would have $\left|\frac{1}{z_1}\right| = 1.$

February 19, 1977

Monomer - dimer problem - (after Hohlmann, Lieb).

Start with a weighted graph G. This means we give a set of vertices S and for each unordered pair of distinct vertices i,j we get a weight $W_{ij} \geq 0$. The edges are then the [redacted] sets $\{i,j\}$ with $W_{ij} > 0$. A dimer on G is a set of edges D such that no two edges in D have a common vertex. The partition function associated to G is (say $S = \{1,\dots,n\}$)

$$P(G; x_1,\dots,x_n) = \sum_{D\in\mathcal{D}} \prod_{(i,j)\in D} W_{ij} \prod_{i\in G-|D|} x_i$$

To simplify suppose $W_{ij} > 0$ for all i,j (hence G is a complete graph). Fix a vertex i. Then the above sum can be broken up according to whether $i \in |D|$ or not.

$$\sum_{\substack{D\in\mathcal{D}\\ i\notin|D|}} \prod_{\sigma\in D} W_\sigma \, x^{G-|D|} = x_i P(G-\{i\}, x_1,\dots,\hat{x}_i,\dots,x_n)$$

$$\sum_{j\neq i} W_{ij} \sum_{\substack{D\in\mathcal{D}\\ (i,j)\in D}} \prod_{\sigma\in D-(i,j)} W_\sigma \, x^{G-|D|} = \sum_{j\in G-\{i\}} W_{ij} \, P(G-\{i,j\}, x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n)$$

So we get the recursion relation

$$P(G,x) = x\,P(G-\{i\},x) + \sum_{j\in G-\{i\}} N_{ij} P(G-\{i,j\},x)$$

where $P(G,x) = P(G,x,\dots,x)$.

Example 1. $\bullet\!\!-\!\!\!\stackrel{|}{-}\!\!\bullet\!\!-\!\!\!\stackrel{|}{-}\!\!\bullet\!\!-\!\!\!\stackrel{|}{-}\!\!\bullet$ n - times.

Let $P_n(x)$ be the partition function of this chain. Then

$$P_n = P_{n-2} + x P_{n-1}, \quad P_0 = 1, \quad P_1(x) = x \qquad P_2(x) = x^2+1$$

$$P_n - x P_{n-1} - P_{n-2} = 0$$

$$P_n = c_1 \lambda_1^n + c_2 \lambda_2^n$$

$$\lambda^2 - x\lambda - 1 = 0 \qquad \lambda = \frac{x \pm \sqrt{x^2+4}}{2}$$

$$P_n = c_1\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n + c_2\left(\frac{x-\sqrt{x^2+4}}{2}\right)^n$$

Put $Q_n(x) = (i)^{-n} P_n(ix)$. Then $Q_0 = 1$, $Q_1 = x$, $Q_2 = x^2-1$ and the recursion relation becomes

$$(i)^{+n} Q_n = i^{n-2} Q_{n-2} + ix\, i^{n-1} Q_{n-1}$$

or

$$Q_n = x Q_{n-1} - Q_{n-2} \qquad \lambda^2 - x\lambda + 1$$

$$Q_n = c_1\left[\frac{x}{2} + i\sqrt{1-\left(\frac{x}{2}\right)^2}\right]^n + c_2\left[\frac{x}{2} - i\sqrt{1-\left(\frac{x}{2}\right)^2}\right]^n$$

$$e^{i\theta} = \frac{x}{2} + i\sqrt{1-\left(\frac{x}{2}\right)^2} \qquad x = 2\cos\theta$$

$$Q_n = c_1 e^{in\theta} + c_2 e^{-in\theta}$$

$$Q_0 = c_1 + c_2 = 1$$

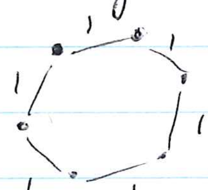
$$Q_1 = c_1 e^{i\theta} + c_2 e^{-i\theta} = x = 2\cos\theta$$

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 2\cos\theta & e^{-i\theta} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ e^{i\theta} & e^{-i\theta} \end{vmatrix}} = \frac{\cos\theta - i\sin\theta - 2\cos\theta}{e^{-i\theta} - e^{i\theta}} = \frac{e^{i\theta}}{2i\sin\theta}$$

$$c_2 = \overline{c_1} = -\frac{e^{-i\theta}}{2i\sin\theta}$$

$$Q_n(x) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{2i\sin\theta} = \frac{\sin(n+1)\theta}{\sin\theta} \qquad \theta = \cos^{-1}\left(\frac{x}{2}\right)$$

which is some sort of Chebycheff polynomial.

<u>Example 2.</u> Ring  n-vertices.

Let $Q_n' = i^{-n} P_n'(ix)$ be the polyn. for this problem. Then

$$Q_n'(x) = x\,Q_{n-1}(x) - 2\,Q_{n-2}(x)$$

$$= 2\cos\theta\,\frac{\sin(n\theta)}{\sin\theta} - 2\,\frac{\sin(n-1)\theta}{\sin\theta}$$

$$= \frac{2}{\sin\theta}\left[\cos\theta\left(\sin\theta\cos(n-1)\theta + \cos\theta\sin(n-1)\theta\right) - \sin(n-1)\theta\right]$$

$$= \frac{2}{\sin\theta}\left[\cos\theta\sin\theta\cos(n-1)\theta - \sin^2\theta\sin(n\theta)\right] = 2\left[\cos\theta\cos(n-1)\theta - \sin\theta\sin(n-1)\theta\right]$$

$$= 2\cos(n\theta)$$

Thus
$$Q_n'(x) = 2\cos(n\theta) \qquad \theta = \cos^{-1}\left(\frac{x}{2}\right)$$
$$= 2^n T_n\left(\frac{x}{2}\right)$$

where
$$T_n(x) = 2^{n-1}\cos\left(n\cos^{-1}(x)\right).$$

<u>Example 3:</u> Complete graph with n-vertices, all $W_{ij} = 1$. Recursion relation is

$$Q_n = xQ_{n-1} - (n-1)Q_{n-2}$$

$Q_0 = 1$

$Q_1 = x$

$Q_2 = x^2 - 1$

$Q_3 = x^3 - x - 2x$

$$Q_{n+1} = xQ_n - nQ_{n-1}$$

$$Q_{n+2} = xQ_{n+1} - nQ_n - Q_n$$

$$\sum_{n\geq 0} Q_{n+2}s^{n-1} = \sum_{n\geq 0} xQ_{n+1}s^{n-1} - \sum_{n\geq 0} nQ_n s^{n-1} - \sum_{n\geq 0} Q_n s^{n-1}$$

$$\frac{f(s) - 1 - xs}{s^3} = \frac{x\left(f(s)-1\right)}{s^2} - f'(s) - \frac{f(s)}{s}$$

$$f(s) - 1 - xs = xsf(s) - xs - s^3f'(s) - s^2f(s)$$

$$s^3f'(s) = f(s)\left[xs - s^2 - 1\right] + 1$$

valid for $n \geq 1$. Thus dividing by $(n-1)!$

$$\sum_{n\geq 0}{}' n \frac{Q_n}{n!} s^{n-1} = \sum_{n\geq 1}{}' x \frac{Q_{n-1}}{(n-1)!} s^{n-1} - \sum_{n\geq 2}{}' \frac{Q_{n-2}}{(n-2)!} s^{n-2} \cdot s$$

So if $f(s) = \sum_{n\geq 0}{}' \frac{Q_n}{n!} s^n$ we get

$$f'(s) = x f(s) - s f(s) \qquad f(0) = 1$$

$$\frac{f'(s)}{f(s)} = x - s$$

$$\log f(s) = xs - \frac{s^2}{2} + \text{cons}$$

$$f(s) = e^{xs - \frac{s^2}{2}} = e^{\frac{1}{2}x^2} e^{-\frac{1}{2}(x - s)^2}$$

Thus $$Q_n = \left(\frac{d}{ds}\right)^n e^{xs - \frac{s^2}{2}} \Big|_{s=0} = e^{\frac{1}{2}x^2} \left(\frac{-d}{dx}\right)^n e^{-\frac{1}{2}x^2}$$

$$= \left(\frac{-d}{dx} + x\right)^n 1$$

$Q_0 = 1$
$Q_1 = x$
$Q_2 = x^2 - 1$
$Q_3 = x^3 - x - 2x = x^3 - 3x$

So $\underline{Q_n(x) = 2^{\frac{n}{2}} H_n(\sqrt{2}x)}$ in this example.

Look at the general recursion formula

$$Q(G, x) = x\, Q(G-i, x) - \sum_{j\neq i} W_{ij} Q(G-i-j, x)$$

From this one sees that $Q(G,x)$ has degree $n = \text{card}(G)$, it is monic, and that it has terms x^{n-2d}, which also is clear from the definition. So $Q(G,x)$ is even for n even and odd for n odd.

Let us argue by induction on n that the roots of $Q(G,x)$ are real & simple and are separated by the roots of $Q(G-i,x)$. Let $\alpha_1 < \cdots < \alpha_{n-1}$ be the roots of $Q(G-i,x)$. By inductive assumption

$$Q(G-i-j\,,\ \alpha_{n-1}) > 0$$

hence as the $W_{ij} > 0$ one has $Q(G, \alpha_{n-1}) < 0$ hence as $Q(G-i, x) > 0$ for $x \gg \alpha_{n-1}$ we see there is at least one root β_n of $Q(G,x)$ with $\beta_n > \alpha_{n-1}$. Similarly $\sum W_{ij} Q(G-i-j, x)$ will have opposite signs at ~~two~~ consecutive roots $\alpha_{i-1}\, \alpha_i$ of $Q(G-i, x)$ so $Q(G,x)$ will have at least one zero β_i between α_{i-1} and α_i. Similar there is at least one root β_1 between $-\infty$ and α_1. So now it is clear one has:

<u>Thm</u>: $Q(G,x)$ has n distinct real roots separated by the roots of $Q(G-i,x)$ provided all $W_{ij} > 0$ for all pairs i,j of vertices

Now suppose some of the W_{ij} are zero. By a limiting argument using Hurwitz one sees $Q(G,x)$ has only real roots. The above argument gives distinct real roots inductively provided one can choose i so that $\exists\, j$ with $W_{ij} > 0$ and then a k different from j, i with $W_{jk} > 0$. Hamiltonian graph: The vertices can be ordered so that $W_{1,2}, W_{2,3}, \ldots, W_{n-1,n}$ are all > 0, i.e. there is a path through the graph hitting each vertex only once. So the above theorem & its proof has the corollaries:

Cor. 1: $Q(G,x)$ has only real roots for any weighted graph.

Cor. 2: If G is Hamiltonian, then $Q(G,x)$ has distinct real roots.

Feb. 20

Relation of the dimer poly to the Ising poly. The dimer polynomial $P(G,x)$ is in a certain sense the high temperature limit of the Ising polynomial $Z(G,z)$. Write the Ising polynomial in symmetric form

$$Z(G,z) = \frac{1}{2^n} \sum_{s\in\{-1,1\}^G} e^{\beta \sum_{[i,j]\in G} J_{ij} s_i s_j} \, z^{\frac{1}{2}\sum s_i}$$

Here β is the inverse of temperature. As $\beta \searrow 0$ the roots of $Z(G,z)$ approach $z=-1$, because

$$\lim_{\beta\to 0} Z(G,z) = \frac{1}{2^n}\sum z^{\frac{1}{2}\sum s_i} = \frac{1}{2^n}\left(z^{\frac{1}{2}}+z^{-\frac{1}{2}}\right)^n = \frac{z^{-n/2}}{2^n}(1+z)^n$$

Now let us use a Cayley transform parameterization of the unit circle

$$z = \frac{x+ia}{x-ia} \qquad x \text{ real}, \quad a \text{ real}$$

$$z = \frac{(x+ia)^2}{x^2+a^2} \qquad z^{\frac{1}{2}} = \frac{x+ia}{\sqrt{x^2+a^2}}$$

$$z^{\frac{s}{2}} = \frac{x + isa}{\sqrt{x^2+a^2}} \qquad s = \pm 1.$$

Then
$$Z(G,z) = (x^2+a^2)^{-n/2} \frac{1}{2^n}\sum_s{}' e^{\beta \sum J_{ij} s_i s_j} \prod_{i=1}^{n} (x + i s_i a)$$

Now notice that $\frac{1}{2^n}\sum_s{}'$ is the Haar integral over the group $\{-1,1\}^n$ and that the characters of this group are the functions $s \mapsto s_{i_1} \cdots s_{i_p}$ for $1 \le i_1 < \cdots < i_p \le n$. Now put $a = \beta^{-1/2}$. ~~Note that~~

$$e^{\beta \sum J_{ij} s_i s_j} = 1 + \beta \sum J_{ij} s_i s_j + \frac{\beta^2}{2!}\left(\sum J_{ij} s_i s_j\right)^2 + \cdots$$

Note that the coeff of β^μ has at most monomials $s_{i_1} \cdots s_{i_p}$ of degree $p \le 2\mu$. To avoid integrating out to zero one must mix a term $i^\nu x^{n-\nu} s_{i_1} \cdots s_{i_\nu} \beta^{-\nu/2}$ with a term $\beta^\mu (\text{const})\, s_{i_1} \cdots s_{i_\nu}$ from $e^{\beta \sum J_{ij} s_i s_j}$; so $\nu \le 2\mu$
$\Rightarrow$ $\mu - \frac{\nu}{2} \ge 0$. Thus the leading term in Z is 0-th order, and one also sees that for $\nu = 2\mu$ one has to mix $\prod J_{i_t j_t} s_{i_t} s_{j_t}$ for a disjoint family of edges. Thus

$$Z(G,s) = (x^2+a^2)^{-n/2}\Bigg(\underbrace{\sum_D \prod_{\{i,j\}\in D} J_{ij}\, x^{n-|D|}\, i^{|D|}}_{\text{dimer polynomial } Q(G,x)} + O(\beta)\Bigg)$$

$$Q(G,x) = i^{-n} p(G, ix) = \sum_D{}' \prod_{\sigma \in D} J_\sigma \underbrace{(ix)^{n-|D|}}_{x^{n-|D|}\, i^{-|D|}} i^{-n} \qquad |D| \text{ even}$$

Transformation of the Ising polynomial.

$$e^{as} = (1+sb)\,c \qquad s = \pm 1$$

$$e^{a} = (1+b)\,c \qquad c = \frac{e^{a}+e^{-a}}{2} = \cosh a$$

$$e^{-a} = (1-b)\,c$$

$$bc = \frac{e^{+a}-e^{-a}}{2} = \sinh a$$

$$b = \tanh a$$

So

$$Z(G,z) = \frac{1}{2^n}\sum_s e^{\sum' J_{ij} s_i s_j}\, z^{\frac{1}{2}\sum' s_i}$$

$$= \frac{1}{2^n}\sum_s{}' \prod_{(i,j)\in G}(\cosh J_{ij}) \prod_{(i,j)\in G}\left(1+\tanh(J_{ij})\, s_i s_j\right) \frac{\prod(1+s_i x)}{(-x^2+1)^{n/2}}$$

here $$z = \frac{1+x}{\sqrt{1-x^2}} \qquad x \in iR$$

$$z^{s} = \frac{1+sx}{\sqrt{1-x^2}} \qquad x \in iR$$

Therefore

$$Z(G,z) = \frac{\text{const}}{(1-x^2)^{n/2}} \frac{1}{2^n}\sum_s{}' \prod_{[i,j]\in G}(1+V_{ij} s_i s_j) \prod_{i=1}^{n}(1+s_i x)$$

where $V_{ij} = \tanh(J_{ij}) \geq 0$ if $J_{ij} \geq 0$. In fact $0 \leq J_{ij} < \infty \iff 0 \leq V_{ij} < 1$. Moreover $V_{ij} = 1$ corresponds to $J_{ij} = \infty$ or if one recalls that that real $Z(G,z)$ has $e^{\sum' J_{ij}(s_i s_j - 1)}$ as factor,

this forces $s_i s_j = 1$ if one is to have a non-trivial contribution.

Consider the linear chain n-vertices and try to calculate its Ising polynomial. The coefficient

$$c_I = \prod_{\substack{i \in I \\ j \in I'}} c_{ij} = c^{\operatorname{card}\{(i,i+1) \mid \text{this pair separated by } I\}}$$

depends only on the partition of $\{1,\dots,n\}$ into two disjoint sets I, I'. Maybe one should think of it as just a set of "simple roots" i.e. those consecutive pairs $(i,i+1)$ separated by I and I'. Thus we get for the Ising polynomial

$$P = \sum_{I,I'}{}' c_I \left(z^I + z^{I'}\right)$$

Take all $z_i = z$. One [redacted] has to determine the [redacted] cardinality of I from its boundary points.

We can write the Ising polynomial up to scalar factors in the form

$$\frac{1}{2^n} \sum \prod_{i=1}^{n-1} \left(1 + V s_i s_{i+1}\right) \prod_{i=1}^{n} \left(1 + s_i x\right) \qquad z^{\frac{1}{2}} = \frac{1+x}{\sqrt{1-x^2}} \qquad x \in i\mathbb{R}$$

which can be rewritten as a sum over subsets of the edges of the graph. Better to work with

$$\frac{1}{2^n} \sum \prod_{i=1}^{n-1} \left(1 + V s_i s_{i+1}\right) \prod_{i=1}^{n} \left(x + s_i \sqrt{-1}\right) \qquad z^{\frac{1}{2} s_i} = \frac{x + s_i \sqrt{-1}}{\sqrt{x^2+1}}$$

$x \in \mathbb{R}$

General setup. The Ising polynomial up to constant factors is

$$\frac{1}{2^n}\sum \prod_{i<j}\left(1+V_{ij}s_is_j\right)\prod_i z_i^{\frac{1}{2}s_i} \qquad 0\le V_{ij}\le 1$$

Now if I substitute

$$z^{\frac{1}{2}s} = \frac{x+s\sqrt{-1}}{\sqrt{x^2+1}} \qquad \text{or} \qquad x = i\,\frac{z+1}{z-1}$$

this becomes

$$\left(x^2+1\right)^{-n/2}\frac{1}{2^n}\sum \prod_{i<j}\left(1+V_{ij}s_is_j\right)\prod_i\left(x+s_i\sqrt{-1}\right)$$

which can be rewritten in the form

$$\left(x^2+1\right)^{-n/2}\sum_D V(D)\frac{1}{2^n}\sum_s s^{|D|}\prod_{i=1}^n\left(x+s_i\sqrt{-1}\right)$$

where D runs over subsets of pairs $(i<j)$, or better just the edges in the graph. $|D|$ denotes the set of vertices which belong to an odd number of the edges in D. So we get upon doing the integration over $\{-1,1\}^n$:

$$\left(x^2+1\right)^{-n/2}\sum_D V(D)\; x^{n-|D|}\,(-1)^{\frac{1}{2}|D|}$$

This generalizes the dimer case where the edges in D are disjoint.

Put

$$P_n(x) = \frac{1}{2^n}\sum_{\{-1,1\}^n}\prod_{i=1}^{n-1}\left(1+Vs_is_{i+1}\right)\prod_{i=1}^n\left(x+s_i\right)$$

$$Q_n(x) = \frac{1}{2^n}\sum s_1\prod_{i=1}^{n-1}\left(1+Vs_is_{i+1}\right)\prod_{i=1}^n\left(x+s_i\right)$$

Then

$$P_n = \frac{1}{2^{n-1}} \sum_{\{-1,1\}^{n-1}} \prod_{i=2}^{n-1} (1+V s_i s_{i+1}) \prod_{i=2}^{n} (x+s_i) \left[\frac{1}{2}\sum_{s_1=+1,-1} (1+V s_1 s_2)(x+s_1)\right]$$

$$\underbrace{\left[\frac{1}{2}\sum_{s_1=+1,-1} (1+V s_1 s_2)(x+s_1)\right]}_{=\,(x+Vs_2)}$$

for Q_n: xVs_2+1

$$\therefore \begin{cases} P_n = x P_{n-1} + V Q_{n-1} \\ Q_n = P_{n-1} + xV Q_{n-1} \end{cases}$$

$$\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \begin{bmatrix} x & V \\ 1 & xV \end{bmatrix} \begin{bmatrix} P_{n-1} \\ Q_{n-1} \end{bmatrix} = \begin{bmatrix} x & V \\ 1 & xV \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P_1 = \int (x+s_1) = x \qquad P_2 = x^2 + V \qquad P_3 = x^3 + (V^2+2V)x$$

$$Q_1 = \int s_1 (x+s_1) = 1 \qquad Q_2 = x + xV$$

Argue directly with the Ising polynomial

$$\sum e^{\sum_{i=1}^{n-1} J_i s_i s_{i+1}} z^{\frac{1}{2}\sum_i (s_i+1)}$$

$$= \sum_{s_1=\pm 1} z^{\frac{1}{2}(s_1+1)} \cdot \sum_{s_2=\pm 1} e^{J s_1 s_2} z^{\frac{1}{2}(s_2+1)} \cdot \sum \cdots e^{J s_{n-1} s_n} z^{\frac{1}{2}(s_n+1)}$$

$$= \begin{pmatrix} z & 1 \end{pmatrix} \left[\begin{pmatrix} e^{J} & e^{-J} \\ e^{-J} & e^{J} \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \right]^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P_1(z) = \begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = z + 1$$

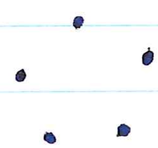
$$P_2(z) = \begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} e^{J} & e^{-J} \\ e^{-J} & e^{J} \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = e^{2J}\left(z^2 + 2e^{-2J} z + 1\right)$$

$$P_3(z) = \begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} e^{J} & e^{-J} \\ e^{-J} & e^{J} \end{pmatrix} \begin{pmatrix} z & \\ & 1 \end{pmatrix} \begin{pmatrix} e^{J} & e^{-J} \\ e^{-J} & e^{J} \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

Note that this formula works even when there are different J-values and also different z_i with evident modifications. We should replace

$$\begin{pmatrix} e^{J} & e^{-J} \\ e^{-J} & e^{J} \end{pmatrix} \longmapsto \begin{pmatrix} 1 & e^{-2J} \\ e^{-2J} & 1 \end{pmatrix}$$

Ring



$$\left(e^{-J}\right)^n \sum_{s_1, s_2, \dots, s_n} \underbrace{z^{\frac{1}{2}(s_1+1)} e^{J s_1 s_2}}\ \underbrace{z^{\frac{1}{2}(s_2+1)} e^{J s_2 s_3}} \cdots\cdots \underbrace{z^{\frac{1}{2}(s_{n-1}+1)} e^{J s_{n-1} s_n}}\ \underbrace{z^{\frac{1}{2}(s_n+1)} e^{J s_n s_1}}$$

$$= \left(e^{-J}\right)^n \operatorname{trace} \left[\begin{pmatrix} z & \\ & 1 \end{pmatrix} \begin{pmatrix} e^{J} & e^{-J} \\ e^{-J} & e^{J} \end{pmatrix}\right]^n$$

$$= \operatorname{trace} \left[\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & e^{-2J} \\ e^{-2J} & 1 \end{pmatrix}\right]^n = \lambda_1^n + \lambda_2^n$$

where λ_1, λ_2 are the eigenvalues of the matrix

$$\begin{pmatrix} z & za \\ a & 1 \end{pmatrix} . \qquad a = e^{-2J}$$

Here's how to see Lee-Yang is true in this cases. Change variable
$$z^{1/2} = \frac{x+\sqrt{-1}}{\sqrt{x^2+1}} \qquad z = \frac{x+i}{x-i} = \frac{x^2-1+2ix}{x^2+1}$$

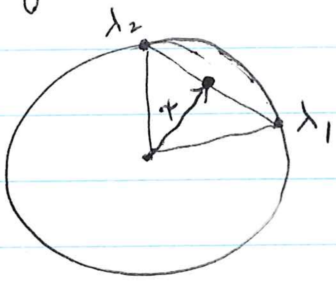
Then up to a multiple:

$$(x-i)^n \operatorname{trace}\left[\begin{pmatrix} z & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & a \\ a & 1\end{pmatrix}\right]^n = \operatorname{trace}\Bigg[\underbrace{\begin{pmatrix} x+i & 0 \\ 0 & x-i\end{pmatrix}\begin{pmatrix} 1 & a \\ a & 1\end{pmatrix}}_{\begin{pmatrix} x+i & a(x+i) \\ a(x-i) & x-i\end{pmatrix}}\Bigg]^n = \lambda_1^n+\lambda_2^n$$

where λ_1, λ_2 are roots of

$$\lambda^2 - 2x\lambda + (x^2+1)(1-a^2) = 0$$

If $\lambda_1^n+\lambda_2^n = 0$, then $|\lambda_1| = |\lambda_2|$ so from $x = \frac{1}{2}(\lambda_1+\lambda_2)$ one sees that $\lambda_1\lambda_2 = (x^2+1)(1-a^2)$ has to be a strictly positive multiple of x^2 (can forget the case $x=0$):



$$(1-a^2)(x^2+1) = (1-a^2)x^2 + (1-a^2) = cx^2 \qquad c>0$$

This forces x^2 to be real. In fact geometrically it is clear that $c \geq 1$, hence as $a^2<1$ one concludes $x^2>0$ forcing x to be real. Hence $\lambda_i = re^{\pm i\theta}$ and $\lambda_1^n+\lambda_2^n = 0$ means $n\theta \in \frac{\pi}{2}+\pi\mathbb{Z}$. Thus for $\theta = \frac{\pi}{2n}+j\frac{\pi}{n}$ $j=0,1,\dots,n-1$ one should get a different root of the Ising polynomial.

$$2x = \lambda_1+\lambda_2 = re^{i\theta}+re^{-i\theta} = 2r\cos\theta$$

$$x = r\cos\theta$$

$$\lambda_1\lambda_2 = r^2 = (x^2+1)\alpha^2 \qquad \alpha^2 = 1-a^2 \qquad \alpha>0$$

$$r^2 = r^2\cos^2\theta\,\alpha^2 + \alpha^2$$

$$(1-\alpha^2\cos^2\theta)r^2 = \alpha^2 \qquad r = \frac{\alpha}{\sqrt{1-\alpha^2\cos^2\theta}}$$

so $$x = \frac{\alpha \cos\theta}{\sqrt{1-\alpha^2\cos^2\theta}}.$$

$$x^2+1 = \frac{\alpha^2\cos^2\theta + 1-\alpha^2\cos^2\theta}{1-\alpha^2\cos^2\theta} = \frac{1}{1-\alpha^2\cos^2\theta}$$

$$z^{\frac{1}{2}} = \sqrt{1-\alpha^2\cos^2\theta}\left(\frac{\alpha\cos\theta}{\sqrt{1-\alpha^2\cos^2\theta}} + i\right)$$

$$z = \left(\alpha\cos\theta + i\sqrt{1-\alpha^2\cos^2\theta}\right)^2$$

$$\theta = \frac{\pi}{2n} + j\frac{\pi}{n} \qquad j = 0, \ldots, n-1$$

are the roots of $P_n(z) = 0$ $\qquad -1 \le a \le 1$

$$P_n(z) = \text{trace}\left[\begin{pmatrix} z & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & a \\ a & 1\end{pmatrix}\right]^n \qquad \alpha^2 = 1-a^2$$

<u>Check</u>: $a = +1$, $\alpha = 0$, eigenvalues of $\begin{pmatrix} z & z \\ 1 & 1\end{pmatrix}$ are $\lambda = 0, 1+z$ so $P_n(z) = (z+1)^n$ and all roots are -1. If $a = 0, \alpha = 1$, $P_n(z) = z^n + 1$ which has the roots

$$e^{i\left(\frac{\pi}{n} + j\frac{2\pi}{n}\right)}$$ as it should.

Complete graph.

$$P_n(z_1, \ldots, z_n) = \sum_{I \not\ni 1}\left(\prod_{\substack{i\in I\\ j \in I'}} c_{ij}\right) z^I + z_1 \sum_{I \ni 1}\left(\prod_{\substack{i \in I,\\ j\in I'}} c_{ij}\right) z^{I-\{1\}}$$

$$= \sum_{I \not\ni 1} \prod_{i \in I, j \in I-\{1\}} c_{ij} \prod_{i>1} c_{i1} z^i + z_1 \sum_{I \ni 1}\left(\prod_{\substack{i\in I-\{1\}\\ j \in I'}} c_{ij}\right)\prod_{j \in I'} c_{1j}\, z^{I-\{1\}}$$

$$= P_{n-1}(c_{21}z_2, \ldots, c_{n1}z_n) + z_1 \prod_{j>1} c_{1j}\, P_{n-1}\left(\frac{z_2}{c_{12}}, \ldots, \frac{z_n}{c_{1n}}\right)$$

$$P_n(z) = P_{n-1}(cz) + zc^{n-1}P_{n-1}\left(\frac{z}{c}\right) = P_{n-1}(cz) + z^n \bar{P}_{n-1}\left(\frac{c}{z}\right)$$

$$P_0(z) = 1$$

$$P_1(z) = 1 + z$$

$$P_2(z) = 1 + cz + z^2\left(1 + \frac{\bar{c}}{z}\right) = 1 + (c+\bar{c})z + z^2$$

$$P_3(z) = 1 + (c+\bar{c})cz + cz^2 + z^3\left(1 + (c+\bar{c})\frac{\bar{c}}{z} + \frac{\bar{c}^2}{z^2}\right)$$

$$= 1 + (c^2 + \bar{c}c + \bar{c}^2)z + (c^2 + c\bar{c} + \bar{c}^2)z^2 + z^3$$

assume c real

$$P_4(z) = 1 + 3c^2cz + 3c^2c^2z + c^3z^3$$
$$+ z^4\left(1 + 3c^2\frac{c}{z} + 3c^2\frac{c^2}{z^2} + \frac{c^3}{z^3}\right)$$

$$= 1 + 4c^3z + 6c^4z^2 + 4c^3z^3 + z^4$$

So one ends up with

$$P_n(z) = \sum_{p=0}^{n} \binom{n}{p} c^{p(n-p)} z^p$$

February 23, 1977:

So we saw on page 33 & before that for the circular ring of length n the (monic + centered) Ising polynomial is

$$P_n(z) = \text{trace}\left[\begin{pmatrix} z^{\frac{1}{2}} & 0 \\ 0 & z^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}\right]^n$$

Now we want to let $n \to \infty$ and approach a limit. Put $z^{\frac{1}{2}} = e^{\frac{1}{n}u}$ and choose a suitable sequence of a_n converging probably to zero like $\frac{1}{n}$. Recall

$$\lim_{n\to\infty}\left(e^{\frac{1}{n}X}e^{\frac{1}{n}Y}\right)^n = e^{X+Y}$$

So it is clear one type of limit to expect is

$$F(u) = \text{trace}\ e^{\begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} + Y}$$

where

$$Y = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & \frac{1}{n}a \\ \frac{1}{n}a & 1 \end{pmatrix}^n = \left(I + \frac{1}{n}\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\right)^n \to e^{\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}}$$

Problem: Calculate $e^{A + uB}$

Suppose $\varepsilon^2 = 0$ and calculate $e^{A+\varepsilon B}$

$$\frac{d}{dt}\, e^{-tA} e^{t(A+\varepsilon B)} = -e^{-tA} A e^{t(A+\varepsilon B)} + e^{-tA}(A+\varepsilon B)e^{t(A+\varepsilon B)}$$

$$= \left(e^{-tA} B e^{tA}\right)\varepsilon$$

so

$$e^{-A} e^{(A+\varepsilon B)} = \int_0^1 \frac{d}{dt}\left(e^{-tA} e^{t(A+\varepsilon B)}\right) dt = \varepsilon\int_0^1 e^{-tA} B e^{tA}\, dt + I$$

or $$e^{A+\varepsilon B} = e^A + \varepsilon e^A \int_0^1 e^{-tA} B e^{tA}\,dt = e^A + \varepsilon e^A \frac{-e^{-ad(A)}+1}{ad(A)}(B)$$

$$e^{A+(u+du)B} = e^{(A+uB)+du\,B}$$

$$= e^{A+uB} + du\, e^{A+uB}\int_0^1 e^{-t(A+uB)} B e^{t(A+uB)}\,dt$$

~~So if~~ $\varphi(u) = e^{A+uB}$ ~~we get~~

$$\varphi'(u) = \varphi(u) + \text{mess}$$

Instead let's compute $F(u) = \text{trace } e^{\begin{pmatrix} u & a \\ a & -u \end{pmatrix}} = e^{\lambda_1} + e^{\lambda_2}$
using eigenvalues $\lambda_1 + \lambda_2 = 0 \qquad \lambda_1\lambda_2 = -u^2 - a^2$

$$\boxed{F(u) = e^{\sqrt{u^2+a^2}} + e^{-\sqrt{u^2+a^2}}}$$

Recall that $\cosh(z) = 0 \iff z = i\left(\frac{\pi}{2} + \pi n\right) \qquad n \in \mathbb{Z}$

Thus

$$\sqrt{u^2+a^2} = i r \qquad r \in \frac{\pi}{2} + \pi\mathbb{Z}$$

$$\Rightarrow \quad u^2 + a^2 = -r^2$$

$$u^2 = -a^2 - r^2 < 0$$

$\Rightarrow$ roots of $F(u)$ are purely imaginary

In fact the roots of $F(u)$ are

$$u = \pm i \sqrt{a^2 + \left(\frac{\pi}{2} + \pi n\right)^2} \qquad n \in \mathbb{Z}$$

$$\sim \pm \pi n i$$

Do the same calculation for the linear chain. $\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet$

Here

$$P_n(z) = (1 \;\; 1)\left[\begin{pmatrix} z^{\frac{1}{2}} & 0 \\ 0 & z^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}\right]^{n-1}\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so in the limit $z^{\frac{1}{2}} = e^{\frac{u}{n}}$ $\quad a \to \frac{a}{n}$ one gets

$$F(u) = (1 \;\; 1)\; e^{\begin{pmatrix} u & a \\ a & -u \end{pmatrix}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

But note that $(1 \;\; 1)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sum_{ij} a_{ij} = \sum_i \left(\sum_j a_{ij} b_{ji}\right)$

if all $b_{jk} = 1$. Thus

$$P_n(z) = \text{trace}\left[\left(\begin{pmatrix} z^{\frac{1}{2}} & 0 \\ 0 & z^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}\right)^{n-1}\begin{pmatrix} z^{\frac{1}{2}} & 0 \\ 0 & z^{-\frac{1}{2}} \end{pmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right]$$

$$\longrightarrow \text{tr}\; e^{\begin{pmatrix} u & a \\ a & -u \end{pmatrix}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = F(u).$$

$$\begin{pmatrix} u & a \\ a & -u \end{pmatrix}\begin{pmatrix} +a & -a \\ -u+\sqrt{u^2+a^2} & u+\sqrt{u^2+a^2} \end{pmatrix} = \begin{pmatrix} a & -a \\ \sqrt{u^2+a^2}-u & \sqrt{u^2+a^2}+u \end{pmatrix}\begin{pmatrix} \sqrt{u^2+a^2} & 0 \\ 0 & -\sqrt{u^2+a^2} \end{pmatrix}$$

$$e^{\begin{pmatrix} u & a \\ a & -u \end{pmatrix}}\Big(\quad '' \quad\Big) = \Big(\quad '' \quad\Big)\begin{pmatrix} e^{\sqrt{u^2+a^2}} & \\ & e^{-\sqrt{u^2+a^2}} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e^A \; T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} T\, e^{\Lambda}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e^A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} T\, e^{\Lambda}\, T^{-1}$$

Calculation seems to yield $\frac{1}{2}F(u) = \cosh\sqrt{u^2+a^2} + a\,\dfrac{\sinh\sqrt{u^2+a^2}}{\sqrt{u^2+a^2}}$

Newton's formula

$$-\log(1-c_1t+c_2t^2) = -\log(1-\lambda_1 t)-\log(1-\lambda_2 t)$$

$$\sum_{n=1}^{\infty}\frac{1}{n}(c_1t-c_2t^2)^n = \sum_{k=1}^{\infty}\frac{\lambda_1^k+\lambda_2^k}{k}t^k$$

$$\|$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=0}^{n}\binom{n}{i}c_1^{n-i}(-c_2)^i t^{n+i} \qquad n+i=k \quad n=k-i$$

$$\|$$

$$\sum_{k=1}^{\infty}\sum_{i=0}^{[\frac{k}{2}]}\frac{k}{k-i}\binom{k-i}{i}c_1^{k-2i}(-c_2)^i\frac{t^k}{k}$$

$$\lambda_1^k+\lambda_2^k = \sum_{i=0}^{[\frac{k}{2}]}\frac{k}{k-i}\binom{k-i}{i}c_1^{k-2i}(-c_2)^i$$

Check: $\lambda_1+\lambda_2 = \frac{1}{1-0}\binom{1-0}{0}c_1^{1-0}(-c_2)^0 = c_1$

$$\lambda_1^2+\lambda_2^2 = \frac{2}{2-0}\binom{2-0}{0}c_1^{2-0} + \frac{2}{1}\binom{2-1}{1}c_1^{2-2}(-c_2)^1$$

$$= c_1^2-2c_2.$$

Now apply this to the matrix $\begin{pmatrix} z^{+1/2} & 0 \\ 0 & z^{-1/2}\end{pmatrix}\begin{pmatrix} 1 & a \\ a & 1\end{pmatrix} = \begin{pmatrix} z^{\frac{1}{2}} & z^{\frac{1}{2}}a \\ z^{-\frac{1}{2}}a & z^{-\frac{1}{2}}\end{pmatrix}$

$$c_1 = z^{\frac{1}{2}}+z^{-\frac{1}{2}} \qquad c_2 = 1-a^2 = \alpha^2$$

$$P_n(z) = \sum_{i=0}^{[\frac{n}{2}]}\frac{n}{n-i}\binom{n-i}{i}\left(z^{\frac{1}{2}}+z^{-\frac{1}{2}}\right)^{n-2i}(-\alpha^2)^i$$

$$\lambda_1^n + \lambda_2^n = \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i}\binom{n-i}{i}\left(z^{\frac12}+z^{-\frac12}\right)^{n-2i}(a^2-1)^i$$

$$= \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i}\,\frac{(n-i)!}{i!\,\cancel{(n-2i)!}}\sum_{j=0}^{n-2i}\frac{\cancel{(n-2i)!}}{j!\,(n-2i-j)!}\, z^{\frac12 j-\frac12(n-2i-j)}\,(a^2-1)^i$$

$$j+i-\frac{n}{2}$$

$k=j+i$ $i=k-j$

$$= z^{-\frac{n}{2}}\sum_{k=0}^{n}\Bigg(\underbrace{\sum_{j=0}^{k}\frac{n}{n-k+j}\,\frac{(n-k+j)!}{(k-j)!\,(j!)(n-2k+j)!}\,(a^2-1)^{k-j}}_{\sum_{\nu=0}^{k}\frac{n}{n-\nu}\frac{(n-\nu)!}{\nu!\,(k-\nu)!\,(n-k-\nu)!}(a^2-1)^{\nu}}\Bigg) z^k$$

$\nu=k-j$

?

Curious point $P_n(z)$ seems to involve only $z^{-\frac{n}{2}} z^k$ $= e^{(-\frac{n}{2}+k)\frac{2u}{n}} = e^{\frac{2k-n}{n}u}$ where $0\le k\le n$, hence

$$-1 \le \frac{2k-n}{n} \le 1$$

It would therefore appear that $F(u) = e^{\sqrt{u^2+a^2}} + e^{-\sqrt{u^2+a^2}}$ is of the form

$$F(u) = \int_{-1}^{1} a(s)\, e^{su}\, ds$$

This is certainly the case for $a=0$.

February 24, 1977

From p. 33 we know the roots of $P_n(z) = tr\left[\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2}\end{pmatrix}\begin{pmatrix} 1 & a \\ a & 1\end{pmatrix}\right]^n$

are
$$z^{\frac12} = \alpha\cos\theta_j + i\sqrt{1-\alpha^2\cos^2\theta_j} \qquad \alpha^2 = 1-a^2$$
$$\theta_j = \frac{1}{n}(2j+1)\frac{\pi}{2}$$

So now we put $z^{\frac12} = e^{\frac{u}{n}}$, $a \mapsto \frac{a}{n}$, whence the roots will be given by $\cos\frac{+iu_j}{n} = \alpha\cos\theta_j$

$$1 + \frac{u_j^2}{2n^2} = \left(1-\frac{a^2}{n^2}\right)^{1/2}\left(1-\frac{1}{2n^2}\left((2j+1)\frac{\pi}{2}\right)^2\right) + O\left(\frac{1}{n^4}\right)$$

so in the limit
$$\frac{u_j^2}{2} = -\frac{a^2}{2} - \frac{1}{2}\left((2j+1)\frac{\pi}{2}\right)^2$$

or the roots are given by
$$\sqrt{u_j^2+a^2} = i\left((2j+1)\frac{\pi}{2}\right)$$

Thus in this limiting process one has
$$\theta_j = \frac{1}{n}(2j+1)\frac{\pi}{2} \longrightarrow 0$$
$$\alpha\cos\theta_j \longrightarrow 1$$
$$\frac{-iu_j}{n} = \cos^{-1}(\alpha\cos\theta_j) \to 0$$

Question: Are there other limiting processes one could use with the same P_n? In other words let $z^{\frac12} = e^{\frac1n u}$ as before but can one choose a_n differently so the ~~roots~~ set of roots converges?

February 25, 1977. Lee-Yang Phys. Rev 87 404–410.

Lattice gas: One has a lattice L (usually $\mathbb{Z}^\nu$) given. Configurations are subsets of L - the positions occupied by the atoms of the gas. For each finite subset Λ of L one can form the grand partition for the finite volume Λ

$$Z_\Lambda(z\beta) = \sum_{i=0}^{N(\Lambda)} z^i \Bigg(\sum_{\substack{I \subset \Lambda \\ \mathrm{card}(I) = i}} e^{-\beta H(I)} \Bigg)$$

where β is inverse temperature and $H(I)$ is the energy of the configuration I. One thinks of there being a reservoir of molecules able to enter the lattice at "activity" z.

The pressure is defined to be

$$\beta p = \lim_{\Lambda \to \infty} \frac{1}{N(\Lambda)} \log (Z_\Lambda)$$

Because $Z_\Lambda(z)$ is a polynomial in z with non-negative coefficients $Z_\Lambda(z)$ does not vanish for $z \in \mathbb{R}_{\geq 0}$. If the roots of Z_Λ do not approach $\mathbb{R}_{\geq 0}$ then p will be analytic in z, β and one says there is no phase transition.

In the situation of the L-Y thm. the roots of $Z_\Lambda(z, \beta)$ lie on $|z| = 1$, and as $\Lambda \to \infty$, they approach a limiting distribution $g(\theta)\, d\theta$ on this circle. One has

$$Z_\Lambda = \prod_{j=1}^{N(\Lambda)} (z - e^{i\theta_j})$$

$$\frac{1}{N(\Lambda)} \log(Z_\Lambda) = \frac{1}{N(\Lambda)} \sum_{j=1}^{N(\Lambda)} \log(z - e^{i\theta_j})$$

$$\longrightarrow \int_0^{2\pi} \log(z - e^{i\theta})\, g(\theta)\, d\theta$$

$$\beta \cdot p = \int_0^{\pi} \log(1 - 2z\cos\theta + z^2)\, g(\theta)\, d\theta$$

For β small ~~[illegible]~~ (high temperature) one has power series expansions of these quantities.

For the gas one has the energy of the subset I given by

$$H(I) = -\sum_{i,j \in I}{}' J_{ij} \qquad \text{with } J_{ij} \geq 0$$

(i.e. the energy decreases if we put in a molecule at the j-th position and the i-th position is already occupied.) Note that

$$+\sum_{i,j \in I}{}' J_{ij} + \sum_{\substack{i \in I \\ j \in I'}} J_{ij} = \sum_{i \in I} \Big(\sum_j{}' J_{ij} \Big) = \sum_{i \in I} \lambda_i$$

and $\lambda_i = \sum_j{}' J_{ij}$ will be independent of i if we assume some sort of translation invariance. In any case it can be absorbed into the activity of the i-th site. Hence the ~~partition function~~ energy becomes

$$H(I) = \sum_{\substack{i \in I \\ j \in I'}} J_{ij} - \sum_{i \in I} \lambda_i \,.$$

February 26, 1977

Review [redacted] Euclid's algorithms and [redacted] continued fractions

Start with a non-zero rational number $\frac{p}{q}$ [redacted] in lowest terms with $q > 0$.

$$p = a_1 q + r_1 \qquad r_1 < q$$

$$q = a_2 r_1 + r_2 \qquad r_2 < r_1$$

$$r_{n-1} = a_{n+1} r_n + r_{n+1}$$

If $r_{n+1} = 0$, then $r_n = $ g.c.d. of $p, q = 1$. Note that the above equations can be expressed as a continued fraction

$$\frac{p}{q} = a_1 + \frac{1}{\frac{q}{r_1}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\left(\frac{r_1}{r_2}\right)}}$$

$$= a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cfrac{1}{a_n + \cfrac{1}{a_{n+1}}}}}.$$

Note that $a_2, \ldots, a_{n+1} \geq 1$ (actually $a_{n+1} \geq 2$) and that $a_1 \geq 1 \Leftrightarrow \frac{p}{q} > 1$.

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q \\ r_1 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n+1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_n \\ r_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} a_{n+1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Next note that if we put

$$\begin{bmatrix} p & u \\ q & v \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n+1} & 1 \\ 1 & 0 \end{bmatrix}$$

Then $pv - qu = (-1)^{n+1}$ and

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is the next to the last "convergent" for the continued fraction.

Now in general if I am given a continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}$$

I put

$$\frac{P_n}{Q_n} = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}$$

more precisely

$$\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

whence one has

$$\begin{bmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$$

Consider next an Ising model linear with different couplings:

$$P = \sum \prod_{i=1}^{n} e^{J_i(s_i s_{i+1} - 1)} z_i^{s_i/2}$$

and assume cyclic boundary conditions:

$$P(z_1, \ldots, z_n) = \operatorname{tr} \text{[redacted]} \left(\begin{bmatrix} z_1^{1/2} & 0 \\ 0 & z_1^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & a_1 \\ a_1 & 1 \end{bmatrix} \cdots\cdots \begin{bmatrix} z_n^{\frac{1}{2}} & 0 \\ 0 & z_n^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & a_n \\ a_n & 1 \end{bmatrix} \right)$$

where $\quad a_i = e^{-2J_{i\,i+1}}$.

[redacted] If you tried to make the a_i periodic of period k and repace $z_i^{1/2}$ by $e^{\frac{1}{n}u_i}$ and a_i by $\frac{1}{n}a_i$, then the result would be

$$\operatorname{tr}\ e^{\begin{pmatrix} u_1 & 0 \\ 0 & -u_1 \end{pmatrix} + \begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix} + \cdots}$$

and so one has essentially the same thing studied before.

It is tempting to speculate that there should be some sort of non-commutative integration process one should investigate in general. Thus

$$\begin{pmatrix} z^{\frac{1}{2}} & 0 \\ 0 & z^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} = \begin{pmatrix} e^{\frac{u}{n}} & 0 \\ 0 & e^{-\frac{u}{n}} \end{pmatrix} \begin{pmatrix} 1 & \frac{\alpha}{n} \\ \frac{\alpha}{n} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{u}{n}} & e^{\frac{u}{n}} \frac{\alpha}{n} \\ e^{-\frac{u}{n}} \frac{\alpha}{n} & e^{-\frac{u}{n}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{n}\begin{pmatrix} u & \alpha \\ \alpha & -u \end{pmatrix}$$

so that

$$e^{\begin{pmatrix} u & \alpha \\ \alpha & -u \end{pmatrix}}$$

is the end $(t=1)$ of the solution of

$$\frac{dA}{dt} = \begin{pmatrix} u & \alpha \\ \alpha & -u \end{pmatrix} A$$

Conjecture: Let $\alpha(t)$ be a real function of t and let $A(t,u)$ be the solution of the above DE with $A(0,u) = I$. Let $F(t,u) = \mathrm{tr}\, A(t,u)$. Then t real $F(t,u) = 0 \Rightarrow \mathrm{Im}(u) = 0$.

Note that one knows

$$\frac{d}{dt}\det(A) = \det(A)\ \mathrm{tr}\begin{pmatrix} u & \alpha \\ \alpha & -u \end{pmatrix} = 0$$

so that $\det(A) = 1$ for all t which agrees with my intuition.

Think of 2×2 matrices as unimodular transf. $z \to \frac{az+b}{cz+d}$. We consider those transformations of the form

$$\begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix} \qquad |a| \leq 1 \qquad |\lambda| < 1$$

and products of these. Then from Lee Yang we know no products of these matrices has trace zero. The geometric reason is that these transformations shrink the unit disk - maybe.

Note that

$$\begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix} \longleftrightarrow (z \longmapsto \lambda z)$$

which shrinks the unit disk if $|\lambda| < 1$. Also

$$z \longmapsto \frac{z+b}{\bar{b}z+1} \qquad \text{carries } |z|=1 \text{ into itself}$$

and sends $0 \longmapsto b$ so that if $|b| \leq 1$ the unit disk goes into itself. Finally note that we can arrange by scaling that the determinant $ad-bc=1$ be 1, so trace $=0$ implies the eigenvalues are $\pm i$. But then the square of the matrix is $-I$, which cannot strictly shrink the unit disk.

Feb 27. So one sees the importance of the semi-group of unimodular matrices which carry the unit disk into itself. Let's classify infinitesimal transf. with this property

$$\begin{bmatrix} 1+\varepsilon a & \varepsilon b \\ \varepsilon c & 1+\varepsilon d \end{bmatrix} e^{i\theta} = \frac{e^{i\theta} + \varepsilon(ae^{i\theta}+b)}{1+\varepsilon(ce^{i\theta}+d)}$$

$$= e^{i\theta}\left[\left(1+\varepsilon(a+be^{-i\theta})\right)\left(1-\varepsilon(ce^{i\theta}+d)\right)\right]$$

$$= e^{i\theta}\left[1+\varepsilon\left[a+be^{-i\theta}-ce^{i\theta}-d\right]\right]$$

This will point inward the unit circle at $e^{i\theta}$ iff

$$\mathrm{Re}\,(a+be^{-i\theta}-ce^{i\theta}-d) \leqslant 0$$

for all θ. Now $a+d=0$ so $-d=a$

$$\mathrm{Re}\,(2a) + \mathrm{Re}\,((\bar{b}-c)e^{i\theta}) \leqslant 0 \quad \text{for all } \theta$$

$$\Updownarrow$$

$$\mathrm{Re}\,(2a) + |\bar{b}-c| \leqslant 0$$

Conjecture: Let $A(t)$ satisfy

$$\frac{dA(t)}{dt} = A(t)\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad A(0) = I$$

where a, b, c are functions of t such that

$$2\,\mathrm{Re}\,(a) + |\bar{b}-c| \leqslant 0.$$

Then $A(t)$ carries the unit disk into itself for all $t \geqslant 0$.

Here is a limitation of the possible functions $F(u)$ one can construct in the form $\mathrm{tr}\, A(t,u)$ where $\dfrac{dA(t)}{dt} = A(t)\begin{pmatrix} au & b \\ \bar{b} & -au \end{pmatrix}$.

Such a matrix $A(1,u)$ one gets by subdividing $0 \leq t \leq 1$ into parts $0 = t_0 < t_1 < \cdots < t_n = 1$, $t = \frac{i}{n}$.

$$A_n = \exp \frac{1}{n}\begin{pmatrix} a(t_0)u & b(t_0) \\ \bar{b}(t_0) & -a(t_0)u \end{pmatrix} \cdot \exp \frac{1}{n}\begin{pmatrix} a(t_1)u & b(t_1) \\ \bar{b}(t_1) & -a(t_1)u \end{pmatrix} \cdots$$

or ~~better~~

$$A_n = \left(I + \frac{1}{n}\begin{pmatrix} a(t_0)u & b(t_0) \\ \bar{b}(t_0) & -a(t_0)u \end{pmatrix}\right) \cdot \left(I + \frac{1}{n}\begin{pmatrix} a(t_1)u & b(t_1) \\ \bar{b}(t_1) & -a(t_1)u \end{pmatrix}\right) \cdots\cdots$$

Then $A(t) = \lim_{n \to \infty} A_n$. This is ~~[illegible]~~ by the successive approximations approach to solving the DE using that

$$A(t+\varepsilon) \doteq A(t) + \varepsilon \frac{dA}{dt}(t) = A(t)\left[1 + \varepsilon \begin{bmatrix} a(t)u & b(u) \\ \bar{b}(u) & -a(t)u \end{bmatrix}\right]$$

Now I should know from calculation that

$$\exp\begin{pmatrix} \frac{1}{n}a(t_0)u & \frac{1}{n}b(t_0) \\ \frac{1}{n}b(t_0) & -\frac{1}{n}a(t_0)u \end{pmatrix}$$

is the L-transform of a measure situated on $-\frac{1}{n}a(t_0) \leq x \leq \frac{1}{n}a(t_0)$
Therefore $A(t)$ should be the L-transform of a measure supported on

$$|x| \leq \int_0^t a(t)\,dt$$

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Suppose we consider a chain of connected oscillators:

Potential energy is $\frac{1}{2}(k_1 x_1^2 + k_2 x_2^2 + \cdots) + \frac{1}{2}\left(a_{12}(x_1-x_2)^2 + a_{23}(x_2-x_3)^2 + \right.$ where $a_{ij} \geq 0$ and $k_i \geq 0$. We can rewrite this form

$$2 P.E. = (b_1 x_1^2 + b_2 x_2^2 + \cdots) - 2(a_{12} x_1 x_2 + a_{23} x_2 x_3 + \cdots)$$

so that it is the quadratic form associated to the real symmetric matrix

$$J_n = \begin{pmatrix} b_1 & -a_{12} & & & & \\ -a_{12} & b_2 & -a_{23} & & & \\ & -a_{23} & b_3 & & & \\ & & & \ddots & & \\ & & & a_{n-2,n-1} & b_{n-1} & -a_{n-1,n} \\ & & & & -a_{n-1,n} & b_n \end{pmatrix}$$

Such a matrix is called a Jacobi matrix. We will simplify by ~~putting~~ $a_{i,i+1} = a_i$.

Suppose we are now given an infinite Jacobi matrix J whose truncations J_n are as above. Put $B_n = \det(J_n)$. One has the recursion relation

$$B_n = b_n B_{n-1} - (+a_{n-1})^2 B_{n-2}$$

Replace J by $J + zI$ and put $B_p(z) = \det(J_p + zI_p)$, so that we have the recursion relation

$$\boxed{B_p(z) = (b_p + z)\,B_{p-1}(z) - (a_{p-1})^2\,B_{p-2}(z)}$$

$$B_1(z) = b_1 + z \qquad B_0(z) = 1$$

$$B_2(z) = (b_2+z)(b_1+z) - a_1^2 .$$

Assertion: Assuming a_i, b_i real and $a_i \neq 0$. More generally one can suppose J is hermitian (replace $(a_{p-1})^2$ above by $|a_{p-1}|^2$). Then the roots of $B_p(z) = \det(J_p + zI_p)$ (which are real because J is hermitian) are of multiplicity 1 (i.e. distinct) and they are separated by the zeroes of $B_{p-1}(z)$.

Continued fraction belonging to J is

$$\frac{1}{b_1 -}\;\frac{a_1^2}{b_2 -}\;\frac{a_2^2}{b_3 -}\;\cdots$$

Put
$$\frac{A_p}{B_p} = \frac{1}{b_1 +}\;\frac{-a_1^2}{b_2 +}\;\cdots\;\frac{-a_{p-1}^2}{b_p} \qquad \frac{\alpha}{\beta + z} = \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}(z)$$

Then
$$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix}\begin{pmatrix} 0 & -a_1^2 \\ 1 & b_2 \end{pmatrix}\cdots\begin{pmatrix} 0 & -a_{n-1}^2 \\ 1 & b_n \end{pmatrix}$$

Thus

$$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} A_{n-2} & A_{n-1} \\ B_{n-2} & B_{n-1} \end{pmatrix} \begin{pmatrix} 0 & -a_{n-1}^2 \\ 1 & b_n \end{pmatrix}$$

$$\Rightarrow \quad B_n = B_{n-2}(-a_{n-1}^2) + B_{n-1} \cdot b_n$$

Also

$$\begin{pmatrix} A_{-1} & A_0 \\ B_{-1} & B_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A_0 & A_1 \\ B_0 & B_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix}$$

So from the recursion formulas we see that

$$B_n = \det(J_n)$$

hence if we set

$$\frac{A_p(z)}{B_p(z)} = \frac{1}{b_1 + z +} \; \frac{-a_1^2}{b_2 + z +} \quad \frac{-a_{p-1}^2}{b_p + z}$$

i.e.

$$\begin{pmatrix} A_{p-1}(z) & A_p(z) \\ B_{p-1}(z) & B_p(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & b_1 + z \end{pmatrix} \cdots \begin{pmatrix} 0 & -a_{p-1}^2 \\ 1 & b_p + z \end{pmatrix}$$

Then

$$B_p(z) = \det(J_p + zI)$$

Continued Fraction Formulas

$$\frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \cdots\ \frac{a_n}{b_n} = \frac{A_n}{B_n}$$

where
$$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix}$$

The point is that this is equivalent to the general 2nd order difference equation:

$$\varphi(n) = b_n \varphi(n-1) + a_n \varphi(n-2)$$

Difference between A_n, B_n:

$$(B_{n-1} \quad B_n) = (0 \quad 1)\begin{pmatrix} 0 & a_1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix}$$

$$(A_{n-1} \quad A_n) = a_1 (0 \quad 1)\begin{pmatrix} 0 & a_2 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix}$$

Relation with orthogonal polynomials & operators in Hilbert space.

Let A be a bdd self-adjoint operator on a Hilbert space H such that $\exists$ a cyclic vector v_0. Then there is a measure μ on $\mathbb{R}$ with support the spectrum of A and an isomorphism $L^2(\mu) \xrightarrow{\sim} H$ such that $1 \mapsto v_0$ and mult. by x corresponds to A. Specifically μ is defined on polynomials by $\int p(x) d\mu = (p(A)v_0, v_0)$ and then extended to all continuous functions by Weierstrass, and the isom. $L^2(\mu) \xrightarrow{\sim} H$ is obtained by completing $p(x) \mapsto p(A)v_0$.

Let F_n = span of $v_0, Av_0, \ldots, A^n v_0$ and let v_n = component of $A^n v_0$ perpendicular to F_n. ~~The~~

$$\cancel{A v_{n-1} = v_n - b_n v_{n-1} + b'_{n-1} v_{n-2}}$$

~~for unique constants b_n, b'_{n-1}. Assume v_{n-1}~~

Assume $v_n \neq 0$ for all n, i.e. H not finite-diml. Then because $Av_n \in F_{n+1}$ is perpendicular to F_{n-2}, $\left((Av_n, F_{n-2}) = (v_n, AF_{n-2}) \subset (v_n, F_{n-1}) = 0 \right)$ we have unique constants c'_n, c''_n such that

$$A v_n = v_{n+1} + c'_n v_n + c''_n v_{n-1} \qquad c''_0 = 0$$

Let $\varphi_n(x)$ be the monic poly of degree n such that $v_n = \varphi_n(A) v_0$. The above equation can be rewritten as a recursion relation

$$x \varphi_n = \varphi_{n+1} + c'_n \varphi_n + c''_n \varphi_{n-1} .$$

Note

$$(Av_n, v_{n-1}) = c''_n \|v_{n-1}\|^2$$

$$\shortparallel$$

$$(v_n, Av_{n-1}) = \|v_n\|^2$$

Thus $c''_n = \left(\dfrac{\|v_n\|}{\|v_{n-1}\|} \right)^2$. Put $a_n = \dfrac{\|v_n\|}{\|v_{n-1}\|}$ $n \geq 1$

and put $b_n = -c'_{n-1}$ Then we get the recursion relation for the φ_n:

$$\varphi_{n+1} = x\varphi_n - c'_n\varphi_n - c''_n \varphi_{n-1}$$

or $$\varphi_n = (x \underbrace{- c'_{n-1}}_{b_n})\varphi_{n-1} - \underbrace{c''_{n-1}}_{a^2_{n-1}} \varphi_{n-2}$$

and initial values

$$\varphi_0 = 1 \qquad \varphi_1 = x + b_1$$

Thus

$$\varphi_n(x) = \det \begin{pmatrix} x+b_1 & -a_1 & & & \\ -a_1 & x+b_2 & -a_2 & & \\ & -a_2 & & & \\ & & & & -a_{n-1} \\ & & & -a_{n-1} & x+b_n \end{pmatrix}$$

On the other hand,

$\|v_n\| = a_n a_{n-1} \cdots a_1$ so that

$$\frac{Av_n}{\|v_n\|} = \frac{v_{n+1}}{\|v_n\|} - b_{n+1}\frac{v_n}{\|v_n\|} + a_n^2 \frac{v_{n-1}}{\|v_n\|}$$

$$A\left(\frac{v_n}{\|v_n\|}\right) = a_n \frac{v_{n-1}}{\|v_{n-1}\|} - b_{n+1}\frac{v_n}{\|v_n\|} + a_{n+1}\frac{v_{n+1}}{\|v_{n+1}\|}$$

Therefore we see that if we put

$$J = \begin{pmatrix} -b_1 & a_1 & \\ a_1 & -b_2 & \\ & & \ddots \end{pmatrix} \qquad \text{infinite}$$

then A is given by the matrix J with respect to the orthonormal basis $\frac{v_n}{\|v_n\|}$, and

$$\varphi_n(x) = \det(xI - J_n).$$

I still have to relate all this to the continued fraction which is the generating function for the moments.

$$c_n = \int x^n \, d\mu$$

$$f(z) = \sum_{n\geq 0} \frac{c_n}{z^{n+1}} = \int \sum_{n\geq 0} \frac{x^n}{z^{n+1}} \, d\mu = \int \frac{1}{z} \frac{1}{1-\frac{x}{z}} \, d\mu$$

$$= \int \frac{d\mu(x)}{z - x}$$

$$= \left((z-A)^{-1} v_0, v_0 \right)$$

But we can compute this coefficient using Cramer's rule. Have to get the formulas straight:

So start with the real Jacobi matrix $J = \begin{pmatrix} -b_1 & a_1 & & \\ a_1 & -b_2 & a_2 & \\ & a_2 & -b_3 & \\ & & & \ddots \end{pmatrix}$

and assume to simplify that the entries are

bounded: $|a_i|, |b_i| \le$ Const. Also assume $a_i > 0$ all i.
Then J gives a bounded self-adjoint operator A on ℓ^2. Put $v_0 = e_1$.
Then we get the measure μ and the orthogonal ~~polynomials~~ polynomials $\varphi_n(x)$ and

$$\frac{\varphi_n(A)v_0}{\|v_n\|} = \frac{v_n}{\|v_n\|} = e_n. \qquad \|v_n\| = a_n \cdots a_1$$

We have

$$\varphi_n(x) = \det(xI - J_n) = \begin{vmatrix} x+b_1 & -a_1 & & \\ -a & & & \\ & & & -a_{n-1} \\ & & -a_{n-1} & x+b_n \end{vmatrix}$$

which we have seen is the denominator of the nth approximant for the continued fraction:

$$\frac{A_n(z)}{B_n(z)} = \frac{-a_0^2}{b_1+z+}\;\frac{-a_1^2}{b_2+z+}\cdots\frac{-a_{n-1}^2}{b_n+z}$$

We saw

$$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} & -a_0^2 \\ 1 & b_1+z \end{pmatrix} \cdots \begin{pmatrix} & -a_{n-1}^2 \\ 1 & b_n+z \end{pmatrix}$$

and that

$$B_n(z) = \det(z - J_n) = \begin{vmatrix} z+b_1 & -a_1 & \\ -a_1 & z+b_2 & \\ & & z+b_n \end{vmatrix}$$

So one has the formula

$$\cancel{\det(zI - J_n) = B_n = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} & -a_0^2 \\ 1 & b_1+z \end{pmatrix} \cdots \begin{pmatrix} & -a_{n-1}^2 \\ 1 & b_n+z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

$$\begin{vmatrix} z_1+b_1 & -a_1 & & \\ -a_1 & z+b_2 & & \\ & & \ddots & \\ & & & z+b_n \end{vmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} & -a_0^2 \\ 1 & b_1+z \end{pmatrix} \cdots \begin{pmatrix} & -a_{n-1}^2 \\ 1 & b_n+z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Also

$$A_n = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -a_0^2 \\ 1 & b_1+z \end{pmatrix} \cdots \begin{pmatrix} 0 & -a_{n-1}^2 \\ 1 & b_n+z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= -a_0^2 \quad \begin{pmatrix} 0, & 1 \end{pmatrix}\begin{pmatrix} & -a_1^2 \\ 1 & b_2+z \end{pmatrix} \cdots \begin{pmatrix} 0 & -a_{n-1}^2 \\ 1 & b_n+z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= -a_0^2 \begin{vmatrix} z+b_2 & -a_2 & \\ -a_2 & & \\ & & z+b_n \end{vmatrix}$$

Therefore if I take $-a_0^2 = 1$ I get the formula

$$\frac{\begin{vmatrix} z+b_2 & -a_2 & & \\ -a_2 & \ddots & & \\ & \ddots & & -a_{n-1} \\ & & -a_{n-1} & z+b_n \end{vmatrix}}{\begin{vmatrix} z+b_1 & -a_1 & & \\ -a_1 & & & \\ & & & -a_{n-1} \\ & & -a_{n-1} & b_n \end{vmatrix}} = \frac{1}{b_1+z+}\,\frac{-a_1^2}{b_1+z+}\cdots\frac{-a_{n-1}^2}{b_n+z}$$

If you use Cramer's rule, the above is

$$\left((zI - J_n)^{-1} v_0, v_0\right) = \int \frac{d\mu_n(x)}{z-x}$$