February 10, 1977

$$Z(s) = \zeta(s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) = 2\sum_1^\infty n^{-s}\int_0^\infty e^{-\pi t^2} t^{+s}\frac{dt}{t}$$

$$= 2\sum_1^\infty \int_0^\infty e^{-\pi n^2 t^2} t^s \frac{dt}{t} = \int_0^\infty [\theta(t) - 1]\, t^s \frac{dt}{t}$$

where $\theta(t) = \sum_{-\infty}^{\infty} e^{-\pi^2 n^2 t^2}$ satisfies $\theta(t^{-1}) = t\,\theta(t)$

Better to work with

$$\varphi(t) = \theta(t^{-1}) = \sum_{-\infty}^{\infty} e^{-\pi^2 n^2/t^2}$$

$$\varphi(t) \searrow 1 \quad \text{very fast} \quad \text{as } t \to 0+$$

$$\frac{\varphi(t)}{t} \searrow 1 \quad \text{very fast} \quad \text{as } t \to +\infty.$$

Thus $\varphi(t)$ is [redacted] analogous to $q^{h^0(L)}$

$$Z(s) = \int_0^\infty [\varphi(t) - 1]\, t^{-s}\frac{dt}{t} \qquad Re(s) > 1$$

$$= \int_0^\infty [\varphi(t) - 1 - t]\, t^{-s}\frac{dt}{t} \qquad 0 < Re(s) < 1$$

$$= \int_0^\infty [\varphi(t) - t]\, t^{-s}\frac{dt}{t} \qquad Re(s) < 0$$

~~Variables~~ $\cancel{u = s - \frac{1}{2}}$ $\cancel{t = e^{+x}}$

$$\cancel{Z(s) = \int_0^\infty \left[t^{-1/2}\varphi(t) - t^{-1/2} - t^{1/2}\right] t^{-u}\frac{dt}{t}}$$

For a curve

$$Z(z) = \frac{1}{q-1}\sum_{L\in \mathrm{Pic}^0}\sum_{n\in\mathbb{Z}} z^n \left(q^{h^0(L(n))} - 1\right) \qquad |z|<\frac{1}{q}$$

I want the symmetrized function

$$\tilde{Z}(z) = z^{1-g} Z(z)$$

$$= \text{cons.} \sum_{L,n} z^{n+1-g}\left(q^{h^0(L(n))}-1\right) \qquad |z|<\frac{1}{q}$$

$$= \text{cons} \sum_{L,n} z^{n+1-g}\left(q^{h^0(L(n))} - q^{n+1-g}\right) \qquad |z|>1$$

$$= \text{cons} \sum_{L,n} z^{n+1-g}\left(q^{h^0(L(n))}-1-q^{n+1-g}\right) \qquad \frac{1}{q}<|z|<1$$

$$= \text{cons.} \sum_{\substack{\deg(L)=g-1\\ n\in\mathbb{Z}}} z^n \left(q^{h^0(L(n))}-1-q^n\right)$$

$$= \text{cons.} \sum_{\substack{\deg(L)=g-1\\ n\in\mathbb{Z}}} \left(q^{-\frac{1}{2}}z\right)^n \left(q^{-\frac{n}{2}} q^{h^0(L(n))} - q^{-\frac{n}{2}} - q^{\frac{n}{2}}\right)$$

Similar to

$$Z(s) = \int_0^\infty \left[t^{-1/2}\varphi(t) - t^{-1/2} - t^{1/2}\right] t^{\frac{1}{2}-s}\frac{dt}{t}$$

$$= \int_{-\infty}^{\infty} \left[e^{-x/2}\varphi(e^x) - e^{-x/2} - e^{x/2}\right] e^{-xu}\,dx \qquad u = s-\frac{1}{2}.$$

Maybe it is simpler to work with $s(1-s)Z(s)$.

$$sZ(s) = \int_0^\infty [\varphi(t)-1-t]\,s t^{-s-1}dt = -(\varphi(t)-1-t)t^{-s}\Big]_0^\infty + \int_0^\infty [\varphi'(t)-1]t^{-s}dt$$

$$(1-s)sZ(s) = \left[(\varphi'(t)-1)t^{1-s}\right]_0^\infty - \int \varphi''(t)t^{1-s}dt$$

$$(1-s)s\,Z(s) = \int_0^\infty -\varphi''(t)\,t^{2-s}\,\frac{dt}{t} = \int_0^\infty \left[-t^{3/2}\varphi''(t)\right] t^{\frac{1}{2}-s}\,\frac{dt}{t}$$

$$= -\int_{-\infty}^{\infty} \psi(x)\, e^{-ux}\, dx \qquad u = s-\tfrac{1}{2}$$

where $\psi(x) = +e^{+\frac{3}{2}x}\varphi''(e^x)$ is even by the functional equation. (Also this can be verified directly).

$$\varphi(t) = \sum{}' e^{-\pi n^2 t^{-2}}$$

$$\varphi'(t) = \sum e^{-\pi n^2 t^{-2}}\left(2\pi n^2 t^{-3}\right)$$

$$\varphi''(t) = \sum{}' e^{-\pi n^2 t^{-2}}\left(4\pi^2 n^4 t^{-6} - 6\pi n^2 t^{-4}\right)$$

Also

$$\varphi(t) = \theta\left(\tfrac{1}{t}\right) = t\,\theta(t) = \sum{}' t e^{-\pi n^2 t^2}$$

$$\varphi'(t) = \sum e^{-\pi n^2 t^2}\left(-2\pi n^2 t^2 + 1\right)$$

$$\varphi''(t) = \sum e^{-\pi n^2 t^2}\left[(-2\pi n^2 t)(-2\pi n^2 t^2 + 1) + (-4\pi n^2 t)\right]$$

$$= \sum e^{-\pi n^2 t^2}\left[4\pi^2 n^4 t^3 - 6\pi n^2 t\right]$$

$$\therefore\quad \varphi''\left(\tfrac{1}{t}\right) = t^3 \varphi''(t)$$

or $$\left(\tfrac{1}{t}\right)^{3/2}\varphi''\left(\tfrac{1}{t}\right) = t^{3/2}\varphi''(t)$$

February 11, 1977

heat equation $\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial t^2}$ on $\mathbb{R}$, $a>0$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\, v(\xi,t)\, d\xi$$

$$u(x,0) = \delta(x)$$

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial t^2} = \frac{1}{2\pi}\int e^{-ix\xi}\left(\frac{\partial v}{\partial t} + a\xi^2 v\right) dt = 0$$

$$\frac{\partial v}{\partial t} + a\xi^2 v = 0 \qquad v(\xi,0) = 1$$

$$v = e^{-a\xi^2 t} = e^{-(at)\xi^2}$$

$(at)^{1/2}\xi + \frac{ix}{2(at)^{1/2}}$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(at)\xi^2 - ix\xi + \frac{x^2}{4at} - \frac{x^2}{4at}}\, d\xi$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4at}} \int_{-\infty}^{\infty} e^{-(at)\xi^2}\, d\xi \qquad \frac{\sqrt{\pi}}{\sqrt{at}}$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi a t}}\, e^{-\frac{x^2}{4at}}}$$

Suppose one looks at the periodic case $u(x+1)=u(x)$.

$$u = e^{\lambda t} e^{i\mu x} \qquad \mu = 2\pi n$$

$$\lambda = -a\mu^2 = -4a\pi^2 n^2$$

So the periodic fundamental solution is

$$\sum_{n\in\mathbb{Z}} e^{-4a\pi^2 n^2 t}\, e^{2\pi i n x}$$

and also $\displaystyle\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{4\pi a t}}\, e^{-\frac{(x-n)^2}{4at}}$

As a check take $4\pi a = 1$, $a = \frac{1}{4\pi}$.

$$\boxed{\sum_n e^{-\pi n^2 t}\, e^{2\pi i n x} = \frac{1}{\sqrt{t}} \sum_n e^{\frac{-\pi(x-n)^2}{t}}}$$

$$t^{3/2}\varphi''(t) = t^{\frac{1}{2}} \sum_{n\neq 0} e^{-\pi n^2 t^2}\left[(2\pi n^2 t^2)^2 - 3(2\pi n^2 t^2)\right]$$

$$p^2 - 3p = (p-3)p > 0 \quad \text{if} \quad p > 3.$$

If $t \geq 1$, then $2\pi n^2 t^2 \geqslant 2\pi > 3$. Therefore

$$t^{3/2}\varphi''(t) > 0 \quad \text{for} \quad t \geq 1$$

and the same holds for $0 < t \leq 1$ by symmetry.

$$-(1-s)sZ(s) = \int_0^\infty t^{1/2}\, t\varphi''(t)\, t^{\frac{1}{2}-s}\, \frac{dt}{t}$$

where $\displaystyle t\varphi''(t) = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t^2}(2\pi n^2 t^2)(2\pi n^2 t^2 - 3)$

is > 0 for $0 < t < \infty$. Thus $-(1-s)sZ(s)$ is the Laplace transform of a positive function $\psi(x) > 0$

$$-(1-s)sZ(s) = \int_{-\infty}^{\infty} \psi(x)\, e^{-xu}\, dx \qquad u = s - \tfrac{1}{2}$$

$$\psi(x) = \psi(-x).$$

So now put $\quad F(u) = \int_{-\infty}^{\infty} \psi(x)\, e^{+xu}\, dx \quad = \quad -(1-s)s\,Z(s).$

Because ψ is even, one has

$$F(-u) = \int_{-\infty}^{\infty} \psi(x)\, e^{-xu}\, dx = \int_{-\infty}^{\infty} \psi(-x)\, e^{xu}\, dx = F(u).$$

and because ψ is real one has $\overline{F(u)} = F(\bar{u})$. Thus if u is purely imaginary $\overline{F(u)} = F(\bar{u}) = F(-u) = F(u)$ so F is real on the imaginary axis.

Because $\psi(x) > 0$ it follows that

$$|F(u)| \leq \int_{-\infty}^{\infty} |\psi(x)||e^{xu}|\, dx = \int_{-\infty}^{\infty} \psi(x)\, e^{x\,\mathrm{Re}(u)}\, dx = F(\mathrm{Re}\, u).$$

Notice also

$$F(u) = \int_{-\infty}^{\infty} \psi(x) \cosh(xu)\, dx$$

also shows F real if $u \in i\mathbb{R}$

$$F'(u) = \int_{-\infty}^{\infty} \psi(x) \sinh(xu)\, x\, dx > 0 \qquad u \in \mathbb{R}_{>0}$$

because $\sinh(xu) = \dfrac{e^{xu} - e^{-xu}}{2} > 0$ for $u > 0$, $x > 0$

Therefore the maximum value of $|F(u)|$ on the circle $|u| = R$ occurs at $u = \pm R$. Stirling's formula gives a good estimate

$$F(u) = -\left(1-\left(\tfrac{1}{2}+u\right)\right)\left(\tfrac{1}{2}+u\right) \zeta\left(u+\tfrac{1}{2}\right) \Gamma\left(\frac{u+\frac{1}{2}}{2}\right) \pi^{-\frac{1}{2}\left(u+\frac{1}{2}\right)}$$

Stirling:

$$\Gamma(n+1) \sim e^{n \log n - n + \frac{1}{2}\log n} \sqrt{2\pi}$$

$$= e^{(n+\frac{1}{2})(\log n - 1)} \sqrt{2\pi e}$$

To replace n by $\frac{u}{2} - \frac{3}{4}$ so that $n+1 = \frac{u}{2} - \frac{3}{4} + 1 = \frac{1}{2}(u+\frac{1}{2})$

$$\left(n+\tfrac{1}{2}\right)\left(\log(n) - 1\right) = \left(\frac{u}{2} - \frac{1}{4}\right)\left[\log\left(u - \frac{3}{2}\right) - \log 2 - 1\right]$$

$$= \frac{1}{2}\left[\left(u - \frac{1}{2}\right)\left[\log u + \log\left(1 - \frac{3}{2u}\right) - \log(2e)\right]\right]$$

$$= \frac{1}{2}\left(u - \frac{1}{2}\right)\log u - \frac{\log(2e)}{2}\left(u - \frac{1}{2}\right) + \frac{1}{2}\left[u - \frac{1}{2} + \frac{3}{2}\right]$$

$$= \frac{1}{2}\left(u - \frac{1}{2}\right)\log u - \left(\log\sqrt{2e}\right)u + \frac{1}{2}u + \text{const.}$$

combine with $\log \pi^{-\frac{u}{2}+\frac{1}{4}} = -\frac{1}{2}u\log\pi + \text{const.}$

so one sees that $F(u)$ has an asymptotic expansion

$$F(u) \sim e^{\frac{1}{2}u\log u + c_1 u + c_2 \log u + \text{const}}$$

where c_1, c_2 are constants; here $u \longmapsto +\infty$.

<u>February 13, 1977.</u>

Try to understand functions which are their own Fourier transforms. I recall that there is an isomorphism of $L^2(\mathbb{R}^1)$ with the Hilbert space $\mathcal{H}$ of entire functions $f(z)$ on $\mathbb{C}$ such that $\int |f|^2 e^{-|z|^2} \frac{dx\,dy}{\pi} < \infty$ and such that Fourier transform on $L^2(\mathbb{R}^1)$ corresponds to $f(z) \mapsto f(iz)$ in the latter.

First use the F.T.

$$(1) \qquad \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} f(x)\,dx \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} \hat{f}(\xi)\,d\xi$$

Then

$$\frac{1}{\sqrt{2\pi}} \int e^{ix\xi} e^{-\frac{1}{2}\xi^2}\,d\xi = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}[\xi - ix]^2 - \frac{x^2}{2}}\,d\xi$$

$$= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \int e^{-\frac{1}{2}\xi^2}\,d\xi = e^{-\frac{x^2}{2}} \int e^{-\pi \xi^2}\,d\xi = e^{-\frac{x^2}{2}}$$

hence with the F.T. (1) $e^{-\frac{x^2}{2}}$ is its own F.T.:

$$(2) \qquad e^{-\frac{1}{2}\xi^2} = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} e^{-\frac{1}{2}x^2}\,dx$$

Next

$$\frac{d}{dx} e^{-\frac{1}{2}x^2} = -x e^{-\frac{1}{2}x^2}$$

hence the annihilation operator is ~~[illegible]~~

$$a = c\left(\frac{d}{dx} + x\right) \qquad c \text{ constant}$$

So

$$a^* = \bar{c}\left(-\frac{d}{dx} + x\right) \qquad [a, a^*] = c\bar{c} \cdot 2 = 1$$

so $c\bar{c} = \frac{1}{2}$. So now we define the isom.

from $\mathcal{H}$ to L^2 to send 1 to $e^{-\frac{x^2}{2}}$ and to commute with $a = \frac{d}{dz}$, $a^* = z$ on $\mathcal{H}$ and

$$a = \frac{\omega}{\sqrt{2}}\left(\frac{d}{dx} + x\right) \qquad a^* = \frac{\bar{\omega}}{\sqrt{2}}\left(-\frac{d}{dx} + x\right) \qquad |\omega| = 1$$

on L^2. Thus

$$f(z) \longmapsto f\left(\frac{\bar{\omega}}{\sqrt{2}}\left(-\frac{d}{dx} + x\right)\right) e^{-\frac{x^2}{2}}$$

$$= f\left(\frac{\bar{\omega}}{\sqrt{2}} e^{\frac{x^2}{2}}\left(-\frac{d}{dx}\right) e^{-\frac{x^2}{2}}\right) e^{-\frac{x^2}{2}}$$

$$z^n \longmapsto \left(\frac{\bar{\omega}}{\sqrt{2}}\right)^n e^{-\frac{x^2}{2}} \underbrace{\left[e^{+x^2}\left(-\frac{d}{dx}\right)^n \left(e^{-x^2}\right)\right]}_{H_n(x)}$$

Check:

$$\frac{1}{i}\frac{d}{d\xi}\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi}\, x u(x)\, dx$$

$$\therefore \quad \boxed{\frac{1}{i}\frac{d}{d\xi} \circ \mathcal{F} = \mathcal{F} \circ x}$$

$$\frac{1}{i}\xi\, \hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int -\frac{d}{dx} e^{ix\xi}\, u(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{ix\xi}\, u'(x)\, dx$$

$$\boxed{\frac{1}{i} x \circ \mathcal{F} = \mathcal{F} \circ \frac{d}{dx}}$$

$$\mathcal{F}\left(-\frac{d}{dx}+x\right)\hat{\ } = \frac{1}{i}\left(-x+\frac{d}{dx}\right)\mathcal{F} = i\left(-\frac{d}{dx}+x\right)\mathcal{F}$$

which agrees with $\quad z f(z) \xmapsto{\mathcal{F}} iz f(iz).$

Example: $e^{-\frac{x^2}{2}} H_4(x) = (16x^4 - 48x^2 + 12)e^{-\frac{x^2}{2}}$ will be fixed under Fourier transform. Hence so will be $e^{-\frac{x^2}{2}}(x^4-3x^2)$. Now if we let $x \mapsto \sqrt{2\pi}\,x$, $\xi \mapsto \sqrt{2\pi}\,\xi$ in (1) we get the F.T.

(3) $\qquad \hat{f}(\xi) = \int e^{2\pi i x\xi} f(x)\,dx \qquad \int e^{-2\pi i\xi}\hat{f}(\xi)\,d\xi = f(x)$

and $\qquad e^{-\frac{x^2}{2}}(x^4-3x^2) \longmapsto e^{-\pi x^2}(2\pi x^2)(2\pi x^2-3)$

which is the function such that

$$2\int_0^\infty e^{-\pi t^2}(2\pi t^2)(2\pi t^2-3)\, t^{+s}\frac{dt}{t} = -(1-s)s\,\pi^{-s/2}\Gamma(s/2)$$

In effect

$$s\,\pi^{-s/2}\Gamma(s/2) = 2\int_0^\infty e^{-\pi t^2} s\, t^{s-1} \frac{dt}{\cancel{t}} = +2\int_0^\infty e^{-\pi t^2}(2\pi t)t^{s+1}\frac{dt}{t}$$

$$= 2\int_0^\infty e^{-\pi/t^2}\, 2\pi t^{-s-3}\frac{dt}{t}$$

$$(1-s)\,s\,\pi^{-s/2}\Gamma(s/2) = 2\int_0^\infty e^{-\pi/t^2}\, 2\pi t^{-3}\,(1-s)t^{-s}\,dt$$

$$\therefore\ -(1-s)s\,\pi^{-s/2}\Gamma(s/2 = +2\int_0^\infty \left(e^{-\pi/t^2}\, 2\pi t^{-3}\right)' t^{1-s}\,dt$$

$$= 2\int_0^\infty e^{-\pi t^{-2}}\left((2\pi t^{-3})^2 - 6\pi t^{-4}\right) t^{2-s}\frac{dt}{t}$$

$$= 2\int_0^\infty e^{-\pi t^2}\left(4\pi^2 t^{\overset{4}{\cancel{6}}} - 6\pi t^{\overset{2}{\cancel{4}}}\right) t^{s-2}\,\frac{dt}{t}$$

$$= 2\int_0^\infty e^{-\pi t^2}\left(2\pi t^2\right)\left(2\pi t^2 - 3\right)\, t^s\,\frac{dt}{t}.$$

<u>February 14, 1977</u>

$$Z(z) = \zeta(s) = \sum_{L\in \mathrm{Pic}^0}\sum_n z^n\, \frac{q^{h^0(L(n))}-1}{q-1}$$

$$P(z) = (1-z)(1-qz)\,Z(z) = \sum_n z^n \underbrace{\sum_{L\in \mathrm{Pic}^0} \frac{q^{h^0(L(n))} - (1+q)\,q^{h^0(L(n-1))} + q\,q^{h^0(L(n-2))}}{q-1}}_{c_n}$$

Recall

$$h^0(L(n)) = 0 \qquad n \leq -1$$

$$h^0(L) = \begin{cases} 0 & L \not\simeq \mathcal{O} \\ 1 & L \simeq \mathcal{O}\end{cases}$$

$$h^0(L(n)) = n+1-g \qquad n \geq 2g-1$$

$$= \begin{cases} g-1 & n = 2g-2 \quad L(n) \not\simeq K \\ g & n = 2g-2 \quad L(n) \simeq K \end{cases}$$

Thus $c_n = 0$ for $n<0$, $n>2g$ and

$$c_0 = \frac{q-(1+q)+q}{q-1} = 1$$

$$c_{2g} = \frac{q\,(q^g - q^{g-1})}{q-1} = q^g$$

Take $g=1$. Then if $r = \mathrm{card}\, Cl(\mathbb{F}_q) = \mathrm{card}\,\mathrm{Pic}^0$ one has

$$c_1 = \frac{rq - (1+q)(r-1+q) + rq}{q-1} = rq - r + 1 - \cancel{q} - \cancel{qr} + \cancel{q} - q^2 + \cancel{rq}$$

$$= r + \frac{-q^2+1}{q-1} = r - q - 1$$

Thus for an elliptic curve

$$P(z) = 1 + (r-q-1)z + qz^2$$

so the Riemann hypothesis is equivalent to the roots being conjugate complex:

$$|r-q-1| \leq 2q^{1/2}$$

$$-(r-q-1) = 2q^{\frac{1}{2}}\cos\theta$$

where the roots are $q^{\frac{1}{2}}e^{\pm i\theta}$. (Lefschetz says $r = 1 - (\text{tr } F \text{ on } H^1) + q$).

Recall $\mathcal{H} \xrightarrow{\sim} L^2$

$$z^n \longmapsto \left[\frac{1}{\sqrt{2}}\left(-\frac{d}{dx}+x\right)\right]^n e^{-x^2/2} \cancel{\tfrac{1}{\sqrt{\pi}}}$$

$$= e^{x^2/2}\left(-\frac{1}{\sqrt{2}}\frac{d}{dx}\right)^n e^{-x^2} \cancel{\tfrac{1}{\sqrt{\pi}}} = \frac{e^{-x^2/2}}{\sqrt{2}^n} H_n(x) \cancel{\tfrac{1}{\sqrt{\pi}}}$$

$$\|z^n\|^2 = n! \implies \frac{e^{-x^2/2} H_n(x)}{\sqrt{2}^n \sqrt{n!} \cancel{\sqrt{\pi}}} \quad \text{orthonormal basis in } L^2.$$

with $\|f\|^2 = \int |f|^2 \frac{dx}{\sqrt{\pi}}$

Note that if $f(z) = \sum a_n \frac{z^n}{n!}$ then $a_n = (f, z^n)$ so

$$f(b) = \sum a_n \frac{b^n}{n!} = \sum (f, z^n)\frac{b^n}{n!} = (f, e^{\bar{b}z})$$

i.e.

$$f(b) = \int f(z)\, e^{b\bar{z}} e^{-|z|^2} dV$$

Now

$$e^{az} = \sum_{n=0}^{\infty} \frac{a^n z^n}{n!} \longmapsto \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{e^{+x^2/2}}{\cancel{\sqrt{\pi}}} \left(-\frac{1}{\sqrt{2}}\frac{d}{dx}\right)^n e^{-x^2}$$

$$= \frac{e^{x^2/2}}{\cancel{\sqrt{\pi}}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{a}{\sqrt{2}}\frac{d}{dx}\right)^n e^{-x^2}$$

$$= \frac{e^{x^2/2}}{\cancel{\sqrt{\pi}}}\, e^{-\left(x-\frac{a}{\sqrt{2}}\right)^2}$$

Thus since

$$f(z) = \sum (f, z^n)\frac{z^n}{n!} = \int \sum_n f(w)\frac{\bar{w}^n z^n}{n!} e^{-|w|^2}\, dV$$

$$= \int f(w)\, e^{\bar{w}z - |w|^2}\, dV$$

we see

$$f(z) \longmapsto \frac{e^{x^2/2}}{\cancel{\sqrt{\pi}}} \int f(w)\, e^{-\left(x-\frac{\bar{w}}{\sqrt{2}}\right)^2 - |w|^2}\, dV$$

So

$$f \longmapsto \hat{f}(x) = \frac{1}{\cancel{\sqrt{\pi}}} \int f(z)\, e^{-\frac{x^2}{2} + \sqrt{2}x\bar{z} - \frac{\bar{z}^2}{2}}\, e^{-z\bar{z}}\, dV$$

Recall translation on $\mathcal{H}$.

$$\int |f(z)|^2 e^{-|z|^2} dV = \int |f(z+\gamma)|^2 e^{-|z+\gamma|^2} dV$$

$$= \int |f(z+\gamma)|^2 e^{-|z|^2 - \gamma\bar{z} - \bar{\gamma}z - |\gamma|^2} dV$$

$$= \int \left| f(z+\gamma)\, e^{-\bar{\gamma}z - \frac{1}{2}|\gamma|^2} \right|^2 e^{-|z|^2} dV$$

Thus

$$f \longmapsto (T_\gamma f)(z) = f(z+\gamma)\, e^{-\bar{\gamma} z - \frac{1}{2}|\gamma|^2}$$

is a unitary operator on $\mathcal{H}$. One has

$$(T_\gamma T_\mu f)(z) = (T_\mu f)(z+\gamma)\, e^{-\bar{\gamma} z - \frac{1}{2}\gamma\bar{\gamma}}$$

$$= f(z+\gamma+\mu)\, e^{-\bar{\mu}(z+\gamma) - \frac{1}{2}\mu\bar{\mu} - \bar{\gamma} z - \frac{1}{2}\gamma\bar{\gamma}}$$

$$e^{+\bar{\mu}\gamma}\, T_\gamma T_\mu f(z) = f(z+\gamma+\mu)\, e^{-\bar{\mu} z - \bar{\gamma} z - \frac{1}{2}\mu\bar{\mu} - \frac{1}{2}\gamma\bar{\gamma}}$$

symmetric

$$\boxed{e^{\bar{\mu}\gamma}\, T_\gamma T_\mu = e^{\mu\bar{\gamma}}\, T_\mu T_\gamma}$$

This shows T_γ, T_μ commute if one is a real multiple of the other or if

$$e^{\gamma\bar{\mu} - \bar{\gamma}\mu} = 1$$

$$\sum_{n\in\mathbb{Z}}' e^{-nz - \frac{n^2}{2}} \longmapsto \sum_{n\in\mathbb{Z}} e^{\frac{x^2}{2}}\, e^{-(x+\frac{n}{\sqrt{2}})^2 - \frac{n^2}{2}}$$

$$= \sum_{n\in\mathbb{Z}}' e^{-\frac{x^2}{2} - \sqrt{2}xn - n^2}$$

$$= \sum e^{-(\frac{x}{\sqrt{2}} + n)^2} = \sum_{n\in\mathbb{Z}}' e^{-\frac{1}{2}(x+\sqrt{2}n)^2}$$

periodic with period $\sqrt{2}$ hence isn't in L^2.

Suppose $f(t)$ such that $\zeta(s)\int_0^\infty f(t)\,t^s\frac{dt}{t}$ satisfies a functional equation

$$\zeta(s)\int_0^\infty f(t)\,t^s\frac{dt}{t} = \int_0^\infty \sum_{n=1}^{\infty} n^{-s} f(t)\,t^s\frac{dt}{t} = \int_0^\infty \Big(\sum_{n=1}^{\infty} f(nt)\Big)\,t^s\frac{dt}{t}$$

Suppose $f(0) = \lim_{t\searrow 0} f(t)$ exists and that we extend f to $\mathbb{R}$ to be even. Then the above becomes

$$\frac{1}{2}\int_0^\infty \Big\{\sum_{n=-\infty}^{\infty} f(nt) - f(0)\Big\}\,t^s\frac{dt}{t} = \frac{1}{2}\int_0^\infty \Big\{\frac{1}{t}\sum_{n\in\mathbb{Z}}\hat{f}\Big(\frac{n}{t}\Big) - f(0)\Big\}\,t^s\frac{dt}{t}$$

$$= \frac{1}{2}\int_0^\infty \Big\{t\sum_{n\in\mathbb{Z}}\hat{f}(nt) - f(0)\Big\}\,t^{-s}\frac{dt}{t}$$

analytic cont.

$$= \frac{1}{2}\int_0^\infty \Big\{\sum \hat{f}(nt) - \hat{f}(0)\Big\}\,t^{1-s}\frac{dt}{t} = \int_0^\infty \sum_{n=1}^{\infty}\hat{f}(nt)\,t^{1-s}\frac{dt}{t}$$

functional equation

$$= \int_0^\infty \sum_{n=1}^{\infty}\hat{f}(nt)\,t^s\frac{dt}{t} = \zeta(s)\int_0^\infty \hat{f}(t)\,t^s\frac{dt}{t}$$

So $\displaystyle\int_0^\infty f(t)\,t^s\frac{dt}{t} = \int_0^\infty \hat{f}(t)\,t^s\frac{dt}{t} \implies f(t) = \hat{f}(t)$.

February 17, 1977

Consider the basic identity

$$\sum e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum e^{-\pi n^2 t^{-1}}$$

and try to prove it by power series expansions around $t = 1$.

Put $t = 1 + \varepsilon$.

$$\sum e^{-n^2 a t} = \sum e^{-n^2 a (1+\varepsilon)}$$

$$= \sum{}' e^{-n^2 a}\left(1 + \frac{(-n^2a)\varepsilon}{1!} + \frac{(-n^2a)^2}{2!}\varepsilon^2 + \cdots\right)$$

$$= \left(\sum{}' e^{-n^2a}\right) + \left(\sum{}' e^{-n^2a}(-n^2a)\right)\frac{\varepsilon}{1!} + \left(\sum e^{-n^2a}(-n^2a)^2\right)\frac{\varepsilon^2}{2!} + \cdots$$

$$t^{-1} = \frac{1}{1+\varepsilon} = 1 - \frac{\varepsilon}{1+\varepsilon}$$

$$\sum{}' e^{-n^2 a t^{-1}} = \sum{}' e^{-n^2 a\left(1 - \frac{\varepsilon}{1+\varepsilon}\right)}$$

$$= \left(\sum{}' e^{-n^2a}\right) - \left(\sum e^{-n^2a}(-n^2a)\right)\frac{\varepsilon}{1+\varepsilon} + \left(\sum{}' e^{-n^2a}(-n^2a)^2\right)\frac{1}{2!}\left(\frac{\varepsilon}{1+\varepsilon}\right)^2$$

$$\frac{1}{\sqrt{t}} = (1+\varepsilon)^{-1/2} = 1 - \frac{1}{2}\varepsilon + \frac{-\frac{1}{2}\cdot -\frac{3}{2}}{2!}\varepsilon^2$$

So

$$\frac{1}{\sqrt{t}}\sum e^{-n^2 a t^{-1}} = \left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2\right)\left(\sum e^{-n^2a} - \sum{}' e^{-n^2a}(-n^2a)\frac{\varepsilon}{1+\varepsilon} + \cdots\right)$$

Compare linear terms we get

$$-\sum e^{-n^2a}(-n^2a) = +\frac{1}{2}\sum e^{-n^2a} + \sum e^{-n^2a}(-n^2a)$$

or $$\sum{}' e^{-n^2a} - 4\sum{}' e^{-n^2a}(+n^2a) = 0$$

and this equation can only hold for $a = \pi$, probably. Next equation is

$$\frac{1}{2}\sum{}' e^{-n^2a}(-n^2a)^2 = \frac{3}{8}\sum{}' e^{-n^2a} + \frac{+1}{2}\sum{}' e^{-n^2a}(-n^2a) + \frac{1}{2}\sum{}' e^{-n^2a^2}(-n^2a^2)^2$$
$$+ \sum{}' e^{-n^2a}(-n^2a)$$

not [redacted] new, but the next one will be new. An interesting consequence of all this is that there are rational numbers g_k such that

$$\sum e^{-n^2a}(-n^2a)^k = g_k \sum{}' e^{-n^2a}$$

for $a = \pi$. This doesn't lead anywhere.

February 18, 1977: Back to Lee-Yang

$$P(z_1, \dots, z_n) = \sum_I \Big(\prod_{\substack{i\in I\\ j\in I'}} c_{ij}\Big) z^I = \sum_{1\notin I} \prod_{\substack{i\in I\\ j\in I'-\{1\}}} c_{ij} \prod_{i\in I} c_{i1} z_i$$
$$+ \sum_{1\in I} \prod_{\substack{i\in I-\{1\}\\ j\in I'}} c_{ij} \prod_{j\in I'} c_{1j}\, z_1 z^{I-\{1\}}$$

If $Q(z_2, \dots, z_n) = \displaystyle\sum_{I\subset\{2,\dots,n\}} \prod_{\substack{i\in I\\ j\in I' \text{ in } \{2,\dots,n\}}} c_{ij}\, z^I$, then

$$P(z_1,\dots,z_n)=\sum_{I\subset\{2,\dots,n\}} \prod_{\substack{i\in I\\ j\in I' \text{ in } \{2,\dots,n\}}} c_{ij} \prod_{i\in I} c_{i1} z_i + z_1 \sum_{I\subset\{2,\dots,n\}} \prod_{\substack{i\in I\\ j\in I' \text{ in } \{2,\dots,n\}}} c_{ij} \frac{\prod_{j=2,\dots,n} c_{1j}}{\prod_{j\in I} c_{1j}} z^I$$

$$= Q(c_{21}z_2,\dots,c_{n1}z_n)+z_1\prod_{j=2}^{n} c_{1j}\, Q\left(\frac{z_2}{c_{12}},\dots,\frac{z_n}{c_{1n}}\right)$$

Now by induction $Q(c_{21}z_2,\dots,c_{n1}z_n)\neq 0$ for $|z_2|\le 1,\dots,|z_n|\le 1$ assuming some $|c_{i1}|<1$; if all $|c_{i1}|=1$ then $c_{1i}^{-1}=\bar{c}_{i1}^{-1}=c_{i1}$ so $P(z_1,\dots,z_n)=Q(c_{21}z_2,\dots,c_{n1}z_n)(1+z_1\prod_{j=2}^{n}c_{1j})$ so the result is clear. Thus

$$\frac{Q\left(\frac{z_2}{c_{12}},\dots,\frac{z_n}{c_{1n}}\right)}{Q(c_{21}z_2,\dots,c_{n1}z_n)}$$

is analytic for $|z_2|,\dots,|z_n|\le 1$, hence the maximum is assumed when $|z_2|=\dots=|z_n|=1$, where

$$\left|\frac{Q\left(\frac{z_2}{c_{12}},\dots,\frac{z_n}{c_{1n}}\right)\prod_2^n c_{1j}}{Q(c_{21}z_2,\dots,c_{n1}z_n)}\right| = \left|\frac{\bar{Q}\left(\frac{1}{c_{12}\bar{z}_2},\dots,\frac{1}{c_{1n}\bar{z}_n}\right)\prod_2^n k_1}{Q(c_{21}z_2,\dots,c_{n1}z_n)}\right|$$

$$= \left|\frac{\bar{Q}(c_{12}\bar{z}_2,\dots,c_{1n}\bar{z}_n)\ \bar{z}_1^{-1}\dots\bar{z}_n^{-1}}{Q(c_{21}z_2,\dots,c_{n1}z_n)}\right| = \left|\frac{Q(\bar{c}_{12}z_2,\dots,\bar{c}_{1n}z_n)}{Q(c_{21}z_2,\dots,c_{n1}z_n)}\right|$$

$$=1$$

So the only possible root

$$\frac{1}{z_1}+\frac{Q\left(\frac{z_2}{c_{12}},\dots,\frac{z_n}{c_{1n}}\right)}{Q(c_{21}z_2,\dots,c_{n1}z_n)}=0$$