March 5, 1977

Let's go back to the one-sided J-matrix problem. Suppose J is a two-sided J-matrix and J_+ is the associated one-sided matrix. We fix λ. The corresponding J_+ eigenfunction is given by

$$(y_n \;\; y_{n+1}) = (y_{n-1} \;\; y_n)\begin{pmatrix} 0 & -\frac{c_{n-1}}{a_n} \\ 1 & \frac{\lambda - b_n}{a_n} \end{pmatrix}$$

starting from $(y_0 \; y_1) = (0 \;\; 1)$. Thus

$$(y_n \;\; y_{n+1}) = (0 \;\; 1)\underbrace{\begin{pmatrix} 0 & -\frac{c_0}{a_1} \\ 1 & \frac{\lambda - b_1}{a_1} \end{pmatrix} \cdots \begin{pmatrix} 0 & -\frac{c_{n-1}}{a_n} \\ 1 & \frac{\lambda - b_n}{a_n} \end{pmatrix}}_{\Phi_n(\lambda)}$$

How is this related to the continued fraction? If we put

$$J_n = \begin{pmatrix} b_1 & a_1 & & \\ c_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & c_{n-1} & b_n \end{pmatrix}$$

$$B_{-1} = 0$$
$$B_0 = 1$$
$$B_1 = z - b_1$$

and $B_n = \det(zI_n - J_n)$, then we get

$$B_n = -(-c_{n-1})(-a_{n-1})B_{n-2} + (z - b_n)B_{n-1}$$

$$B_n = -c_{n-1}a_{n-1}B_{n-2} + (z - b_n)B_{n-1}$$

or

$$\cancel{\begin{pmatrix} B_n & B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} & -c_0 a_0 \\ 1 & z-b_1 \end{pmatrix} \cdots \begin{pmatrix} & -c_{n-1} a_{n-1} \\ 1 & z-b_n \end{pmatrix}}$$

$$\frac{B_n}{a_1 \cdots a_n} = -\frac{c_{n-1}}{a_n} \frac{B_{n-2}}{a_1 \cdots a_{n-2}} + \frac{z-b_n}{a_n} \frac{B_{n-1}}{a_1 \cdots a_{n-1}}$$

Consequently $\boxed{y_{n+1}(\lambda) = \dfrac{\det(\lambda I_n - J_n)}{a_1 \cdots a_n}}$

This is also clear because we know that both sides are polys. of degree n with same leading terms and the same number of distinct roots. Note J_n has simple eigenvalues, for there is a cyclic vector.

~~When is this function y_n bounded in n?~~

~~We have fractional linear transformations $\Phi_n(\lambda)^t$ carrying $\mathrm{Im}(z) > 0$ strictly into itself if $\mathrm{Im}(\lambda) > 0$,~~

$$\cancel{\lim_{n \to \infty} \Phi_n(\lambda)^t \{\mathrm{Im}\, z > 0\}}$$

~~is either a limit circle or a limit point z_∞. In the latter case~~

$$\cancel{\frac{y_n(\lambda)}{y_{n+1}(\lambda)} \longrightarrow z_\infty}$$

Suppose J periodic with period r and we are in the real symm. case. There are $(r-1)$ distinct λ such that $y_r(\lambda) = \dfrac{\det(\lambda I_{r-1} - J_{r-1})}{a_1 \cdots a_{r-1}} = 0.$

For each of these $\Phi(\lambda)$ has the eigenvector $(0 \ 1)$ which is <u>real</u>, hence the eigenvalue z must be real. This implies that λ is outside the interior of the bands.

<u>Problem</u>: For each λ such that $y_n(\lambda) = 0$ determine whether $|y_{n+1}(\lambda)|$ is ≥ 1 or < 1.

If $|y_{n+1}(\lambda)| = 1$, then $y_{n+1}(\lambda) = \pm 1$ and so we are in the situation where $z = 1$ or -1, hence there is a periodic or ~~half~~-periodic solution. So it is not in L^2.

Isospectral deformation. Suppose $L(t)$ is a one parameter family of operators of the form

$$L(t) = U(t) L_0 U(t)^{-1}$$

Then

$$L_t = U_t L_0 U^{-1} - U L_0 U^{-1} U_t U^{-1}$$

where $L_t = \frac{\partial}{\partial t} L$ etc. So

$$L_t = B L - L B = [B, L]$$

where $B = U_t U^{-1}$. If $U(t)$ is unitary, then

$$U U^* = 1 \implies U_t U^* + U U_t^* = 0$$

$$\implies B + B^* = 0 \implies B \text{ skew-adjoint}$$

Lax applies this to $L = \partial^2 + q$ ($\partial = \frac{\partial}{\partial x}$, q a function of x, t) when $L_t = q_t$. He tries to construct a B which works which is a differential operator.

If $B = a\partial + b$ is skew-adjoint, then

$$B^* = -\partial a + b = -a\partial - a_x + b = -a\partial - b$$

$$\Longrightarrow \quad a_x = 2b \quad \text{or} \quad b = \tfrac{1}{2} a_x$$

So if $B = \partial^3 + a\partial + \frac{1}{2} a_x$, then

$$\begin{aligned}[B, L] = \left[\partial^3 + a\partial + \tfrac{1}{2} a_x, \partial^2 + q\right] &= 3q_x \partial^2 + 3q_{xx} \partial + q_{xxx} \\ &\quad - 2a_x \partial^2 - a_{xx} \partial + a q_x \\ &\quad - a_{xx} \partial - \tfrac{1}{2} a_{xxx} \\ &= (3q_x - 2a_x)\partial^2 + (3q_{xx} - 2a_{xx})\partial + (q_{xxx} - \tfrac{1}{2} a_{xxx} + a q_x)\end{aligned}$$

So if $a = \frac{3}{2} q$ we have

$$[B, L] = \tfrac{1}{4} q_{xxx} + \tfrac{3}{2} q q_x$$

This should be equal to q_t. Thus if

$$\boxed{\tfrac{1}{4} q_{xxx} + \tfrac{3}{2} q q_x = q_t}$$

up to scaling this is the KdV equation.

one has at least in some formal sense that as t varies the operators $L = \partial^2 + q$ are all conjugate and hence have the same spectra.

More Lax:

Let M be the (infinite-dim'l) manifold of functions $u(x)$ say for periodic functions. Consider functions $F: M \to R$ say for example

$$F(u) = \int_0^1 L(x, u, u_x)\, dx$$

Then one defines its derivative in the direction v

$$F(u+\varepsilon v) = \int_0^1 L(x, u+\varepsilon v, u_x + \varepsilon v_x)\, dx$$

$$F(u+\varepsilon v) - F(u) = \varepsilon \int_0^1 \left(\frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_x} v_x \right) dx \quad + O(\varepsilon^2)$$

$$I. \quad \frac{d}{d\varepsilon} F(u+\varepsilon v)\Big|_{\varepsilon=0} = \int_0^1 \left(\frac{\partial L}{\partial u} \cdot v + \frac{\partial L}{\partial u_x} v_x \right) dx$$

and one expresses this as an inner product

$$\frac{d}{d\varepsilon} F(u+\varepsilon v)\Big|_{\varepsilon=0} = \left(G_F(u), v \right)$$

where G_F is the gradient of F. In this example

$$G_F = \frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x} \right)$$

Now what.

March 6, 1977.

Let us relate unimodular transf shrinking the UHP $\operatorname{Im} w \geq 0$ with those shrinking the unit disk $|z| \leq 1$.

Let M be the monoid of unimodular transformations carrying $\operatorname{Im} w \geq 0$ into itself. Then

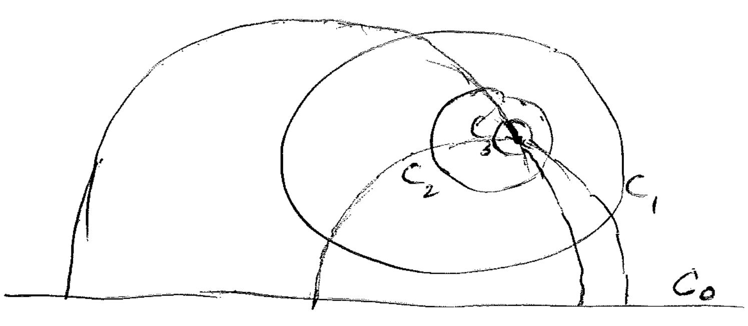
$$M \supset PSL_2(\mathbb{R})$$

in fact, $PSL_2(R)$ is the set of invertible elements of M. Also M contains the translations $w \mapsto w+\tau$ with $\operatorname{Im}\tau \geq 0$

Consider the set $\mathcal{C}$ of circles in $\operatorname{Im} w > 0$; ~~that~~ don't include lines $\operatorname{Im}(w) =$ constant for they meet $\operatorname{Im}(w) = 0$ at ∞. $PSL_2(\mathbb{R})$ acts on $\mathcal{C}$. One can successively reflect to obtain a ~~dec~~ decreasing sequence of circles

$$C_0 = \mathbb{R} \cup \{\infty\}, \quad C_1 = C, \quad C_2 = \text{reflection of } C_0 \text{ in } C_1,$$

etc.



which converge to a point in the UHP. To see this draw the circles orthogonal to C_0 and C_1. So it's now clear that if we move this limit point to $w = i$, then ~~[illegible]~~ the rotations around i leave $\mathcal{C}$ fixed. The only invariant of C is its radius i.e. the non-Euclidean distance from C_1 to $w=i$. Conclude

$$PSL_2(\mathbb{R}) \backslash \mathcal{C} \simeq \mathbb{R}_{>0}$$

and the stabilizer of any circle $\sim S^1$.

~~So now we can classify any unimodular transf. carrying $\operatorname{Im} w \geq 0$ strictly into itself. Look at the corresponding unit disk pictures. If $\theta D \subset \operatorname{Int}(D)$, then $\theta(\partial D)$ is a circle inside D, and there is a unimodular transformation φ preserving ∂D (this corresp. to an elt. of $PSL_2(\mathbb{R})$) such that $\varphi\theta(\partial D) = \rho\,\partial D$ with $0<\rho<1$. φ is unique up to a rotation. Hence $\rho^{-1}\varphi\theta$ preserves ∂D and the origin so it NO is a rotation. Thus~~

$$\cancel{\theta = \lambda\varphi^{-1}}$$

~~where φ^{-1} preserves ∂D and $0<|\lambda|<1$. We can normalize things by requiring $0<\lambda<1$.~~

So now let $\theta \in PSL_2(\mathbb{C})$ take the unit disk D into its interior. Choose a transformation φ_1 preserving ∂D such that $\varphi_1^{-1}\theta(\partial D)$ is concentric with ∂D, hence $\varphi_1^{-1}\theta(\partial D) = \lambda\,\partial D$ with $0<|\lambda|<1$, whence

$$\lambda^{-1}\varphi_1^{-1}\theta = \varphi_2 \quad \text{preserving } \partial D$$

40 $\boxed{\theta = \varphi_1\lambda\varphi_2}$.

Note that φ_1 is unique up to a rotation, hence if [redacted] H is the subgroup preserving ∂D, then one has

$$H \times^{S^1} \{\lambda \mid |\lambda| < 1\} \times^{S^1} H$$

for the monoid $M = \{\theta \in PSL_2(\mathbb{C}) \mid \theta(D) \subset \mathrm{Int}(D)\}$. Observe that one gets the [redacted] right dimension for M: $3 + $ [redacted] $1 + (3-1) = 6$.

March 7, 1977

$M \subset PSL_2(\mathbb{C})$ the monoid consisting of $\theta \ni \theta(D) \subset \mathrm{Int}(D)$, Γ the subgroup preserving ∂D (hence $\Gamma \simeq SL_2(\mathbb{R})$). Have seen

$$\Gamma \times^{S^1} \{\lambda \mid |\lambda| < 1\} \times^{S^1} \Gamma \xrightarrow{\sim} M.$$

Objects of interest: $\bar{M} = \{\theta \in PSL_2(\mathbb{C}) \mid \theta(D) \subseteq D\}$. $\bar{M}$ should be the closure of M. $\bar{M}$ consists of M, those θ carrying ∂D to a circle tangent to D and interior to D, and Γ. What I am going to be interested in are holomorphic maps

$$f : \mathbb{C} \longrightarrow PSL_2(\mathbb{C})$$

such that

$$f(\operatorname{Im} z \geq 0) \subset \bar{M}$$

$$f(\operatorname{Im} z \leq 0) \subset (\bar{M})^{-1}$$

Such f are determined by the restriction to $\mathbb{R}$ which is a map $f: \mathbb{R} \longrightarrow \Gamma^0 \simeq PSL_2(\mathbb{R})$.

First problem is to get ∂D related to $\mathbb{R}$. So I want to convert the matrices occurring in Lee-Yang to the ones occurring in the Jacobi problem. Any isom

$$|y| < 1 \longleftrightarrow \operatorname{Im}(w) > 0$$

will be unique up to an element of $PSL_2(\mathbb{R})$. The simplest is given by

$$y : \begin{array}{ccc} 1 & \longleftrightarrow & 0 \\ i & \longleftrightarrow & 1 \\ -1 & \longleftrightarrow & \infty \end{array} : w \qquad \begin{array}{l} y(-iw+1) = +iw+1 \\ -(yi+i)w = 1-y \end{array}$$

$$\left\{ \begin{array}{l} y = \dfrac{+iw+1}{-iw+1} \\[2ex] w = \dfrac{1}{i}\dfrac{y-1}{y+1} \end{array} \right. \qquad w = -\frac{1-y}{i(1+y)} = -i\,\frac{y-1}{y+1}$$

$$y = 0 \Leftrightarrow w = i$$

~~Thus multiplication by λ^2 in the y plane~~

$$\frac{1}{2i}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}\begin{pmatrix} \lambda i & \lambda \\ -\lambda^{-1} i & \lambda^{-1} \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} \lambda i + \lambda^{-1} i & \lambda - \lambda^{-1} \\ -\lambda + \lambda^{-1} & \lambda i + i\lambda^{-1} \end{pmatrix}$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ has i as fixpt

$$\frac{ai+b}{ci+d} = i$$

$$\Rightarrow \quad ai+b = -c+di \quad \Rightarrow \quad a=d, \; b=-c$$

$$\text{and} \quad ad-bc = a^2+b^2 = 1.$$

Thus the rotation matrices $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ leave i fixed. Observe that for $\theta = \pi$ this gives $-I$ which acts as identity on UHP. Thus

$$y \mapsto e^{-i\theta} y \qquad \text{corresponds to}$$

$$w \mapsto \frac{\left(\cos\frac{\theta}{2}\right) w + \left(\sin\frac{\theta}{2}\right)}{-\left(\sin\frac{\theta}{2}\right) w + \left(\cos\frac{\theta}{2}\right)}$$

Thus the transformations leading to Lee-Yang polys:

$$\begin{pmatrix} z^{\frac{1}{2}} & 0 \\ 0 & z^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$$

$z = e^{i\theta}$

can be more generally any elt. of Γ

will correspond to the transformations

$$\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1-a & 0 \\ 0 & 1+a \end{pmatrix}$$

$w \mapsto \left(\frac{1-a}{1+a}\right) w$ so need $-1 < a < 1$

can be more generally any elt. of $SL_2(\mathbb{R})$.

I am interested in all matrices of the form

$$\varphi(\theta) = \left(R_{(\theta/2)} A_1\right)\left(R_{(\theta/2)} A_2\right) \cdots \left(R_{\theta/2} A_n\right)$$

where $A_1, \ldots, A_n \in SL_2(\mathbb{R})$ and $R_{(\theta/2)} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$.

~~Suppose $\varphi(\theta) = I$, i.e. $A_1 \cdots A_n = I$. Then~~ Putting

$$B_i = A_i A_{i+1} \cdots A_n$$

we have $A_i = B_i B_{i+1}^{-1}$ for $i = 1, \ldots, n-1$ and $A_n = B_n = B_n B_1^{-1}$ if $B_1 = A_1 \cdots A_n = I$.

$$\varphi(\theta) = R_{\theta/2} B_1 \left(B_2^{-1} R_{\theta/2} B_2\right) \cdots \left(B_n^{-1} R_{\theta/2} B_n\right)$$

Consequently we see that $\varphi(\theta)$ is $\underset{= \varphi(0)}{B_1}$ times conjugates $B_i^{-1} R_{(\theta/2)} B_i$ of the basic loop $\theta \mapsto R_{(\theta/2)}$ in $PSL_2(\mathbb{R})$.

Let $M = \{\theta \in PSL_2(\mathbb{C}) \mid \theta \bar{H} \subset \operatorname{Int} \bar{H}\}$ $\qquad \bar{H} = \{w \mid \operatorname{Im}(w) \geq 0\}$

and let $\bar{M}$ = closure of M. To describe elements of $\partial M - PSL_2(\mathbb{R})$. These are θ carrying $P_1(\mathbb{R}) = \partial H$ to a "generalized" circle in $\bar{H}$ tangent to ∂H at one point. Changing θ to $\varphi^{-1}\theta$ with $\varphi \in \Gamma$, we can assume $\varphi^{-1}\theta(\partial H)$ tangent to ∂H at ∞, hence $\varphi^{-1}\theta(\partial H) = \{w \mid \operatorname{Im}(w) = c > 0\}$. Thus we see that if $\tau_c(w) = w + c$, then $\varphi^{-1}\theta(\partial H) = \tau_c(\partial H)$. So $\theta = \varphi \tau_c \varphi_1$ some $\varphi_1 \in \Gamma$.

Put $B=\left\{\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix}\in\Gamma \,\middle|\, a>0,\ b\in\mathbb{R}\right\}$. Then we get the description

$$\Gamma \times^{B} \left\{\begin{pmatrix} a & \tau \\ & a^{-1}\end{pmatrix} \,\middle|\, \operatorname{Im}\tau>0\right\} \times^{B} \Gamma$$

for those $\theta\in G=PSL_2(\mathbb{C})$ such that $\theta\in\bar{M}-M-\Gamma$. Now

$$B\backslash\Gamma/B = B\backslash P_1(\mathbb{R}) = 2\text{-pts.}$$

Given a linear Ising chain with periodic conditions does there exist an eigenvalue problem whose characteristic roots are the roots of the partition function?

Then the partition function is the trace of a certain 2×2 matrix which one can adjust to have determinant $+1$. For the trace to vanish means then that it has the eigenvalues $+1, -1$.

Look at the linear Ising chain with the same constants

$$P(z) = tr\left[\begin{pmatrix} z^{\frac12} & 0\\ 0 & z^{-\frac12}\end{pmatrix}\begin{pmatrix} 1 & a\\ a & 1\end{pmatrix}\right]^n = \lambda_+^n+\lambda_-^n .$$

Make this matrix of determinant $+1$

$$P(z) = tr\Bigg(\underbrace{\begin{pmatrix} z^{\frac12} & 0\\ 0 & z^{-\frac12}\end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1-a^2}} & \frac{a}{\sqrt{1-a^2}}\\ \frac{a}{\sqrt{1-a^2}} & \frac{1}{\sqrt{1-a^2}}\end{pmatrix}}_{\text{has trace } \frac{z^{\frac12}+z^{-\frac12}}{\sqrt{1-a^2}}}\Bigg)^n = \lambda_+^n+\lambda_-^n \qquad \lambda_+\lambda_-=1$$

$P(z)=0 \quad \Rightarrow \quad (\lambda_+/\lambda_-)^{n} = -1 \quad , \quad \lambda_+/\lambda_- = e^{+\frac{2j+1}{n}\pi i}$. Thus

$$\lambda_\pm = e^{\pm\frac{2j+1}{2n}\pi i}$$

so the roots of $P(z)$ are given by $\quad z = e^{i\varphi}$ where

$$\frac{\cos(\varphi/2)}{\sqrt{1-a^2}} = \cos\left(\frac{2j+1}{2n}\pi\right)$$

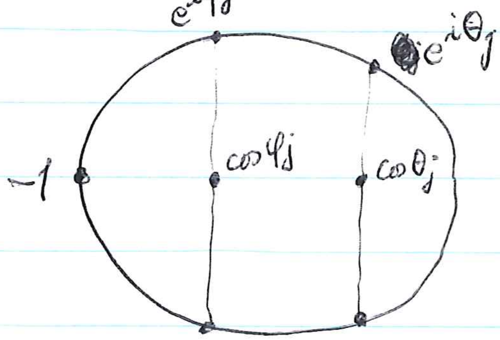
$$\cos(\varphi/2) = \sqrt{1-a^2}\,\cos\left(\frac{2j+1}{2n}\pi\right)$$

$$\cos\varphi = 2\cos^2\left(\frac{\varphi}{2}\right) - 1 = 2(1-a^2)\cos^2\left(\frac{2j+1}{2n}\pi\right) - 1$$

$$= (1-a^2)\left(1+\cos\left(\frac{2j+1}{n}\pi\right)\right) - 1$$

$$\boxed{\cos\varphi_j = -a^2 + (1-a^2)\cos\left(\frac{2j+1}{n}\pi\right)}$$

Visualize this as homotopy with parameter $t = a^2$ moving the root pair $\exp\left(\pm i\left(\frac{2j+1}{n}\right)\pi\right)$ to -1.



If $\quad n$ is even

$$P(z) = z^{-\frac{n}{2}} \prod_{j=1}^{n/2} \left(z^2 - 2\left(-a^2+(1-a^2)\cos\left(\frac{2j+1}{n}\pi\right)\right)z + 1\right)$$

and if

n is odd

$$P(z) = z^{-\frac{n}{2}}(z+1)\prod_{j=1}^{[n/2]}\left(z^2 - 2\left(-a^2 + (1-a^2)\cos\left(\tfrac{2j+1}{n}\pi\right)\right)z + 1\right)$$

What is the Heilman-Lieb dimer limit of this partition function. Here $a = e^{-J\beta}$ and lets $\beta \to 0$ whence $a \nearrow 1$. Then all the roots tend to $z = -1$.

$$z^{1} - 2\left(\left(1-a^2\right)\left(1+\cos\left(\tfrac{2j+1}{n}\pi\right)\right) - 1\right) + z^{-1}$$

$$= \left(z + 2 + z^{-1}\right) - 2\left(1-a^2\right)\left(1 + \cos\left(\tfrac{2j+1}{n}\pi\right)\right)$$

$$1 - a^2 = 1 - e^{-2J\beta} = 2J\beta - 2J^2\beta^2 + \cdots$$

So to have an interesting limit one wants $\dfrac{z+2+z^{-1}}{\beta} = \dfrac{\left(z^{\frac{1}{2}} + z^{-\frac{1}{2}}\right)^2}{\beta}$ to have a limit. So if one puts

$$\frac{z^{\frac{1}{2}} + z^{-\frac{1}{2}}}{\beta^{1/2}} = 2x$$

$$\cos\frac{\varphi}{2} = \beta^{1/2}x \qquad \sin\frac{\varphi}{2} = \sqrt{1-\beta x}$$

$$\begin{cases} z^{\frac{1}{2}} = \beta^{1/2}x + i\sqrt{1-\beta x} \\ z^{-\frac{1}{2}} = \beta^{1/2}x - i\sqrt{1-\beta x} \end{cases}$$

$\to i$ as $\beta \to 0$

so $z \to -1$

Put $n = 2m + \varepsilon$ $\qquad \varepsilon = 0, 1$

$$P(z) = \underset{2\beta^{1/2}x}{\underset{\shortparallel}{\left(z^{\frac{1}{2}}+z^{-\frac{1}{2}}\right)}}^{\varepsilon} \prod_{j=1}^{m} \left(\underset{4\beta x^2}{\underset{\shortparallel}{z+2+z^{-1}}} - 2(2J\beta)\left(1+\cos\tfrac{2j+1}{n}\pi\right)\right)$$

$$= 2^n \beta^{n/2} x^{\varepsilon} \prod_{j=1}^{m} \left(x^2 - 2J\left(\frac{1+\cos\left(\frac{2j+1}{n}\pi\right)}{2}\right)\right)$$

$$= 2^n \beta^{n/2} \prod_{j=1}^{m} \left(x - \sqrt{2J}\cos\left(\frac{2j+1}{2n}\pi\right)\right)$$

which is essentially the Chebycheff poly. $\cos\left(n\cos^{-1}\left(\frac{x}{\sqrt{2J}}\right)\right)$.

Is it generally true that the dimer polynomial is obtained from an eigenvalue problem?

Review limit procedure before

$z = e^{i\varphi}$ $\qquad \cos\frac{\varphi_j}{2} = \sqrt{1-a^2}\,\cos\left(\frac{2j+1}{2n}\pi\right)$

Now we put $z^{\frac{1}{2}} = e^{iu/n}$ i.e. $\frac{u_j}{n} = \frac{\varphi_j}{2}$ and let $a \mapsto \frac{a}{n}$

$$\cos\left(\frac{u_j}{n}\right) = \left(1-\frac{a^2}{n^2}\right)^{1/2}\cos\left((2j+1)\frac{\pi}{2}\frac{1}{n}\right)$$

$$\Big\downarrow$$

$$1-\frac{1}{2}\left(\frac{u_j}{n}\right)^2 = \left(1-\frac{1}{2}\frac{a^2}{n^2}\right)\left(1-\frac{1}{2}\left((2j+1)\frac{\pi}{2}\right)^2\frac{1}{n^2}\right)$$

$$u_j^2 = a^2 + \left((2j+1)\frac{\pi}{2}\right)^2$$

Limiting Roots are $\qquad u_j = \pm\sqrt{a^2+\left((2j+1)\frac{\pi}{2}\right)^2}$

An important point to notice is that letting $a \to 0$ corresponds to $\beta \to \infty$, that is, to the <u>low-temperature</u> limit. Now if we fix the ~~parameter~~ particular Ising system under consideration then the ~~limit~~ low-temp limit is $1+z^n$.

Recall $\overline{M} = \{ \Theta \in SL_2(\mathbb{C}) \mid \Theta(\mathrm{Im}(w) \geq 0) \subset \mathrm{Im}(w) \geq 0 \}$.
Consider a 1-parameter subgroup e^{tA} in $SL_2(\mathbb{C})$. It induces a flow in $P_1(\mathbb{C})$ ~~which~~ which sends w to $e^{tA}w$ at time t. Thus we get a vector field on $P_1(\mathbb{C})$ and we can ask when the flow ~~maps~~ carries the UHP H into itself.

If $e^{tA} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ then

$$e^{tA} \cdot w = \frac{a(t)w + b(t)}{c(t)w + d(t)} \qquad A(0) = I$$

$$\frac{d}{dt} A(t)\, w \Big|_{t=0} = \frac{\big(c(t)w + d(t)\big)\big(a'(t)w + b'(t)\big) - \big(a(t)w + b(t)\big)\big(c'(t)w + d'(t)\big)}{\big(c(t)w + d(t)\big)^2}\Bigg|_{t=0}$$

$$= \frac{1\,\big(a'(0)w + b'(0)\big) - w\big(c'(0)w + d'(0)\big)}{1^2}$$

$$= -c'(0)\,w^2 + \big(a'(0) - d'(0)\big)w + b'(0)$$

Now $\begin{pmatrix} a'(0) & b'(0) \\ c'(0) & d'(0) \end{pmatrix} = \frac{d}{dt} e^{tA}\Big|_{t=0} = A$

So if $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ we want

$$w \in \mathbb{R} \implies \mathrm{Im}\big(-\gamma w^2 + (\alpha - \delta)w + \beta\big) \geq 0$$

$$-w^2(\mathrm{Im}\,\gamma) + w(\mathrm{Im}\,\alpha - \mathrm{Im}\,\delta) + \mathrm{Im}\,\beta \geq 0$$

which is the case iff

$$(\operatorname{Im}\alpha - \operatorname{Im}\delta)^2 \leq -4(\operatorname{Im}\gamma)(\operatorname{Im}\beta) \qquad \operatorname{Im}\gamma \leq 0 \qquad \operatorname{Im}\beta \geq 0$$

If A has trace zero, then $\operatorname{Im}\delta = -\operatorname{Im}\alpha$, so this becomes

$$(*) \qquad (\operatorname{Im}\alpha)^2 \leq (-\operatorname{Im}\gamma)(\operatorname{Im}\beta) \quad \text{and } \operatorname{Im}\gamma \leq 0, \ \operatorname{Im}\beta \geq 0.$$

Thus if $A = A_0 + iB_0$ with A_0, B_0 real, then A_0 can be arbitrary; $B_0 = \begin{pmatrix} p & q \\ -r & -p \end{pmatrix}$ with $r \geq 0$, $q \geq 0$ and $\det(B_0) = -p^2 + qr \geq 0$.

<u>March 9, 1977.</u>

To classify all $u \mapsto A(u)$ holomorphic such that $\operatorname{tr} A(u) = 0$, $u \in \mathbb{R} \Rightarrow A(u)$ real, $\operatorname{Im}(u) > 0 \Rightarrow A$ satisfies $(*)$ above, and $\operatorname{Im}(u) < 0 \Rightarrow A(u)$ satisfies the opposite of $(*)$.

$$\overline{A(\bar{u})} = A(u) \implies \operatorname{Im} A(\bar{u}) = -\operatorname{Im} A(u).$$ Suppose $A(u) = \begin{pmatrix} a(u) & b(u) \\ c(u) & -a(u) \end{pmatrix}$. Then

$$\operatorname{Im}(u) \begin{matrix} >0 \\ =0 \\ <0 \end{matrix} \implies \operatorname{Im}(b(u)) \begin{matrix} \geq 0 \\ = 0 \\ \leq 0 \end{matrix}$$

The open mapping thm. $\Rightarrow$ if $b \not\equiv 0$, then $\operatorname{Im}(u) > 0$

$\Rightarrow \operatorname{Im}(b(u)) > 0$. Thus $b^{-1}(\mathbb{R}) = \mathbb{R}$. The map $b: \mathbb{R} \to \mathbb{R}$ is etale: suppose $b'(x) = 0$; translate b so that $x = 0$ and $b = 0$; then $b(z) = z^n(a_0 + a_1 z + \cdots)$ around the origin looks like $z \mapsto z^n$, which does not preserve $\operatorname{Im} z > 0$ for $n > 1$. By Picard an entire function misses at most one value, hence $b: \mathbb{R} \to \mathbb{R}$ must be an isomorphism. Similarly by symmetry of $b(\mathbb{C})$ under conjugation, one sees that $b(\mathbb{C}) = \mathbb{C}$. b has a single zero, so

$$b(z) = z e^{h(z)}$$

with $h(z)$ entire, translating so that the zero is at zero. Scaling we can assume $h(0) = 1$. Note that h is real: $\overline{h(\bar{z})} = h(z)$.

Look at the critical values of the map $b: \mathbb{C} \to \mathbb{C}$. Note that the tangent space map is either 0 or an isomorphism, and is never of rank 1. Thus the inverse image of a smooth embedded curve in $\mathbb{C}$ under b which avoids the critical points is a smooth codim. 1 submanifold of $\mathbb{C}$ etale over the given curve.

Wait: the big Picard theorem says that in a nbd of essential singularity at most one value is omitted, I think. This means that if b is not a polynomial, then for some $a \in \mathbb{R}$ $b(z) - a$ has infinitely many zeroes.

So it seems that the only possibility for $b(z)$ is

$$b(u) = g_1 u + h_1 \qquad h_1 \in \mathbb{R},\ g_1 \geq 0. \quad \text{similarly}$$
$$c(u) = -g_2 u + h_2 \qquad h_2 \in \mathbb{R},\ g_2 \geq 0.$$

$$\operatorname{Im}(a(u))^2 \leq g_1 g_2 \operatorname{Im}(u)^2$$

$$|\operatorname{Im}(a(u))| \leq \sqrt{g_1 g_2}\, |\operatorname{Im}(u)|.$$

$$\left|e^{+ia(u)}\right| = e^{\operatorname{Re}(ia(u))} = e^{-\operatorname{Im}(a(u))} \leq e^{|\operatorname{Im}(a(u))|} \leq e^{c|\operatorname{Im}(u)|}$$
$$\leq e^{c|u|}$$

so $e^{ia(u)}$ is an entire function of exp. type without zeroes hence is $e^{P(u)}$, $\deg P(u) \leq 1$. $\therefore$ $a(u)$ linear in u.

Therefore it seems that the only possibilites for a holom. function

$$u \longmapsto \begin{pmatrix} a(u) & b(u) \\ c(u) & -a(u) \end{pmatrix}$$

of the good type is linear

$$\underbrace{\begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix}}_{\text{arbitrary}} + u \underbrace{\begin{pmatrix} p & q \\ -r & -p \end{pmatrix}}_{q, r \geq 0,\ qr - p^2 \geq 0.}$$

In the past we've seen the following examples for $\begin{pmatrix} p & q \\ -r & -p \end{pmatrix}$:

, $\begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}$

But note

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} +b & d \\ -a & -c \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} ab+cd & b^2+d^2 \\ -a^2-c^2 & -ab-cd \end{pmatrix}$$

Observe that any $\begin{pmatrix} p & q \\ -r & -p \end{pmatrix}$ with $-p^2+qr = 1$ and $q \geq 0$, $r \geq 0$ is in this form. In effect take vector (a,c) of length $r^{\frac{1}{2}}$, (b,d) of length $q^{1/2}$ and with angle θ between them given by

$$\cos\theta = \frac{p}{q^{1/2}r^{1/2}}$$

$$-p^2+qr \geq 0$$
$$\Rightarrow \quad p^2 \leq qr$$
$$-1 \leq \frac{p}{\sqrt{qr}} \leq 1$$

so that $(a,c)\cdot(b,d) = \frac{p}{q^{1/2}r^{1/2}}\, q^{1/2}r^{1/2} = p$. Then from the identity

$$(ab+cd)^2 + (ad-bc)^2 = a^2b^2+c^2d^2+a^2d^2+b^2c^2 = (a^2+c^2)(b^2+d^2)$$

one gets $ad-bc = \pm 1$, and you can makes the sign $+1$ by interchanging (a,C), (b,d).

Next $$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & +d \\ 0 & -c \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix}$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} +b & 0 \\ -a & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} +ab & b^2 \\ -a^2 & -ab \end{pmatrix}$$

Now given $\begin{pmatrix} p & q \\ -r & -p \end{pmatrix}$ $q, r \geq 0$ and $p^2 = qr$
put

$$d = \sqrt{q} \qquad c = \pm\sqrt{r} \qquad cd = p$$

At least if not both q, r are zero, then not both c, d are zero so we can find a, b with $ad - bc = 1$.
Summary.

<u>Proposition:</u> Let $A \in sl_2(\mathbb{C})$. Then the flow on $\mathbb{P}_1(\mathbb{C})$ induced by e^{tA} carries H into itself for $t \geq 0$ iff $\mathrm{Im}(A)$ has the form

$$(*) \quad \begin{cases} \mathrm{Im}(A) = \begin{pmatrix} p & q \\ -r & -p \end{pmatrix} \\ q \geq 0,\ r \geq 0 \text{ and } p^2 \leq qr. \end{cases}$$

The group $SL_2(\mathbb{R})$ acts on the non-zero matrices of the form $(*)$ and a cross-section for the orbits is the set

$$\begin{pmatrix} 0 & 1 \\ -r & 0 \end{pmatrix} \qquad r \geq 0.$$