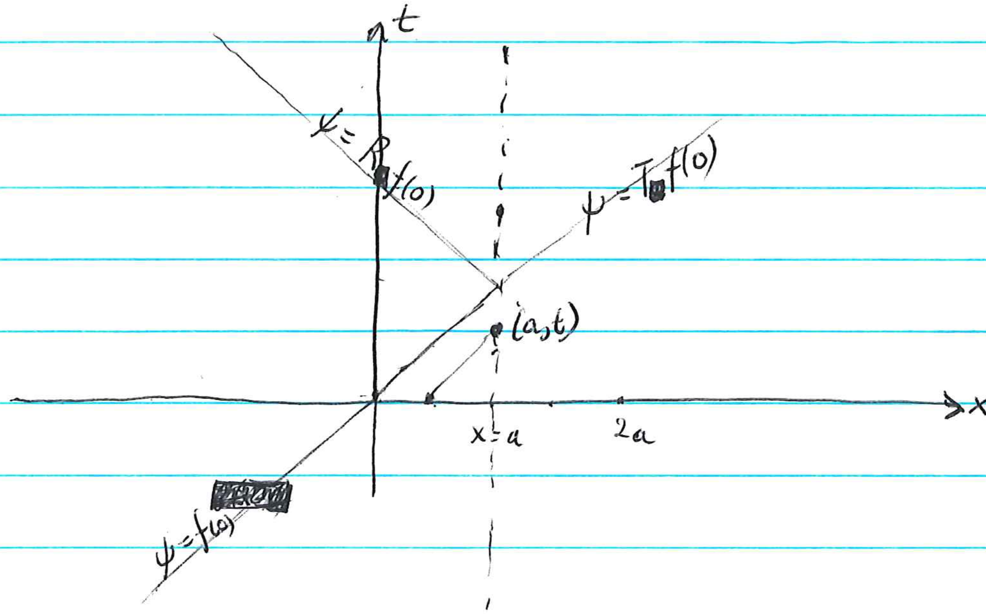
March 5, 1978:

Suppose we have an interface at $x = a$. Then an incoming wave $f(x-t)$ from the left splits into [redacted] reflected and transmitted waves



The transmitted wave is T[redacted] $f(x-t)$. The reflected wave is of the form $g(x+t)$ where

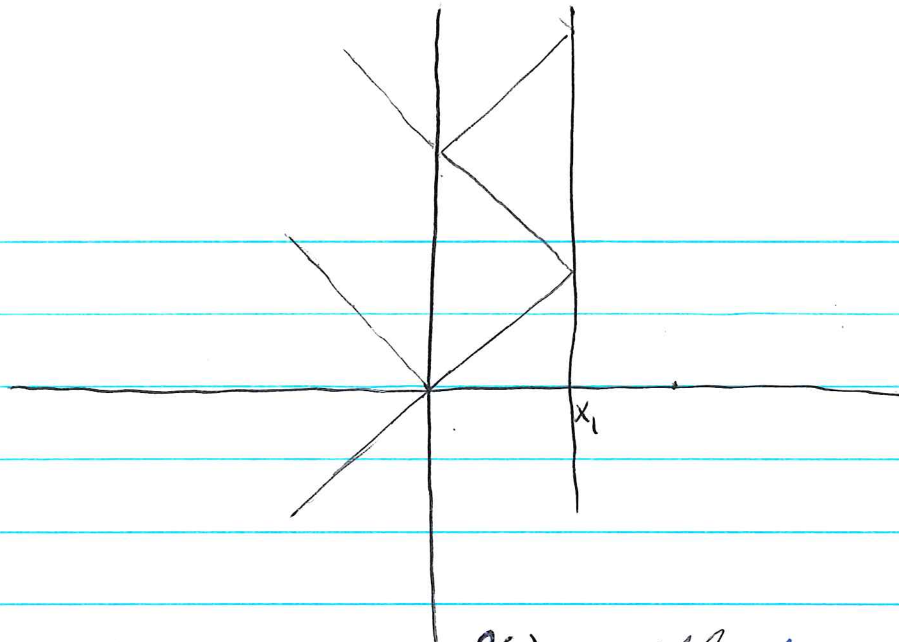
$$g(a+t) = R_{[redacted]} f(a-t)$$

[redacted] hence the reflected wave is

$$g(x+t) = R_{[redacted]} f(2a-x-t)$$

Hence if we use the $t=0$ descriptions we find that the incoming wave $f(x)$ splits into the transmitted wave T[redacted]$f(x)$ and the reflected wave R[redacted]$f(2a-x)$.

Next suppose we have interfaces at $x = x_0 = 0$ and $x = x_1$. Then we have multiple reflections.



An incoming wave $f(x)$ will give rise to the reflected wave

$$R_0 f(-x) + T_0 R_1 T_0 f(2x_1 - x) + T_0 R_1 \tilde{R}_0 R_1 T_0 f(4x_1 - x) + \cdots$$

Let

$$f(x) = \int e^{i\lambda x} a(\lambda) \frac{d\lambda}{2\pi}$$

so that

$$f(x-t) = \int e^{-i\lambda t} e^{i\lambda x} a(\lambda) \frac{d\lambda}{2\pi}$$

The reflected wave $(Rf)(-x-t)$ is defined so that

$$\psi(x,t) = \begin{pmatrix} f(x-t) \\ Rf(-x-t) \end{pmatrix} = \int e^{-i\lambda t} \begin{pmatrix} e^{i\lambda x} a(\lambda) \\ e^{-i\lambda x} R(\lambda) a(\lambda) \end{pmatrix} \frac{d\lambda}{2\pi}$$

is a solution for $x < 0$. So we have

$$(Rf)(-x) = R_0 f(-x) + T_0 R_1 T_0 f(2x_1 - x) + \cdots$$

or

$$Rf(x) = R_0 f(x) + T_0 R_1 T_0 f(2x_1 + x) + \cdots\cdots$$

hence

$$R(\lambda) = R_0 + T_0 R_1 T_0 e^{2i\lambda x_1} + T_0 R_1 \tilde{R}_0 R_1 T_0 e^{4i\lambda x_1} + \cdots$$

$$= R_0 + \frac{T_0^2 R_1 e^{2i\lambda x_1}}{1-\tilde{R}_0 R_1 e^{2i\lambda x_1}} = \frac{R_0 + (-R_0\tilde{R}_0 + T_0^2) R_1 e^{2i\lambda x_1}}{1-\tilde{R}_0 R_1 e^{2i\lambda x}}$$

$$= \frac{R_0 - (\det S_0) R_1 e^{2i\lambda x_1}}{1-\tilde{R}_0 R_1 e^{2i\lambda x_1}}$$

which is what we obtained before. [redacted]

Problem: Can one write down more or less explicitly an infinite[redacted] series for the reflection coefficient in the case of many interfaces.

Let us suppose that the interfaces are located at the points $0 = x_0 < x_1 < x_2 < \cdots\cdots$. The reflection coefficient will be a sum of all possible reflected paths. We can think of a path as a walk [redacted] through the points $0, 1, 2, \cdots\cdots$ which comes back to zero such as

$$0 \ 1 \ 2 \ 1 \ 2 \ 3 \ 2 \ 1 \ 0$$

Notice that the number of $i, i+1$ steps is the same as the number of $i+1, i$ steps. The length of the path is [redacted] the sum for $i = 0, 1, 2, \ldots$ of twice [redacted] $x_{i+1} - x_i$ for each $(i, i+1)$ step in the path. The reflection coefficient will be an infinite sum over paths

$$\sum_{\gamma} c(\gamma) e^{i\ell(\gamma)\lambda}$$

where $\ell(\gamma)$ is the length of the path and where $c(\gamma)$ is a product of reflection and [redacted] transmission coefficients at vertices along the path.

What possible lengths can occur [redacted] assuming that [redacted] the distances $x_{i+1} - x_i$ are independent over $\mathbb{Q}$? Suppose that we have a path with n_1 0-1 steps n_2 1-2 steps etc., are there any restrictions on $n_1, n_2, \ldots$ Clearly if $n_i > 0$, then so must be $n_1, n_2, \ldots, n_{i-1}$. Conversely suppose given $n_1, n_2, \ldots, n_i$ all ≥ 1. It can be realized by a path as follows. First go [redacted] back and forth between 0, 1 so as to use up n_1, then use up n_2, etc. For example suppose $(n_1, n_2, n_3) = (3, 2, 1)$. Then we have the path

$$0\ 1\ 0\ 1\ 0\ 1\ 2\ 1\ 2\ 3\ 2\ 1\ 0$$

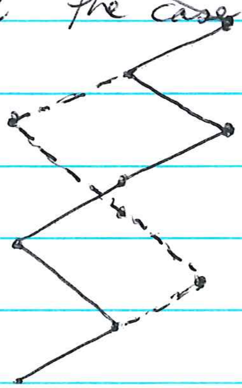
n_1 times $\quad n_2$ times $\quad n_3$ times

So the possible lengths for paths are

$$2\{n_1(x_1 - x_0) + n_2(x_2 - x_1) + \cdots + n_i(x_{i+1} - x_i)\}$$

where $n_1, n_2, \ldots, n_i$ [redacted] are positive integers.

It would be nice if all paths γ with the same multiplicities have the same coefficient $c(\gamma)$. But this isn't the case for the paths



$$0\ 1\ 0\ 1\ 2\ 1\ 2\ 1\ 0$$
$$0\ 1\ 2\ 1\ 0\ 1\ 2\ 1\ 0$$

which have coeffs.

$$T_0(R_1 \tilde{R}_0 T_1 R_2 \tilde{R}_1) R_2 \tilde{T}_1 \tilde{T}_0$$
$$T_0(T_1 R_2 \tilde{T}_1 \tilde{R}_0 T_1) R_2 \tilde{T}_1 \tilde{T}_0$$

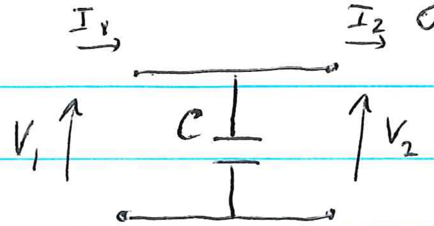
The inner groups differ by the factors

$$\begin{array}{ccc} R_1\tilde{R}_1 & \text{versus} & T_1\tilde{T}_1 \\ \parallel & & \parallel \\ -\dfrac{\tilde{\beta}_1\beta_1}{\alpha_1^2} & & \dfrac{1}{\alpha_1^2} \end{array}$$

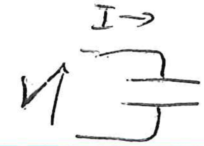
So these are different.

March 11, 1978 (Fields Medal letter - March 6)

Review electricity:



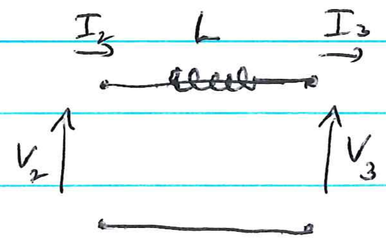
For a capacitance



$$Q = CV$$

$$I = C\frac{dV}{dt} = Ci\omega V$$

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} V_2 \\ Ci\omega V_2 + I_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Ci\omega & 1 \end{pmatrix}\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$



for an inductance

$$V = L\frac{dI}{dt} = Li\omega I$$

$$\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} Li\omega I_3 + V_3 \\ I_3 \end{pmatrix} = \begin{pmatrix} 1 & Li\omega \\ 0 & 1 \end{pmatrix}\begin{pmatrix} V_3 \\ I_3 \end{pmatrix}$$

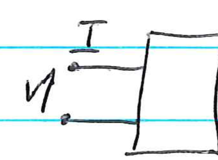
So for the shunt capacitance we have transfer matrix

$$\begin{pmatrix} iV_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C\omega & 1 \end{pmatrix}\begin{pmatrix} iV_2 \\ I_2 \end{pmatrix}$$

and for the series inductance we have

$$\begin{pmatrix} iV_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} 1 & -L\omega \\ 0 & 1 \end{pmatrix}\begin{pmatrix} iV_3 \\ I_3 \end{pmatrix}$$

Recall ~~Notice~~ that the power flow into a 1-port



is $Re(V\bar{I})$, (up to $\frac{1}{2}$ factor)

$$\frac{1}{2i}\begin{pmatrix} iV \\ I \end{pmatrix}^* P \begin{pmatrix} iV \\ I \end{pmatrix} = Im\left(\frac{iV}{I}\right)\cdot |I|^2 = Re(V\bar{I})$$

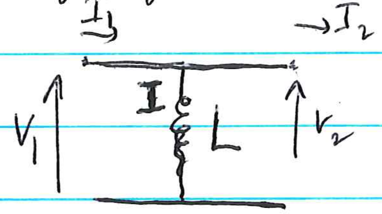
Convention: Time-dependence is denoted $e^{i\omega t}$, hence frequencies ω with $\mathrm{Im}\,\omega < 0$ constitute the physically significant ones (experiments involve voltages vanishing for $t \ll 0$, and $e^{i\omega t}$ for $\mathrm{Im}\,\omega < 0$ can be approximated by these).

So for $\mathrm{Im}\,\omega < 0$ a 2-port must use up power, hence

$$\mathrm{Im}(iV_1\bar{I}_1) - \mathrm{Im}(iV_2\bar{I}_2) \geq 0$$

consequently the transfer matrix shrinks the UHP in the physical domain $\mathrm{Im}(\omega) < 0$.

Note for future research: The 2-port



$V_2 = Li\omega I$

with transfer matrix $\begin{pmatrix} iV_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{Li\omega} & 1 \end{pmatrix} \begin{pmatrix} iV_2 \\ I_2 \end{pmatrix}$

This transfer matrix has entries in $\mathbb{C}[\omega, \frac{1}{\omega}]$, so it appears that there might exist a theory of Nevanlinna matrices whose entries would be analytic off 0.

March 12, 1978:

Let A be a closed symmetric operator in a Hilbert space $\mathcal{H}$. Let the elements of $\mathcal{H}\oplus\mathcal{H} =$ [redacted] $\mathcal{H}^{\oplus 2}$ be denoted as column vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. For A to be closed means its graph $\Gamma_A = \left\{ \begin{pmatrix} x \\ Ax \end{pmatrix} \,\middle|\, x \in \mathcal{D}_A \right\}$ is a closed subspace, and for A to be symmetric means that

$$\begin{pmatrix} y \\ Ay \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ Ax \end{pmatrix} = \begin{pmatrix} y \\ Ay \end{pmatrix}^* \cdot \begin{pmatrix} -Ax \\ x \end{pmatrix} = \text{[redacted]}\ (x, Ay) - (Ax, y) = 0$$

for all $x, y \in \mathcal{D}_A$, in other words that the subspaces Γ_A and $P\Gamma_A$ are orthogonal where $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is equivalent to require that Γ_A be isotropic for the hermitian form

$$X, Y \mapsto \frac{1}{i} Y^* P X$$

on $\mathcal{H}^{\oplus 2}$.

Suppose A densely-defined. Then [redacted] the orthogonal complement [redacted] $(P\Gamma_A)^\perp$ of $P\Gamma_A$ in $\mathcal{H}^{\oplus 2}$ is of the form Γ_{A^*}, where A^* is an operator called the adjoint of A. Since A is symmetric

$$\Gamma_{A^*} = (P\Gamma_A)^\perp \supset \Gamma_A$$

Since $P^2 = -$ identity the subspace $\Gamma_A \oplus P\Gamma_A$ is P stable, hence so is its orthogonal complement

$$(\Gamma_A \oplus P\Gamma_A)^\perp = \Gamma_A^\perp \cap (P\Gamma_A)^\perp = \Gamma_A^\perp \cap \Gamma_{A^*}$$

which we can decompose into $+i$ and $-i$ eigenspaces

$$P \begin{pmatrix} x \\ A^*x \end{pmatrix} = \text{[redacted]} \begin{pmatrix} -A^*x \\ x \end{pmatrix} = i \begin{pmatrix} x \\ A^*x \end{pmatrix} \iff A^*x = -ix$$

As usual equip $\mathcal{D}_{A^*}$ with the norm $\|x\|^2 + \|A^*x\|^2$ so that $\mathcal{D}_{A^*} \xrightarrow{\sim} \Gamma_{A^*}$, $x \longmapsto \binom{x}{A^*x}$ is a unitary isom.

Let $\mathcal{N}_\lambda = \mathrm{Ker}\{A^* - \lambda : \mathcal{D}_{A^*} \to \mathcal{H}\}$. Then what we have produced is an orthogonal decomposition

$$\Gamma_{A^*} = \Gamma_A \oplus (\Gamma_A^\perp \cap \Gamma_{A^*})$$

$$= \Gamma_A \oplus \underbrace{\left\{\binom{x}{-ix} \,\middle|\, x \in \mathcal{N}_{-i}\right\}}_{\substack{+i \text{ eigenspace} \\ \text{for } P}} \oplus \underbrace{\left\{\binom{x}{ix} \,\middle|\, x \in \mathcal{N}_i\right\}}_{\substack{-i \text{ eigenspace} \\ \text{for } P}}$$

This can be written as an orthogonal decomposition

$$\mathcal{D}_{A^*} = \mathcal{D}_A \oplus \mathcal{N}_{-i} \oplus \mathcal{N}_i$$

We next want to understand how $\mathcal{N}_\lambda$ sits inside $\mathcal{D}_{A^*}$ with respect to this decomposition.

<u>March 14, 1978:</u>

A closed symmetric densely-defined operator with deficiency indices $(1,1)$. Put

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and consider the skew-hermitian form

$$P(x,y) = \left(J\binom{x}{A^*x}, \binom{y}{A^*y} \right) = (A^*x, y) - (x, A^*y)$$

on $\mathcal{D}_{A^*}$. It's kernel is $\mathcal{D}_A$, so P is a "power form" on the 2-dim'l space $\mathcal{D}_{A^*}/\mathcal{D}_A \xrightarrow{\sim} \mathcal{N}_i \oplus \mathcal{N}_{-i}$.

Notice that $\mathcal{N}_\lambda = \mathrm{Ker}(A^* - \lambda : \mathcal{D}_A \to \mathcal{H})$ gives us a line in $\mathcal{D}_{A^*}$ disjoint from $\mathcal{D}_A$ when λ is non-real. If $u_\lambda \in \mathcal{N}_\lambda$, then

$$\frac{1}{2i} P(u_\lambda, u_\lambda) = \frac{1}{2i}(\lambda - \bar{\lambda}) \cdot |u_\lambda|^2 = (\mathrm{Im}\,\lambda)|u_\lambda|^2$$

so that $(\mathrm{Im}\,\lambda) > 0$ means positive power flow.

Let's review why $\mathcal{N}_\lambda$ is 1-diml in general. We know because A is closed that $(A \pm i) : \mathcal{D}_A \longrightarrow \mathcal{H}$ (which is isometric because A is symmetric) has closed image. $((A-\lambda)\mathcal{D}_A)^\perp = \mathcal{N}_{\bar{\lambda}}$. So we have an unitary isom.

$$T : \mathcal{D}_A \oplus \mathcal{N}_{-i} \xrightarrow{\sim} \mathcal{H}$$

$$x \oplus y \longmapsto (A-i)x + y$$

On the other hand we we have the map

$$K : \mathcal{D}_A \oplus \mathcal{N}_{-i} \longrightarrow \mathcal{H} \qquad K(x \oplus 0) = (A+i)x$$

of norm ≤ 1. Hence for any u $|z| < 1$ we know $T + zK$ is an isomorphism varying analytically in z.

$$\begin{aligned}(T+K)(x \oplus y) &= y + (A-i)x + z(A+i)x \\ &= y + (1+z)Ax - i(1-z)x \\ &= y + (1+z)\left\{A - \frac{1}{i}\frac{z-1}{z+1}\right\}x\end{aligned}$$

So we see that $\mathcal{N}_{-i}$ is complementary to the subspace

$(A-\lambda)\mathcal{D}_A$

where $\lambda = \frac{1}{i}\frac{z-1}{z+1}$. As z ranges over the disk λ ranges over the UHP.

$$z = \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}^{-1}(\lambda) = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}(\lambda) = \frac{i\lambda+1}{-i\lambda+1} = \frac{\lambda-i}{\lambda+i}$$

Notice that because $T+zK$ is an isomorphism an element x of $\mathcal{H}$ can be expressed uniquely in the form

$$x = (A-\lambda)\,\alpha(\lambda, x) + \hat{x}(\lambda)\,u_{-i}$$

with $\alpha(\lambda, x) \in \mathcal{D}_A$, $\hat{x}(\lambda) \in \mathbb{C}$, and where u_{-i} generates $\mathcal{N}_{-i}$. Moreover this expression is analytic in λ so that $\hat{x}(\lambda)$ is an analytic function in the UHP. If $u_{\bar\lambda}$ generates $\mathcal{N}_{+\bar\lambda} = \ker(A^* - \bar\lambda) = ((A-\lambda)\mathcal{D}_A)^\perp$, then we have

$$(x, u_{\bar\lambda}) = \hat{x}(\lambda)(u_{-i}, u_{\bar\lambda})$$

or

$$\hat{x}(\lambda) = \frac{(x, u_{\bar\lambda})}{(u_{-i}, u_{\bar\lambda})}$$

So we have a natural map of $\mathcal{H}$ to analytic functions in the UHP such that u_{-i} goes to 1. The problem is now to see if the norm ~~on~~ $\mathcal{H}$ leads to a norm on these analytic functions

March 16, 1978

To understand structure of closed symmetric operators with deficiency indices (1,1).

First suppose we look at the case where the indices are (1,0). This means that

$$(A+i) : \mathcal{D}_A \xrightarrow{\sim} \mathcal{H}$$

and that $(A-i) : \mathcal{D}_A \hookrightarrow \mathcal{H}$ is an isometric embedding with cokernel one-dimensional. Thus $V = (A-i)(A+i)^{-1}$ is an isometric embedding with one-diml. cokernel. If $u_{-i} \in \mathcal{N}_{-i} = \mathrm{Ker}(A^*+i) = \left((A-i)\mathcal{D}_A\right)^\perp$ is a unit vector, then we know that $\mathcal{H}$ splits as a direct sum

$$\mathcal{H} = \left[\mathbb{C}u_{-i} \oplus \mathbb{C}Vu_{-i} \oplus \cdots\cdots \right] \oplus \bigcap_{n\geq 0} V^n\mathcal{H}.$$

The first factor is isomorphic to the space of [redacted] analytic functions $\sum_{n\geq 0} a_n z^n$ in the disk with $\sum |a_n|^2 < \infty$ with $V =$ multiplication by z; V is unitary on the second factor. Call the Hilbert space of analytic functions $H^2(\frac{d\theta}{2\pi})$.

To each $x \in \mathcal{H}$ we have defined an analytic function $\hat{x}(\lambda)$ on the UHP by

$$x \equiv \hat{x}(\lambda)\, u_{-i} \qquad \mathrm{mod}\ (A-\lambda)\mathcal{D}_A$$

or

$$\hat{x}(\lambda) = \frac{(x, \phi_{\bar\lambda})}{(u_{-i}, \phi_{\bar\lambda})}$$

where $\phi_{\bar\lambda}$ spans $\mathcal{N}_{\bar\lambda}$. Note that if we arrange $(u_{-i}, \phi_{\bar\lambda}) = 1$, then we have $\hat{x}(\lambda) = (x, \phi_{\bar\lambda})$. If $x \in \mathcal{D}_A$

$$\widehat{Ax}(\lambda) = (Ax, \phi_{\bar\lambda}) = (x, A^*\phi_{\bar\lambda}) = \lambda(x, \phi_{\bar\lambda}) = \lambda\hat{x}(\lambda)$$

and similarly $\widehat{Vx}(\lambda) = \dfrac{\lambda - i}{\lambda + i}\,\hat{x}(\lambda)$.

Therefore we get an isomorphism of $\mathcal{H}/\bigcap_{n\geq 0} V^n\mathcal{H}$ with the Hilbert space of analytic functions in the UHP in which one has [redacted] the orthonormal basis

$$\left(\frac{\lambda - i}{\lambda + i}\right)^n \qquad n \geq 0.$$

This is just the space $\mathcal{H}^2\left(\frac{d\theta}{2\pi}\right)$ moved to the UHP by the Cayley transform: $z = \dfrac{\lambda - i}{\lambda + i}$

$$\frac{d\theta}{2\pi} = \frac{dz}{2\pi i z} = \frac{1}{2\pi i}\left\{\frac{d\lambda}{\lambda - i} - \frac{d\lambda}{\lambda + i}\right\} = \frac{1}{\pi}\,\frac{d\lambda}{\lambda^2 + 1}.$$

Denote this new Hilbert space

$$\mathcal{H}^2\left(\frac{d\lambda}{\pi(\lambda^2+1)}\right)$$

Summarizing we get:

<u>Prop:</u> Let A be a closed densely-defined symmetric operator with deficiency indices $(1,0)$. [redacted] Let u_{-i} be a unit vector in $\mathcal{N}_{-i} = \left((A-i)\mathcal{D}_A\right)^\perp$ and let $\phi_{\bar\lambda}$ span $\mathcal{N}_{\bar\lambda} = \left((A-\lambda)\mathcal{D}_A\right)^\perp$ for any $\lambda \in$ UHP. Then we get [redacted] a projection

$$\mathcal{H} \longrightarrow \mathcal{H}^2\left(\frac{d\lambda}{\pi(\lambda^2+1)}\right)$$

$$x \longmapsto \hat{x}(\lambda) = \frac{(x, \phi_{\bar\lambda})}{(u_{-i}, \phi_{\bar\lambda})}$$

on whose kernel A is self-adjoint.

The notation might be simpler if we worked exclusively with the disk. Thus suppose we are given an isometric embedding $V: \mathcal{H} \to \mathcal{H}$ whose cokernel has dim 1, and let u_0 denote a unit vector orthogonal to $V\mathcal{H}$. Then by looking at

$$\mathcal{H} \oplus \mathbb{C} \xrightarrow{V + \cdot u_0} \mathcal{H}$$

which is unitary, we see for $|z| < 1$ that $(V-z)\mathcal{H}$ is complementary to $\mathbb{C}u_0 = \mathcal{N}_0$. So we we put $\mathcal{N}_{\bar{z}} = (V-z)\mathcal{H}^{\perp}$, this is 1-dimensional. We can define $\hat{x}(z)$ by

$$x \equiv \hat{x}(z) u_0 \quad \mathrm{mod}((V-z)\mathcal{H})$$

or

$$(x, \phi_{\bar{z}}) = \hat{x}(z)(u_0, \phi_{\bar{z}})$$

$$\hat{x}(z) = \frac{(x, \phi_{\bar{z}})}{(u_0, \phi_{\bar{z}})}$$

where $\phi_{\bar{z}}$ spans $\mathcal{N}_{\bar{z}}$. Thus as above we find

$$(*) \qquad \begin{aligned} \mathcal{H} &\longrightarrow\!\!\!\!\rightarrow \mathcal{H}^2\left(\tfrac{d\theta}{2\pi}\right) \\ x &\longmapsto \hat{x}(z) \end{aligned}$$

Note that if $\mathcal{H} = \mathcal{H}^2\left(\frac{d\theta}{2\pi}\right)$ with $V =$ mult. by ζ, then from

$$f(\zeta) = (\zeta - z) g + \hat{f}(z) \underbrace{u_0}_{=1}$$

we see that $\hat{f} = f$.

We want the adjoint to $(*)$. It obviously sends $f(\zeta)$ holomorphic on disk to $f(V)u_0$. It is given by the Cauchy formula

$$f(V)u_0 = \frac{1}{2\pi i} \oint f(z)(z-V)^{-1} u_0 \, dz$$

where the integral is taken over the circle of radius $1+\varepsilon$.

Suppose now that V is an isometry in $\mathcal{H}$ with deficiency indices $(1,1)$. Let u_0 be a unit vector orthogonal to $V\mathcal{D}_V$, and choose a unitary extension U of V, that is a unit vector $U^{-1}u_0$ orthogonal to $\mathcal{D}_V$. Then u_0 and U determine a prob. measure $d\mu$ on S^1 such that the U-stable space spanned by u_0 is isomorphic to $L^2(S^1, d\mu)$.

Let us therefore suppose $\mathcal{H} = L^2(S^1, d\mu)$ and that V is the isometry given by multiplying by $\zeta = e^{i\theta}$ with range

$$R_V = \{ f \in \mathcal{H} \mid (f, u_0) = \int f \, d\mu = 0 \}$$

and

$$\mathcal{D}_V = \{ f \in \mathcal{H} \mid \zeta f \in R_V, \text{ i.e. } \int \zeta f \, d\mu = 0 \}.$$

We have seen that $(V - z) : \mathcal{D}_V \to \mathcal{H}$ for $|z| < 1$ is an embedding with closed range of codim 1; let $\mathcal{N}_{\bar{z}}$ denote the orthogonal complement and let $\phi_{\bar{z}}$ span $\mathcal{N}_{\bar{z}}$. For each $x \in \mathcal{H}$ we can define a function $\hat{x}(z)$ for $|z| < 1$ by

$$x \equiv \hat{x}(z) \, u_0 \mod (V - z)\mathcal{D}_V$$

namely

$$\hat{x}(z) = \frac{(x, \phi_{\bar{z}})}{(u_0, \phi_{\bar{z}})}$$

This function is analytic in the disk, and our task now is to understand what it is for $\mathcal{H} = L^2(S^1, d\mu)$.

If $x(\zeta)$ is a Laurent poly function on S^1, then $\hat{x}(z)$ is that complex number such that

$$\frac{x(\zeta) - \hat{x}(z)}{\zeta - z}$$

belongs to $\mathcal{D}_V$ i.e. is such that

$$\int \xi \, \frac{x(\xi) - \hat{x}(z)}{\xi - z} \, d\mu = 0$$

Recall

$$\int_{S^1} \xi \frac{f(\xi)}{\xi - z} \frac{d\theta}{2\pi} = \frac{1}{2\pi i} \oint \frac{f(\xi)\, d\xi}{\xi - z}$$

$$\|$$

$$\int f(\xi) \frac{1}{1 - \xi^{-1} z} \frac{d\theta}{2\pi} = \int f(\xi) \sum_{n \geq 0} \xi^{-n} z^n \frac{d\theta}{2\pi} = \sum_{n \geq 0} a_n z^n$$

where $\displaystyle a_n = \int f(\xi)\, \xi^{-n} \frac{d\theta}{2\pi} = \frac{1}{2\pi i} \int f(\xi) \frac{d\xi}{\xi^{n+1}}$

hence

$$\frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - z} \, d\xi$$

is the "analytic half" of f. This means you write f as the sum of an analytic f^+ inside S^1 and f^- analytic outside S^1 vanishing at ∞. Then f^+ is the analytic-half.

If $\displaystyle d\mu = \rho(\xi) \frac{d\theta}{2\pi}$ then

$$(x\rho)^+(z) = \int \frac{\xi x \rho}{\xi - z} \frac{d\theta}{2\pi} = \hat{x}(z) \int \xi \frac{\rho}{\xi - z} \frac{d\theta}{2\pi} = \hat{x}(z)\, \rho^+(z)$$

or

$$\hat{x}(z) = \frac{(x\rho)^+}{\rho^+}(z)$$

Note that $\displaystyle \mathrm{Re}\left(\frac{1}{1 - \xi^{-1} z}\right) \geq \frac{1}{2}$ for $|z| < 1$, $|\xi| = 1$

$|1 - \xi^{-1} z|^2$

hence $\rho^+ = \int \frac{d\mu}{1-\zeta^{-1}z}$ satisfies $\operatorname{Re} \rho^+(z) \geq \int \frac{1}{2} d\mu = \frac{1}{2}$

and so $\rho^+(z)$ doesn't vanish for $|z|<1$.

March 17, 1978:

Let A be closed symmetric densely-defined with deficiency indices $(1,1)$. Choose unit vectors

$$u_i \in \mathcal{N}_i = \left((A+i)\mathcal{D}_A\right)^\perp \qquad u_{-i} \in \mathcal{N}_{-i} = \left((A-i)\mathcal{D}_A\right)^\perp$$

and let $\tilde{A}$ be the self-adjoint extension whose Cayley transform $U = \frac{\tilde{A}-i}{\tilde{A}+i}$ sends u_i to u_{-i}

$$\begin{array}{ccccc} V = \frac{A-i}{A+i} : & (A+i)\mathcal{D}_A & \longrightarrow & (A-i)\mathcal{D}_A \\ & \oplus & U & \oplus \\ & \mathcal{N}_i & \xrightarrow[u_i \mapsto u_{-i}]{} & \mathcal{N}_{-i} \end{array}$$

Suppose $\mathcal{H}$ generated by u_i and the operator $\tilde{A}$, whence we can realize $\mathcal{H}$ as $L^2(\mathbb{R}, d\mu)$ and $\tilde{A}$ as multiplication by λ once we fix a suitable cyclic vector (which might be outside L^2) since

$$\frac{\tilde{A}-i}{\tilde{A}+i}(u_i) = u_{-i}$$

we might try representing u_i by the function $\frac{1}{\lambda-i}$ and u_{-i} by $\frac{1}{\lambda+i}$. If we do this, then

$$\mathcal{D}_A = \left\{ f \;\middle|\; \int (\lambda^2+1)|f|^2 d\mu < \infty, \ \int (\lambda-i) f \, \underbrace{\overline{\frac{1}{\lambda+i}}}_{u_{-i}} d\mu = 0 \right\}$$

Thus $\mathcal{D}_A = \{ f \in L^2(\mathbb{R}, d\mu) \mid \lambda f \in L^2 \text{ and } \int f \, d\mu = 0 \}$

<u>Question</u>: Let $d\mu$ be a measure on $\mathbb{R}$ such that $\int \frac{d\mu}{\lambda^2+1} < \infty$, and let $\tilde{A}$ be the self-adjoint operator given by multiplying by λ. Is the functional $f \mapsto \int f \, d\mu$ defined on $\mathcal{D}_{\tilde{A}}$?

Yes, because $u_{-i} = \frac{1}{\lambda + i} \in L^2(d\mu)$ and

$$\int f \, d\mu = ((\lambda - i)f, u_{-i}) \qquad \text{i.e.}$$

$$\int f \, d\mu = \int (\lambda - i) f \cdot \frac{1}{\lambda - i} \, d\mu \leq \left(\int (\lambda^2+1)|f|^2 d\mu \right)^{1/2} \left(\int \frac{d\mu}{\lambda^2+1} \right)^{1/2}$$

So at this point I have a sort of canonical model for a closed densely-defined symmetric operator A with def. indices $(1,1)$, where a self-adjoint extension $\tilde{A}$ has been chosen, namely

$$\mathcal{H} = L^2(\mathbb{R}, d\mu) \qquad \text{where} \qquad \int \frac{d\mu}{x^2+1} = 1$$

$$\tilde{A} = \text{mult. by } x$$

$$\mathcal{D}_A = \{ f \in \mathcal{H} \mid x f \in \mathcal{H} \text{ and } \int f \, d\mu = 0 \}$$

$$u_{-i} = \frac{1}{x+i} \qquad\qquad u_i = \frac{1}{x-i}$$

It remains to calculate the transforms of $f \in \mathcal{H}$. In the UHP:

$$f = (A - \lambda) g + \hat{f}(\lambda) u_{-i} \qquad\qquad g \in \mathcal{D}_A$$

or

$$\int \left\{ f(x) - \frac{\hat{f}(\lambda)}{x+i} \right\} \frac{1}{x - \lambda} \, d\mu = 0$$

or since
$$\frac{\lambda+i}{(x+i)(x-\lambda)} = \frac{1}{x-\lambda} - \frac{1}{x+i}$$
we have
$$\int \frac{f\,d\mu}{x-\lambda} = \frac{\hat{f}(\lambda)}{\lambda+i}\left[\int \frac{d\mu}{x-\lambda} - \int \frac{d\mu}{x+i}\right]$$

March 18, 1978:

Suppose A closed symmetric operator in $\mathcal{H}$, but not necessarily densely defined. $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathcal{H}^{\oplus 2}$. A closed means $\Gamma_A = \left\{ \begin{pmatrix} x \\ Ax \end{pmatrix} \,\middle|\, x \in \mathcal{D}_A \right\}$ closed in $\mathcal{H}^{\oplus 2}$, A-symmetric means
$$\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ Ax \end{pmatrix}, \begin{pmatrix} y \\ Ay \end{pmatrix} \right) = (Ax, y) - (x, Ay) = 0$$
i.e. that Γ_A is orthogonal to $J\Gamma_A$.

Identify $\mathcal{D}_A$ with Γ_A. Then
$$\mathcal{D}_A \xrightarrow{A \pm i} \mathcal{H}$$
are isometric hence have closed ranges. Let $\mathcal{N}_{\bar{\lambda}}$ = orthogonal complement of $(A-\lambda)\mathcal{D}_A$. We have a unitary isomorphism
$$T : \mathcal{D}_A \oplus \mathcal{N}_{-i} \xrightarrow{(A-i)+in} \mathcal{H}$$
and $K(x,u) = (A+i)x$ which has norm ≤ 1, hence $T + zK$ is an isomorphism for all $|z| < 1$. This shows that $(A-\lambda)\mathcal{D}_A$ is closed in $\mathcal{H}$ for all λ in UHP and that $(A-\lambda)\mathcal{D}_A$ is complementary to $\mathcal{N}_{-i}$, with analogous results in the LHP. Furthermore it should be true that $\mathcal{N}_\lambda$ varies holomorphically in λ.

Next from the fact that $\Gamma_A \oplus J\Gamma_A$ is J-stable we find $\Gamma_A^{\perp} \cap (J\Gamma_A)^{\perp}$ is J-stable hence it splits into $\pm i$ eigenspaces. Suppose [redacted] $\begin{pmatrix} x \\ ix \end{pmatrix}$ is an eigenvector with eigenvalues i in $(J\Gamma_A)^{\perp}$ i.e

$$(Ay, x) = (y, ix) \qquad \forall y \in \mathcal{D}_A$$

This holds $\iff$ $x \in ((A+i)$ [redacted] $\mathcal{D}_A)^{\perp} = \mathcal{N}_i$. So we get an orthogonal decomp.

$$\Gamma_A^{\perp} \cap (J\Gamma_A)^{\perp} \simeq \text{[redacted]}\ \mathcal{N}_i \oplus \mathcal{N}_{-i}$$

Suppose the deficiency indices are $(1,1)$ and choose unit vectors $u_i \in \mathcal{N}_i$, $u_{-i} \in \mathcal{N}_{-i}$. [redacted]

[redacted] Then we can define a self-adjoint extension $\tilde{A}$ of A by extending the Cayley transform $\frac{A-i}{A+i}$ to a unitary operator $U = \frac{\tilde{A}-i}{\tilde{A}+i}$ with $U(u_i) = u_{-i}$. This means that $\mathcal{D}_{\tilde{A}} = \mathcal{D}_A + \mathbb{C}v$ where v is an element of $\mathcal{D}_{A^*}$ such that

$$(A^*+i)v = u_i$$
$$(A^*-i)v = u_{-i}$$

[redacted]

It is easily seen that $v = \frac{u_i - u_{-i}}{2i}$. Thus

$$\Gamma_{\tilde{A}} = \Gamma_A + \mathbb{C}\begin{pmatrix} \frac{u_i - u_{-i}}{2i} \\ \\ \frac{u_i + u_{-i}}{2} \end{pmatrix}$$

Note
$$P\left(\frac{u_i - u_{-i}}{2i}, \frac{u_i - u_{-i}}{2i}\right) = P\left(\frac{u_i + u_{-i}}{2}, \frac{u_i + u_{-i}}{2}\right)$$

$$P\left(\frac{u_i + u_i}{2}, \frac{u_i - u_i}{2i}\right) = -1$$

So now we can define u_λ to be the unique element of $\mathcal{N}_\lambda$ such that

$$u_\lambda - \left(\frac{u_i + u_{-i}}{2} + m(\lambda) \frac{u_i - u_{-i}}{2i} \right) \in \mathcal{D}_A$$

for some constant $m(\lambda)$. Because $\mathcal{N}_\lambda \not\subset \mathcal{D}_{\bar{A}}$ for λ non-real and because the image of $\mathcal{N}_\lambda$ in $\mathcal{N}_i \oplus \mathcal{N}_{-i}$ varies holomorphically, it seems that $m(\lambda)$ is holomorphic for non-real λ, and also u_λ.

$$(\bar{z} - \lambda)(u_{\bar{z}}, u_{\bar{\lambda}}) = P(u_{\bar{z}}, u_{\bar{\lambda}}) = P\left(\frac{u_i + u_{-i}}{2} + m(\bar{z}) \frac{u_i - u_{-i}}{2i}, \frac{u_i + u_{-i}}{2} + m(\bar{\lambda}) \frac{u_i - u_{-i}}{2i} \right)$$

$$= -\overline{m(\bar{\lambda})} + m(\bar{z})$$

Setting $\bar{z} = \lambda$ we see that $\overline{m(\bar{\lambda})} = m(\lambda)$. Also

$$(u_{\bar{z}}, u_{\bar{\lambda}}) = \frac{m(\lambda) - m(\bar{z})}{\lambda - \bar{z}}$$

which gives $\dfrac{\operatorname{Im} m(\lambda)}{\operatorname{Im} \lambda} = \|u_{\bar{\lambda}}\|^2 > 0$.

Finally note that $m(i) = i$.

By Riesz-Herglotz we know there is a unique measure $d\mu$ on $\mathbb{R}$ and $p \geq 0$ with

$$m(\lambda) = p\lambda + \int \left\{ \frac{1}{x - \lambda} - \frac{x}{x^2 + 1} \right\} d\mu + \text{real constant}$$

Since $\operatorname{Re} \dfrac{1}{x - i} = \dfrac{x}{x^2 + 1}$ it follows from $m(i) = i$ that the real constant is zero and also

$$p + \int \frac{d\mu}{x^2 + 1} = 1.$$

Suppose $p = 0$, and $d\mu$ is a measure on $\mathbb{R}$ with

$$\int \frac{d\mu}{x^2+1} = 1$$

Then suppose $\mathcal{H} = L^2(\mathbb{R}, d\mu)$, $\tilde{A}$ = multiplication by x, and $\mathcal{D}_A = \{ f \in \mathcal{D}_{\tilde{A}} \mid \int f \, d\mu = 0 \}$. Here U = multiplication by $\frac{x-i}{x+i}$. Also for $f \in \mathcal{D}_A$

$$\left((A-i)f, \frac{1}{x+i} \right) = \int (x-i) f \overline{\frac{1}{x+i}} \, d\mu = \int f \, d\mu = 0$$

so that u_{-i} can be chosen to be $\frac{1}{x+i}$, whence $u_i = \frac{1}{x-i}$. Then from the fact that $u_\lambda = \frac{c}{x-\lambda}$ and from $u_\lambda - \frac{u_i + u_{-i}}{2} \in \mathcal{D}_{\tilde{A}}$ we see

$$u_\lambda = \frac{1}{x-\lambda}$$

hence

$$\frac{u_i + u_{-i}}{2} = \frac{1}{2}\left(\frac{1}{x-i} + \frac{1}{x+i} \right) = \frac{x}{x^2+1}$$

$$\frac{u_i - u_{-i}}{2i} = \frac{1}{x^2+1}$$

and from

$$\left(\frac{1}{x-\lambda} - \frac{x}{x^2+1} \right) - m(\lambda) \frac{1}{x^2+1} \in \mathcal{D}_A$$

we get

$$m(\lambda) = \int \left(\frac{1}{x-\lambda} - \frac{x}{x^2+1} \right) d\mu$$

as well as

$$(u_{\bar{z}}, u_{\bar{\lambda}}) = \int \frac{1}{x-\bar{z}} \frac{1}{x-\lambda} \, d\mu$$

$$= \int \frac{1}{\lambda - \bar{z}} \left[-\frac{1}{x-\bar{z}} + \frac{1}{x-\lambda} \right] d\mu = \frac{m(\lambda) - m(\bar{z})}{\lambda - \bar{z}}.$$