January 1, 1978

Review: Consider a D-system $Lu = P\frac{du}{dx} + Qu = \lambda u$ with $P = \begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix}$. [redacted]

$$\frac{1}{i}(Pu, u) = -|u_1|^2 + |u_2|^2 > 0$$ describes the disk $\left|\frac{u_1}{u_2}\right| < 1$ in $\mathbb{P}^1$.

since [redacted]

$$\frac{d}{dx}(Pu, u) = (Lu, u) - (u, Lu) = (\lambda - \bar{\lambda})|u|^2$$

we have

$$\frac{1}{i}(Pu,u)(\ell) = \frac{1}{i}(Pu,u)(0) + 2\,\mathrm{Im}\,\lambda \int_0^\ell |u|^2 dx$$

hence propagation from 0 to $\ell > 0$ shrinks the disk for $\mathrm{Im}(\lambda) > 0$.

Suppose given the boundary condition [redacted] $u_1 = u_2$ at 0, $u_1 = e^{i\theta} u_2$ at ℓ. Let $\varphi(x,\lambda)$, $\psi(x,\lambda)$ denote the solutions with

$$\varphi(0,\lambda) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \psi(\ell,\lambda) = \begin{pmatrix} e^{i\theta/2} \\ e^{i\theta/2} \end{pmatrix}$$

Calculation leads to the following formula for the Green's matrix

$$G(x,y,\lambda) = \begin{cases} \frac{i}{W}\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}\begin{pmatrix} \psi_2(y) & \psi_1(y) \end{pmatrix} & x < y \\ \frac{i}{W}\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}\begin{pmatrix} \phi_2(y) & \phi_1(y) \end{pmatrix} & x > y \end{cases}$$

$W = W(\phi, \psi)$

$(\lambda - L)G = \delta$ or $-PG\Big]_{x=y-}^{x=y+} = I$ or $G(y_+) - G(y_-) = -P^{-1} = P$

since $\psi_2(x,\lambda) = \overline{\psi_1(x,\bar\lambda)}$ this can be written

$$G(x,y,\lambda) = \begin{cases} \frac{i}{W}\ \varphi(x,\lambda)\,\psi(y,\bar\lambda)^* & x<y \\ \frac{i}{W}\ \psi(x,\lambda)\,\varphi(y,\bar\lambda)^* & x>y \end{cases}$$

Now

$$W = W(\varphi(\lambda),\psi(\lambda)) = \begin{vmatrix} \varphi_1(x,\lambda) & \psi_1(x,\lambda) \\ \varphi_2(x,\lambda) & \psi_2(x,\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \overline{\varphi_2(x,\bar\lambda)} & \psi_1(x,\lambda) \\ \overline{\varphi_1(x,\bar\lambda)} & \psi_2(x,\lambda) \end{vmatrix} = \frac{1}{i}\,\varphi(x,\bar\lambda)^* P\,\psi(x,\lambda)$$

$$= \frac{1}{i}\big(P\psi(\lambda),\varphi(\bar\lambda)\big)(x)$$

also $W = i\big(P\varphi(\lambda),\psi(\bar\lambda)\big)(x)$
is independent of x and it is an entire function of λ. Its zeroes are the eigenvalues of the self-adjoint problem defined by L and the given boundary values.

suppose λ_0 is an eigenvalue. $\lambda_0 = \bar\lambda_0$

$$W = W(\varphi(\lambda),\psi(\lambda)) = W(\varphi(\lambda),\psi(\lambda))(b) = W(\varphi(\lambda),\psi(\lambda_0))(b)$$

$$= i\big(P\varphi(\lambda),\psi(\bar\lambda_0)\big)(b)$$

$$= \underbrace{i\big(P\varphi(\lambda),\psi(\bar\lambda_0)\big)(0)}_{=0 \text{ because } \lambda_0 \text{ eigenvalue}} + i(\lambda-\bar\lambda_0)\int_0^b \big(\varphi(\lambda),\psi(\bar\lambda_0)\big)\,dx$$

so $$\lim_{\lambda\to\lambda_0} \frac{W}{\lambda-\lambda_0} = i\int_0^b \big(\varphi(\lambda_0),\psi(\bar\lambda_0)\big)\,dx$$

Hence

$$\lim_{\lambda\to\lambda_0} (\lambda-\lambda_0)\,G(x,y,\lambda) = \frac{\varphi(x,\lambda_0)\,\psi(y,\lambda_0)^*}{\big(\varphi(\lambda_0),\psi(\lambda_0)\big)_{[0,b]}} \qquad x<y$$

$$= \frac{\varphi(x,\lambda_0)\,\varphi(y,\lambda_0)^*}{\|\varphi(\lambda_0)\|^2_{[0,b]}}$$

and similarly for $x>y$.

Project: Rewrite the above using only the solution matrix of the [redacted] Dirac system.
The de Branges idea: Construct a Hilbert space of pairs of entire functions belonging to the solution matrix over the interval $0 \leq x \leq l$

Check details of the following: suppose given a matrix $K(\lambda, z)$, $(\lambda, z) \in \Omega \times \Omega$, where Ω is [redacted] a set. [redacted] Assume this matrix is hermitian and ≥ 0; [redacted] let $\mathcal{H}$ be the associated Hilbert, [redacted] that is, "the" Hilbert space with generators J_z [redacted] for $z \in \Omega$ such that

$$(J_z, J_\lambda) = K(\lambda, z).$$

[redacted] For each $h \in \mathcal{H}$ we can define a function on Ω by

$$h(z) = (h, J_z)$$

Note that $h(z)=0$ for all $z \in \Omega \Rightarrow h=0$.
[redacted] We have

$$|h(z)| \leq \|h\| \cdot \|J_z\| \qquad \text{where } \|J_z\|^2 = K(z,z)$$

Let S be a subset of Ω on which the function $z \mapsto K(z,z)$

is bounded. Given $h \in \mathcal{H}$ and $\varepsilon > 0$ we know that because the J_z are dense in $\mathcal{H}$ for $z \in \Omega$, there exists a finite linear combination $\sum_{i=1}^{n} c_i J_{z_i}$ within ε of h. Hence

$$\left| h(\lambda) - \sum_{i=1}^{n} c_i J_{z_i}(\lambda) \right| < \varepsilon \cdot K(\lambda, \lambda)$$

which means that the function $\lambda \mapsto h(\lambda)$ can be uniformly approximated on S by linear combinations of the functions $J_z(\lambda) = K(\lambda, z)$.

So now suppose Ω is an open subset of $\mathbb{C}$ and that for each z the function $J_z(\lambda) = K(\lambda, z)$ is holomorphic on Ω. Provided $K(z,z)$ is bounded on compact sets, for example, if it is continuous, then the above shows the function $\lambda \mapsto h(\lambda)$ can be uniformly approximated on compact subsets of Ω by holomorphic functions, hence $\lambda \mapsto h(\lambda)$ is holomorphic. Thus to each element of $\mathcal{H}$ is associated as holomorphic function on Ω which determines it. Hence we can identify $\mathcal{H}$ with a space of holomorphic functions on Ω.

<u>Example</u>: Consider $Lu = \left(-\frac{d^2}{dx^2} + q \right) u = \lambda u$ note: λ on $0 \le x \le l$ with boundary condition given at $x = 0$. In $L^2(0,l)$ we consider the elements $\varphi_{\bar z}$ for $z \in \mathbb{C}$. Then

$$(\varphi_{\bar z}, \varphi_{\bar \lambda})_{[0,l]} = \int_0^l \varphi(x, \bar z)\, \varphi(x, \lambda)\, dx = (\varphi_\lambda, \varphi_z)_{[0,l]}$$

$$(\lambda - \bar z)(\varphi_\lambda, \varphi_z)_{[0,l]} = \left\{ (L\varphi_\lambda, \varphi_z) - (\varphi_\lambda, L\varphi_z) \right\}_{[0,l]}$$

$$= \int_0^l \left\{ -\frac{d^2}{dx^2}\varphi_\lambda \cdot \varphi_{\bar z} + \varphi_\lambda \cdot \frac{d^2 \varphi_{\bar z}}{dx} \right\} dx = \left[-\frac{d}{dx}\varphi_\lambda \cdot \varphi_{\bar z} + \varphi_\lambda \frac{d\varphi_{\bar z}}{dx} \right]_0^l$$

$$= \begin{vmatrix} \varphi_\lambda & \varphi_{\bar z} \\ \varphi_\lambda' & \varphi_{\bar z}' \end{vmatrix} (\ell)$$

Thus $\quad (\varphi_\lambda, \varphi_z)_{[0,\ell]} = \dfrac{W(\varphi_\lambda, \varphi_{\bar z})(\ell)}{\lambda - \bar z}$

For example if $q = 0$ and $\varphi(x,\lambda) = \dfrac{\sin(\sqrt{\lambda}\, x)}{\sqrt{\lambda}}$ then

$$(\varphi_\lambda, \varphi_z)_{[0,\ell]} = \frac{1}{\lambda - \bar z} \begin{vmatrix} \frac{\sin\sqrt{\lambda}\,\ell}{\sqrt{\lambda}} & \frac{\sin\sqrt{\bar z}\,\ell}{\sqrt{\bar z}} \\ \cos\sqrt{\lambda}\,\ell & \cos\sqrt{\bar z}\,\ell \end{vmatrix}$$

Recall the formula

$$J_z(\lambda) = \frac{-1}{\lambda - \bar z} \begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z) \end{vmatrix} \qquad \text{for } d\mu = \frac{d\lambda}{\pi |E(\lambda)|^2}$$

for the de Branges space based on $E = A - iB$. Hence we get the deB space based on

$$E(\lambda) = \cos\sqrt{\lambda}\,\ell - i\,\frac{\sin\sqrt{\lambda}\,\ell}{\sqrt{\lambda}}$$

If on the other hand $\varphi(x,\lambda) = \cos\sqrt{\lambda}\,x$, then we get

$$E(\lambda) = \cos(\sqrt{\lambda}\,\ell) - i\sqrt{\lambda}\,\sin(\sqrt{\lambda}\,\ell)$$

[redacted] Back in March 9, 1977 we looked at [redacted] fractional linear transformations which shrink the upper half-plane. Especially we looked ~~at~~ systems

$$\frac{du}{dx} = A(\lambda)\,u \qquad\qquad A(\lambda) = \alpha + \lambda\beta$$

where the solution matrix preserves [redacted], expands, shrinks the UHP according as $\mathrm{Im}\,\lambda = 0, < 0, > 0$ respectively.

The basic calculation amounted to looking at matrices of the form

$$I + \varepsilon\lambda\beta \qquad \varepsilon^2 = 0 \qquad tr(\beta) = 0$$

with this property. So if $x \in \mathbb{R}$

$$(I + \varepsilon\lambda\beta)(x) = \frac{(1+\varepsilon\lambda\beta_{11})x + \varepsilon\lambda\beta_{12}}{\varepsilon\lambda\beta_{21}x + (1+\varepsilon\lambda\beta_{22})}$$

$$= \left[x + \varepsilon\lambda(\beta_{11}x + \beta_{12})\right]\left[1 - \varepsilon\lambda(\beta_{21}x + \beta_{22})\right]$$

$$= x + \varepsilon\lambda\left[\beta_{11}x + \beta_{12} - \beta_{21}x^2 - \beta_{22}x\right]$$

If $\operatorname{Im}\lambda > 0$ we want this to point into the UHP. Hence for any x real we want

$$\beta_{12} + \underbrace{(\beta_{11} - \beta_{22})}_{= 2\beta_{11}} x - \beta_{21}x^2 \geq 0$$

i.e. $\beta_{12} \geq 0$, $\beta_{21} \leq 0$, $\beta_{11}^2 \leq -\beta_{21}\beta_{12}$. Thus β has the form

$$\beta = \begin{pmatrix} p & q \\ -r & -p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} r & p \\ p & q \end{pmatrix}$$

where $r \geq 0$, $q \geq 0$, $p^2 \leq qr$, so the last matrix is ≥ 0.

Notice also that a real matrix α of trace 0 is of the form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & -\alpha_{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\alpha_{21} & +\alpha_{11} \\ \alpha_{11} & \alpha_{12} \end{pmatrix}$$

arbitrary real symmetric.

hence a system

$$\frac{du}{dx} = (\alpha + \lambda\beta)\, u$$

with requisite shrinking property is a self-adjoint system

$$P \frac{du}{dx} + Qu = \lambda R u$$

where $\quad P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad Q, R$ are real + symmetric,

so $\quad tr(P^{-1}Q) = tr(P^{-1}R) = 0$

hence the solution matrix is unimodular, and finally $R \geq 0$.

So in the class of Nevanlinna matrices is included solution matrices for SL systems

(*) $$\left(-\frac{d^2}{dx^2} + q\right) u = \lambda u$$

Rewrite this as

$$\frac{d}{dx}\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q-\lambda & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}$$

or

$$L\begin{pmatrix} u \\ u' \end{pmatrix} = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx} + \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} u \\ u' \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}$$

Hence if $\phi, \tilde{\phi}$ are the solutions of (*) corresponding to different boundary conditions at 0, (*) then

(*) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ resp.

$$S(\lambda) = \begin{pmatrix} \phi(x,\lambda) & \tilde{\phi}(x,\lambda) \\ \phi'(x,\lambda) & \tilde{\phi}'(x,\lambda) \end{pmatrix}$$

is a Nevanlinna matrix all $x > 0$.

January 2, 1978:

To fix the ideas consider a S-L system on $0 \le x \le \ell$

1) $$Lu = -\frac{d^2u}{dx^2} + qu = \lambda u$$

I can write it in the form

2) $$\frac{d}{dx}\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q-\lambda & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \lambda-q & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}$$

or $$P\frac{d\tilde{u}}{dx} + Q\tilde{u} = \lambda R\tilde{u} \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Notice that the Hilbert space associated to the matrix measure $R\,dx$ is just $L^2((0,\ell))$. Let $M(x,\lambda)$ be the solution matrix for the system 2) on $0 \le x \le \ell$. For each $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$ we have the solution $\varphi_a(x,\lambda) = M(x,\lambda)a$ and we can form the transform

$$\hat{f}_a(\lambda) = \int_0^\ell (Rf, \varphi_a(\bar{\lambda}))\,dx \quad = \quad \text{inner product of } f, \varphi_a(\bar{\lambda}) \text{ in } L^2[(0,\ell), Rdx].$$

Then $\hat{f} : a \mapsto \hat{f}_a$ is a homomorphism of $\mathbb{R}^2$ into entire functions associated to each f in $L^2([0,\ell], Rdx)$ which we identify with $L^2(0,\ell)$. So in this way we get a Hilbert space space consisting vector entire functions. The goal now is to understand the point-evaluator which characterizes this Hilbert space.

It's natural to work with entire functions with values in a vector-space. Hence we should think of $\hat{f}$

$$\hat{f}_a(\lambda) = a^* \int_0^\ell M(x,\bar{\lambda})^* Rf\,dx$$

~~as~~ being an entire function with values in the space of conjugation linear functionals on the space $\mathbb{C}^2$ of ~~initial~~ values at $x=0$. Since $\mathbb{C}^2$ comes equipped with an inner product

we can think of $\hat{f}(\lambda)$ as an entire function with values in the initial value space of the system.
Thus $\hat{f}(\lambda)$ is the column vector of entire functions

$$\hat{f}(\lambda) = \int_0^\ell \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix} f(x)\,dx \qquad \text{where} \qquad M(x,\lambda) = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{pmatrix}$$

So we know what the space of transforms consists of. We next have to describe the inner product. It should suffice to give the point evaluator. Represent the functional

$$\hat{f} \longmapsto \hat{f}_a(z) = (Rf, M(\bar{z})a)_{[0,\ell]} = (\hat{f}, \widehat{M(\bar{z})a})$$

$$\widehat{M(\bar{z})a}_b(\lambda) = \big(RM(\bar{z})a, M(\bar{\lambda})b\big)_{[0,\ell]}$$

Recall that $\quad (\bar{z}-\lambda)\big(RM(\bar{z})a, M(\bar{\lambda})b\big)_{[0,\ell]}$

$$= \Big\{\big(LM(\bar{z})a, M(\bar{\lambda})b\big) - \big(M(\bar{z})a, LM(\bar{\lambda})b\big)\Big\}_{[0,\ell]}$$

$$= \big(PM(\bar{z})a, M(\bar{\lambda})b\big)\Big]_0^\ell$$

$$= \big(b^* M(\bar{\lambda})^* P M(\bar{z})a\big)\Big]_0^\ell$$

Better approach. The Hilbert space of transforms consists of entire functions $\hat{f} : \mathbb{C} \longrightarrow \mathbb{C}^2$. For each $a \in \mathbb{C}^2$ and $z \in \mathbb{C}$ we have a linear functional

$$\hat{f} \longmapsto (\hat{f}(z), a)$$

which is representable by $J_{z,a}$ since

$$\left(\hat{f}, J_{z,a}\right) = \left(\hat{f}(z), a\right) = \left(Rf, M(\bar{z})a\right)_{[0,\ell]}$$

it follows that

$$J_{z,a} = \widehat{M(\bar{z})a}$$

hence

$$\left(J_{z,a}, J_{\lambda,b}\right) = \left(J_{z,a}(\lambda), b\right) = \left(\widehat{M(\bar{z})a}(\lambda), b\right)$$

$$= \left(RM(\bar{z})a, M(\bar{\lambda})b\right)_{[0,\ell]}$$

$$= \left[b^* M(\bar{\lambda})^* P M(\bar{z})a\right]_0^\ell \Big/ \bar{z}-\lambda$$

The formulas might be prettier if instead one wants the basic map to be

$$(z, a) \longmapsto \varphi_a(z)\text{[redacted]} = M(z)a$$

for then the inner products are

inner product in $L^2(R\,dx)$

[redacted] $\langle M(\lambda)a, M(z)b \rangle$

$$= \left[b^* M(z)^* P M(\lambda) a\right]_0^\ell \Big/ (\lambda - \bar{z})$$

or

$$\langle M(\lambda)a, M(z)b\rangle = b^* \frac{M(\ell, z)^* P M(\ell, \lambda) - P}{\lambda - \bar{z}} a \text{ [redacted]}$$

Question: Given a Nevanlinna matrix [redacted] $M(\lambda)$ does one always get a Hilbert space of entire functions defined in this way?

Interesting point related to the fact [redacted] already noted that a [redacted] Toeplitz matrix (c_{i-j}) is positive semi-definite $\Longleftrightarrow$ $g(z) = c_0 + 2\sum_{n\geq 1}' c_{-n} z^n$ has a positive real part in the disk.

Take $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\varphi = \varphi_a = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_1'(x,\lambda) \end{pmatrix}$. Drop the a, we have

$$M(x,\lambda)a = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_1'(x,\lambda) \end{pmatrix}$$

where $\varphi_1(x,\lambda)$ is the solution of $Lu = -u'' + qu = \lambda u$ with initial values $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We know that in this case we get a Hilbert space of entire functions with point evaluator

$$\left(J_z, J_\lambda\right) = \int_0^\ell \varphi_1(x,\lambda)\,\varphi_1(x,\bar z)\,dx$$

$$= \frac{1}{\lambda - \bar z}\left(P\varphi(\lambda), \varphi(z)\right)(\ell)$$

$$= \frac{1}{\lambda - \bar z}\begin{vmatrix} \varphi_1(\lambda) & \varphi_1(\bar z) \\ \varphi_1'(\lambda) & \varphi_1'(\bar z) \end{vmatrix}(\ell)$$

Conversely suppose we are given entire functions $A(\lambda)$, $B(\lambda)$ real on the real axis such that for all $\operatorname{Im}\lambda \neq 0$

1) $$\frac{1}{\lambda - \bar\lambda}\begin{vmatrix} A(\lambda) & A(\bar\lambda) \\ B(\lambda) & B(\bar\lambda) \end{vmatrix} > 0 \qquad \text{for } \operatorname{Im}\lambda \neq 0$$

or equivalently

$$0 < \frac{\dfrac{A(\lambda)}{B(\lambda)} - \dfrac{A(\bar\lambda)}{B(\bar\lambda)}}{\lambda - \bar\lambda}\,|B(\lambda)|^2 = \frac{\operatorname{Im}\left(\frac{A}{B}(\lambda)\right)}{\operatorname{Im}(\lambda)}\,|B(\lambda)|^2$$

for $\operatorname{Im}\lambda \neq 0$. Notice that $A(\lambda) = 0 \Rightarrow 0 = \overline{A(\lambda)} = A(\bar\lambda)$ so that A, B have only real zeroes. Thus 1) amounts to A, B having only real zeroes and

$$\frac{\operatorname{Im}\left(\frac{A}{B}(\lambda)\right)}{\operatorname{Im}\lambda} > 0 \qquad \text{for } \operatorname{Im}\lambda \neq 0.$$

note change in notation

But then if we put $E(\lambda) = A(\lambda) + iB(\lambda)$, then it should follow E is a deBranges function.

$$\frac{E^\#}{E} = \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\frac{1}{2i(\lambda-\bar{z})}\begin{vmatrix} E^\#(\lambda) & E^\#(\bar{z}) \\ E(\lambda) & E(\bar{z}) \end{vmatrix} = \frac{\cancel{2i}}{\cancel{2i}(\lambda-\bar{z})}\begin{vmatrix} A(\lambda) & A(\bar{z}) \\ B(\lambda) & B(\bar{z}) \end{vmatrix} \qquad \text{etc.}$$

So by deBranges' theory we know that the matrix

$$K(\lambda, z) = \frac{1}{\lambda - \bar{z}}\begin{vmatrix} A(\lambda) & A(\bar{z}) \\ B(\lambda) & B(\bar{z}) \end{vmatrix}$$

has to be ≥ 0.

Remark: If $\alpha \in SL_2(\mathbb{R})$, then $\alpha^t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ because $SL_2(\mathbb{R}) = Sp_2(\mathbb{R})$. In fact

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & -ad+bc \\ ad-bc & 0 \end{pmatrix}$$

so that $\alpha^t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \iff \det\alpha = 1$.

Consequently if $M(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ B(\lambda) & D(\lambda) \end{pmatrix}$ is a matrix of entire functions which are real (i.e. real on $\mathbb{R}$) and $\det M(\lambda) = 1$, then

$$M(z)^* P M(\lambda) - P \quad = \quad M(\bar{z})^t P M(\lambda) - P$$

vanishes for $\bar{z} = \lambda$ and hence

$$K(\lambda, z) = \frac{M(z)^* P M(\lambda) - P}{\lambda - \bar{z}}$$

is an entire matrix function of λ for any value of z.

Question: Is $K(\lambda,\lambda) \geq 0$ when M is a Nevanlinna matrix?

To show $K(\lambda,\lambda) \geq 0$ we must prove

$$(K(\lambda,\lambda)a, a) \geq 0$$

for all $a \in \mathbb{C}^2$. If a is real, then $(Pa, a) = 0$, so

$$(K(\lambda,\lambda)a, a) = \frac{\frac{1}{i}(PM(\lambda)a, M(\lambda)a)}{2 \operatorname{Im} \lambda} \geq 0$$

because M Nevanlinna $\Rightarrow \operatorname{Im}(M(\lambda)a)$ [redacted] ≥ 0 for $\operatorname{Im}(\lambda) > 0$.
If a is complex, say $\alpha + i\beta$, then

$$(PM(\lambda)\alpha, M(\lambda)\alpha) + (PM(\lambda)\alpha, iM(\lambda)\beta) + (iPM(\lambda)\beta, M(\lambda)\alpha) + (PM(\lambda)\beta, M(\lambda)\beta)$$

$$\cancel{(P\alpha, \alpha)} + (P\alpha, i\beta) + (iP\beta, \alpha) + \cancel{(P\beta, \beta)}$$

?

January 3, 1978

Lee Yang thm: Let $P(z_1, \dots, z_n) = \sum a_I z^I$ where I runs over subsets of $\{1, \dots, n\}$ and

$$a_I = \prod_{\substack{i \in I \\ j \in I'}} c_{ij} \qquad I' = \{1, \dots, n\} - I$$

and c_{ij} are numbers of modulus ≤ 1 given for $i \neq j$ such that $c_{ij} = \overline{c_{ji}}$. The theorem asserts that if $|z_1|, \dots, |z_n| < 1$, then $P(z_1, \dots, z_n) \neq 0$.

Question: Does $P(z_1, \dots, z_n) = \det\left(\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} + T\right)$ for some unitary matrix T of determinant $= 1$?

Example: If $n = 2$ $\quad P(z_1, z_2) = 1 + a z_1 + \bar{a} z_2 + z_1 z_2 \qquad a = c_{12}$

and

$$\det\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\right) = \begin{vmatrix} x + a & b \\ -\bar{b} & y + \bar{a} \end{vmatrix} = xy + \bar{a}x + ay + \underbrace{|a|^2 + |b|^2}_{1}$$

First reduction: Look at the effect of the substitution $z_i \mapsto \varepsilon_i z_i$ where $|\varepsilon_i| = 1$. I claim by such a substitution I can make $P(\varepsilon_1 z_1, \dots, \varepsilon_n z_n)$ a Lee-Yang poly with $0 \leq c_{ij} \leq 1$. In effect choose θ_{ij}:

$$c_{ij} = |c_{ij}| \, e^{\sqrt{-1}\,\theta_{ij}} \qquad \text{with } \theta_{ij} = -\theta_{ji}$$

put $\theta_{ii} = 0$

Then

$$\prod_{\substack{i \in I \\ j \in I'}} c_{ij} = \prod_{\substack{i \in I \\ j \in I'}} |c_{ij}| \; e^{\sqrt{-1} \sum_{\substack{i \in I \\ j \in I'}} \theta_{ij}}$$

$$\sum_{\substack{i \in I \\ j \in I'}} \theta_{ij} = \sum_{\substack{i \in I \\ j \in I'}} \theta_{ij} + \underbrace{\sum_{\substack{i \in I \\ j \in I}} \theta_{ij}}_{=0 \text{ by anti-symmetry}} = \sum_{i \in I} \left(\sum_{j=1}^{n} \theta_{ij} \right)$$

So if I put $\varepsilon_i = \exp\left(\sqrt{-1}\sum_{j=1}^{n}\theta_{ij}\right)$ I have

$$a_I = \prod_{\substack{i\in I\\ j\in I'}} c_{ij} = \prod_{\substack{i\in I\\ j\in I'}} |c_{ij}| \cdot \varepsilon^I$$

and so the claim is clear. Note $\varepsilon_1 \cdots \varepsilon_n = 1$.

On the other side

$$\det\left(\begin{pmatrix}\varepsilon_1 z_1 & & \\ & \ddots & \\ & & \varepsilon_n z_n\end{pmatrix} + T\right) = \det\left(\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n\end{pmatrix} + T\begin{bmatrix}\varepsilon_1^{-1} & & \\ & \ddots & \\ & & \varepsilon_n^{-1}\end{bmatrix}\right) \cdot \underbrace{\prod \varepsilon_i}_{= 1}$$

consequently modulo this substitution the poly depends only on the class of T in $SU_n /$ diag max. torus $\doteq$ flag manifold. In fact if we use

$$\det\left(\underline{\varepsilon}\,\underline{z}\,\underline{\eta} + T\right) = \det\left(\underline{z} + \underline{\varepsilon}^{-1} T \underline{\eta}^{-1}\right)$$

we see the poly depends only on the image of T in

$$\text{diag. tor} \backslash SU_n / \text{diag tor} = \text{diag tor.} \backslash \text{flag manifold.}$$

The latter has real dimensional $(n^2 - n) - (n-1) = n^2 - 2n + 1 = (n-1)^2$. The set of possible real c_{ij} has dimension $\frac{n(n-1)}{2}$ so this makes one suspect that the possible T-polys. are richer thatn Lee-Yang polynomials.

Symmetry of Lee-Yang polys.

$$\overline{a_I} = \prod_{\substack{i\in I\\ j\in I'}} \overline{c_{ij}} = \prod_{\substack{j\in I'\\ i\in I}} c_{ji} = a_{I'}$$

hence

$$\overline{P(z)} = \sum a_{I'} \bar{z}^I = \prod_{i=1}^{n} \bar{z}_i \sum a_{I'} (\bar{z}^{I'})^{-1} = \overline{\left(\prod_1^n z_i\right) P(z^{-1})}$$

or [redacted] [redacted]

$$\left(\prod_1^n z_i\right)\cdot P^*(z) = P(z) \qquad \text{where } f^*(z) = \overline{f(\bar{z}^{-1})}$$

Next observe the same is true for the other polys: $\det(T+z)$

$$\prod_1^n z_i \, \overline{\det(\bar{z}^{-1}+T)} = \prod_1^n z_i \, \det(z^{-1}+\bar{T})$$

$$= \det(z)\det(z^{-1}+T^*) \qquad T^* = \bar{T}^t = T^{-1}$$

as T unitary

$$= \det(I + zT^{-1})$$

$$= \det(T+z) \qquad \text{since } \det(T) = 1$$

Gradually I am coming to the viewpoint that the right framework for Lee-Yang is polynomials of the form

$$P(z_1, \ldots, z_n) = [redacted] \det\left(I - \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} T\right)$$

where T is unitary. Notice that if $|z_i| < 1$ then $z\cdot T$ is a contraction operator so all eigenvalues are inside the disk and so $P(z) \neq 0$. But more is true: suppose that $|z_i| \leq 1$ and at least one z_i has modulus < 1. Then $\|zT\| \leq 1$. What does it mean for zT to have ~~the~~ eigenvalue 1?

Write $V = V_1 \oplus V_2$ where $|z| < 1$ on V_1 and $|z| = 1$ on V_2. Then $\|zv\| = \|v\| \iff v \in V_2$. Hence $\|zTv\| = \|v\|$ [redacted] $\Rightarrow \|zTv\| = \|Tv\| \Rightarrow Tv \in V_2$. Hence $zTv = v \Rightarrow v, Tv \in V_2$. So it's clear that if zT has the eigenvalue 1, then this eigenspace is a subspace of V_2 stable under T. Hence [redacted] it's clear that if T leaves no non-zero subspace of V_2 invariant, then zT can have the eigenvalue 1 (in fact any eigenvalue on S^1.)

The above argument is what should explain statements such as if all $|c_{ij}| < 1$, then $P(z) \neq 0$ if all $|z_i| \leq 1$ and some $|z_i| < 1$.

Let's start with a LY poly $P(z) = P(z_1, \dots, z_n)$ and try to write it in the form $\det(I + zT)$ where T is unitary. Suppose the $c_{ij} \geq 0$. Write P as a linear polynomial in z_1:

$$P(z) = \sum_{J \subset \{2,\dots,n\}} \prod_{i \in J} c_{i1} \cdot \prod_{\substack{i \in J \\ j \in J'}} c_{ij} \, z^J + \sum_{J \subset \{2,\dots,n\}} \prod_{j \in J'} c_{1j} \prod_{\substack{i \in J \\ j \in J'}} c_{ij} \, z_1 z^J$$

$$= Q(c_{21} z_2, \dots, c_{n1} z_n) + \prod_{j=2}^{n} c_{1j} \, Q\left(\frac{z_2}{c_{12}}, \dots, \frac{z_n}{c_{1n}}\right) z_1$$

$$= Q(c_{21} z_2, \dots, c_{n1} z_n) + Q\left(\frac{c_{21}}{z_2}, \dots, \frac{c_{n1}}{z_n}\right) z_1 z_2 \cdots z_n$$

Suppose T orthogonal, & $\det T = 1$ to simplify. Then

$$P_1(z) = \det(1 + zT) = z_1 \cdots z_n \det(z^{-1} + T) = z_1 \cdots z_n \det(z^{-1} T^{-1} + I)$$

$$= z_1 \cdots z_n \det\left(I + \tfrac{1}{z} T\right) \qquad \text{since } T^t = T^{-1}$$

It follows that if

$$P_1(z) = A(z_2, \dots, z_n) + B(z_2, \dots, z_n) z_1$$

then

$$A\left(\tfrac{1}{z'}\right) + B\left(\tfrac{1}{z'}\right)\frac{1}{z_1} = \frac{1}{z_1 z'}\left(A(z') + B(z') z_1\right)$$

so

$$B(z') = z' A\left(\tfrac{1}{z'}\right) \qquad \text{and} \qquad P_1(z) = A(z') + A\left(\tfrac{1}{z'}\right) z' z_1$$

$$P_1(z) = \begin{vmatrix} 1+z_1 t_{11} & z_1 t_{12} & \cdots \\ z_2 t_{21} & 1+z_2 t_{22} & \cdots \\ & & \end{vmatrix}$$

Set $z_1 = 0$ to find $A(z')$.

You get $A(z') = \det(1+z'S)$

where $S = \begin{pmatrix} t_{22} & \cdots & t_{2n} \\ \vdots & & \\ t_{n2} & \cdots & t_{nn} \end{pmatrix}$.

The idea is to find T so that $P(z) = P_1(z)$, using by induction the fact that

$$Q(z') = \det(1+z'T')$$

for some orthogonal T' of det 1. If this is to work we must have

$$Q(c'z') = \det(1+c'z'T') = \det(1+z'S)$$

where $c' = \begin{pmatrix} c_{21} & & \\ & \ddots & \\ & & c_{n1} \end{pmatrix}$. So the problem becomes this: Given c' diagonal $0 < c' \leq I$ and T' orthogonal of determinant 1 can you find an orthogonal T with

$$\det(1+c'z'T') = \det(1+z'S)$$

$S = [2,n] \times [2,n]$ block of T.

This equation is probably equivalent to $c'T'$ and S being conjugate by diagonal matrices. First possibility to try is $S = (c')^{1/2} T' (c')^{1/2}$

This is a contraction operator. If there is an orthogonal matrix T extending it, then we get

$$\det(1+zT) = \det(1+z'S) + \det\left(1+\tfrac{1}{z'}S\right) z' z_1 = Q(c'z') + Q\left(\tfrac{c'}{z'}\right) z' z_1 = P(z).$$

which is what we want

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Let T be a unitary $n\times n$ matrix and S its lower $[2,n]\times[2,n]$ block.

$$\begin{pmatrix} t_{11} & \\ t_{21} & \boxed{\begin{matrix} & & \\ & & \\ & & \end{matrix}} \\ \vdots & \\ t_{n1} & \end{pmatrix}$$

The rows of T are orthonormal vectors. Hence if $v_i = (t_{i2}, \ldots, t_{in})$ one has

$$t_{i1}\bar{t}_{j1} + (v_i, v_j) = \delta_{ij} \qquad 2 \le i,j \le n$$

i.e.

$$\delta_{ij} - (v_i, v_j) = \begin{pmatrix} t_{21} \\ \vdots \\ t_{2n} \end{pmatrix} \begin{pmatrix} \bar{t}_{21} & \cdots & \bar{t}_{2n} \end{pmatrix}$$

Hence we see that

$$I - SS^* = vv^* \qquad v = \begin{pmatrix} t_{21} \\ \vdots \\ t_{2n} \end{pmatrix}$$

Thus a necessary condition that a $(n-1)\times(n-1)$ matrix S come from a unitary T in the above way is that $I - SS^*$ be of rank 1 and ≥ 0.

Conversely given S, v with $I - SS^* = vv^*$ we reverse the above procedure to get a $(n-1)\times n$ matrix (v, S) whose rows are orthogonal unit vectors. Then there is a unique unit vector up to a scalar of modulus 1 orthogonal to these rows, so we get a unique choice for T having prescribed determinant.

So the question is whether given a unitary $(n-1)\times(n-1)$ matrix U and a diagonal matrix $0 < c \leq 1$ can we find a diagonal invertible matrix d such that

$$d\,c\,U\,d^{-1} = S$$

extends in the above way to a unitary $n\times n$ matrix T. So we want

$$I - SS^* = I - d\,c\,U\,d^{-1}(d^*)^{-1}U^{-1}c\,d^*$$

to be of the form vv^*. Put $d^{-1} = \delta$. Then we want

$$\delta\delta^* - (cU)\,\delta\delta^*\,(U^*c)$$

to be ≥ 0 of rank 1. Thus we might as well assume the diagonal entries of δ are > 0. We want

$$c^{-2}(\delta^2) - U(\delta^2)U^*$$

to be ≥ 0 of rank 1. Not clear why δ^2 should $\exists$ with this property.

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