February 4, 1977.

Ising inequalities, etc. (from Simon's book).

Start with an Ising model: Configuration space $\{-1,1\}^n$ with energy

$$H(\sigma) = -\sum_{i<j} a_{ij}\sigma_i\sigma_j$$

where $a_{ij} \geq 0$ (energy decreases when spins are aligned). Thus we get the prob. measure

$$\mu(\sigma) = e^{-H(\sigma)}/Z \qquad \sigma \in \{-1,1\}^n$$

where

$$Z = \sum_{\sigma\in\{-1,1\}^n} e^{-H(\sigma)}$$

Next suppose we consider a real function $\{-1,1\}^n$

$$\sigma \longmapsto \sum h_i\sigma_i$$

This gives us a measure on $\mathbb{R}$, the image of the μ-measure under the function. The characteristic function of this measure is

$$\sum_\sigma e^{it\sum h_i\sigma_i} e^{-H(\sigma)}/Z$$

Put $u = it$ and let

$$F(u) = \sum_\sigma e^{u\sum h_i\sigma_i} e^{-H(\sigma)}/Z.$$

The claim then is that $F(u) \neq 0$ if $\mathrm{Re}(u) > 0$ and the $h_i > 0$. This is a consequence of the Lee-Yang circle theorem, as follows: Put $z_i = e^{+2uh_i}$

$$ZF(u) = \sum_{\sigma\in\{-1,1\}^n} \prod_i z_i^{+\frac{1}{2}(\sigma_i+1)} \prod_{i<j} \left(e^{a_{ij}}\right)^{\sigma_i\sigma_j} \cdot (z_1\cdots z_n)^{-1/2}$$

Lee-Yang says that for

$$P(z_1,\dots,z_n) = \sum_\sigma \prod_{i<j} (x_{ij})^{-\sigma_i\sigma_j} \cdot \prod_i z_i^{\frac{1}{2}(\sigma_i+1)} \qquad \begin{matrix} -1 < x_{ij} < +1 \\ x_{ij} \neq 0 \end{matrix}$$

one has $|z_1| \le 1, \dots, |z_{n-1}| \le 1, |z_n| < 1 \Rightarrow P \neq 0$, so it works at least if the $a_{ij} > 0$. Rest by a limit process using Hurwitz thm.

I want to review my previous understanding of the ζ function.

Let C be a curve (complete non-singular) over a finite field $\mathbb{F}_q = H^0(C, \mathcal{O}_C)$. Then

$$\zeta_C(s) = \prod_{P\in C} \left(1 - \frac{1}{N(P)^s}\right)^{-1} = \sum_{D\geq 0} \frac{1}{N(D)^s}$$

where

$$N(P) = \text{card } k(P) = q^{\deg(P)}$$

$$N(D) = q^{\deg D}$$

So if we put $z = q^{-s}$ one has

$$\zeta_C(s) = Z(z) = \sum_{D\geq 0} z^{\deg(D)}$$

Rewrite this sum over divisor classes

$$Z(z) = \sum_{L \in \mathrm{Pic}(C)} z^{\deg L} \sum_{\substack{D \geq 0 \\ \mathcal{O}(D^{-1}) \simeq L}}' 1$$

$$= \sum_{L \in \mathrm{Pic}(C)} z^{\deg L} \; \frac{q^{h^0(L)} - 1}{q - 1}$$

Now use R-R

$$h^0(L) - h^0(K \otimes L^{-1}) = \deg(L) + 1 - g$$

Also recall that one has the analytic continuation:

$$\sum_{n \geq 0}' z^n = -\sum_{n<0}' z^n$$

Proof:

$$\underset{\text{for } |z|<1}{\overset{\shortparallel}{}} \qquad\qquad \underset{\text{for } |z|>1}{\overset{\shortparallel}{}}$$

$$\frac{1}{1-z} \underset{\text{for all } z}{=} \frac{-z^{-1}}{1-z^{-1}}$$

Assume known that C has a line bundle $\mathcal{O}(1)$ of degree 1 whence any L of degree n is of the form $L'(n)$ for a unique $L' \in \mathrm{Pic}^0(C)$. Then

$$Z(z) = \sum_{L \in \mathrm{Pic}^0} \sum_n z^n \; \frac{q^{h_0(L(n))} - 1}{q - 1}$$

If we use the formal relations

$$\sum_n q^n z^n = 0 \qquad \sum_n z^n = 0$$

which can be justified by analytic continuation we have

$$Z(z) = \sum_{L\in \mathrm{Pic}^0} \sum_n z^n \, \frac{q^{h_0(L(n))} - q^{n+1-g}}{q-1}$$

$$= \frac{q^{1-g}}{q-1} \sum_{L\in \mathrm{Pic}^0} {\sum_n}' (qz)^n \left(q^{\overset{= h_0(K\otimes L^{-1}(-n))}{h_0(L(n))-n-1+g}} - 1 \right)$$

$$= \frac{q^{1-g}}{q-1} {\sum_{L\in \mathrm{Pic}^0}}' (qz)^{2g-2} {\sum_n}' \left((qz)^{-1}\right)^{2g-2-n} \left(q^{h_0(K\otimes L^{-1}(-n))} - 1 \right)$$

$$= \frac{q^{g-1}}{\cancel{q-1}} z^{2g-2} Z\!\left(\frac{1}{qz}\right)$$

which is the functional equation. Next I show

$$Z(z) = \frac{P(z)}{(1-z)(1-qz)}$$

where $\deg P(z) = 2g$. The point is that

$$(1-z)(1-qz)Z(z) = \frac{1}{q-1} \sum_{L\in \mathrm{Pic}^0} {\sum_n}' z^n \underbrace{\left(q^{h_0(L(n))} - (1+q) q^{h_0(L(n-1))} + q\, q^{h_0(L(n-2))} \right)}$$

Now from R-R one knows that

$$h_0(L(n)) = 0 \quad \text{if } n<0$$

$$h_0(L(n)) = n+1-g \quad \text{if } n>2g-2$$

Hence it is easily seen that because $x^2-(1+q)x+q = (x-1)(x-q)$, that

$$q^{h_0(L(n))} - (1+q)\,q^{h_0(L(n-1))} + q\,q^{h_0(L(n-2))} = 0$$

if $n<0$ or if $n>2g$, hence $(1-z)(1-qz)\,Z(z)$ is a poly of degree $2g$.

Do the same for the Riemann ζ-function.

$$\Gamma(s) = \int_0^\infty e^{-t}\,t^s\,\frac{dt}{t} = \pi^s\int_0^\infty e^{-\pi t}\,t^s\,\frac{dt}{t}$$

$$= 2\pi^s\int_0^\infty e^{-\pi t^2}\,t^{2s}\,\frac{dt}{t}$$

~~Recall that if~~

$$\cancel{\theta(y) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}}$$

~~then one has~~ $\cancel{\theta\left(\frac{1}{t}\right)}$

$$Z(s) = \zeta(s)\,\Gamma(s/2)\,\pi^{-s/2} = \sum_{n=1}^{\infty}\frac{1}{n^s}\,2\int_0^\infty e^{-\pi t^2}\,t^s\,\frac{dt}{t}$$

$$= 2\sum_{1}^{\infty}\int_{0}^{\infty} e^{-\pi n^2 t^2}\, t^s \,\frac{dt}{t}$$

$$= \int_{0}^{\infty} [\theta(t) - 1]\, t^s \,\frac{dt}{t}$$

where $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2}$. Recall Poisson identity

$$\theta(\tfrac{1}{t}) = t\,\theta(t)$$

So

$$Z(s) = \int_{0}^{\infty} [\theta(t) - 1]\, t^s \frac{dt}{t} = \int_{0}^{\infty} [\theta(\tfrac{1}{t}) - 1]\, t^{-s} \frac{dt}{t}$$

$$= \int_{0}^{\infty} [t\,\theta(t) - 1]\, t^{-s} \frac{dt}{t}$$

Using the formal identity $\int_0^\infty t^\alpha\, dt = 0 \quad \alpha \neq 0$ which can be justified by analytic continuation (better view both sides as distributions and $\int_0^\infty t^\alpha dt = \text{const.}\, \delta(\alpha)$), one gets

$$Z(s) = \int_0^\infty [t\,\theta(t) - t]\, t^{-s}\frac{dt}{t} = \int_0^\infty [\theta(t) - 1]\, t^{1-s}\frac{dt}{t} = Z(1-s)$$

which is the functional equation.

It's better to work with

$$\varphi(t) = \theta(\tfrac{1}{t}) = \sum_{n\in\mathbb{Z}} e^{-\pi^2 n^2/t^2}$$

which resembles $q^{hd(L(n))}$ in that it has the asymptotic behavior:

$$\varphi(t) \sim t \qquad \text{as} \qquad t \longrightarrow \infty$$

$$\varphi(t) \sim 1 \qquad \text{as} \qquad t \longrightarrow 0+$$

and these approaches are very fast:

$$\frac{\varphi(t)}{t} = \frac{1}{t}\theta\left(\frac{1}{t}\right) = \theta(t) \longrightarrow 1 \quad \text{very fast}$$

So

$$Z(s) = \int_0^\infty \left[\varphi(t) - 1\right] t^{-s-1} dt \qquad \mathrm{Re}(s) > 1$$

Integrate by part

$$Z(s) = \left[(\varphi(t)-1)\frac{t^{-s}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty \varphi'(t)\, t^{-s} dt$$

$$= \frac{1}{s}\left[\varphi'(t)\frac{t^{-s+1}}{-s+1}\right]_0^\infty - \frac{1}{s(1-s)}\int_0^\infty \varphi''(t)\, t^{-s+1} dt$$

$$\therefore \quad s(1-s)Z(s) = \int_0^\infty (-\varphi''(t))\, t^{-s+1} dt = \frac{1}{2}\int_0^\infty -\varphi''(t^{1/2})\, t^{-\frac{s}{2}} dt$$

Replace s by $1-s$

$$(1-s)s\,Z(s) = \int_0^\infty (-\varphi''(t))\, t^{s} dt$$

and this should converge for all s in $\mathbb{C}$.

$$\varphi(t) = t\,\theta(t) = \sum e^{-\pi n^2 t^2}\, t$$

$$\varphi'(t) = \sum e^{-\pi n^2 t^2}\left(1 - 2\pi n^2 t^2\right)$$

$$\varphi''(t) = \sum e^{-\pi n^2 t^2}\left[-4\pi n^2 t + (1-2\pi n^2 t^2)(-2\pi n^2 t)\right]$$

$$= \sum e^{-\pi n^2 t^2}\left[4\pi^2 n^4 t^3 - 6\pi n^2 t\right]$$

$$-\varphi''(t) = t \sum e^{-\pi n^2 t^2}\, 2\pi n^2 \left(2\pi n^2 t^2 - 3\right)$$

$$(1-s)(s)Z(s) = \int_0^\infty \left(-\varphi''(t)\right) t^{s-\frac{1}{2}}\, t^{3/2} \frac{dt}{t}$$

$$= \int_0^\infty \left(-t^{3/2}\varphi''(t)\right) t^{s-\frac{1}{2}} \frac{dt}{t}$$

Put $\gamma(t) = -t^{3/2}\varphi''(t)$. Then one can show

$$\gamma(t) = \gamma\left(\tfrac{1}{t}\right).$$

corresponding to symmetry of $(1-s)s\,Z(s)$ around s.
So we get the integral representation

$$(1-s)\,s\,Z(s) = \int_{-\infty}^{\infty} \gamma(e^u)\, e^{(s-\frac{1}{2})u}\, du$$

where $\gamma(e^u)$ is even in u. Too messy. You probably want a more sophisticated smoothing out of $\varphi(t)$, corresponding to a product

$$f(1-s)\, f(s)\, Z(s)$$

with f chosen very shrewdly.

February 5, 1977.

distribution of n-Bernouilli trials is

$$k \longmapsto \binom{n}{k} p^k (1-p)^{n-k}$$

Let $p = \frac{\lambda}{n}$ and let $n \to \infty$, i.e. you make more and more trials but the expected number of successes np is fixed.

$$\frac{n(n-1)\cdots(n-k+1)}{k! \quad n^k} (\lambda)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$\longrightarrow \frac{\lambda^k}{k!} e^{-\lambda}$$

Characteristic function of this distribution is

$$\int e^{iut} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \delta(t-k)$$

$$= \sum \frac{e^{iuk} \lambda^k}{k!} e^{-\lambda} = e^{\lambda(e^{iu}-1)}$$

So we have the formal relation:

$$\boxed{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta(t-k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iut} e^{\lambda e^{iu}} du}$$

Compare this with expression for the Γ-function

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t} = \int_{-\infty}^{\infty} e^{-e^v} e^{sv}\, dv$$

$$= i \int_{-i\infty}^{+i\infty} e^{-e^{iu}} e^{isu}\, du$$

So we want to compare:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \delta(t+k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{+itu} e^{-e^{iu}}\, du$$

$$\Gamma(t) = i \int_{-i\infty}^{i\infty} e^{itu} e^{-e^{iu}}\, du$$

One should notice also that $\Gamma(t)$ has simple poles at $t = -k$ with residue $\frac{(-1)^k}{k!}$, because

$$\Gamma(t-k) = \frac{\Gamma(t)}{(t-k)(t-k+1)\cdots(t-1)}$$

$\Gamma(t) \sim \frac{1}{t}$ as $t \to 0$

February 6, 1977.

Lee-Yang thm. (Asano proof).

Consider the class $\mathcal{a}$ of polynomials $P(z_1, \ldots, z_n)$ with complex coefficients of degree ≤ 1 in each of the variables such that $|z_i| < 1$ all $i \Rightarrow P(z_1, \ldots, z_n) \neq 0$.

If P, Q have different variables and are in $\mathcal{a}$, then $PQ \in \mathcal{a}$.

If $P \in \mathcal{a}$ and x, y are two of the variables of P, then

$$P = A(z) + B(z)x + C(z)y + D(z)xy$$

where $z = (z_1, \ldots, z_k)$ are the remaining variables. The Asano-contraction of P is the poly $\tilde{P}(w, z) = A(z) + D(z)w$. The claim is that $\tilde{P} \in \mathcal{a}$. In effect we fix z with $|z| < 1$, then $P(x, x, z) = A + (B + C)x + Dx^2$ has its two roots outside $|x| < 1$, hence the product of these roots $\frac{A}{D}$ has absolute value ≥ 1. Hence the root of $\tilde{P}$, $w = -\frac{A}{D}$ has absolute value ≥ 1.

Now start with example.

$$P(z_1, z_2) = \frac{1}{a} + a z_1 + a z_2 + \frac{1}{a} z_1 z_2$$

$$P(z_1, z_2) = \frac{1}{a}\left(1 + a^2(z_1 + z_2) + z_1 z_2\right).$$

Better consider the polynomial

$$P(x, y) = 1 + ax + \bar{a}y + xy$$

where $|a| < 1$. Then $1 + ax + \bar{a}y + xy = 1 \Rightarrow y = -\frac{1 + ax}{x + \bar{a}}$

Note that if $x\bar{x}=1$, then $\overline{-\frac{1+ax}{x+\bar{a}}} = -\frac{1+\bar{a}\bar{x}}{\bar{x}+a} = -\frac{x+\bar{a}}{1+\bar{a}x} = \left(-\frac{1+\bar{a}x}{x+\bar{a}}\right)^{-1}$

hence $x \longmapsto -\frac{1+ax}{x+\bar{a}}$ is a fractional linear transf. preserving $|x|=1$. As $-a^{-1}$ which is outside $|x|=1$ gets mapped inside it follows $|x|<1 \Rightarrow |y|>1$ when $P(x,y)=0$. So $P(x,y)$ is in the good class.

Note that if $1+2bz+z^2$ has both roots outside of $|z| \leq 1$, then because the product of the roots is 1 they have to lie on $|z|=1$ and be of the form $e^{i\theta}, e^{-i\theta}$, hence $b=\cos\theta$ satisfies $-1 \leq b \leq 1$. Thus if $1+ax+by+xy$ is in the class $\mathcal{A}$, we see that for any ζ with $|\zeta|=1$, that

$$-1 \leq a\zeta + b\zeta^{-1} \leq 1$$
$$\parallel$$
$$(a+b)\cos\theta + (ai-bi)\sin\theta$$

So $a+b \in \mathbb{R}$, $ai-bi \in \mathbb{R} \Rightarrow a-b \in \mathbb{R}i$.

$$a+b = \alpha \qquad a = \frac{\alpha+\beta i}{2} \qquad b = \frac{\alpha-\beta i}{2}$$
$$a-b = \beta i$$

$\therefore$ $b=\bar{a}$, $-1 \leq \alpha\cos\theta - \beta\sin\theta \leq 1 \iff \alpha^2+\beta^2 \leq 1$ by Cauchy-Schwarz. Note

$$|a|=1 \Rightarrow 1+ax+\bar{a}y+xy = (1+ax)(1+a^{-1}y)$$

which vanishes only if $x = -\frac{1}{a}$ or $y=-a$ so it does not vanish for $|x|<1$ and $|y|<1$. Thus we have proved:

<u>Prop</u>: The polys $1 + ax + by + xy$ not vanishing for $|x|<1$, $|y|<1$ are precisely those with $b = \bar{a}$ and $|a| \leq 1$.

In Simon's book Lee-Yang is stated for polynomials

$$\sum_{\sigma\in\{-1,1\}^n} \Big(\prod_{i<j} X_{ij}^{-\sigma_i\sigma_j}\Big) z_1^{\frac{1}{2}(\sigma_1+1)}\cdots\cdots z_n^{\frac{1}{2}(\sigma_n+1)} \qquad -1 < \overset{\neq 0}{X_{ij}} < 1$$

However note that if ε_{ij} is the sign (± 1) of X_{ij}, then

$$\prod_{i<j} X_{ij}^{-\sigma_i\sigma_j} = \prod_{i<j} \left(\varepsilon_{ij}\,|X_{ij}|\right)^{-\sigma_i\sigma_j}$$

$$= \prod_{i<j}\varepsilon_{ij} \prod_{i<j} |X_{ij}|^{-\sigma_i\sigma_j}$$

independent of σ

since $\sigma_i\sigma_j = \pm 1$ always. Thus the signs of the X_{ij} don't matter in Simon's formulation.

First note that

$$\sigma \longmapsto \prod_{i<j} X_{ij}^{-\sigma_i\sigma_j}$$

is some sort of quadratic function on $\{-1,1\}^n$ with values in $\mathbb{R}^*$. ???

~~Newman's basic examples of Lee Yang polynomials are as follows:~~

Let u, u_i etc denote complex variables which are going to be related to z, z_i by expressions of the form $z = e^{au}$ where $a \in \mathbb{R}$ and $a < 0$ so that $Re(u) > 0 \iff |z| = e^{a Re(u)} < 1$. LY in dim. 1 says that $1+z$ has zeroes outside of $|z|>1$.

$$1+z = 1+e^{au} = e^{au/2}\left(e^{au/2}+e^{-au/2}\right)$$

Change notation $z = e^{iau}$ with $a>0$ so that $|z| = e^{-a\,Im(u)} < 1 \iff Im(u) > 0$. Then

$$1+z = e^{iau/2}\left(e^{iau/2}+e^{-iau/2}\right)$$

$$= 2e^{iau/2}\, Cos\left(\frac{au}{2}\right)$$

So we see $\cos\left(\frac{au}{2}\right) \neq 0$ for $Im(u) > 0$.

In dim 2, the LY polys. are

$$1+\alpha z_1 + \bar{\alpha} z_2 + z_1 z_2 \qquad |a| \leq 1.$$

and by modifying z_1 by $e^{-i\arg(\alpha)}$ one can suppose α is real $-1 \leq \alpha \leq 1$. So we get ~~[illegible]~~ up to an exponential factor

$$e^{-\frac{i}{2}(a_1u_1+a_2u_2)} + \alpha\left(e^{\frac{i}{2}(a_1u_1-a_2u_2)}+e^{\frac{i}{2}(a_2u_2-a_1u_1)}\right) + e^{+\frac{i}{2}(a_1u_1+a_2u_2)}$$

$$= 2\left(\cos\tfrac{1}{2}(a_1u_1+a_2u_2) + \alpha\cos\tfrac{1}{2}(a_1u_1-a_2u_2)\right) \qquad -1\leq\alpha\leq 1$$

Better notation maybe is $z_j = e^{2 i u_j}$. Then the functions we know don't vanish for $\mathrm{Im}(u_i) > 0$

$$\cos(u)$$

$$\cos(u_1 + u_2) + \alpha \cos(u_1 - u_2) \qquad -1 \leq \alpha \leq 1.$$

Now we set $u_i = a_i u + b_i$ $\quad a_i > 0,\ b_i \in \mathbb{R}$.

You get

$$\cos(\lambda u + \beta) + \alpha\left(\cos(\lambda' u + \beta')\right)$$

where $\lambda + \lambda' > 0$, $\lambda - \lambda' > 0$ $\quad \beta$ and β' are arbitrary real numbers.

February 8, 1977

Lee-Yang thm.

$$P(z_1, \ldots, z_n) = \sum_{I \subset \{1, \ldots, n\}} c_I z^I \qquad z^I = z_{i_1} \cdots z_{i_k} \text{ if } \{i_1, \ldots, i_k\} = I, \; i_1 < \cdots < i_k$$

$$c_I = \prod_{\substack{i \in I \\ j \in I'}} c_{ij}$$

where c_{ij} are complex numbers defined for $i \neq j$ such that $c_{ij} = \bar{c}_{ji}$ and $|c_{ij}| \leq 1$. Thus for $n=2$ we get

$$1 + c_{12} z_1 + \bar{c}_{12} z_2 + z_1 z_2$$

which I've seen doesn't vanish for $|z_1| < 1$, $|z_2| < 1$. Note that

$$c_{I'} = \prod_{\substack{i \in I' \\ j \in I}} c_{ij} = \prod_{\substack{j \in I' \\ i \in I}} \bar{c}_{ji} = \overline{c_I}$$

hence

$$z_1 \cdots z_n \, P(z_1^{-1}, \ldots, z_n^{-1}) = \sum c_I z^{I'} = \sum \bar{c}_I z^I = \bar{P}(z).$$

Suppose to begin with that $c_{ij} \neq 0$. Then we can write

$$c_{ij} = \varepsilon_{ij} e^{-a_{ij}} \qquad a_{ij} \geq 0 \qquad |\varepsilon_{ij}| = 1.$$

Put $\sigma_i = \begin{cases} +1 & i \in I \\ -1 & i \in I' \end{cases}$ whence $z^I = \prod_{i=1}^{n} z_i^{\frac{1}{2}(\sigma_i + 1)}$

Put $x_i = \frac{1}{2}(\sigma_i + 1) = \begin{cases} 1 & i \in I \\ 0 & i \in I' \end{cases}$. Then

$$c_I = \prod_{i,j} c_{ij}^{x_i(1-x_j)} = \prod_{i<j} c_{ij}^{x_i(1-x_j)} \, \bar{c}_{ij}^{\,x_j(1-x_i)}$$

$$= \prod_{i<j} \varepsilon_{ij}^{x_i(1-x_j) - x_j(1-x_i)} \prod_{i<j} e^{-a_{ij}[x_i(1-x_j) + x_j(1-x_i)]}$$

$$= \prod_{i<j} \varepsilon_{ij}^{+x_i} \prod_{i<j} \varepsilon_{ij}^{-x_j} \prod_{i<j} e^{-a_{ij}\frac{1}{4}[(\sigma_i+1)(1-\sigma_j) + (\sigma_j+1)(1-\sigma_i)]}$$

$$= \prod_{i<j} \varepsilon_{ij}^{x_i} \prod_{i>j} \varepsilon_{ji}^{-x_i} \prod_{i<j} e^{\frac{1}{4}a_{ij}\sigma_i\sigma_j - \frac{1}{2}a_{ij}}$$

$$= \prod_i \Big(\prod_j \varepsilon_{ij}\Big)^{x_i} e^{\sum_{i<j} \frac{1}{4}a_{ij}\sigma_i\sigma_j - \frac{1}{2}a_{ij}}$$

Now notice that [redacted] we can absorb the first term into $z^I = \prod_i z_i^{x_i}$ by replacing z_i by $(\prod_j \varepsilon_{ij})^{-1} z_i$. Therefore in formulating the Lee-Yang theorem at least for non-zero c_{ij} we can suppose $0 < c_{ij} \leq 1$ and hence that we are in the ferro-magnetic situation.

$\blacksquare$

February 9, 1977

Next consider what happens when some of the c_{ij} are zero. If $c_{12}=0$, then

$$c_I = 0 \qquad \text{if } I \text{ separates } 1,2$$

so

$$P(z_1,\dots,z_n) = \sum_{\{1,2\}\subset I'}{}' c_I z^I + \Big(\sum_{\{1,2\}\subset I}{}' c_I z^{I-\{1,2\}}\Big) z_1 z_2$$

~~already appears in contracted form. The effect of the contraction therefore is that~~ hence $P(z) = Q(z_1 z_2, z_3, \dots, z_n)$ where Q is of the same sort but with $c^Q_{1i} = c_{1i}c_{2i}$. Hence it is clear that if we partition the variables according to the equivalence relation generated by the relation $c_{ij}=0$, then P is obtained from a similar P on the equivalence classes.

Of interest to me is the class of polynomials $P(z)$ of one variable with cx. coefficients such that the roots are on $|z|=1$ and stable under inversion $\lambda \mapsto \lambda^{-1}$! Observe that then the non-real roots occur in pairs $e^{i\theta}, e^{-i\theta}$; since

$$(z-e^{i\theta})(z-e^{-i\theta}) = z^2 + (2\cos\theta)z + 1$$

one sees P has the form

$$P(z) = (z-1)^p (z+1)^q \prod_{i=1}^{m} (z^2 + 2a_i z + 1)$$

with $-1 \le a_i \le 1$ and where $p,q = 0$ or 1. Moreover

P has the symmetry property:

$$z^n P\left(\frac{1}{z}\right) = (-1)^p P(z) \qquad n = p+q+2m = \deg P.$$

Next suppose $z = e^{-2u}$ so that $\mathrm{Re}(u) > 0 \iff |z| = e^{-2\mathrm{Re}(u)} < 1$. Corresponding to the above poly. P one has the [redacted] "trig." poly:

$$F(u) = e^{nu} P(e^{-2u}) = (-1)^p (e^{u} - e^{-u})^p (e^{u} + e^{-u})^q \prod_{i=1}^{m} (e^{2u} + 2a_i + e^{-2u})$$

which has the symmetry

$$F(-u) = (-1)^p F(u)$$

hence is either even or odd. $F(u)$ is the Laplace transform of a signed measure on $\mathbb{R}$ [redacted] supported at the integers between $-n$ and $+n$. For example

$$e^{2u} + 2a_i + e^{-2u} = \int_{\mathbb{R}} e^{ux} \left(\delta(x-2) + 2a_i\,\delta(0) + \delta(x+2)\right) dx$$

Thus a generalization of the class of real polys. with roots on $|z| = 1$ stable under $\lambda \mapsto \lambda^{-1}$ is the class of signed measures μ on $\mathbb{R}$ with finite support which are even ~~or~~ odd such that

$$\int_{\mathbb{R}} e^{ux}\, d\mu(x)$$

has its roots on $\mathrm{Re}(u) = 0$. For example take a Lee-Yang polynomial

$$P(z_1, \ldots, z_n) = \sum{}' c_I z^I$$

$$c_I = \prod_{\substack{i \in I \\ j \in I'}} c_{ij} \qquad 0 \leq c_{ij} \leq 1 \qquad c_{ij} = c_{ji}$$

Then put

$$F(u) = P(\varepsilon_1 e^{-\mu_1 u}, \ldots, \varepsilon_n e^{-\mu_n u}) \, e^{(\mu_1 + \cdots + \mu_n) u}$$

where $\mu_1, \ldots, \mu_n \geq 0$ and $|\varepsilon_i| = 1$ (F will be real if $\varepsilon_i = \pm 1$). Thus if no $c_{ij} = 0$

$$F(u) = (\text{const}) \sum_{\sigma \in \{-1,1\}^n} e^{\frac{1}{4} \sum_{i<j} a_{ij} \sigma_i \sigma_j} \prod_i \varepsilon_i^{\frac{-\sigma_i + 1}{2}} e^{\sum_i \mu_i \sigma_i u}$$

So the corresponding signed measure on $\mathbb{R}$ is

$$\boxed{\mu(x) = (\text{const}) \sum_{\sigma \in \{-1,1\}^n} e^{\frac{1}{4} \sum a_{ij} \sigma_i \sigma_j} \prod_i \varepsilon_i^{\frac{-\sigma_i + 1}{2}} \delta\left(x - \sum_{i=1}^n \mu_i \sigma_i\right)}$$

Note that $\mu(-x)$ is given by the same expression but with σ_i changes to $-\sigma_i$. Since

$$\prod_i \varepsilon_i^{\frac{+\sigma_i + 1}{2}} = \prod \varepsilon_i^{+\sigma_i + \frac{-\sigma_i + 1}{2}} \qquad \text{and } \varepsilon_i^{\sigma_i} = \varepsilon_i \text{ if } \varepsilon_i = \pm 1.$$

one has

$$\mu(-x) = \left(\prod \varepsilon_i\right) \mu(x)$$

so μ is either even or odd.