March 10, 1977

Question: 2nd order differential equations

Go back to the discrete string with potential energy

$$V = \frac{1}{2}\sum_n a_n (y_n - y_{n+1})^2 + \frac{1}{2}\sum k_n y_n^2$$

$$= \sum{}' \frac{1}{2} b_n y_n^2 - \sum a_n y_n y_{n+1} \qquad b_n = a_n + a_{n-1} + k_n$$

If all particles have mass ρ we get the kinetic energy

$$T = \frac{1}{2} m \sum y_n^2$$

hence the DE of motion

$$m\ddot{y}_n = a_n(y_{n+1} - y_n) + a_{n-1}(y_{n-1} - y_n) - k_n y_n$$

$$= a_n y_{n+1} - b_n y_n + a_{n-1} y_{n-1}$$

The limiting case of the continuous string is obtained by putting $x = n\Delta x$, $m = \rho\Delta x$, $a_n = \frac{a((n+1)\Delta x)}{\Delta x}$

$k_n = q(n\Delta x)\Delta x$.

$$\ddot{y}_n = \frac{1}{\Delta x}\left[a((n+1)\Delta x)\,\frac{y_{n+1} - y_n}{\Delta x} - a(n\Delta x)\,\frac{y_n - y_{n-1}}{\Delta x}\right] - q(n\Delta x)\,y_n$$

which gives in the limit

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial y}{\partial x}\right) - q(x)\,y$$

The corresponding eigenvalue problem is

$$\lambda y = \Big[\underbrace{-\frac{d}{dx}\, a \frac{d}{dx}}_{\text{self-adjoint}} + q\Big] y$$

($\sqrt{\lambda}$ is the frequency). Write as a system:

$$\frac{d}{dx}\left(a\frac{dy}{dx}\right) + (\lambda - q)\, y = 0$$

$$a\frac{d^2y}{dx^2} + a_x\frac{dy}{dx} + (\lambda - q)\, y = 0$$

Note λ has changed sign

$$\frac{d^2y}{dx^2} = -\frac{a_x}{a}\frac{dy}{dx} + \frac{(-\lambda+q)}{a}\, y$$

$$\frac{d}{dx}\begin{pmatrix} y \\ \frac{dy}{dx} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-\lambda+q}{a} & -\frac{a_x}{a} \end{pmatrix}\begin{pmatrix} y \\ \frac{dy}{dx} \end{pmatrix}$$

Better to write the system as

$$\frac{d}{dx}\begin{pmatrix} y \\ a\frac{dy}{dx} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{a} \\ -\lambda+q & 0 \end{pmatrix}\begin{pmatrix} y \\ a\frac{dy}{dx} \end{pmatrix}$$

for then the trace of the matrix is zero.

The problem now is to consider D.E.'s of the form

$$\frac{dX}{dt} = (A+\lambda B)X$$

where A, B are real matrix functions of t of trace 0 and $B = \begin{pmatrix} p & q \\ -r & -p \end{pmatrix}$ satisfies $q \geq 0$, $r \geq 0$, $-p^2 + qr \geq 0$. [redacted]
If $\Phi(t,\lambda)$ is the solution matrix

$$\frac{d\Phi}{dt} = [redacted] (A+\lambda B)\Phi \qquad \Phi(0) = I$$

then one would [redacted] like to describe the possible [redacted] matrix functions of λ one obtains by taking $\Phi(t,\lambda)$ for fixed t and for $t \to +\infty$.

Example: suppose $A+\lambda B$ constant in t. Then

$$\Phi(t,\lambda) = e^{t(A+\lambda B)}$$

and taking $t=1$, one finds

$$tr(\Phi(1,\lambda)) = tr\left(e^{A+\lambda B}\right) = e^{i\sqrt{\det(A+\lambda B)}} + e^{-i\sqrt{\det(A+\lambda B)}}$$

$$= 2\cos\left(\sqrt{\det(A+\lambda B)}\right)$$

If this lies in $[-2,2]$, then $\sqrt{\det(A+\lambda B)}$ has to be real, i.e. $\det(A+\lambda B) \geq 0$. ~~thus this is equivalent to~~ Note $\det(B) \geq 0$, so one knows $\det(A+\lambda B) \geq 0 \Rightarrow \lambda \in R$, once one finds a λ such that $\det(A+\lambda B) \leq 0$. If $B = \begin{pmatrix} p & q \\ -r & -p \end{pmatrix}$ [redacted] and $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ one can choose λ such that

$$c - \lambda r = [redacted]\ b + \lambda q\ [redacted]. \qquad c - b = \lambda(q+r) \qquad q+r>0$$

hence $A+\lambda B = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$ so $\det(A+\lambda\beta) = -\alpha^2-\beta^2 \le 0$.

Observe that there are two different cases for the distribution of zeroes.

$$\operatorname{tr}(\Phi(1,\lambda)) = 2\cos\left(\sqrt{\det(A+\lambda B)}\right) = 0$$

$$\iff \sqrt{\det(A+\lambda B)} = (2j+1)\frac{\pi}{2}$$

$$\iff \det(A+\lambda B) = \left[(2j+1)\frac{\pi}{2}\right]^2$$

But this is quadratic in λ if $\det(B)>0$ and linear in λ if $\det(B)=0$ and $B\neq 0$. In the former

$$\lambda_j \sim (\det B)^{-\frac{1}{2}}(2j+1)\frac{\pi}{2} \sim (\det B)^{-\frac{1}{2}} j\pi$$

and in the latter

$$\lambda_j \sim (\text{const})\,[j\pi]^2 .$$

Note that the latter case is the one occurring with SL systems.

<u>March 11, 1977</u>:

Consider the DE $\frac{d^2y}{dx^2} + \lambda y = 0$

$$\frac{d}{dx}\begin{pmatrix} y \\ \frac{dy}{dx} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}\begin{pmatrix} y \\ \frac{dy}{dx} \end{pmatrix}$$

$$\Phi(t,\lambda) = e^{t\begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} e^{t\begin{pmatrix} 0 & \lambda^{\frac{1}{2}} \\ -\lambda^{\frac{1}{2}} & 0 \end{pmatrix}} \begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \lambda^{\frac{1}{2}}t & \sin \lambda^{\frac{1}{2}}t \\ -\sin \lambda^{\frac{1}{2}}t & \cos \lambda^{\frac{1}{2}}t \end{pmatrix} \begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Phi(t,\lambda) = \begin{pmatrix} \cos \lambda^{\frac{1}{2}}t & \lambda^{-\frac{1}{2}} \sin \lambda^{\frac{1}{2}}t \\ -\lambda^{\frac{1}{2}} \sin \lambda^{\frac{1}{2}}t & \cos \lambda^{\frac{1}{2}}t \end{pmatrix}$$

an entire function of λ.

This is not an even function of λ so that you can't approximate by LY polynomials without shifting along the circle.

To do the approximation, first replace $\lambda\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ by

$$\lambda\begin{pmatrix} 0 & \varepsilon^2 \\ -1 & 0 \end{pmatrix} = \lambda\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}\begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1+\varepsilon^2\lambda \\ -\lambda & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \varepsilon^{-1}+\lambda\varepsilon \\ -\lambda\varepsilon & 0 \end{pmatrix}\begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

so

$$\operatorname{tr}\left(e^{t\begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}}\right) = \lim_{\varepsilon \to 0} \operatorname{tr}\left(e^{t\begin{pmatrix} 0 & \varepsilon^{-1}+\lambda\varepsilon \\ -\lambda\varepsilon & 0 \end{pmatrix}}\right)$$

Next $$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 & \varepsilon^{-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \varepsilon^{-1}\sin\theta & \varepsilon^{-1}\cos\theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\cos\theta\sin\theta & \varepsilon^{-1}\cos^2\theta \\ -\varepsilon^{-1}\sin^2\theta & -\varepsilon^{-1}\cos\theta\sin\theta \end{pmatrix}$$

and conjugation by $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ leaves $\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}$ alone.

So if we take $\theta = 45^\circ$ we find

$$\operatorname{tr} e^{t\begin{pmatrix}0 & \varepsilon^{-1}+\lambda\varepsilon\\ -\lambda\varepsilon & 0\end{pmatrix}} = \operatorname{tr} e^{t\left[\begin{pmatrix}\frac12\varepsilon^{-1} & \frac12\varepsilon^{-1}\\ -\frac12\varepsilon^{-1} & -\frac12\varepsilon^{-1}\end{pmatrix}+\begin{pmatrix}0 & \lambda\varepsilon\\ -\lambda\varepsilon & 0\end{pmatrix}\right]}$$

$$= \operatorname{tr} e^{t\left[\begin{pmatrix}0 & \lambda\varepsilon+\frac12\varepsilon^{-1}\\ -\lambda\varepsilon-\frac12\varepsilon^{-1} & 0\end{pmatrix}+\begin{pmatrix}\frac12\varepsilon^{-1} & 0\\ 0 & -\frac12\varepsilon^{-1}\end{pmatrix}\right]}$$

$$= 2\cos\left(t\sqrt{-\tfrac14\varepsilon^{-2}+(\lambda\varepsilon+\tfrac12\varepsilon^{-1})^2}\right)$$

$$= 2\cos\left(t\sqrt{\lambda^2\varepsilon^2+\lambda}\right) \longrightarrow 2\cos(t\sqrt{\lambda})$$

as $\varepsilon \to 0$.

Now we have seen that

$$\operatorname{tr} e^{\begin{pmatrix}a & u\\ -u & -a\end{pmatrix}} = 2\cos\left(\sqrt{u^2-a^2}\right)$$

is the limit of partition functions of Ising models. Specifically

$$\operatorname{tr} e^{\begin{pmatrix}a & u\\ -u & -a\end{pmatrix}} =$$

$$= \lim_{n\to\infty} \operatorname{tr}\left[\begin{pmatrix}\cos\frac{u}{n} & \sin\frac{u}{n}\\ -\sin\frac{u}{n} & \cos\frac{u}{n}\end{pmatrix}\begin{pmatrix}e^{+\frac{a}{n}} & 0\\ 0 & e^{-\frac{a}{n}}\end{pmatrix}\right]^n$$

Somehow we want $\frac{a}{n} = \frac{1}{2\varepsilon n}$ $\qquad \frac{u}{n} = \frac{\lambda\varepsilon + 1/2\varepsilon}{n} = \frac{\lambda\varepsilon^2+1/2}{\varepsilon n}$

and we want $\varepsilon n \to \infty$ with $\varepsilon \to 0$.

The real way to see what's happening is this. One we have a limit of Ising model partition functions such as $\cos(\sqrt{u^2-a^2})$, then we also can get the same thing with u replaced by $mu+n$ with m positive. [redacted] So we can [redacted] replace u^2-a^2 by $\left(\varepsilon u+\frac{1}{2\varepsilon}\right)^2-\left(\frac{1}{2\varepsilon}\right)^2 = \varepsilon^2u^2+u$ and so obtain u in the limit.

To understand Bessel functions a bit.

$$\Gamma(s) = \int_0^\infty e^{-t}\, t^s \frac{dt}{t} = \lambda^s \int_0^\infty e^{-\lambda t}\, t^s \frac{dt}{t} = \lambda^s \int_0^\infty e^{-\lambda/t}\, t^{-s} \frac{dt}{t}$$

$$\therefore \quad \frac{\Gamma(s)}{\lambda^s} = \int_0^\infty e^{-\lambda t}\, t^s \frac{dt}{t} \qquad \cancel{\frac{\Gamma(v-s)}{\lambda^{v-s}} = \int_0^\infty e^{-\lambda/t}\, t^{s-v} \frac{dt}{t}}$$

$Re(s) > 0$ $\qquad$ ~~$Re(s) < Re(v)$~~

~~Suppose $Re(v) > 0$~~

$$\cancel{\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{\Gamma(s)\Gamma(v-s)}{\lambda^v}\, ds = \int_0^\infty e^{-\lambda t}\frac{dt}{t}\int_0^\infty \frac{e^{-\lambda/t'}}{t'^v}\frac{dt'}{t'}\,\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} t/t}$$

$$\frac{\Gamma(s)}{\lambda^s} = \int_0^\infty e^{-\lambda/t}\, t^{-s}\frac{dt}{t}$$

$$\frac{\Gamma(s-v)}{\lambda^{s-v}} = \int_0^\infty e^{-\lambda/t}\, t^{v-s}\frac{dt}{t}$$

$$\frac{\Gamma(s)\Gamma(s-\nu)}{\lambda^{2s-\nu}} = \int_0^\infty e^{-\lambda t-(\lambda/t')}\, t'^{\nu}\, \frac{dt}{t}\frac{dt'}{t'}\left(t/t'\right)^s$$

But $$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} (t/t')^s \, ds = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{\alpha(a+ix)}\, i\, dx = \delta(\alpha)$$

so $$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(s)\Gamma(s-\nu)}{\lambda^{2s-\nu}}\, ds = \int_0^\infty e^{-\lambda\left(t+\frac{1}{t}\right)} t^{\nu}\frac{dt}{t} .$$

Bessel's DE is

$$\underbrace{r^2\frac{du}{dr^2} + r\frac{du}{dr}}_{\left(r\frac{d}{dr}\right)^2 u} + (r^2-n^2)u = 0 \qquad \text{on} \quad 0<r<\infty$$

Thus if we put $r = e^x$ $\quad -\infty < x < \infty$, then

$$\frac{d}{dx} = \frac{dr}{dx}\frac{d}{dr} = r\frac{d}{dr}$$

and the DE becomes

$$\frac{d^2u}{dx^2} + \left(e^{2x}-n^2\right)u = 0 .$$

Heading toward $x \to -\infty$, e^{2x} becomes negligible, so one has solutions asymptotic to $e^{nx} = r^n$, $e^{-nx} = r^{-n}$ as $r \to 0$. These should be J_n and J_{-n} roughly.

More interesting is to change r to iz in

Bessel's D.E.:

$$(*) \qquad \left[-\left(z\frac{d}{dz}\right)^2 + z^2\right]u = \lambda u$$

change n to in also

This time we get upon putting $z = e^x$ the DE

$$\left[-\frac{d^2}{dx^2} + e^{2x}\right]u = \lambda u.$$

$-\frac{d^2}{dx^2} + e^{2x}$ is formally a positive operator. Because e^{2x} grows fast as $x \to +\infty$ we expect any eigenvalue problem on $[x_0, \infty)$ to have a discrete spectrum. In fact it is known that $(*)$ has a unique non-trivial solution which is zero at $z \to +\infty$:

$$u = \int_{-\infty}^{\infty} e^{-z\left(\frac{e^t + e^{-t}}{2}\right)} e^{i\sqrt{\lambda}t}\, dt$$

March 12, 1977: Problem is to calculate the inverse Fourier transform of $\cos(\sqrt{u^2-a^2})$. The Paley-Wiener theorem ought to say this function is the characteristic function of a distribution supported in $-1 \leq x \leq 1$, which is something I know already anyway.

$$\cos(\sqrt{u^2-a^2}) - \cos(u) = \cos(u-\varepsilon) - \cos u \qquad \varepsilon = u - \sqrt{u^2-a^2}$$

$$= \cos u \cos\varepsilon + \sin u \sin\varepsilon - \cos u$$

$$= \cos u\left[\cos\varepsilon - 1\right] + \sin u \sin\varepsilon$$

$$\varepsilon = u\left(1-\left(1-\frac{a^2}{u^2}\right)^{1/2}\right) = u\left[1-\left(1-\frac{1}{2}\frac{a^2}{u^2}+\frac{\frac{1}{2}(-\frac{1}{2})}{2!}\frac{a^4}{u^4} \mp \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}\frac{a^6}{u^6}+\dots\right)\right]$$

$$= u\left(\frac{1}{2}\frac{a^2}{u^2}+\frac{1}{8}\frac{a^4}{u^4}+\cdots\right) = \frac{a^2}{2u}+\frac{a^4}{8u^3}+\cdots$$

$$\cos\varepsilon - 1 = -\frac{\varepsilon^2}{2}+\frac{\varepsilon^4}{24} = -\frac{1}{2}\left(\frac{a^2}{2u}\right)^2 + O\left(\frac{1}{u^4}\right)$$

$$\sin\varepsilon = \varepsilon - \frac{\varepsilon^3}{6} = \frac{a^2}{2u}+\frac{a^4}{8u^3}-\frac{1}{6}\frac{a^6}{(2u)^3} + O\left(\frac{1}{u^4}\right)$$

$$\cos\left(\sqrt{u^2-a^2}\right)-\cos(u) = \cos u\left(-\frac{a^4}{8u^2}\right) + \sin u\left(\frac{a^2}{2u}+\frac{6a^4-a^6}{48u^3}\right) + O\left(\frac{1}{u^4}\right)$$

The idea is that the function with value 1 in $[-1,1]$ and 0 outside has Fourier transform

$$\int_{-1}^{1} e^{-ixu}\,dx = \frac{e^{iu}-e^{-iu}}{iu} = 2\,\frac{\sin u}{u}$$

More easy question: Look at

$$e^{i\sqrt{u^2-a^2}}$$

which is analytic in a half space, hence it can be the Fourier transform of a ~~[illegible]~~ function with support in a half-line. Recall

$$\int_0^\infty e^{ixu}\,\varphi(x)\,dx$$

tends to be analytic for $\operatorname{Im}(u) > \text{constant}$ and it vanishes as $\operatorname{Im}(u) \nearrow \infty$.

$\sqrt{u^2-a^2}$ has Riemann surface obtained by cutting from

$-a$ to $+a$, hence it is analytic for $\mathrm{Im}(u) > 0$. Put $u = is$ so that $\mathrm{Im}(u) > 0 \longleftrightarrow \mathrm{Re}(s) > 0$.

$$e^{i\sqrt{u^2-a^2}} = e^{i(-s^2-a^2)^{1/2}} = e^{-\sqrt{s^2+a^2}} \qquad \text{like } e^{iu} = e^{-s}.$$

From Laplace transform tables

$$e^{-s} - e^{-\sqrt{s^2+a^2}} = \int_0^\infty e^{-st}\varphi(t)\,dt$$

where
$$\varphi(t) = \begin{cases} 0 & t < 1 \\ \dfrac{a}{\sqrt{t^2-1}} J_1\left(a\sqrt{t^2-1}\right) & t > 1. \end{cases}$$

Idea: suppose $f(s)$ analytic around ∞ and $f(\infty) = 0$, so that

$$f(s) = \frac{c_0}{s} + \frac{c_1}{s^2} + \cdots$$

with $\quad |c_j| \leq \mathrm{Const}\cdot(R^j) \quad$ some R. Since

$$\frac{\Gamma(n+1)}{s^{n+1}} = \int_0^\infty e^{-st} t^n\,dt$$

one has at least formally that

$$f(s) = \sum_{n\geq 0} c_n \frac{1}{n!} \int_0^\infty e^{-st} t^n\,dt$$

$$= \int_0^\infty e^{-st} \sum_{n\geq 0} \frac{c_n t^n}{n!}\,dt$$

However $\varphi(t) = \sum_{n\geq 0} \frac{c_n t^n}{n!}$ is an entire function of exp. type

$$|\varphi(t)| \leq \text{Const} \sum_{n\geq 0} \frac{R^n |t^n|}{n!} = (\text{Const})\, e^{R|t|}$$

so there's no problem with the analysis.

Consider $e^{k\sqrt{s^2+a^2}}$, where always we choose the branch of $\sqrt{s^2+a^2}$ so as to single-valued off the cut from $-ia$ to ia and to be asymptotic to s for s-large. Then

$$e^{k\sqrt{s^2+a^2}} = e^{ks}\{1+f(s)\}$$

where $f(s)$ is analytic around ∞ and $f(\infty)=0$. Thus I know that there is an entire function of exp. type $\varphi(t)$ with

$$f(s) = \int_0^\infty e^{-st}\varphi(t)\,dt$$

$$e^{ks}f(s) = \int_0^\infty e^{-s(t-k)}\varphi(t)\,dt = \int_{-k}^\infty e^{-st}\varphi(t+k)\,dt$$

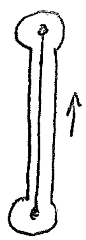
so

$$e^{k\sqrt{s^2+a^2}} = \int_{-k}^{\infty} e^{-st}\{\delta(t+k)+\varphi(t+k)\}\,dt$$

To calculate the inverse transform of $e^{k\sqrt{s^2+a^2}}$ we use the integral

$$g(t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{k\sqrt{s^2+a^2}}\, e^{st}\, ds$$

where $\varepsilon > 0$. Note that if $t < -k$, then as we push ε to ∞ the integrand disappears giving 0. The same holds for $t > -k$ if we push ε toward $-\infty$, except that because $\sqrt{s^2+a^2}$ is not single-valued for s on the cup $[-ia, ia]$ we pick up an integral around this cut. No contribution from the circles because the integrand is bounded. Put



$$s = ai\sin(\theta) \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$ds = ai\cos\theta \, d\theta$$

$$\sqrt{s^2+a^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$$

So the upward integral on the left is

$$\frac{1}{2\pi i}\int_{-\pi/2}^{\pi/2} e^{ka\cos\theta} e^{(ai\sin\theta)t} \, ai\cos\theta \, d\theta$$

$$\frac{1}{2\pi i}\int_{\pi/2}^{3\pi/2} e^{ka\cos\theta} e^{(ai\sin\theta)t} \, ai\cos\theta \, d\theta .$$

So for $t > -k$ or better $Re(t) > Re(-k)$ one has

$$g(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{(ka\cos\theta + tai\sin\theta)} \, a\cos\theta \, d\theta$$

$$ka\cos\theta + tai\sin\theta = i\sqrt{(t^2-k^2)}\,a\sin(\theta \, \alpha)$$

where α is a complex number with $\cos(\alpha) = \frac{t}{\sqrt{t^2-k^2}}$ $\sin(\alpha) = \frac{ki}{\sqrt{t^2-k^2}}$

$$\therefore \quad e^{k\sqrt{s^2+a^2}} - e^{ks} = \int_{-k}^{\infty} e^{-st} \overbrace{\frac{1}{2\pi i}\int_{0_+-ia}^{0_++ia} \left(e^{k\sqrt{s^2+a^2}} - e^{-k\sqrt{s^2+a^2}}\right) e^{st}\, ds}^{\varphi_k(t)}$$

$$e^{-k\sqrt{s^2+a^2}} - e^{-ks} = \int_{k}^{\infty} e^{-st} \frac{1}{2\pi i}\int_{0_+-ia}^{0_++ia} \left(e^{-k\sqrt{s^2+a^2}} - e^{+k\sqrt{s^2+a^2}}\right) e^{st}\, ds$$

$$\therefore \quad e^{k\sqrt{s^2+a^2}} + e^{-k\sqrt{s^2+a^2}} - e^{ks} - e^{-ks} = \int_{-k}^{k} e^{-st}\, \varphi_k(t)\, dt$$

where

$$\boxed{\varphi_k(t) = \frac{1}{2\pi i}\int_{0_+-ia}^{0_++ia} \left(e^{+k\sqrt{s^2+a^2}} - e^{-k\sqrt{s^2+a^2}}\right) e^{st}\, ds}$$

Note that $\varphi_k(t)$ makes sense for k complex. Set

$$s = ia\sin\theta$$

$$\sqrt{s^2+a^2} = \sqrt{a^2(1-\sin^2\theta)} = a\cos\theta \qquad \text{if} \quad s \in 0_+ + i\mathbb{R}$$

Then as before we will get

$$\varphi_k(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{ka\cos\theta + ita\sin\theta}\, a\cos\theta\, d\theta.$$

To evaluate this I will take k to be purely imaginary $k = il$

$$la\cos\theta + ta\sin\theta = a\sqrt{t^2+l^2}\,\sin(\theta+\alpha)$$

$$\cos\alpha = \frac{t}{\sqrt{t^2+l^2}} \qquad \sin\alpha = \frac{l}{\sqrt{t^2+l^2}}$$

Actually we can suppose k purely imaginary if we want. NO

$$g(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{ia\sqrt{t^2-k^2}\sin\theta}(\cos\theta\cos\alpha + \sin\theta\sin\alpha)\,d\theta. \qquad ?$$

Instead start with the function

$$e^{k\sqrt{s^2+a^2}-ks} - 1$$

which we know is analytic off the cut from $-ia$ to ia and which vanishes at ∞. This is true for all k

$$\text{Put} \quad g(t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\left(e^{k\sqrt{s^2+a^2}-ks}-1\right)e^{st}\,ds \qquad \begin{matrix} t \text{ real} \\ \varepsilon>0 \end{matrix}$$

If $t<0$, then moving $\varepsilon \to +\infty$ the integrand goes to zero so $g(t)=0$. If $t>0$, then moving ε past zero, we get the integral around the cut, and then pushing ε to $-\infty$ we get 0. So we find

$$g(t) = \frac{1}{2\pi i}\int_{0_+-ia}^{0_++ia}\left(e^{k\sqrt{s^2+a^2}} - e^{-k\sqrt{s^2+a^2}}\right)e^{s(t-k)}\,ds$$

Now

$$e^{k\sqrt{s^2+a^2}-ks} - 1 = \int_0^\infty e^{-st}g(t)\,dt$$

$$e^{k\sqrt{s^2+a^2}} - e^{ks} = \int_0^\infty e^{-s(t-k)}g(t)\,dt = \int_{-k}^\infty e^{-st}g(t+k)\,dt$$

provided k real.

$$\varphi_k(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{ia\sqrt{t^2+\ell^2}\sin\theta}\, a\cos(\theta-\alpha)\, d\theta$$

$$= \frac{a}{2\pi}\int_0^{2\pi} e^{i a\sqrt{t^2+\ell^2}\sin\theta}\left(\cos\theta\frac{t}{\sqrt{t^2+\ell^2}} + \sin\theta\frac{\ell}{\sqrt{t^2+\ell^2}}\right) d\theta$$

Now $\sin\theta$ is fixed under $\theta \mapsto \pi - \theta$ and $\cos(\pi-\theta) = -\cos\theta$ so the first part will integrate to zero. $\therefore$

$$\varphi_k(t) = \frac{a}{2\pi}\int_0^{2\pi} e^{i a\sqrt{t^2+\ell^2}\sin\theta}\frac{\ell}{\sqrt{t^2+\ell^2}}\sin\theta\, d\theta$$

$$= \frac{a}{2\pi}\int_0^{2\pi} e^{i a\sqrt{t^2+\ell^2}\sin\theta}\frac{\ell}{\sqrt{t^2+\ell^2}}\frac{e^{i\theta}}{i}\, d\theta$$

$$= -\frac{ak}{\sqrt{t^2-k^2}}\frac{1}{2\pi}\int_0^{2\pi} e^{i a\sqrt{t^2-k^2}\sin\theta + i\theta}\, d\theta$$

$$\therefore \quad \boxed{\varphi_k(t) = -\frac{ak}{\sqrt{t^2-k^2}} J_1\left(a\sqrt{t^2-k^2}\right)}$$

Because
$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin\theta + in\theta}\, d\theta$$

So we get for k real

$$\therefore \quad \boxed{e^{-k\sqrt{s^2+a^2}} - e^{-ks} = \int_k^\infty e^{-st}\frac{(-ak)}{\sqrt{t^2-k^2}} J_1\left(a\sqrt{t^2-k^2}\right) dt}$$

$$e^{ik\sqrt{u^2-a^2}} - e^{iku} = \int_k^\infty e^{iut}\frac{(-ak)}{\sqrt{t^2-k^2}} J_1\left(a\sqrt{t^2-k^2}\right) dt$$

$$\boxed{\cos k\sqrt{u^2-a^2} = \cos ku + \frac{1}{2}\int_{-k}^{k} e^{iut}\,\frac{ak}{\sqrt{t^2-k^2}}\,J_1\!\left(a\sqrt{t^2-k^2}\right)dt}$$

<u>March 14, 1977</u> L-function for $\mathbb{Z}[i]$.

Put $B=\mathbb{Z}[i]$. An odd prime p in $\mathbb{Z}$ splits into 2 primes in B when X^2+1 has a root mod p, i.e. when -1 is a quad. residue mod p, i.e. when the Legendre symbol

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$$

is $+1$, i.e. when $p\equiv 1 \bmod 4$. The factor of ζ_B belonging to the primes over any odd prime p of $\mathbb{Z}$ is therefore

$$\left(1-\frac{1}{p^s}\right)^{-1}\left(1-\left(\frac{-1}{p}\right)\frac{1}{p^s}\right)^{-1} = \begin{cases} \left(1-\frac{1}{p^{2s}}\right)^{-1} & p \text{ doesn't split} \\ \left(1-\frac{1}{p^s}\right)^{-2} & p \text{ splits} \end{cases}$$

Hence

$$\zeta_B^{-1} = \left(1-\frac{1}{2^s}\right)\prod_{p \text{ odd}}\left(1-p^{-s}\right)\left(1-\left(\frac{-1}{p}\right)p^{-s}\right)$$

So $\zeta_B = \zeta_{\mathbb{Z}}\cdot L$ where

$$L(s) = \prod_{p \text{ odd}}\left(1-\left(\frac{-1}{p}\right)p^{-s}\right)^{-1} = \sum_{n\geq 1}\left(\frac{-1}{n}\right)n^{-s}$$

where $\left(\frac{-1}{n}\right) = 0$ if n is even by convention.

$$\left(\frac{-1}{n}\right) = \prod_i \left(\frac{-1}{p_i}\right)^{a_i} \quad \text{if} \quad n = \prod p_i^{a_i} \quad \text{odd primes.}$$

$$= \begin{cases} +1 & \text{if } n \text{ is of the form } a^2+b^2 \quad \text{and } n \text{ odd} \\ -1 & \text{if } n \text{ is not of the form } a^2+b^2 \quad \text{———} \end{cases}.$$

By Unique Factorization in B one has

$$\zeta_B(s) = \frac{1}{4} \sum_{(m,n)\neq 0} (m^2+n^2)^{-s}$$

$$\Gamma(s) = 2\pi^s \int_0^\infty e^{-\pi t^2} t^{2s} \frac{dt}{t}$$

$$\zeta_B(s)\Gamma(s)\pi^{-s} = \frac{2}{4} \sum_{(m,n)\neq 0} \int_0^\infty e^{-\pi t^2} (m^2+n^2)^{-s} t^{2s} \frac{dt}{t}$$

$$\int_0^\infty e^{-\pi(m^2+n^2)t^2} t^{2s} \frac{dt}{t}$$

$$= \frac{1}{2} \int_0^\infty \left(\sum_{m,n} e^{-\pi(m^2+n^2)t^2} - 1 \right) t^{2s} \frac{dt}{t}$$

$$= \frac{1}{2} \int_0^\infty \left(\theta(t)^2 - 1 \right) t^{2s} \frac{dt}{t} \qquad \theta(t) = \sum_n e^{-\pi n^2 t^2}$$

Recall

$$\zeta_{\mathbb{Z}}(s)\Gamma(s/2)\pi^{-s/2} = \int_0^\infty (\theta(t) - 1) t^s \frac{dt}{t}$$

Now

$$\frac{\zeta_B(s)\Gamma(s)\pi^{-s}}{\zeta_{\mathbb{Z}}(s)\Gamma(s/2)\pi^{-s/2}} = L(s)\pi^{-s/2} \frac{\Gamma(s)}{\Gamma(s/2)} = L(s)\pi^{-\frac{s}{2}}\left(\pi^{-1/2} 2^{s-1} \Gamma\left(\frac{s+1}{2}\right)\right)$$

$$= L(s)\Gamma\left(\frac{s+1}{2}\right)\pi^{-\left(\frac{s+1}{2}\right)} 2^{s-1}$$

$$L(s)\,\Gamma\left(\frac{s+1}{2}\right)\pi^{-\left(\frac{s+1}{2}\right)}2^{s-1} = L(s)\,2\int_0^\infty e^{-\pi t^2}\,t^{s+1}\,\frac{dt}{t}\,2^{s-1}$$

$$= 2\sum_{\substack{\cancel{n\in\mathbb{Z}}\\ n\geq 1}}{}' \left(\frac{-1}{n}\right) n^{-s}\int_0^\infty e^{-\pi t^2/4}\,t^{s+1}\,\frac{dt}{t}\cdot 2^{-2}$$

$$= \frac{1}{2}\sum_{n\geq 1}{}'\left(\frac{-1}{n}\right)\int_0^\infty e^{-\pi n^2t^2/4}\,nt\,t^s\,\frac{dt}{t}$$

$$= \frac{1}{2}\int_0^\infty \left(\sum_{\substack{\cancel{n\in\mathbb{Z}}\\ n\geq 1}}{}'\left(\frac{-1}{n}\right) nt\,e^{-\pi n^2t^2/4}\right) t^s\,\frac{dt}{t}$$

so for some reason it should be true that

$$\rho(t) = \sum_{\substack{\cancel{n\in\mathbb{Z}}\\ n\geq 1}}{}'\left(\frac{-1}{n}\right) nt\,e^{-\pi n^2t^2/4}$$

satisfies $\rho\left(\frac{1}{t}\right) = t\,\rho(t)$. However note that if n_1, n_2 are two odd integers then

$$(n_1-1)(n_2-1) \equiv 0 \quad (4)$$

$$\Rightarrow \quad n_1n_2 - 1 \equiv n_1 - 1 + n_2 - 1 \quad (4)$$

$$\Rightarrow \quad (-1)^{\frac{n_1n_2-1}{2}} = (-1)^{\frac{n_1-1}{2}}(-1)^{\frac{n_2-1}{2}}$$

$$\therefore \quad \left(\frac{-1}{n}\right) = (-1)^{\frac{n-1}{2}} \quad \text{for } n \text{ odd and } \geq 1$$

$$\therefore \quad \left(\frac{-1}{n}\right) = \sin\left(\frac{n\pi}{2}\right) \quad \text{for all integers } n \geq 1$$

Recall Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} e^{2\pi i n x} \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i n \xi} f(\xi)\, d\xi}_{\hat{f}(n)}$$

$$e^{-\pi x^2} = \int e^{-2\pi i x \xi} e^{-\pi \xi^2} d\xi$$

$$\frac{i}{2\pi}\frac{d}{dx}\left(e^{-\pi x^2}\right) = \int e^{-2\pi i x \xi}\left(\xi e^{-\pi \xi^2}\right) d\xi$$

$$\|$$

$$-i x e^{-\pi x^2}$$

$$-i\frac{xt}{2}\, e^{-\pi\left(\frac{x^2t^2}{4}\right)} = \frac{2}{t}\int e^{-2\pi i \frac{xt}{2}\frac{2\xi}{t}} \underbrace{\left(\frac{2\xi}{t} e^{-\pi 4\xi^2/t^2}\right)}_{f(\xi)} d\xi$$

$$\frac{2}{t}\sum_{n\in\mathbb{Z}} \frac{2(x+n)}{t} e^{-\pi 4(x+n)^2/t^2} = \sum_{n\in\mathbb{Z}} e^{2\pi i n x}\left(-\frac{i n t}{2} e^{-\pi\frac{n^2t^2}{4}}\right)$$

Put $x = \frac{1}{4}$

$$\sum_{n\in\mathbb{Z}} \frac{4\left(\frac{1}{4}+n\right)}{t^2} e^{-4\pi\left(\frac{1}{4}+n\right)^2/t^2} = \sum_{n\in\mathbb{Z}} \sin\left(\frac{\pi n}{2}\right)\frac{nt}{2} e^{-\pi\frac{n^2t^2}{4}}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\frac{1}{t}\sum_{m\equiv 1\,(4)} \frac{m}{t} e^{-\pi\frac{m^2/t^2}{4}} \qquad \frac{1}{2}\sum_{m\equiv 1(4)} mt\, e^{-\pi\frac{m^2t^2}{4}}$$

$$-\frac{1}{2}\sum_{m\equiv 3(4)} mt\, e^{-\pi\frac{m^2t^2}{4}}$$

But observe: $\displaystyle\sum_{m\equiv 1\,(2)} mt\,e^{-\pi\frac{m^2t^2}{4}} = 0$ by symmetry.

Thus we find the formulas

$$L(s)\,\Gamma\!\left(\frac{s+1}{2}\right)\pi^{-\left(\frac{s+1}{2}\right)}2^{s-1} = \frac{1}{2}\int_0^\infty \rho(t)\,t^s\,\frac{dt}{t}$$

where

$$\rho(t) = \sum_{n\geq 1}\left(\frac{-1}{n}\right) nt\,e^{-\pi n^2t^2/4}$$

$$= \sum_{m\equiv 1\,(4)} mt\,e^{-\pi m^2t^2/4} \qquad m \text{ both } + \text{ and } -$$

satisfies $\rho\left(\frac{1}{t}\right) = t\rho(t)$

It's clear ~~that $\rho(t) > 0$ and~~ that $\rho(t)\to 0$ fast as $t\to+\infty$, hence also $\rho\left(\frac{1}{t}\right)\to 0$ fast as $t\searrow 0$.

Claim $\rho(t) > 0$. Enough to do for $t\geq 1$. $\rho(t)$ is an alternating series the first term of which is >0. Suffices to show the terms decrease in size i.e. for $t\geq 1$

$$nt\,e^{-\pi n^2t^2/4} > (n+2)t\,e^{-\pi(n+2)^2t^2/4} \qquad \frac{n^2+4n+4}{4}$$

$$e^{\pi(n+1)t^2} > \frac{n+2}{n} = 1+\frac{2}{n}$$

increases in n, t $\qquad$ decreases in n

So can check for $t=1$, $n=1$.

$$e^{\pi\cdot 2} > 1+2 = 3 \qquad \text{Yes.}$$