Vx=(A-i)(A+i)^{-1}x$$

so

$$\left(\frac{1-V}{2i}\right)x=(A+i)^{-1}x$$

$$\left(\frac{1+V}{2}\right)x=A(A+i)^{-1}x$$

so

$$\left(\frac{1-V}{2i}\right)(A+i)\,y=y \qquad y\in\mathcal{D}_A.$$

In other words $\mathcal{D}_A=$ [redacted] $(1-V)\mathcal{D}_V$ whence $\Rightarrow$ is clear. Conversely suppose $(1-V)\mathcal{D}_V$ dense in $\mathcal{H}$. First we show $1-V$ injective. If $Vh=h$, $h\in\mathcal{D}_V$ then $\forall f\in\mathcal{D}_V$

$$(h,(1-V)f)=(h,f)-(h,Vf)=(h,f)-(Vh,Vf)$$
$$=(h,f)-(h,f)=0$$

$\therefore$ $h=0$ as $(1-V)\mathcal{D}_V$ is dense. Now define A by

$$A = \left(\frac{1+V}{2}\right)\left(\frac{1-V}{2i}\right)^{-1} \quad \text{on} \quad (1-V)\mathcal{D}_V$$

This is a well-defined operator on $(1-V)\mathcal{D}_V$. One has

$$\Gamma_A = \left\{ \begin{pmatrix} \left(\frac{1-V}{2i}\right)x \\ \left(\frac{1+V}{2}\right)x \end{pmatrix} \;\middle|\; x \in \mathcal{D}_V \right\}$$

and
$$\left\|\left(\frac{1-V}{2i}\right)x\right\|^2 + \left\|\frac{1+V}{2}x\right\|^2 = \frac{1}{2}\left\{ \|x\|^2 + \|Vx\|^2 \right\} = \|x\|^2$$

so Γ_A is closed. Symmetry:

$$4i\left\{ \left(\left(\frac{1-V}{2i}\right)x, \left(\frac{1+V}{2}\right)y\right) - \left(\frac{1+V}{2}x, \frac{1-V}{2i}y\right) \right\}$$
$$= (x-Vx, y+Vy) + (x+Vx, y-Vy)$$
$$= (x,y) - (Vx,Vy) + (x,y) - (Vx,Vy) = 0 .$$

QED.

Suppose V of type $(1,1)$ without unitary component. I've seen we can identify $\mathcal{H}$ with $L^2(S^1, d\nu)$ where $V =$ mult. by z on $\mathcal{D}_V = \{ f \in \mathcal{H} \mid \int z f d\nu = 0 \}$. Suppose $f \perp (1-V)\mathcal{D}_V$. This means

$$(f, (1-z)g) = 0 \qquad \forall g \in \mathcal{H} \ni (g, z^{-1}) = 0$$
$$\|$$
$$((1-z^{-1})f, g) = ((z-1)f, zg)$$

hence $(z-1)f \perp \{ zg \in \mathcal{H} / (zg, 1) = 0 \}$. Thus

$$(z-1)f = c \qquad c \text{ constant}$$

Assuming $f \neq 0$ two cases are possible: $(z-1)f = 0$ whence 1 is an atom for $d\nu$. If $c \neq 0$, then $\frac{1}{z-1} \in L^2(S^1, d\mu)$.

So

(*) $(1-V)\mathcal{D}_V$ dense in $\mathcal{H}$ $\Longleftarrow$ 1 not an atom for $d\nu$ and $\frac{1}{z-1} \notin L^2(S^1, d\nu)$

It is clear that if 1 is an atom for $d\nu$ then $(1-V)\mathcal{D}_V \subset (1-U)\mathcal{H} < \mathcal{H}$ fails to be dense. If 1 is not an atom, but $\frac{1}{z-1} \in L^2(S^1, d\nu)$, then for $g \in \mathcal{D}_V$

$$\left(\frac{1}{1-z}, (1-z)g\right) = \left(\frac{1}{1-z}(1-z^{-1}), g\right) = -(1, zg) = 0$$

So the above (*) is $\Longleftrightarrow$.

The interesting point is that although $d\nu$ depends upon choosing a unitary extension of V, the condition $(1-V)\mathcal{D}_V$ dense depends only on V. And we have seen that it means V is the Cayley transform of a d.d. symm. op. A.

Suppose we replace $L^2(S^1, d\nu)$ with the isomorphic space $L^2(\mathbb{R}, d\mu)$ where $1 = u_{-i}$ in the former goes to $\frac{1}{x+i}$ in the latter. Then

$$\frac{2i}{z-1} \longmapsto (x+i)\cdot\frac{1}{x+i} = 1$$

Hence for the A arising from $L^2(\mathbb{R}, d\mu)$ where $d\mu$ is a measure $\ni \int \frac{d\mu}{x^2+1} = 1$ we have

$$\mathcal{D}_A \text{ dense} \Longleftrightarrow \int d\mu = \infty.$$

(Check: $\Rightarrow$ obvious, for otherwise $1 \in L^2$ and $1 \perp \mathcal{D}_A$. $\Leftarrow$ Let $f \perp \mathcal{D}_A$, i.e. $(f,g) = 0$ for all $g \in L^2 \ni xg \in L^2$ and $\int g = 0$. Restricting f to a finite interval $[a,b]$ we have $(f,g) = 0$ for all g with support in $[a,b] \ni \int g = 0$, hence f has to be const.

on $[a,b]$, hence constant globally, so $\int d\mu < \infty$ if $f \neq 0$.

Possible consequence of the above: First note that replacing V by $\zeta^{-1} V$ with $|\zeta| = 1$ we can conclude

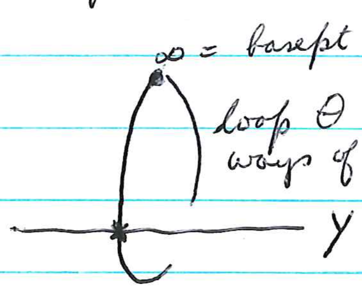
$(1-\zeta^{-1}V)\mathcal{D}_V$ dense in $\mathcal{H}$ $\iff$ ζ not an atom for any $d\nu$ belonging to V and $\frac{1}{1-\zeta^{-1}z} \notin L^2(S^1; d\nu)$.

This somehow amounts to a characterization of continuous spectrum in some sense.

The idea of a partial isometry $\mathcal{D}_V \underset{V}{\overset{in}{\rightrightarrows}} \mathcal{H}$ reminds me of Waldhausen[redacted]'s way of treating π_1 for a [redacted] manifold X and a [redacted] codimension 1 [redacted] submanifold Y. Two cases according to whether Y disconnects X. [redacted] of $X = X^+ \cup_Y X^-$, then in the free product situation

$$\pi_1(X) \xleftarrow{\sim} \pi_1(X^+) *_{\pi_1(Y)} \pi_1(X^-).$$

But if $X-Y$ is connected one has the following. One assumes the normal line bundle of Y in X is trivial, so that we have two ways of pushing [redacted] Y into $X-Y$.



$$\begin{array}{ccc} \pi_1 Y & \xrightarrow{+} & \pi_1(X-Y) \\ {\scriptstyle -}\downarrow & \overset{\theta}{\nearrow} & \downarrow \\ \pi_1(X-Y) & \longrightarrow & \pi_1(X) \end{array}$$

What this suggests is looking for a Hilbert spaces $\tilde{\mathcal{H}}$ with a unitary operator U and an isometric embedding $i : \mathcal{H} \to \tilde{\mathcal{H}}$ such that

$$U\, i(x) = i(Vx) \qquad \forall x \in \mathcal{D}_V :$$

i.e.

$$\begin{array}{ccc} \mathcal{D}_V & \xrightarrow{in} & \mathcal{H} \\ V\downarrow & \overset{U}{\swarrow} & \downarrow i \\ \mathcal{H} & \xrightarrow[i]{} & \tilde{\mathcal{H}} \end{array}$$

commutes.

For example, if i is an isomorphism, then we have simply a unitary operator on $\mathcal{H}$ extending V.

Examples: 1) Suppose $\mathcal{D}_V = \mathcal{H}$ so that V is an isometric embedding. Then there seems to be a unique possibility for $\tilde{\mathcal{H}}$, namely $L^2(S^1; \mathcal{N})$ where $\mathcal{N} = \mathcal{H} \ominus V\mathcal{H}$. In fact we know

$$\mathcal{H} = H^2(S^1; \mathcal{N}) = \bigoplus_{n \geq 0}^{(\ell^2)} V^n \mathcal{N}$$

and if this is embedded in $\tilde{\mathcal{H}}$ then $\{U^n \mathcal{N} \mid n \in \mathbb{Z}\}$ have to be orthogonal subspaces.

2) Next suppose $R_V = \mathcal{H}$ so that V^{-1} is an isometric embedding. Then

$$\tilde{\mathcal{H}} = L^2(S^1; \mathcal{N}) \qquad \mathcal{N} = \mathcal{H} \ominus \mathcal{D}_V$$

$$\mathcal{H} = H^{2-}(S^1; \mathcal{N}) = \bigoplus_{n \leq 0} V^n \mathcal{N}$$

Recall from studying Waldhausen that given a diagram $F_{-1} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} F_0$ of vector spaces over k

with α injective we get a $k[T]$-module M with an exact sequence

$$0 \longrightarrow k[T] \otimes F_{-1} \xrightarrow{T \otimes \alpha - 1 \otimes \beta} k[T] \otimes F_0 \longrightarrow M \longrightarrow 0.$$

Moreover $F_{-1} \overset{\alpha}{\subset} F_0 \subset M$ is the beginning of a filtration with

$$F_p = F_0 + TF_0 + \cdots + T^p F_0$$

such that mult. by T gives an isomorphism $F_p/F_{p-1} \xrightarrow{\sim} F_{p+1}/F_p$

for $p \geq 0$.

So it clear now how to obtain [redacted] from

$$\mathcal{D}_V \underset{V}{\overset{\hookrightarrow}{\rightrightarrows}} \mathcal{H}$$

a universal $(\tilde{\mathcal{H}}, U)$ with U invertible extending V. $\tilde{\mathcal{H}}$ will be [redacted] given by an exact sequence

$$0 \longrightarrow \mathbb{C}[U,U^{-1}] \otimes \mathcal{D}_V \xrightarrow{U \otimes in - id \otimes V} \mathbb{C}[U,U^{-1}] \otimes \mathcal{H} \longrightarrow \tilde{\mathcal{H}} \longrightarrow 0$$

This $\tilde{\mathcal{H}}$ is purely algebraic. Its positive part $\sum_{n \geq 0} U^n \mathcal{H}$ has a filtration

$$F_p(\tilde{\mathcal{H}}^+) = \sum_{0 \leq n \leq p} U^n \mathcal{H}$$

and $\mathcal{H}/\mathcal{D}_V \xrightarrow{\sim} F_p/F_{p-1}$. In other words, in order to obtain $\tilde{\mathcal{H}}^+$ one adds to $\mathcal{H}$ new elements $U^n(u_i)$, $n \geq 1$. To get $\tilde{\mathcal{H}}$ one adds to $\mathcal{H}$ independent elements $U^n(u_i)$ $n \geq 1$ and $U^{-n}(u_{-i})$ for $n \geq 1$. This algebraic $\tilde{\mathcal{H}}$ doesn't come with a unique inner product such that U is unitary. For example if $\mathcal{H} = \mathbb{C}$, with $\mathcal{D}_V = 0$, then $\tilde{\mathcal{H}} = \mathbb{C}[U, U^{-1}]$ and we can define [redacted] an inner product using any prob. measure on S^1. There is an [redacted] obvious choice for inner product, if it works, viz, to require these new basis elements to form an orthonormal basis for the orthogonal complement to $\mathcal{H}$ in $\tilde{\mathcal{H}}$.

March 31, 1978:

Let's go back to a contraction op. T on $\mathcal{H}$, with unitary extension U on $\tilde{\mathcal{H}}$ and try again to calculate the scattering operator. Put

$$T_p = \begin{cases} T^p & p \geq 0 \\ T^{*-p} & p \leq 0 \end{cases}$$

and recall that $\tilde{\mathcal{H}}$ is obtained by completing the space of Laurent polys. $\sum z^n \alpha_n$ with coeffs. in $\mathcal{H}$ with respect to the norm

$$\|\sum z^n \alpha_n\|^2_{\tilde{\mathcal{H}}} = \sum_{n,m} (T_{n-m}\alpha_n, \alpha_m)_{\mathcal{H}}$$

$$= \int_{S^1} \Big(\sum_p T_p z^{-p} \cdot \sum_n z^n \alpha_n , \sum z^m \alpha_m \Big)_{\mathcal{H}} \frac{d\theta}{2\pi}$$

where I suppose that $\|T\| < 1$ so that the following series converges

$$\sum_p T_p z^{-p} = \sum_{p \geq 0} T^p z^{-p} + \sum_{p \geq 1} T^{*p} z^p$$

$$= (1 - z^{-1}T)^{-1} + zT^*(1 - zT^*)^{-1}$$

$$= (1 - z^{-1}T)^{-1}\big[1 - zT^* + (1 - z^{-1}T)zT^*\big](1 - zT^*)^{-1}$$

$$= (1 - z^{-1}T)^{-1}\big[1 - TT^*\big](1 - zT^*)^{-1}$$

Put

$$\varphi(z) = (1 - TT^*)^{1/2}(1 - zT^*)^{-1}.$$

Then we have an isomorphism

$$L^2(S^1; \mathcal{H}) \xrightarrow{\ \sim\ } \tilde{\mathcal{H}}$$

$$\varphi(z)\alpha(z) \longleftarrow\!\shortmid\ \alpha(z)$$

compatible with multiplication by z. With respect to this isom., the canonical embedding $i: \mathcal{H} \to \tilde{\mathcal{H}}$ becomes

$$i(h) = \varphi(z)\, h.$$

Thus

$$(i^*\beta, h) = (\beta, ih) = (\beta, \varphi \cdot h) = (\varphi^*\beta, h)_{\tilde{\mathcal{H}}}$$

hence

$$i^*\beta(z) = \int \varphi^*(z)\beta(z)\, \frac{d\theta}{2\pi}.$$

So

$$i^* U^n i(h) = \int \varphi^*(z)\, z^n \varphi(z) \frac{d\theta}{2\pi}(h)$$

$$= \int \sum T_p z^{-p} z^n \frac{d\theta}{2\pi}(h) = T_n(h)$$

as it should be.

The outgoing subspace generated by $i\mathcal{H}$ is

$$\mathcal{D}_0 = \left\{ \varphi(z)\alpha(z) \;\middle|\; \alpha(z) = \sum_{n \geq 0} z^n \alpha_n \right\} = \varphi H^2(S^1; \mathcal{H})$$

Because φ is holomorphic an invertible for $|z| \leq 1$ we have

$$\mathcal{D}_0 = \varphi H^2 = H^2$$

Hence

$$\mathcal{D}_1 = \mathcal{D}_0 \ominus i\mathcal{H} = \left\{ \alpha \in H^2 \;\middle|\; i^*\alpha = \int \varphi^* \alpha \frac{d\theta}{2\pi} = 0 \right\}$$

Better: What you've done with $\varphi(z) = (1 - TT^*)^{1/2}(1 - zT^*)^{-1}$ is to get an isom

$$L^2(S^1; \mathcal{H}) \xleftarrow{\sim} \tilde{\mathcal{H}}$$

$$\varphi(z)\alpha(z) \longleftarrow\!\shortmid\ \alpha$$

such that $\mathcal{D}_0$ goes to H^2. In other words you have constructed the outgoing spectral representation. But if you use

$$\psi(z) = (1-T^*T)^{1/2}(1-z^{-1}T)^{-1}$$

then you get an isomorphism

$$L^2(S^1;\mathcal{H}) \xrightarrow{\sim} \tilde{\mathcal{H}}$$

$$\psi(z)\alpha \longleftarrow\!\shortmid \alpha$$

such that $\mathcal{D}_0^- = \overline{\sum_i U^{-n}i\mathcal{H}}$ goes to H^{2-}. This is the incoming spectral representation. The scattering matrix is the operator

$$L^2(S^1;\mathcal{H}) \simeq \tilde{\mathcal{H}} \simeq L^2(S^1;\mathcal{H})$$

$$\alpha \longmapsto \psi(z)^{-1}\alpha \longmapsto \varphi(z)\psi(z)^{-1}\alpha$$

$$S(z) = \varphi(z)\psi(z)^{-1} = (1-TT^*)^{1/2}(1-zT^*)^{-1}(1-z^{-1}T)(1-T^*T)^{-1/2}$$

This has unitary values on S^1 because

$$S(z)^{-1} = \psi(z)\varphi(z)^{-1} \qquad S(z)^* = (\psi(z)^{-1})^*\varphi(z)^*$$

and

$$\psi(z)^*\psi(z) = (1-\bar{z}^{-1}T^*)^{-1}(1-T^*T)(1-z^{-1}T)^{-1}$$

$$\varphi(z)^*\varphi(z) = (1-\bar{z}T)^{-1}(1-TT^*)(1-zT^*)^{-1}$$

are equal by the basic calculation of $\sum T_p z^{-p}$.

Suppose V is a partial isometry on $\mathcal{H}$ with deficiency indices $(1,1)$, let u_i be a unit vector orthogonal to $\mathcal{D}_V$ and u_{-i} a unit vector orthogonal to $\mathcal{R}_V$. Let T be the contraction operator on V given by

$$\begin{cases} T(x) = Vx & \text{if } x \in \mathcal{D}_V \\ T(u_i) = 0 \end{cases}$$

Let $(\tilde{\mathcal{H}}, U, i)$ be the unitary operator generated by T. I claim that $\{U^n(u_i), U^{-n}(u_{-i}); n \geq 1\}$ is an orthonormal basis for $\tilde{\mathcal{H}} \ominus i\mathcal{H}$.

We have $i^* U^n i = T^n$ for $n \geq 0$, hence

$$i^* U^n(u_i) = T^n u_i = 0 \qquad \text{for } n \geq 1$$

so that $U^n(u_i) \perp i\mathcal{H}$ for $n \geq 1$. It follows that

$$\left(U^{n+p}(u_i), U^p(u_i)\right) = \left(U^n(u_i), u_i\right) = 0$$

for $n \geq 1$ showing that the set $U^n(u_i)$, $n \geq 1$ is orthonormal and $\perp i\mathcal{H}$. A similar thing holds for $U^{-n}(u_{-i})$, $n \geq 1$ (note: $T^* u_{-i} = 0$). Finally

$$\left(U^n(u_i), U^{-m}(u_{-i})\right) = \left(U^{n+m} u_i, u_{-i}\right) = 0$$

QED.

The above shows that $(\tilde{\mathcal{H}}, U, i)$ (= unitary operator generated by T) is in some sense the simplest unitary extension of $(\mathcal{H}, V)$ in a larger Hilbert space.

Next I should understand a little better the unitary operator generated by a contraction operator such that its spectrum is inside S^1. For example if [redacted] T has no unitary components, then T has no eigenvalues (discrete spectrum) on S^1, hence if $\dim(\mathcal{H}) < \infty$ the spectrum of T is inside S^1. Also if $T^n \to 0$, or equivalently if $\|T^n\| < 1$ for some $n \geq 1$, then its spectrum lies inside S^1.

Under this assumption the operators $(1-zT^*)^{-1}$ and $(1-z^{-1}T)^{-1}$ are analytic on S^1, and the calculations

$$\sum T_p z^{-p} = (1-zT^*)^{-1}(1-T^*T)(1-z^{-1}T)^{-1}$$

$$= (1-z^{-1}T)^{-1}(1-TT^*)(1-zT^*)^{-1}$$

are valid as analytic functions defined near S^1.

Note: The spectral radius of T is $\lim_{n\to\infty} \|T^n\|^{1/n} = \inf_n \|T^n\|^{1/n}$, so that for the spectrum of T to be [redacted] inside S^1 is equivalent to $\|T^n\| < 1$ for some $n \geq 1$, or that $T^n \to 0$ as $n \to \infty$.

Review the construction of $\tilde{\mathcal{H}}$. We start with $\mathcal{A}(S^1; \mathcal{H}) = \{$analytic functions $\alpha(z) = \sum \alpha_n z^n$ on S^1 with values in $\mathcal{H}\}$ and equip this with the inner product

$$\|\alpha\|^2_{\tilde{\mathcal{H}}} = \int \left(\sum T_p z^{-p} \alpha(z), \alpha(z) \right)_{\mathcal{H}} \frac{d\theta}{2\pi}$$

and complete to get $\tilde{\mathcal{H}}$. Let $\rho: \mathcal{H} \longrightarrow \mathcal{N}_1$ be the completion of $\mathcal{H}$ with respect to the inner product

$$\|x\|^2_{\mathcal{N}_1} = \left((1-TT^*)x, x\right)_{\mathcal{H}}$$

Then we have

$$\|\alpha\|^2_{\tilde{\mathcal{H}}} = \int \left\| (1-zT^*)^{-1}\alpha(z)\right\|^2_{\mathcal{N}_1} \frac{d\theta}{2\pi}$$

Now $1-zT^*$ acts invertibly on $\mathcal{a}(S^1;\mathcal{H}_1)$.

$$\begin{array}{ccccc}
\mathcal{a}(S^1;\mathcal{H}) & \xrightarrow[\sim]{1-zT^*} & \mathcal{a}(S^1;\mathcal{H}) & \xrightarrow{\rho(1-zT^*)^{-1}} & L^2(S^1;\mathcal{N}_1) \\
 & & \downarrow \text{in} & \nearrow & \\
 & & \tilde{\mathcal{H}} & &
\end{array}$$

The point is that $\mathcal{a}(S^1,\mathcal{N}_1)$ is dense in $L^2(S^1;\mathcal{N}_1)$ so because we have:

$$\begin{array}{ccc}
\mathcal{a}(S^1;\mathcal{H}) & \xrightarrow[\text{dense}]{\rho} & \mathcal{a}(S^1;\mathcal{N}_1) \\
{\scriptstyle 1-zT^*}\ \downarrow \wr \ {\scriptstyle \text{auto}} & & \downarrow \text{dense} \\
\mathcal{a}(S^1;\mathcal{H}) & \xrightarrow[\rho(1-zT^*)^{-1}]{} & L^2(S^1;\mathcal{N}_1) \\
{\scriptstyle \text{dense}} \downarrow & \nearrow & \\
\tilde{\mathcal{H}} & &
\end{array}$$

So we get an isomorphism of $\tilde{\mathcal{H}}$ with $L^2(S^1;\mathcal{N}_1)$ which sends $\alpha(z) \in \mathcal{a}(S^1;\mathcal{H})$ to $\rho(1-zT^*)^{-1}\alpha(z)$. Moreover this isomorphism carries $\mathcal{a}_{hol}(S^1;\mathcal{H})$ to $H^2(S^1;\mathcal{N}_1)$ and hence it is the outgoing spectral representation.

Better: Define $i : \mathcal{H} \longrightarrow L^2(S^1;\mathcal{N}_1)$ by

$$i(h) = \rho(1-zT^*)^{-1}h = \sum_{n\geq 0} z^n \rho(T^{*n}h)$$

Then $(i^* U^n i\, h, h') = (U^n i h, i h') = \left(z^n \rho (1-zT^*)^{-1} h, \rho (1-zT^*)^{-1} h'\right)_{L^2(S^1; \eta)}$

$$= \left(z^n (1-TT^*)(1-zT^*)^{-1} h, (1-zT^*)^{-1} h'\right)_{L^2(S^1, \mathcal{H})}$$

$$= \left(z^n \sum T_p z^{-p} h, h'\right) = (T_n h, h').$$

So by the defining property of $\tilde{\mathcal{H}}$ i induces an embedding

$$\tilde{\mathcal{H}} \longrightarrow L^2(S^1; \eta_1)$$

which is an isomorphism because as $(1-zT^*)$ is invertible one has

$$\overline{\sum_{n\geq 0} z^n i \mathcal{H}} = H^2(S^1, \eta_1).$$

Similarly if $\rho_2 : \mathcal{H} \longrightarrow \eta_2$ is the completion of $\mathcal{H}$ with respect to the norm $\|x\|^2_{\eta_2} = ((I - T^*T)x, x)$, we can use the embedding

$$\mathcal{H} \xrightarrow{i_2} L^2(S^1, \eta_2)$$

$$h \longmapsto \rho_2 (1-z^{-1}T)^{-1} h$$

to obtain an isomorphism

$$\tilde{\mathcal{H}} \xrightarrow{\sim} L^2(S^1, \eta_2)$$

with

$$\overline{\sum_{n\geq 0} z^{-n} \mathcal{H}} \simeq H^2_-(S^1, \eta_2)$$

The scattering operator can be understood as follows. Start from the basic identity

$$(1-z^{-1}T)^{-1}(1-TT^*)(1-zT^*)^{-1} = (1-zT^*)^{-1}(1-T^*T)(1-z^{-1}T)^{-1}$$

which yields for $x, y \in \mathcal{H}$ and $z \in S^1$

$$\left((1-TT^*)(1-zT^*)^{-1}x,\ (1-zT^*)^{-1}y\right) = \left((1-T^*T)(1-z^{-1}T)^{-1}x,\ (1-z^{-1}T)^{-1}y\right)$$

using the fact that $1-z^{-1}T$ is invertible on $\mathcal{H}$ we get

$$\left((1-TT^*)\,S(z)x,\ S(z)y\right) = \left([redacted](1-T^*T)x,\ y\right)$$

where

$$\boxed{S(z) = (1-zT^*)^{-1}(1-z^{-1}T)}$$

It follows that for $|z|=1$, $S(z)$ induces an isomorphism between $\mathcal{N}_2$ = completion wrt $1-T^*T$ and $\mathcal{N}_1$ = completion wrt $1-TT^*$.

It might be more natural to multiply $S(z)$ by z so as to get

$$zS(z) = (1-zT^*)^{-1}(z-T)$$

which is evidently holomorphic in the disk. This scattering operator probably corresponds to the one transforming $\mathcal{D} = \sum_{n\geq 0} U^n i\mathcal{H}$ to $\mathcal{D}\ominus i\mathcal{H}$.

Can any of this be applied to [redacted] the T associated to a partial isometry V of type $(1,1)$? The problem seems to be whether T has its spectrum inside S^1. If $\mathcal{H}$ is finite-dimensional, there is no problem on this score.

Let's see how much sense we can make out of S. [redacted] [redacted] $T=V$ except on $\langle u_i\rangle$, consequently [redacted] $1-T^*T=0$ on $\mathcal{D}_V$, in fact $1-T^*T$ = projection on $\langle u_i\rangle$, so

$$\mathcal{N}_2 = \mathcal{H}/\mathcal{D}_V \cong \langle u_i\rangle \qquad \mathcal{N}_1 = \mathcal{H}/\mathcal{R}_V \cong \langle u_{-i}\rangle$$

[redacted]

For $\mathcal{H}$ finite-dimensional, we know that [redacted] for any z [redacted] the operator $z - V$ has image of codim 1. Precisely $(z-V)\mathcal{D}_V$ [redacted] is a hyperplane in $\mathcal{H}$. In effect if $h \in \mathcal{D}_V$ and $(z-V)h = 0$, then [redacted] because h is isometric $|z| = 1$ and so $\langle h \rangle$ would be a unitary component of V. Thus $(z-V)\mathcal{D}_V$ is a hyperplane.

I want to compute $S(z)u_i$. Since $T(u_i) = 0$

$$S(z)u_i = (1 - zT^*)^{-1} u_i \text{ projected onto } \langle u_{-i} \rangle$$

hence I want

$$(S(z)\, u_i, u_{-i}) = ((1 - zT^*)^{-1} u_i, u_{-i}). \quad ?$$

Assume z such that $(z-V)\mathcal{D}_V$ is closed of codim 1 and let [redacted] e_z be a unit vector orthogonal to this hyperplane. Then

$$0 = ((z-V)\mathcal{D}_V, \text{[redacted]}\; e_z) = ((z-T)\mathcal{D}_V, e_z)$$

$$= (\mathcal{D}_V, (z^{-1} - T^*) e_z)$$

hence $\quad (z^{-1} - T^*) e_z = c u_i \qquad c = \text{constant}$

or $\quad e_z = c\,(z^{-1} - T^*)^{-1} u_i$. We have used $z \in S^1$ [redacted].

Better: Let $e_{\bar z} \perp (z-V)\mathcal{D}_V$. Then

$$0 = ((z-V)\mathcal{D}_V, e_{\bar z}) = ((z-T)\mathcal{D}_V, e_{\bar z}) = \text{[redacted]}\; (\mathcal{D}_V, (\bar z - T^*) e_{\bar z})$$

so $\quad e_{\bar z} = c\,(\bar z - T^*)^{-1} u_i \qquad$ assuming $(\bar z - T^*)^{-1}$ exists.

This argument is reversible and seems only to use that T is a contraction operator extending V. In any case what it shows is that [redacted] $(1-zT^*)^{-1}u_i$ is perpendicular to $(1-\bar{z}V)\mathcal{D}_V$:

$$\left((1-\bar{z}V)\mathcal{D}_V, (1-zT^*)^{-1}u_i\right) = \left((1-\bar{z}T)\mathcal{D}_V, (1-zT^*)^{-1}u_i\right)$$
$$= (\mathcal{D}_V, u_i) = 0.$$

Therefore the scattering matrix

$$(S(z)u_i, u_{-i}) = \left((1-zT^*)^{-1}u_i, u_{-i}\right)$$

is the analytic function one obtains by taking the z-line in $\langle u_i\rangle \oplus \langle u_{-i}\rangle$ and composing its coefficients.

Important thing to examine:

1) The above seems to be valid for any contraction T extending V. S seems not to depend on the choice of T.

2) How do singularities of V [redacted] affect things? Better: It seems that there's a connection between [redacted] those z on S^1 where $(1-\bar{z}V)\mathcal{D}_V$ is not closed and [redacted] with the spectrum of T. [redacted] Spectrum T inside $S^1 \iff \mathcal{H}$ finite-dimensional?? Yes: $\Leftarrow$ we've seen. $\Rightarrow$: V^n is defined on a subspace of codim $\leq n$, hence if $\|T^n\| \to 0$ one must have $\mathcal{H}$ finite-dimensional.

April 2, 1978

T = the contraction belonging a partial isom. V of type $(1,1)$. (no unitary component)
$(\tilde{\mathcal{H}}, U)$ the unitary operator generated by T.

I've seen that one has orthogonal decomposition

$$\tilde{\mathcal{H}} = \langle \dots, \underset{e_{-2}}{U^{-2}u_{-i}}, \underset{e_{-1}}{U^{-1}u_{-i}} \rangle \oplus \mathcal{H} \oplus \langle \underset{e_1}{U u_i}, \underset{e_2}{U^2 u_i}, \dots \rangle$$

hence we have two embeddings

$$L^2(S^1) \xrightarrow{\text{inc.}} \tilde{\mathcal{H}} \xleftarrow{\text{out}} L^2(S^1)$$

$$z^n \longmapsto U^n u_i \quad , \quad U^n u_{-i} \longleftarrow z^n$$

and I'd like to understand when these are isos. and what the scattering operator is.

When $T^n \to 0$ i.e. $\mathcal{H}$ finite-dimensional I have available the operator $(1-zT^*)^{-1}$ for z in S^1. TT^* is the projection on R_V so $1-TT^* : h \longmapsto (h, u_{-i})u_{-i}$. Thus if we define $i : \mathcal{H} \longrightarrow L^2(S^1)$ by

$$i(h) = ((1-zT^*)^{-1}h, u_{-i})$$

then i induces an isom: $i_\# : \tilde{\mathcal{H}} \longrightarrow L^2(S^1)$ sending $\overline{\bigoplus_{n\geq 0} U^n \mathcal{H}}$ to $H^2(S^1)$. It follows that $i_\#$ is the inverse of the "out" map above. So so we get

$$\begin{array}{ccccc}
 & ((1-z^{-1}T)^{-1}h, u_i) \leftarrow\!\shortmid\, h & \mathcal{H} & h \mapsto ((1-zT^*)^{-1}h, u_{-i}) & \\
 & \swarrow & \downarrow & \searrow & \\
L^2(S^1) & \xleftarrow{\ \sim\ } & \tilde{\mathcal{H}} & \xrightarrow{\ \sim\ } & L^2(S^1)
\end{array}$$

$$z^n \longmapsto U^n u_i \quad , \quad U^n u_{-i} \longleftarrow z^n$$

To get the scattering operator, we [redacted] take $h = u_i$ whence

$$((1-z^{-1}T)^{-1}u_i, u_i) \longleftarrow\!\shortmid\; \mathcal{U}_i \longmapsto ((1-zT^*)^{-1}u_i, u_{-i})$$

$$((1-z^{-1}T)^{-1}u_i, u_i) = (u_i, u_i) = 1$$

hence

$$\boxed{S(z) = ((1-zT^*)^{-1}u_i,\ u_{-i})}$$

Note this is an analytic function in z with the property

$$S(\mathcal{U})\, u_{-i} = u_i .$$

So far we have been assuming $T^n \to 0$, i.e. [redacted] $\mathcal{H}$ finite-dimensional. What happens when this restriction is relaxed?

First point is that $(1-zT^*)^{-1}u_i$ is analytic for $|z|<1$ in general and possibly analytically continues to points of S^1. We have

$$\left(\left(1-\bar{z}V\right)\mathcal{D}_V, (1-zT^*)^{-1}u_i\right) = \left((1-\bar{z}T)\mathcal{D}_V, (1-zT^*)^{-1}u_i\right)$$

$$= (\mathcal{D}_V, u_i) = 0$$

Also $\quad ((1-zT^*)^{-1}u_i, u_i) = (u_i, (1-\bar{z}T)^{-1}u_i) = (u_i, u_i) = 1.$
Thus we can describe $(1-zT^*)^{-1}u_i$ as the unique element of $\mathcal{H}$ perp. to $(1-\bar{z}V)\mathcal{D}_V$ whose inner product with u_i is 1. Note that this [redacted] description depends on V alone, and might be useful for ~~other~~ T extending V.

There's no relation between invariant subspaces for T and the chain of invariant subspaces we want to find for

V. In effect one can have [redacted] non-nilpotent T [redacted] (the finite diml. T correspond to polys. with roots inside S^1 with one root 0) and then the invariant subspaces for T don't form a chain.