Marchenko equation:

Consider
$$\left(-\frac{d^2}{dx^2} + q(x)\right)\psi = \lambda\psi$$

on an interval (α, ∞), where α can be $-\infty$.

Assume i) $q(x) \to 0$ fast enough as $x \to +\infty$ so that one has a! solution $\psi(x,k) \sim e^{ikx}$ as $x \to +\infty$ for each k with $\operatorname{Im} k) \geq 0$ (except $k=0$); here $\lambda = k^2$.

ii) As $x \to \alpha'$ one has either the limit point case, or one gives a boundary condition, so that one has a solution $\phi(x,k)$ determined up to a non-zero scalar multiple. $\phi(x,k)$ should be definable analytic for $k \in$ UHP. Assume it has nice bdry values for k real $\neq 0$.

~~The Green's function is~~

$$\cancel{G_k(x,x') = \frac{\phi(x_<,k)\,\psi(x_>,k)}{W(\phi,\psi)}}$$

For example suppose $\alpha = -\infty$ and $q \to 0$ as $x \to -\infty$ so that we can define $\phi(x,k)$ by $\phi(x,k) \sim e^{-ikx}$ as $x \to -\infty$.

Now define $A(k)$, $B(k)$ for k real $\neq 0$ by

$$\phi(x,k) = A(k)\,\psi(x,-k) + B(k)\,\psi(x,k)$$

The ratio $R(k) = \dfrac{B(k)}{A(k)}$ is well-defined and is called the *reflection coefficient*.

The Green's function (kernel for $\left(k^2 + \frac{d^2}{dx^2} - q\right)^{-1}$ on L^2) is given by

$$G_k(x,x') = \frac{\phi(x_<,k)\,\psi(x_>,k)}{W(\phi(x,k),\,\psi(x,k))}.$$

Notice that $W = W(\phi(x,k), \psi(x,k))$ is analytic in the UHP with 0's at the bound states

But we have for k real

$$W = W(A(k)\psi(x,-k), \psi(x,k)) \qquad \text{constant in } x$$
$$= A(k)\, W(e^{-ikx}, e^{ikx}) \qquad (\text{take } x \to +\infty)$$
$$= A(k)\, 2ik.$$

Thus $A(k)$ ~~[illegible]~~ has an analytic extension ~~to~~ the UHP.

Completeness Relation

$$\frac{1}{2\pi i} \oint G_{\sqrt{\lambda}}(x,x')\, d\lambda = \delta(x-x')$$

contour enclosing spectrum

~~[illegible]~~ One expects the spectrum to consist of $\lambda \geq 0$ and some discrete spectra for $\lambda < 0$. The ~~[illegible]~~ discrete part is evaluated by residues; the continuous part by changing to $k = \sqrt{\lambda}$:

$$\frac{1}{2\pi i}\int_{+\infty}^{-\infty} \frac{\phi(x_<,k)\psi(x_>,k)}{A(k)\,2ik}\, 2k\,dk$$

So the completeness relation becomes

$$\delta(x-x') = \sum_{\substack{A(k)=0 \\ \mathrm{Im}(k)>0}} \frac{\psi(x,k)\psi(x',k)}{\|\psi(x,k)\|^2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi(x_<,k)\psi(x_>,k)}{A(k)}\, dk$$

To simplify suppose no discrete spectrum. Then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi(x_<,k)\psi(x_>,k)}{A(k)}\, dk = \delta(x-x')$$

~~The~~ The next point is that we know

$$\psi(x,k) = e^{ikx} + \int_x^\infty dy\, T(x,y) e^{iky}$$

from the way we know $\exists$ soln. $\psi(x,k) \sim e^{ikx}$ as $x \to +\infty$.
Moreover this equation can be inverted to express e^{ikx} in terms of the $\psi(y,k)$ for $y \geq x$. Thus

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\phi(x,k)}{A(k)}}_{\psi(x,-k)+R(k)\psi(x,k)} e^{iky}\, dk = 0 \qquad x < y$$

so

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left\{ e^{-ikx} + \int_{z>x} dz\, T(x,z) e^{-ikz} + R(k)\left[e^{ikx} + \int_{z>x} dz\, T(x,z) e^{ikz}\right]\right\} e^{iky} = 0 \quad \text{for } y>x$$

and interchanging order of integration yields:

$$\boxed{T(x,y) + \hat{R}(x+y) + \int_{z>x} dz\, T(x,z)\hat{R}(z+y) = 0 \quad \text{for } y > x}$$

This is the Marchenko equation. Here

$$\hat{R}(x) = \frac{1}{2\pi}\int dk\, e^{ikx} .$$

With bound states, one gets same equation with $\hat{R}(x)$ replaced by

$$\sum C_j e^{-\kappa_j x} + \hat{R}(x)$$

$-\kappa_j^2$ are the bound energies and $\frac{1}{C_j} = \| \psi(x, i\kappa_j)\|^2$.

Review adding a bound state. Start with

$$\left(-\frac{d^2}{dx^2}+q\right)=\left(\frac{d}{dx}+p\right)\left(-\frac{d}{dx}+p\right)$$

holds if $\quad q = p^2+p'$

One can get a p by solving $\left(-\frac{d}{dx}+p\right)u=0$

i.e $\quad p=\frac{u'}{u}\quad$ where u is an non-vanishing soln of $-u''+qu=0$.

So now start with a Schroedinger eqn. on the line

$$\left(-\frac{d^2}{dx^2}+q\right)u = k^2 u$$

where $q\to 0$ fast as $x\to\infty$. Let $-\beta^2<$ spectrum of this operator and let u be a non-vanishing solution of

$$\left(-\frac{d^2}{dx^2}+q+\beta^2\right)u=0.$$

One knows such a u exists, because $-\beta^2<$ spec. In fact there's a whole 1-parameter family, not counting scalar multiples:

$$u=\underbrace{\psi(x,+i\beta)}_{\sim e^{-\beta x}\text{ at }+\infty}+c\,\underbrace{\phi(x,-i\beta)}_{\sim e^{+\beta x}\text{ at }-\infty}$$

with $c>0$. Both $\psi(x,i\beta)$, $\phi(x,-i\beta)$ can't vanish (by some version of min-max).

I assume neither $\psi(x,i\beta)$ nor $\phi(x,-i\beta)$ occur for u; then I know that

$$u \sim \begin{cases} \text{const } e^{\beta x} & x \to +\infty \\ \text{const } e^{-\beta x} & x \to -\infty \end{cases}$$

So now put $p = \frac{u'}{u}$. Then

$$\left(-\frac{d^2}{dx^2} + q + \beta^2\right) = \left(\frac{d}{dx} + p\right)\left(-\frac{d}{dx} + p\right),$$

So that if we define $\tilde{q}$ by

$$-\frac{d^2}{dx^2} + \tilde{q} + \beta^2 = \left(-\frac{d}{dx} + p\right)\left(\frac{d}{dx} + p\right)$$

The operator $-\frac{d^2}{dx^2} + \tilde{q}$ has the bound state $\frac{1}{u}$.

Also
$$\tilde{q} + \beta^2 = p^2 - p' = q + \beta^2 - 2p'$$
$$\tilde{q} = q - 2p' \to 0 \text{ fast at } \infty.$$

Also we know the operators $\frac{d}{dx} + p$, $-\frac{d}{dx} + p$ intertwine $-\frac{d^2}{dx^2} + q$ and $-\frac{d^2}{dx^2} + \tilde{q}$:

$$\left(-\frac{d^2}{dx^2} + \tilde{q}\right)\left(-\frac{d}{dx} + p\right) = \left(-\frac{d}{dx} + p\right)\left(-\frac{d^2}{dx^2} + q\right)$$

so the spectra are the same except for things killed by these operators: Only $\frac{1}{u}$ is involved.

So if $\psi(x,k)$ (resp $\phi(x,k)$) is the eigenfn for $-\frac{d^2}{dx^2} + q$ with

$\psi(x,k) \sim e^{-ikx}$ as $x \to \infty$

resp. $\phi(x,k) \sim e^{-ikx}$ as $x \to -\infty$

we have

$$\left(-\frac{d}{dx} + p\right)\begin{cases}\psi(x,k) \\ \phi(x,k)\end{cases} \begin{matrix} \sim (-ik + \beta)e^{-ikx} \\ \sim (+ik - \beta)e^{-ikx}\end{matrix} \quad \text{as } x \to +\infty$$

is an eigenfunction of $-\frac{d^2}{dx^2}+\tilde{q}$. Thus

$$\tilde{\psi}(x,k) = \frac{1}{-ik+\beta}\left(-\frac{d}{dx}+p\right)\psi(x,k)$$

$$\tilde{\phi}(x,k) = \frac{1}{ik-\beta}\left(-\frac{d}{dx}+p\right)\phi(x,k)$$

Since

$$\phi(x,k) = A(k)\psi(x,-k) + B(k)\psi(x,k)$$

we get on applying $-\frac{d}{dx}+p$

$$(+ik-\beta)e^{-ikx} \longleftrightarrow A(k)(+ik+\beta)e^{-ikx} + B(k)(-ik+\beta)e^{ikx}$$

and hence we conclude

$$\begin{cases} \tilde{A}(k) = \left(\dfrac{+ik+\beta}{ik-\beta}\right)A(k) \\ \tilde{B}(k) = -B(k) \end{cases}$$

note $\tilde{A}(k)$ has a zero at $k=i\beta \in$ UHP.

Therefore we have

$$\tilde{R}(k) = \frac{\tilde{B}}{\tilde{A}} = \frac{-B(k)}{\frac{(\beta+ik)}{(-\beta+ik)}A(k)} = \frac{\beta-ik}{\beta+ik}R(k)$$

$$\boxed{\tilde{R}(k) = \frac{\beta-ik}{\beta+ik}R(k)}$$

Goal: to understand the ψ-solution.

First consider a Dirac system on $\mathbb{R}$ with p of compact support. Let $\psi^{\pm}(x,\lambda)$ be the unique solutions such that for $x >> 0$

$$\psi^{+}(x,\lambda) = e^{i\lambda x}\begin{pmatrix}1\\0\end{pmatrix} \qquad \psi^{-}(x,\lambda) = e^{-i\lambda x}\begin{pmatrix}0\\1\end{pmatrix}$$

Notice that this depends on the choice of origin.

Let $\phi, \tilde{\phi}$ denote the solutions whose value at 0 are respectively

$$\begin{pmatrix}1\\1\end{pmatrix} \qquad \begin{pmatrix}i\\-i\end{pmatrix}$$

The solution $a\phi + b\tilde{\phi}$ has the value $\begin{pmatrix}a+ib\\a-ib\end{pmatrix}$ at $x=0$. In order for $a\phi + b\tilde{\phi}$ to be a good φ-solution we want

$$\overline{a+ib} = a-ib$$

that is

$$\bar{a} - i\bar{b} = a - ib, \qquad 0 = (a-\bar{a}) - (b-\bar{b})i$$

$$\text{or } \operatorname{Im}(a) = \frac{a-\bar{a}}{2i} = \frac{(b-\bar{b})}{2} = i\operatorname{Im}(b)$$

so a, b have to be real. Therefore for (a,b) real $\neq 0$ we have a φ-solution $\varphi = a\phi + b\tilde{\phi}$ and hence a spectral measure.

Suppose

$$\phi = B(\lambda)\psi^{+}(x,\lambda) + A(\lambda)\psi^{-}(x,\lambda)$$

$$\tilde{\phi} = \tilde{B}(\lambda)\psi^{+}(x,\lambda) + \tilde{A}(\lambda)\psi^{-}(x,\lambda)$$

Then clearly

$$\varphi = a\phi + b\tilde{\phi} = (aB + b\tilde{B})\psi^{+} + (aA + b\tilde{A})\psi^{-}$$

Hence we know that the spectral measure belonging to $\varphi = a\phi + b\tilde{\phi}$ is

$$d\mu(\lambda) = \frac{d\lambda}{2\pi\left|aB + b\tilde{B}\right|^2}$$

Problem: Is it possible to develop a deB-like theory using the ψ-solution instead of the φ-solution? The idea would be to form transforms of functions with support in $[\alpha, \infty)$.

December 30, 1977. Marchenko equation.

$$G(x,x',\lambda) = \frac{\varphi(x_<,\lambda)\,\psi^+(x_>,\lambda)}{W(\varphi(\lambda),\psi^+(\lambda))}$$

$$W(\varphi(\lambda),\psi^+(\lambda)) = A(\lambda)\,W(\psi^-(\lambda),\psi^+(\lambda)) = A(\lambda)\,2i\lambda$$

The completeness and orthogonality relations are assuming no bound states

$$\delta(x-x') = \frac{1}{2\pi i}\oint_{R\to\infty}(\lambda^2-L)^{-1}_{(x,x')}\,d\lambda^2 = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} G(x,x';\lambda)\,2\lambda\,d\lambda$$

$$= \frac{-1}{2\pi i}\int_{-\infty}^{\infty}\frac{\varphi(x_<)\,\psi^+(x_>)}{2i\lambda A(\lambda)}\,2\lambda\,d\lambda$$

or

$$\boxed{\delta(x-x') = \int_{-\infty}^{\infty}\frac{\varphi(x_<)\,\psi^+(x_>)}{A(\lambda)}\,\frac{d\lambda}{2\pi}}$$

$$= \int_0^{\infty}\varphi(x_<,\lambda)\left(\frac{\psi^+(x_>,\lambda)}{A(\lambda)} + \frac{\psi^+(x_>,-\lambda)}{A(-\lambda)}\right)\frac{d\lambda}{2\pi}$$

$$= \int_0^\infty \varphi(x_<,\lambda)\,\varphi(x_>,\lambda)\,\frac{d\lambda}{2\pi\,A(\lambda)A(-\lambda)}$$

showing that $\quad d\mu(\lambda) = \dfrac{d\lambda}{2\pi\,|A(\lambda)|^2}$.

In order to obtain the Gelfand-Levitan equation put $x' = y < x$ and use the basic repn.

$$(*) \qquad \varphi(x,\lambda) = \varphi^\circ(x,\lambda) + \int_0^x K(x,y)\,\varphi^\circ(y,\lambda)\,dy$$

and its inverse. Because $\varphi^\circ(y,\lambda)$ is a linear combination of $\varphi(y',\lambda)$ for $y' \le y$ we obtain from the orthogonality:

$$0 = \int_0^\infty \varphi(x,\lambda)\,\varphi^\circ(y,\lambda)\,d\mu(\lambda) \qquad\qquad y < x$$

or putting in $(*)$:

$$0 = \int_0^\infty \left(\varphi^\circ(x,\lambda) + \int_0^x K(x,t)\,\varphi^\circ(t,\lambda)\,dy\right)\varphi^\circ(y,\lambda)\,d\mu$$

$$= \Omega(x,y) + \int_0^x K(x,t)\,\{\delta(t-y) + \Omega(t,y)\}\,dt$$

$$= \Omega(x,y) + K(x,y) + \int_0^x K(x,t)\,\Omega(t,y)\,dt$$

where $\quad \Omega(x,y) = \displaystyle\int_0^\infty \varphi^\circ(x,\lambda)\,\varphi^\circ(y,\lambda)\left(d\mu(\lambda) - \tfrac{2}{\pi}d\lambda\right)$

spectral measure
if $\varphi^\circ(x,\lambda) = \cos\lambda x$

To obtain the Marchenko equation you go back to

$$(*)\qquad \delta(x-x') = \int_{-\infty}^{\infty}\left[\psi^-(x_<,\lambda) + \frac{A(-\lambda)}{A(\lambda)}\psi^+(x_<,\lambda)\right]\psi^+(x_>,\lambda)\,\frac{d\lambda}{2\pi}$$

and use the representation

$$\psi^+(x,\lambda) = e^{i\lambda x} + \int_x^{\infty} v(x,y)e^{i\lambda y}\,dy$$

and its inverse which shows $e^{i\lambda x}$ is a linear comb. of the $\psi^+(x',\lambda)$ for $x' \geq x$. Hence the orthogonality part of the above gives for $y > x$

$$0 = \int_{-\infty}^{\infty}\left(\psi^-(x,\lambda) + \frac{A(-\lambda)}{A(\lambda)}\psi^+(x,\lambda)\right)e^{i\lambda y}\,\frac{d\lambda}{2\pi}$$

$$= \int_{-\infty}^{\infty}\underbrace{\psi^+(x,\lambda)}_{e^{i\lambda x} + \int_x^{\infty} v(x,t)\,e^{i\lambda t}\,dt}e^{-i\lambda y}\,\frac{d\lambda}{2\pi} + \int_{-\infty}^{\infty}\frac{A(-\lambda)}{A(\lambda)}\psi^+(x,\lambda)e^{i\lambda y}\,\frac{d\lambda}{2\pi}$$

$$0 = \delta(x-y) + \int_x^{\infty} v(x,t)\,\delta(t-y)\,dt + \int_{-\infty}^{\infty}\frac{A(-\lambda)}{A(\lambda)}e^{i\lambda(x+y)}\,\frac{d\lambda}{2\pi} + \int_x^{\infty} v(x,t)\int_{-\infty}^{\infty}\frac{A(-\lambda)}{A(\lambda)}e^{i\lambda(t+y)}\frac{d\lambda}{2\pi}\,dt$$

$$\boxed{\begin{aligned} &0 = v(x,y) + F(x+y) + \int_x^{\infty} v(x,t)\,F(t+y)\,dt \qquad \text{for } y > x \\ &\text{where}\quad F(x) = \int_{-\infty}^{\infty}\frac{A(-\lambda)}{A(\lambda)}e^{i\lambda x}\,\frac{d\lambda}{2\pi}\end{aligned}}$$