so in the limit as $n \to \infty$ we get

$$\int \frac{d\mu(x)}{z-x} = \frac{1}{b_1+z+}\;\frac{-a_1^2}{b_2+z+}\cdots\cdots\frac{-a_{n-1}^2}{b_n+z+}\cdots\cdots$$

<u>March 1, 1977</u>:

Periodic Jacobi matrices

$$\tilde{J}_n = \begin{pmatrix} +b_1 & -a_1 & & & -a_n \\ -a_1 & b_2 & -a_2 & & \\ & -a_2 & & & \\ & & & & \\ & & & +b_{n-1} & -a_{n-1} \\ -a_n & & & -a_{n-1} & -b_n \end{pmatrix}$$

Compute $\det(\tilde{J}_n)$ using minor expansion along first column.

$$b_1 \begin{vmatrix} b_2 & -a_2 & & \\ -a_2 & \ddots & \ddots & \\ & \ddots & \ddots & -a_{n-1} \\ & & -a_{n-1} & b_n \end{vmatrix} + a_1 \begin{vmatrix} -a_1 & 0 & & & -a_n \\ -a_2 & b_3 & -a_3 & & \\ & -a_3 & b_4 & \ddots & \\ & & \ddots & \ddots & -a_{n-1} \\ & & & -a_{n-1} & b_n \end{vmatrix} + (-1)^n(-a_n) \begin{vmatrix} -a_1 & & & & -a_n \\ b_2 & -a_2 & & & \\ & \ddots & \ddots & & \\ & & -a_{n-2} & b_{n-1} & -a_{n-1} \end{vmatrix}$$

$$= \begin{cases} b_1 \begin{vmatrix} b_2 & -a_2 & & \\ -a_2 & \ddots & \ddots & \\ & \ddots & \ddots & -a_{n-1} \\ & & -a_{n-1} & b_n \end{vmatrix} + a_1(-a_1) \begin{vmatrix} b_3 & -a_3 & & \\ -a_3 & b_4 & \ddots & \\ & \ddots & \ddots & -a_{n-1} \\ & & -a_{n-1} & b_n \end{vmatrix} + a_1(-1)^{n-2}(-a_n) \begin{vmatrix} -a_2 & \ddots & \ddots \\ & \ddots & \ddots \\ 0 & & -a_{n-1} \end{vmatrix} \text{ triang.} \\ + (-1)^n(-a_n)(-a_1) \begin{vmatrix} -a_2 & & 0 \\ & \ddots & \\ & b_{n-1} & -a_{n-1} \end{vmatrix} \text{ triang.} + (-1)^n(-a_n)(-1)^{n-1}(-a_n) \begin{vmatrix} b_2 & -a_2 & & \\ -a_2 & b_3 & \ddots & \\ & \ddots & \ddots & \\ & & & b_{n-1} \end{vmatrix} \end{cases}$$

So it's clear we get a formula relating $\det(\tilde{J}_n)$ to ordinary Jacobi determinants

$$\det(\tilde{J}_n) = b_1 \begin{vmatrix} b_2 & -a_2 & & \\ -a_2 & b_3 & -a_3 & \\ & -a_3 & \ddots & -a_{n-1} \\ & & -a_{n-1} & b_n \end{vmatrix} - a_1^2 \text{[redacted]} \begin{vmatrix} b_3 & -a_3 & & \\ -a_3 & b_4 & -a_4 & \\ & -a_4 & \ddots & -a_{n-1} \\ & & -a_{n-1} & b_n \end{vmatrix}$$

$$-a_n^2 \begin{vmatrix} b_2 & -a_2 & & \\ -a_2 & b_3 & \ddots & \\ & \ddots & \ddots & -a_{n-1} \\ & & -a_{n-1} & b_{n-1} \end{vmatrix} \;\; \bar{\text{[redacted]}}\; a_1 a_2^{\text{[redacted]}} a_3 a_4 \cdots a_n \;\; \bar{\text{[redacted]}}\; a_1 a_2 a_3 \cdots a_n$$

Simpler derivation would be to observe that as a function of $a_n$ $\det(\tilde{J}_n)$ is obviously quadratic

$$\det(\tilde{J}_n) = A + Ba_n + Ca_n^2$$

where $A = \text{[redacted]} \begin{vmatrix} b_1 & -a_1 & & \\ -a_1 & b_2 & \ddots & \\ & \ddots & \ddots & \\ & & & b_n \end{vmatrix}$ is obtained by putting $a_n = 0$

where $C = - \begin{vmatrix} b_2 & -a_2 & & \\ -a_2 & b_3 & \ddots & \\ & \ddots & \ddots & \\ & & & b_{n-1} \end{vmatrix}$ by minors

also $B = 2(-1)(-1)^{n-1}(-a_1)\cdots(-a_{n-1}) = -2a_1 \cdots a_{n-1}$

$$\begin{array}{l} \begin{vmatrix} -a_1 & \cdots & & -a_n \\ b_2 & \ddots & & \\ & \ddots & \ddots & \text{[redacted]} \\ & & b_{n-1} & -a_{n-1} \end{vmatrix} \\ -a_n \end{array}$$

Orthogonal functions on $\overset{S^1}{|z|=1}$. Let $d\mu(\theta)$ be a measure on $|z|=1$ whence we get a Hilbert space $H = L^2(S^1, d\mu)$ with unitary operator $U =$ mult. by $z$ and cyclic vector $v_0 = 1$. Now put

$$v_n = \text{component of } U^n v_0 \text{ perpendicular to } v_0, Uv_0, \ldots, U^{n-1}v_0$$

and let $\varphi_n(z)$ be the monic polynomial $\ni$ $v_n = \varphi_n(U)v_0$.

$$(Uv_n, Uv_i) = (v_n, v_i) = 0 \qquad \text{for} \quad i = 0, \ldots, n-1$$

Thus

$$Uv_n = v_{n+1} - c_{n+1} v_0 \qquad -c_{n+1} = (Uv_n, v_0)/\|v_0\|^2$$

So the way to express this is to work with $U^{-n}v_n \longleftrightarrow \varphi_n(z)/z^n = 1 + c_1 z^{-1} + \cdots + c_n z^{-n}$. Put

$$f(z) = 1 + c_1 z^{-1} + c_2 z^{-2} + \cdots$$

formal power series. Put $D^- =$ closed span in $H$ of $1, z^{-1}, z^{-2}, \ldots = v_0, U^{-1}v_0, U^{-2}v_0, \ldots$. There are two cases. If $U^{-1}D^- < D^-$, then since $U^{-1}D^- + \mathbb{C}v_0 = D^-$, one sees that $f(z) \longleftrightarrow f(U)v_0$ is the component of $v_0$ perpendicular to $U^{-1}D^-$. The other case is $U^{-1}D^- = D^-$ whence $U^{-n}v_n \longrightarrow 0$ in $H$.

Let us also assume that we are in the first case and that $f$ is a cyclic vector for $U$. This is equivalent to the requirement that $D^-$ be an incoming subspace for $U$.

Then by sending $z^n \longrightarrow \frac{U^n f}{\|f\|}$ we get an isomorphism $L^2(S_1, \frac{d\theta}{2\pi})$ with $H$.

$$p(z) \longmapsto \frac{p(z) f(z)}{\|f\|}$$

$$\|f\| \frac{z^n}{f(z)} \longleftarrow z^n$$

so

$$\int z^{n-m} \, d\mu(\theta) = \|f\|^2 \int \frac{z^{n-m}}{|f(z)|^2} \frac{d\theta}{2\pi}.$$

all $n, m$ so

$$d\mu(\theta) = \|f\|^2 \frac{1}{|f(z)|^2} \frac{d\theta}{2\pi}$$

which shows $d\mu$ is absolutely continuous with respect to Lebesgue measure. If

$$d\mu(\theta) = g(\theta) \frac{d\theta}{2\pi}$$

then we have factored $g(\theta) = \frac{\|f\|}{f(z)} \cdot \overline{\frac{\|f\|}{f(z)}}$

Goal: Take a suitable limit to get a suitable 2nd order DE.

First try might be to replace the formula

$$\begin{pmatrix} A_{p-1} & A_p \\ B_{p-1} & B_p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & z+b_1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & -a_{p-1}^2 \\ 1 & z+b_p \end{pmatrix}$$

by a DE.

$$\begin{pmatrix} B_{p-1} & B_p \end{pmatrix} = \begin{pmatrix} B_{p-2} & B_{p-1} \end{pmatrix} \begin{pmatrix} 0 & -a_{p-1}^2 \\ 1 & z+b_p \end{pmatrix}$$

One difficulty is that the matrix ↑ is far from the identity matrix

$$\begin{pmatrix} B_{p-1}-B_{p-2} & B_p-B_{p-1} \end{pmatrix} = \begin{pmatrix} B_{p-2} & B_{p-1} \end{pmatrix} \begin{pmatrix} -1 & -a_{p-1}^2 \\ 1 & z+b_p-1 \end{pmatrix}$$

---

Let $J$ be an infinite periodic Jacobi matrix

$$\begin{matrix} -b_1 & a_1 & & \\ a_1 & -b_2 & \ddots & \\ & \ddots & \ddots & \\ & & & \ddots \end{matrix}$$

with period $n$: $a_{i+n} = a_i$, $b_{i+n} = b_i$. Then the continued fraction associated to $J$ is periodic so if $f(z)$ is the limit of the continued fraction, then $f(z)$ should be quadratic over $\mathbb{C}[z]$. Precisely we have

$$\frac{A_p}{B_p} = \begin{pmatrix} 0 & 1 \\ 1 & z+b_1 \end{pmatrix} \begin{pmatrix} 0 & -a_1^2 \\ 1 & z+b_2 \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & -a_{p-1}^2 \\ 1 & z+b_p \end{pmatrix} (0)$$

so if we let $\Phi(z) = \begin{pmatrix} 0 & -a_1^2 \\ 1 & z+b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & -a_n^2 \\ 1 & z+b_{n+1} \end{pmatrix}$

then $g(z) = \begin{pmatrix} 0 & 1 \\ 1 & z+b_1 \end{pmatrix}^{-1}(f(z)) = \lim_{\substack{\longrightarrow \\ n}} \left(\Phi(z)\right)^n$

Thus $g(z)$ is fixed under the fractional linear transformation $\Phi(z)$, so $g(z)$ is quadratic over the field of rational functions. So we get a hyperelliptic function field.

Next I should look at a doubly-infinite periodic J-matrix. Then on the Hilbert space of square summable sequences we have the bounded self-adjoint operator $J$ and a unitary operator $T$ given by shift $n$-steps and these commute: $TJT^{-1} = J$.

<u>March 3, 1976</u>. Look at a doubly-infinite J-matrix

$$\begin{matrix} \ddots & & & & \\ a_0 & -b_1 & a_1 & & \\ & a_1 & -b_2 & a_2 & \\ & & a_2 & -b_3 & \\ & & & & \ddots \end{matrix}$$

which is periodic with period $h$: $a_{i+h} = a_i$, $b_{i+h} = b_i$.

[redacted]

View this matrix as acting on $\bigoplus_{\mathbb{Z}} \mathbb{C}$ with standard basis $e_n$ $n \in \mathbb{Z}$. Maybe I should think of this as $\mathbb{C}[z, z^{-1}]$, $e_n = z^n$, so that $z e_n = e_{n+1}$. Now the periodicity condition says that for period $r$

$$J(z^r e_i) = J e_{i+r}$$

$$= + a_{i+r-1} e_{i+r-1} - b_{i+r} e_{i+r} + a_{i+r} e_{i+r+1}$$

$$= z^r \left( a_{i-1} e_{i-1} - b_i e_i + a_i e_{i+1} \right)$$

$$= z^r J e_i$$

so $J \cdot z^r = z^r \cdot J$. This means that $J$ is given by an $r \times r$ matrix over $\mathbb{C}[z^r, z^{-r}]$. Specifically take the basis $e_1, \dots, e_r$ for $\mathbb{C}[z, z^{-1}]$ over $\mathbb{C}[z^r, z^{-r}]$, then

$$J e_1 = a_0 e_0 - b_1 e_1 + a_1 e_2$$

$$= - b_1 e_1 + a_1 e_2 \qquad + a_0 z^{-r} e_r$$

$$J e_2 = a_1 e_1 - b_2 e_2 + a_2 e_3$$

$$J e_{r-1} = a_{r-2} e_{r-2} - b_{r-1} e_{r-1} + a_{r-1} e_r$$

$$J e_r = a_{r-1} e_{r-1} - b_r e_r + a_r e_{r+1}$$

$$= a_{r-1} e_{r-1} - b_r e_r + a_r z^r e_1$$

so we get the following matrix

$$\begin{pmatrix} -b_1 & a_1 & & & & a_n z^{-n} \\ a_1 & -b_2 & a_2 & & & \\ & a_2 & -b_3 & a_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-2} & -b_{n-1} & a_{n-1} \\ a_n z^n & & & & a_{n-1} & -b_n \end{pmatrix}$$

So specializing to $z=1$ I get the general "finite" Jacobi matrix which is periodic.

---

How to calculate the spectrum of the periodic Jacobi matrix. You set it up as a periodic solutions of the 2nd order difference equation defined by the infinite matrix. So we want to find a vector $y_n$ $n \in \mathbb{Z}$ such that $Jy = \lambda y$ i.e.

$$a_{n-1} y_{n-1} - b_n y_n + a_n y_{n+1} = \lambda y_n$$

Now because all $a_i > 0$ one gets solutions of this D.E. by recursion starting from arbitrary values for $y_0$ and $y_1$.

$$y_{n+1} = a_n^{-1}\left((\lambda + b_n) y_n - a_{n-1} y_{n-1}\right)$$

$$y_{n+1} = y_{n-1}\left(-\frac{a_{n-1}}{a_n}\right) + y_n\left(\frac{\lambda+b_n}{a_n}\right)$$

$$(y_n \;\; y_{n+1}) = (y_{n-1} \;\; y_n)\begin{pmatrix} 0 & -\frac{a_{n-1}}{a_n} \\ 1 & \frac{\lambda+b_n}{a_n} \end{pmatrix}$$

so

$$(y_r \;\; y_{r+1}) = (y_0 \;\; y_1)\begin{pmatrix} 0 & -\frac{a_0}{a_1} \\ 1 & \frac{\lambda+b_1}{a_1} \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & -\frac{a_{r-1}}{a_r} \\ 1 & \frac{\lambda+b_r}{a_r} \end{pmatrix}$$

For a periodic solution we want $(y_r \; y_{r+1}) = (y_0 \; y_1)$. Let $\Phi(\lambda)$ be the $r$-fold product matrix. Then $\Phi(\lambda)$ is a $2\times 2$ matrix of degree $r$ in $\lambda$ whose determinant is

$$\frac{a_0}{a_1}\cdots\cdots\frac{a_{r-1}}{a_r} = \frac{a_0}{a_r} = 1 .$$

Hence if it has the eigenvalue 1, both eigenvalues are 1, and this is the case iff $\operatorname{tr}\Phi(\lambda) = 2$. As

$$\operatorname{tr}\Phi(\lambda) \equiv \operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & \frac{\lambda}{a_1} \end{pmatrix}\cdots\begin{pmatrix} 0 & 0 \\ 0 & \frac{\lambda}{a_n} \end{pmatrix} \equiv \frac{\lambda^n}{a_1\cdots a_n}$$

modulo lower terms it follows that

$$a_1\cdots a_n\left(\operatorname{tr}(\Phi(\lambda)) - 2\right) = \det(\lambda I_r - \tilde{J}_r).$$

This is clear if ~~all~~ all the eigenvalues are distinct and true probably in general by specialization. Also I can probably prove it from the recurrence formula.

One sees also that the eigenvalues of $\tilde{J}_r$ have at most multiplicity 2. This happens exactly when $\Phi(\lambda) = I$.

$$\lambda \text{ eigenvalue of } \tilde{J}_r \iff tr(\Phi(\lambda) - I) = 0$$
$$\lambda \text{ double eigenvalue of ''} \iff \Phi(\lambda) = I.$$

---

Generalization. Suppose we take a periodic Jacobi matrix which is not symmetric

$$\begin{pmatrix} +b_1 & a_1 & & & & c_1 \\ c_2 & +b_2 & a_2 & & & \\ & c_3 & +b_3 & a_3 & & \\ & & & & & \\ & & & c_{r-1} & b_{r-1} & a_{r-1} \\ a_r & & & & c_r & +b_r \end{pmatrix}$$

$$a_n y_{n+1} + b_n y_n + c_n y_{n-1} = \lambda y_n$$

$$y_{n+1} = a_n^{-1}\left( -c_n y_{n-1} + (\lambda - b_n) y_n \right)$$
$$= \frac{-c_n}{a_n} y_{n-1} + \frac{\lambda - b_n}{a_n} y_n$$

$$(y_n \;\; y_{n+1}) = (y_{n-1} \;\; y_n) \begin{pmatrix} 0 & -\frac{c_n}{a_n} \\ 1 & \frac{\lambda - b_n}{a_n} \end{pmatrix}$$

$$(y_r \;\; y_{r+1}) = (y_0 \;\; y_1) \begin{pmatrix} 0 & -\frac{c_1}{a_1} \\ 1 & \frac{\lambda - b_1}{a_1} \end{pmatrix} \cdots \begin{pmatrix} 0 & -\frac{c_r}{a_r} \\ 1 & \frac{\lambda - b_r}{a_r} \end{pmatrix}$$

$$\Phi(\lambda) = \begin{pmatrix} 0 & -\frac{c_1}{a_1} \\ 1 & \frac{\lambda - b_1}{a_1} \end{pmatrix} \cdots \begin{pmatrix} 0 & -\frac{c_n}{a_n} \\ 1 & \frac{\lambda - b_n}{a_n} \end{pmatrix}$$

$$\det \Phi(\lambda) = \frac{c_1 \cdots c_n}{a_1 \cdots a_n}$$

If $\tilde{J}$ is hermitian $\overline{a_i} = c_{i+1}$ so that $\det \Phi(\lambda) = \frac{\overline{a_1 \cdots a_n}}{a_1 \cdots a_n}$ is of absolute value 1. Put

$$\omega = \frac{c_1 \cdots c_n}{a_1 \cdots a_n} = \det \Phi(\lambda).$$

Now $\lambda$ is an eigenvalue for $\tilde{J}$ iff 1 is an eigenvalue for $\Phi(\lambda)$. In this case the eigenvalues are $1, \omega$. Thus 1 is an eigenvalue for $\Phi(\lambda) \iff \operatorname{tr} \Phi(\lambda) = 1 + \omega$.
Thus

$$a_1 \cdots a_n \left( \operatorname{tr}(\Phi(\lambda)) - 1 - \omega \right) = \det(\lambda I - \tilde{J})$$

If I put

$$\Psi(\lambda) = \begin{pmatrix} 0 & -c_1 \\ a_1 & \lambda - b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -c_n \\ a_n & \lambda - b_n \end{pmatrix}$$

then we find

$$\begin{cases} \operatorname{tr}(\Psi(\lambda)) - a_1 \cdots a_n - c_1 \cdots c_n = \det(\lambda I - \tilde{J}) \\ \det \Psi(\lambda) = a_1 \cdots a_n c_1 \cdots c_n \end{cases}$$

Review: We label according to the simple root and so use the notation

$$\tilde{J} = \begin{pmatrix} b_1 & a_1 & & & c_r \\ c_1 & b_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{r-1} \\ a_r & & & c_{r-1} & b_r \end{pmatrix}$$

for the finite periodic matrix and $J$ for the infinite periodic matrix. We can identify $\tilde{J}$ acting on $\mathbb{C}^r$ with $J$ acting on $r$-periodic vectors $y = (y_n)$ in $\mathbb{C}^\infty$.

$$(Jy)_n = c_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n$$

$$y_{n+1} = -\frac{c_{n-1}}{a_n} y_{n-1} + \frac{\lambda - b_n}{a_n} y_n$$

$$\begin{pmatrix} y_n & y_{n+1} \end{pmatrix} = \begin{pmatrix} y_{n-1} & y_n \end{pmatrix} \begin{pmatrix} 0 & -\frac{c_{n-1}}{a_n} \\ 1 & \frac{\lambda - b_n}{a_n} \end{pmatrix}$$

Thus

$$\begin{pmatrix} y_r & y_{r+1} \end{pmatrix} = \begin{pmatrix} y_0 & y_1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -\frac{c_0}{a_1} \\ 1 & \frac{\lambda - b_1}{a_1} \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & -\frac{c_{r-1}}{a_r} \\ 1 & \frac{\lambda - b_r}{a_r} \end{pmatrix}}_{\Phi(\lambda)}$$

$$\det \Phi(\lambda) = \frac{c_1 \cdots c_r}{a_1 \cdots a_r}$$

$$a_1 \cdots a_r \left( tr(\Phi(\lambda)) - 1 - \frac{c_1 \cdots c_r}{a_1 \cdots a_r} \right) = \det\left(\lambda I_r - \tilde{J}\right)$$

Fix a non-zero complex number $z$. Then we can consider $J$ acting on vectors $y = (y_n)$ such that $y_{n+r} = z y_n$. This operator is equivalent to the operator on $\mathbb{C}^r$ given by the matrix

$$\tilde{J}_z = \begin{pmatrix} b_1 & a_1 & & & c_r z^{-1} \\ c_1 & b_2 & a_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_{r-1} \\ a_r z & & & c_{r-1} & b_r \end{pmatrix}$$

i.e.
$$(Jy)_1 = c_0 y_0 + b_1 y_1 + a_1 y_2 = b_1 y_1 + a_1 y_2 \cdots + c_0 z^{-1} y_r$$
$$(Jy)_r = c_{r-1} y_{r-1} + b_r y_r + a_r y_{r+1} = a_r z\, y_1 + c_{r-1} y_{r-1} + b_r y_r$$

The eigenvalues of $\tilde{J}_z$ are those $\lambda$ such that $\Phi(\lambda)$ has the eigenvalue $z$ since

$$(y_r \;\; y_{r+1}) = z(y_0 \;\; y_1)$$
$$\shortparallel$$
$$(y_0, y_1)\Phi(\lambda).$$

Thus
$$\det(\lambda I_r - \tilde{J}_z) = a_1 \cdots a_r \left( \operatorname{tr} \Phi(\lambda) - z - \frac{c_1 \cdots c_r}{a_1 \cdots a_r} \frac{1}{z} \right)$$
$$= -a_1 \cdots a_r z^{-1} \left( z^2 - (\operatorname{tr} \Phi(\lambda)) z + \det \Phi(\lambda) \right)$$
$$= (-a_1 \cdots a_r z^{-1}) \det(z I_2 - \Phi(\lambda))$$

So now I understand the spectrum of $J$ in the hermitian case $c_i = \bar{a}_i$. The joint spectrum of $J$ and the translation operator

$$(Ty)_n = y_{n+r} \qquad \text{(shifts backwards } r\text{-steps)}$$

is the set of pairs $(z, \lambda)$ with $|z| = 1$ and

$$\det\left(\lambda I_r - \tilde{J}_z\right) = 0$$

which forces $\lambda$ to be real. Now the spectrum of $J$ is obtained by taking the image under $(z,\lambda) \mapsto \lambda$.

Suppose we change variables $y_n' = d_n y_n$ where $d_n \neq 0$. Then $a_n' = d_{n+1} a_n d_n^{-1}$, $c_n' = d_{n+1}^{-1} c_n d_n$. Thus we can alter $a_1, \dots, a_r$ in any fashion we wish provided we don't change the product $a_1 \cdots a_r$. The new matrix is $J' = (d_n)^{-1} J (d_n)$, so if $J$ is hermitian, $J'$ will be hermitian if $|d_n| = 1$ for all $n$.

So I can arrange that $a_1, \dots, a_{r-1} > 0$ and then if $a_r = e^{i\theta}|a_r|$, I can replace $z$ by $e^{+i\theta} z$. So without altering spectrum in the hermitian case we can always suppose $a_1, \dots, a_r > 0$ and $c_i = a_i$, whence we have $\det(\Phi(\lambda)) = 1$ identically.

A similar conclusion holds in the algebraic case – we can arrange that $a_1 = a_2 = \cdots = a_r = 1$ by changing $z$ to $cz$, $c \neq 0$ and $y_n$ to $d_n y_n$.

So suppose $a_i = c_i > 0$. For each $|z| = 1$ we have $n$ real values of $\lambda$ in the spectrum, and since $\det \Phi(\lambda) = 1$, for each $\lambda$ we have 2 values of $z$ which are inverses of each other. So the spectrum of $J$ consists of those $\lambda$ such that $-2 \leq tr(\Phi(\lambda)) \leq 2$.

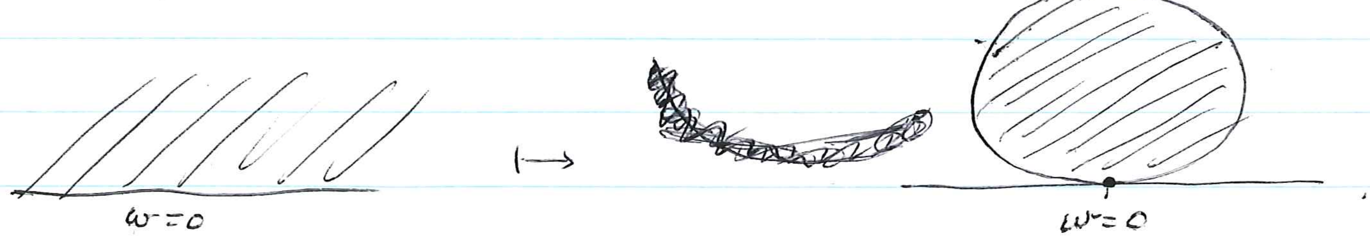
I'd like to prove that $-2 \leq tr(\Phi(\lambda)) \leq 2$ forces $\lambda$ to be real. This is clear because the condition forces $|z| = 1$, hence $\tilde{J}_z$ is hermitian, and so the eigenvalues are real. Another proof goes as follows. Consider the fractional linear transformation belonging to a matrix

$$\begin{pmatrix} 0 & -\alpha \\ 1 & t \end{pmatrix} \qquad \begin{matrix} \alpha > 0 \\ Im(t) > 0 \end{matrix}$$

i.e.

$$w \longmapsto \frac{-\alpha}{w + t}$$

Look at the image of $\mathbb{R}$. $w \in \mathbb{R} \Rightarrow Im(w+t) > 0$ $\Rightarrow Im\left(\frac{1}{w+t}\right) < 0 \Rightarrow Im\left(\frac{-\alpha}{w+t}\right) > 0$. Hence this fractional linear transformation carries $Im(w) \geq 0$ strictly into itself

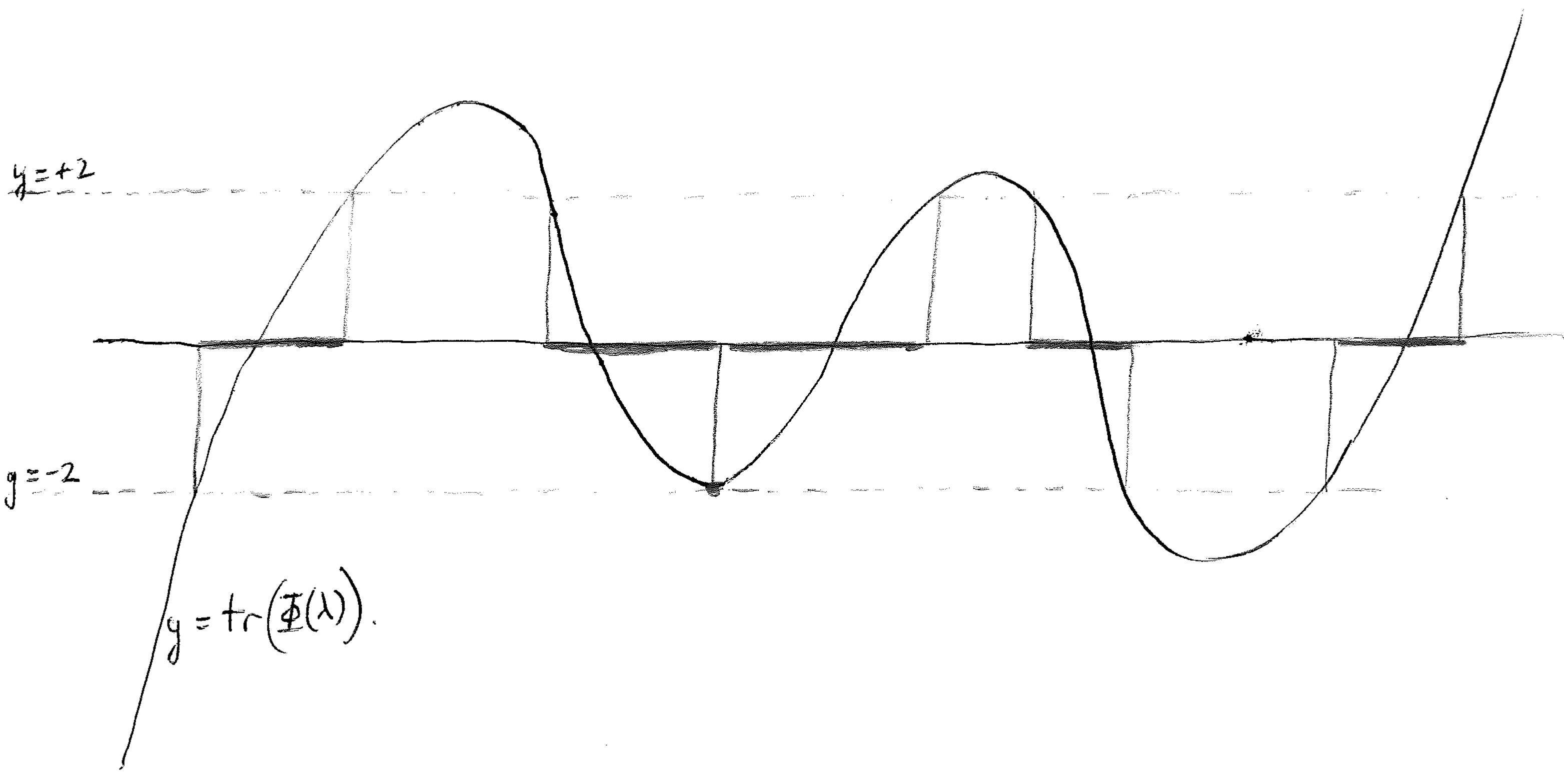



So no product of matrices of this form can be elliptic i.e. have eigenvalues on the unit circles. Hence

$$\Phi(\lambda) = \begin{pmatrix} 0 & -\frac{c_0}{a_1} \\ 1 & \frac{\lambda - b_1}{a_1} \end{pmatrix} \cdots \begin{pmatrix} 0 & -\frac{c_{n-1}}{a_n} \\ 1 & \frac{\lambda - b_n}{a_n} \end{pmatrix}$$

cannot have its trace on $[-2,2]$ for $\mathrm{Im}(\lambda) > 0$, and a similar argument works for $\mathrm{Im}(\lambda) < 0$.

Hence the graph of $y = \mathrm{tr}(\Phi(\lambda))$ looks as follows:




Note that when $\lambda$ is a multiple eigenvalue it has multiplicity 2, because eigen~~functions~~vectors for $J$ are given by eigenvectors for $\Phi(\lambda)$. So the multiple case occurs when $z = \pm 1$ is a double root of $\det(\lambda I - \tilde{J}_z)$. This happens when the graph is tangent to $y = \pm 2$.

Look at the curve $C_a$ defined by

$$\det(\lambda I - \tilde{J}_2) = 0$$

$$z^2 - \operatorname{tr}\Phi(\lambda)\, z + \omega = 0 \qquad \omega = \frac{c_1 \cdots c_n}{a_1 \cdots a_n}$$

$$z = \frac{1}{2\omega}\operatorname{tr}\Phi(\lambda) \pm \frac{1}{\omega}\sqrt{\left(\tfrac{1}{2}(\operatorname{tr}\Phi(\lambda))\right)^2 - \omega}$$

This is an affine curve. To complete it we let $\lambda \to \infty$

$$\operatorname{tr}\Phi(\lambda) \sim \frac{1}{a_1 \cdots a_n}\lambda^n$$

So the two roots $z_1, z_2$ as $\lambda \to \infty$ are

$$z_1 \sim \frac{1}{a_1 \cdots a_n}\lambda^n \qquad z_2 \sim \frac{c_1 \cdots c_n}{\lambda^n}.$$

Thus we have to add two points $\lambda = \infty, z = \infty$ and $\lambda = \infty, z = 0$ to the affine curve $C_a$ to obtain $C$. Over $C_a$ we have a line bundle whose fibre over $\lambda, z$ is the corresponding eigenspace, i.e. the null-space of $\lambda I - \tilde{J}_2$, which can be identified with the null-space of $zI - \Phi(\lambda)$. It is necessary to assume that $\operatorname{tr}\Phi(\lambda) = \pm 2\sqrt{\omega}$ has no double roots, which turns out to be the same as requiring $C_a$ to be non-singular. In effect a singular point occurs when

$$\left.\begin{aligned} z^2 - \operatorname{tr}\Phi(\lambda) z + \omega &= 0 \\ 2z - \operatorname{tr}\Phi(\lambda) &= 0 \\ -\operatorname{tr}\Phi'(\lambda)\, z &= 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \omega &= z^2 \\ z &= \pm\sqrt{\omega} \neq 0 \\ \operatorname{tr}\Phi(\lambda) &= \pm 2\sqrt{\omega}, \end{aligned}$$

Put $\varphi(\lambda) = \operatorname{tr} \Phi(\lambda)$ so $C_a$ is

$$z^2 - \varphi(\lambda) z + \omega = 0$$

So we have to assume $C_a$ non-singular (i.e. $\pm 2\sqrt{\omega}$ regular values of $\varphi(\lambda)$).

Now because

$$\Phi(\lambda) \sim \begin{pmatrix} 0 & 0 \\ 0 & \frac{\lambda^n}{a_1 \cdots a_n} \end{pmatrix} \qquad \text{as } \lambda \longrightarrow \infty$$

it is clear that we have a limiting eigenspace for $(\lambda, z) \longrightarrow (\infty, \infty)$ or $(\infty, 0)$, so it should be the case that the line bundle we have defined over $C_a$ extends over $C$. This line bundle appears as a sub-line bundle of $\mathcal{O}^2$. For each $(\lambda, z) \in C_a$ we have a unique eigenfunction $(y_0, y_1)$ of $\Phi(\lambda)$ with eigenvalue $z$, up to scalars. Hence $\frac{y_1}{y_0}$ is a well-defined rational function on $C_a$, call it $f(\lambda, z)$.

One has

$$f(\infty, 0) = \frac{0}{1} = 0$$

$$f(\infty, \infty) = \frac{1}{0} = \infty$$

Thinking of $\Phi(\lambda)^t$ as a linear fractional transformation, it is clear that $f(\lambda, z)$ for the two values of $z$ are just the fixpoints for $\Phi(\lambda)^t$. In effect:

$$z(y_0 \ y_1) = (y_0 \ y_1)\, \Phi(\lambda)$$

$$z \begin{pmatrix} y_1 \\ y_0 \end{pmatrix} = \Phi(\lambda)^t \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}$$

Try to relate periodic J-matrices with ones occurring for orthogonal polynomials. Observe that if we consider eigenfunctions $y = (y_n)$ with $y_0 = 0$, then the first equations is

$$(Jy)_1 = b_1 y_1 + a_1 y_2 = \lambda y_1$$
$$(Jy)_2 = c_1 y_1 + b_2 y_2 + a_2 y_3 = \lambda y_2$$

Hence we get the one-sided J-matrix

$$\begin{vmatrix} b_1 & a_1 \\ c_1 & b_1 \end{vmatrix}$$

If $J$ is periodic as before then if $(Ty)_n = y_{n+r}$, one has

$$(JTy)_n = c_{n-1}(Ty)_{n-1} + b_n(Ty)_n + a_n(Ty)_{n+1}$$

no good for $n=1$ as $(Ty)_0 = 0$

$$= c_{n-1}\, y_{n-1+r} + b_n y_{n+r} + a_n y_{n+r+1}$$

$$(TJy)_n = (Jy)_{n+r}$$
$$= c_{n+r-1}\, y_{n+r-1} + b_{n+r}\, y_{n+r} + a_{n+r}\, y_{n+r+1}$$

Thus $\quad TJ = JT \quad$ NO

on the subspace of $(y_n)$ with $y_n = 0$ for $n \leq 0$.

Make precise the point that $(Ty)_n = 0$ for $n \leq 0$ and $(Ty)_n = y_{n+r}$ for $n \geq 1$.

What is the relation between the spectrum of the one-sided

J-matrix and the infinite J-matrix? $\lambda$ given, we can identify eigenfunctions for the one-sided problem with eigenfunctions for the two-sided J which vanish at 0. In effect 2-sided solutions of $Jy = \lambda y$ are determined by $y_0$ and $y_1$. Given a one-sided solution $(y_1, y_2, \ldots$ one extends it by putting $y_0 = 0$ whence

$$b_1 y_1 + a_1 y_2 = \lambda y_1 \implies (Jy)_1 = c_0 y_0 + b_1 y_1 + a_1 y_2 = \lambda y_1$$

and then extends negatively.

So the spectrum of J one-sided and two-sided are the same on the algebraic level. But now if $\lambda$ is fixed we have $\Phi(\lambda)$ working on the two-dimensional space of eigenfunctions for $\lambda$. If $\operatorname{tr}\Phi(\lambda)$ is in $[-2,2]$ so that the two $z$-values are conjugate points on $|z|=1$, then we get bounded solutions for each $z$-value and a suitable linear combination will then be a generalized eigenvector for the one-sided J. So in this case $\lambda$ will be in the spectrum of the one-sided J. If $\operatorname{tr}\Phi(\lambda) \notin [-2,2]$, then one of the $z$-values is of abs. value $> 1$ and the other of abs. value $< 1$, so there is a unique possible eigenvector y for J with eigenvalue $\lambda$ and this will be square integrable provided it exists, i.e. $y_0 = 0$, or equivalently

$$\Phi(\lambda)^t = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad \text{or} \quad \Phi(\lambda) = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

~~But $\det(\Phi(\lambda)) = 1$ and the entries of $\Phi(\lambda)$ are polys, so~~

~~this can happen~~