December 14, 1977

I want to discuss the wave equation approach to

$$Lu = -\frac{\partial^2 u}{\partial x^2} + Vu = \lambda^2 u$$

Suppose this equation is considered on $0 \leq x \leq \ell$ with self-adjoint boundary conditions given at the endpts., say

$$u(0) = 0$$
$$u(\ell) = 0$$

to fix the ideas. Introduce the solution $\phi(x,\lambda)$ with $\phi(0,\lambda) = 0$, $\frac{d\phi}{dx}(0,\lambda) = 1$; assume the eigenvalues λ^2 are all >0 and let λ_n be the n-th positive one. ~~Let $d\mu(\lambda)$ be the spectral measure~~

Introduce the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - Vu = -Lu$$

with the same boundary conditions: If $u(x,t)$ is a solution its energy

$$E(u) = \left\| \frac{\partial u}{\partial t} \right\|^2 + (Lu, u)$$

is constant in time:

$$\frac{d}{dt} E(u) = (\ddot{u}, \dot{u}) + (\dot{u}, \ddot{u}) + (L\dot{u}, u) + (Lu, \dot{u}) = 0$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{(\dot{u}, Lu)}$$

Examples of solutions of the wave equation are $e^{\pm i\lambda t}\phi(x,\lambda)$ for λ an eigenvalue. Using completeness of the eigenfunctions for L one sees that linear combinations of these suffice to give all possible Cauchy data.

To see what's happening it might be easier to take the case of a finite range potential on $0 \le x < \infty$. If I put

$$u(x,t) = \int e^{-i\lambda t}\, \phi(x,\lambda)\, \alpha(\lambda)\, d\lambda$$

then for x large

$$u(x,t) = \int e^{-i\lambda t}\left(A(\lambda)e^{-i\lambda x} + B(\lambda)e^{i\lambda x}\right)\alpha(\lambda)\, d\lambda$$

$$= \widehat{A\alpha}(-x-t) + \widehat{B\alpha}(x-t)$$

so

$$u(x,t) \sim \widehat{A\alpha}(-x-t) \qquad \text{for} \qquad t \to -\infty$$

Now

$$E(u) = \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial x}\right\|^2 + (Vu,u)$$

so using fact that $E(u)$ is time-independent, we can let $t \to -\infty$ and get

$$E(u) = 2\left\|\frac{d}{dx}\widehat{A\alpha}\right\|^2_{(-\infty,\infty)} = 4\pi \int |A(\lambda)\alpha(\lambda)|^2 d\lambda$$

If $\alpha(\lambda)$ is odd, then $u(x,0)=0$, so

$$E(u) = \left\|\int \phi(x,\lambda)\,\lambda\alpha(\lambda)\, d\lambda\right\|^2_{(0,\infty)} = 4\pi \int |A(\lambda)\alpha(\lambda)|^2 d\lambda .$$

but this is less interesting now, because what I want to exploit is the isomorphism between the Hilbert space $L^2(\mathbb{R}, 4\pi\,|A(\lambda)|^2 d\lambda)$ and the Hilbert space of solutions of the wave equation in the energy norm. Maybe I should write

$$E\left(\int e^{-i\lambda t}\phi(x,\lambda)\alpha(\lambda)\,d\lambda\right) = \int |\alpha(\lambda)|^2\, 4\pi\, |A(\lambda)|^2 d\lambda$$

so that the spectral measure in this setup is $\dfrac{d\lambda}{4\pi\,|A(\lambda)|^2} = d\mu(\lambda)$.

So far one has solved the wave equation globally, but the other side of the theory is to get at the solution locally using a fundamental solution. <u>Question</u>: What is the fundamental solution for the wave equation? I guess I want to take δ function-like Cauchy data.

Better: let $u(x,t)$ be a solution of the wave equation. Then we ought to be able to express u in terms of its Cauchy data at $t=0$

$$u(x,t) = \int_0^\infty G_1(x,t,x',0)\, u(x',0) + G_2(x,t,x',0)\, \frac{\partial u}{\partial t}(x',0) \quad dx$$

where $G = (G_1, G_2)$ is some sort of Riemann's Green's function.

December 16, 1977

Gelfand–Levitan approach: let L, L_0 be J-matrices (one-sided infinite) and let $\{\phi_n\}$ $\{\mathring{\phi}_n\}$ be the associated orthogonal polys. (Start from $\phi_0 = 1$ so ϕ_n is of degree n). These two bases for $\mathbb{C}[\lambda]$ are related by a triangular matrix

$$\phi_n = \sum_{m \le n} k_{mn} \mathring{\phi}_m$$

with $K_{nn} > 0$. K appears:

We can also interpret the above formula as

$$\phi_\lambda = K^t \mathring{\phi}_\lambda$$

where ϕ_λ denotes the infinite column vector with components $\phi(n,\lambda)$. If $V = C_0(N)$ is the space of column vectors with finite support, then $\phi_\lambda \in V^\vee$. One has

$$L K^t \mathring{\phi}_\lambda = L \phi_\lambda = \lambda \phi_\lambda = \lambda K^t \mathring{\phi}_\lambda = K^t L_0 \mathring{\phi}_\lambda$$

and since the components of $\mathring{\phi}_\lambda$ are independent this gives

$$L K^t = K^t L_0 \qquad \text{or} \qquad KL = L_0 K$$

Another interpretation is that we have two triples (V, L, e_0), (V, L_0, e_0) consisting of an inf. diml space, operator + cyclic vector, hence a unique isom. between them:

$$\begin{tikzcd} & (\mathbb{C}[\lambda], \lambda, 1) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ (V, L, e_0) \arrow[rr, "K"] & & (V, L_0, e_0) \end{tikzcd}$$

$p \mapsto p(L)e_0$

$$\begin{tikzcd} \phi_n \arrow[d] & = & \sum k_{mn} \mathring{\phi}_m \arrow[d] \\ e_n \arrow[rr, mapsto] & & \sum k_{mn} e_m \end{tikzcd}$$

Now the inverse problem of scattering involves starting with the spectral measure $d\mu$ and constructing the sequence $\phi(n,\lambda)$ and the J-matrix L. Thus I want to do Gram-Schmidt relative to $\overset{\circ}{\phi}_0, \overset{\circ}{\phi}_1, \ldots$ and the inner product determined by $d\mu$.

Set

$$\overset{\circ}{\phi}_n = \sum_{n' \leq n} h_{n'n} \phi_{n'}$$

so that

$$(h_{n'n}) = (k_{n'n})^{-1}$$

Then if we have already found $\phi_{n'}$ for $n'<n$ we know $h_{in'}$ $k_{in'}$ for $n'<n$, hence

$$h_{n'n} = (\overset{\circ}{\phi}_n, \phi_{n'}) = (\overset{\circ}{\phi}_n, \sum_{i \leq n'} k_{in'} \overset{\circ}{\phi}_i)$$

$$= \sum_{i \leq n'} \alpha_{in} k_{in'} \qquad \alpha_{in} = (\phi_n^\circ, \phi_i^\circ)$$

gives us the $h_{n'n}$ for $n' \leq n$, so we can then find the $k_{n'n}$ for $n' \leq n$. In any case one sees one is solving an equation

$$K^{-1} = K^t \alpha \qquad \text{or} \qquad \begin{aligned} I &= K K^t \alpha \\ I &= K^t \alpha K \end{aligned}$$

or $\quad H^t H = \alpha$. Here's how to find h_{in}:

$$h_{0n} h_{00} = \alpha_{n0}$$

$$h_{0n} h_{01} + h_{1n} h_{11} = \alpha_{11}$$

$$h_{0n} h_{02} + h_{1n} h_{12} + h_{2n} h_{22} = \alpha_{22}$$

To understand G-L all you have to do is recast these as integral equations

Unfortunately this is no good because you are getting a non-linear equation for the matrix K and the Gelfand-Levitan equations are linear. So instead you proceed as follows. Let's separate the process of orthogonalization and normalizing. Put

$$\tilde{\phi}_n = \mathring{\phi}_n + \sum_{m<n} k_{mn} \mathring{\phi}_m$$

so that $\tilde{\phi}_n$ is a multiple of ϕ_n. Then we get n-linear equations

$$0 = (\tilde{\phi}_n, \mathring{\phi}_j) = (\mathring{\phi}_n, \mathring{\phi}_j) + \sum_{m<n} k_{mn} (\mathring{\phi}_m, \mathring{\phi}_j)$$

for the n-unknowns $k_{0n}, \dots, k_{n-1,n}$. In fact

$$(\mathring{\phi}_m, \mathring{\phi}_j) = \int \mathring{\phi}_m \mathring{\phi}_j \, d\mu = \int \mathring{\phi}_m \mathring{\phi}_j [d\mu_0 + d\mu - d\mu_0]$$

$$= \delta_{mj} + \Omega_{mj} \qquad \Omega_{mj} = \int \mathring{\phi}_m \mathring{\phi}_j (d\mu - d\mu_0)$$

and [redacted] in this notation we get the equations

$$\Omega_{nj} + \sum_{m<n} k_{mn} \Omega_{mj} + k_{\text{[redacted]}jn} = 0 \qquad j<n$$

Discrete version of Gelfand-Levitan: What this does is to start from the spectral measure $d\mu$ and construct the J-matrix L as a perturbation of a given one L_0. Let $\{\phi_n^\circ\}$, $d\mu_0$ be the orthonormal system and spectral measure respectively for L_0. We orthogonalize $\{\phi_n^\circ\}$ wrt the inner product $(f,g) = \int f\bar{g}\, d\mu$ using Gram-Schmidt. Thus we put

$$\tilde{\phi}_n = \phi_n^\circ + \sum_{m<n} K_{nm}\phi_m^\circ$$

where the K_{nm} are unique such that $\forall j<n$

$$0 = (\tilde{\phi}_n, \phi_j^\circ) = (\phi_n^\circ, \phi_j^\circ) + \sum_{m<n} K_{nm}(\phi_m^\circ, \phi_j^\circ)$$

Putting

$$\Omega_{mj} = \int \phi_m^\circ \phi_j^\circ (d\mu - d\mu_0)$$

so that

$$(\phi_m^\circ, \phi_j^\circ) = \Omega_{mj} + \delta_{mj}$$

we get the Gelfand-Levitan equation for K:

$$K_{nj} + \Omega_{nj} + \sum_{m<n} K_{nm}\Omega_{mj} = 0 \qquad j<n$$

This is a system of n equations in the n unknowns K_{nm}, $m<n$. Since

$$(\tilde{\phi}_n, \tilde{\phi}_n) = (\tilde{\phi}_n, \phi_n^\circ) = (\phi_n^\circ, \phi_n^\circ) + \sum_{m<n} K_{nm}(\phi_m^\circ, \phi_n^\circ)$$

$$= 1 + \Omega_{nn} + \sum_{m<n} K_{nm}\Omega_{mn}$$

we can now normalize the $\tilde{\phi}_n$ to get the desired orthonormal basis $\{\phi_n\}$. If K_{nn} is defined so that the G-L equation holds for $j=n$:

$$K_{nn} + \Omega_{nn} + \sum_{m<n} K_{nm}\Omega_{mn} = 0$$

then we have

$$\|\Phi_n\|^2 = 1 - K_{nn}$$

Suppose now that we suppose $d\mu$ is even and $L_0 = \frac{1}{2}T + \frac{1}{2}T^{-1}$ i.e.

$$\lambda\phi^0_n = \frac{1}{2}\phi^0_{n+1} + \frac{1}{2}\phi^0_{n-1}$$

$\theta = \cos^{-1}(\lambda)$

starting from $\phi^0_{-1} = 0$, $\phi^0_0 = 1$. $\therefore$ $\phi^0_n(\lambda) = \frac{\sin(n+1)\theta}{\sin\theta}$

Then the coefficients in the recursion relation for the ϕ_n:

$$\lambda\phi_n = a_n\phi_{n+1} + a_{n-1}\phi_{n-1}$$

can be determined by looking at the highest terms. As Φ_n and ϕ^0_n have the same highest terms we get

$$a_n\phi_{n+1} \equiv \lambda\phi_n = \frac{\lambda\Phi_n}{\|\Phi_n\|} \equiv \frac{\lambda\phi^0_n}{\|\Phi_n\|} \equiv \frac{\frac{1}{2}\phi^0_{n+1}}{\|\Phi_n\|} \equiv \frac{\|\Phi_{n+1}\|}{2\|\Phi_n\|}\phi_{n+1}$$

so

$$a_n = \frac{\|\Phi_{n+1}\|}{2\|\Phi_n\|} = \frac{1}{2}\cdot\left(\frac{1-K_{n+1,n+1}}{1-K_{n,n}}\right)^{1/2}$$

Algebraic analysis. Let U denote an automorphism of $V = \mathbb{C}e_0 + \mathbb{C}e_1 + \cdots$ preserving the standard flag $F_nV = \mathbb{C}e_0 + \cdots + \mathbb{C}e_n$; hence U is given by an upper-triangular matrix. Let L_0 be a J-matrix. If UL_0U^{-1} is hermitian, then

$$(U^{-1})^* L_0 U^* = UL_0U^{-1} \quad \text{or}$$

U^*U commutes with L_0. (Ignore for the moment the difficulty of U^* not being an auto. of V).
Conversely if U^*U commutes with L_0, then UL_0U^{-1} is hermitian and since it carries F_nV into $F_{n+1}V$ it is a [redacted] hermitian tridiagonal matrix. If U has positive diagonal entries, then because

$$(UL_0U^{-1})(e_n) = (UL_0)(u_{nn}^{-1}e_n \mod F_{n-1})$$
$$= U(u_{nn}^{-1}a_n e_{n+1} \mod F_n)$$
$$= u_{n+1,n+1}u_{nn}^{-1}a_n e_{n+1} \mod F_n$$

we see UL_0U^{-1} has to be a J-matrix. One sees that the diagonal elements of U can't be the identity or else $L=L_0$. Hence something quite different happens in the discrete case, from the continuous case. [redacted]

Next consider $L_0 = -\frac{d^2}{dx^2}$ $\quad L = -\frac{d^2}{dx^2} + q$
and let U be an operator of the form

$$(Uf)(x) = f(x) + \int_0^x K(x,y)f(y)\,dy$$

Let's see if we can arrange $UL_0 = LU$.

$$-(UL_0f)(x) = +\frac{d^2f}{dx^2} + \int_0^x K(x,y)\frac{d^2f}{dy^2}(y)\,dy$$
$$= \frac{d^2f}{dx^2} + \int_0^x \left\{\frac{\partial}{\partial y}\left[K(x,y)\frac{df}{dy}(y) - \frac{\partial K}{\partial y}(x,y)f(y)\right] + \frac{\partial^2K}{\partial y^2}(x,y)f(y)\right\}dy$$
$$= \frac{d^2f}{dx^2} + \left[K(x,y)\frac{df}{dy}(y) - \frac{\partial K}{\partial y}(x,y)f(y)\right]_0^x + \int_0^x \frac{\partial^2 K}{\partial y^2}(x,y)f(y)\,dy$$

$$\frac{d}{dx}(Uf) = \frac{df}{dx} + K(x,x)f(x) + \int_0^x \frac{\partial K}{\partial x}(x,y)f(y)\,dy$$

$$\frac{d^2}{dx^2}(Uf) = \frac{d^2 f}{dx^2}(x) + \frac{d}{dx}\left[K(x,x)f(x)\right] + \frac{\partial K}{\partial x}(x,x)f(x) + \int_0^x \frac{\partial^2 K}{\partial x^2}(x,y)f(y)\,dy$$

$$-q(Uf) = -q(x)f(x) - \int_0^x q(x)K(x,y)f(y)\,dy$$

Since $\frac{d}{dx}K(x,x) = \frac{\partial K}{\partial x}(x,x) + \frac{\partial K}{\partial y}(x,x)$

$$(-LUf + UL_0 f)(x) = \left\{-q(x) + \frac{d}{dx}K(x,x) + \frac{\partial K}{\partial x}(x,x) + \frac{\partial K}{\partial y}(x,x)\right\}f(x)$$

$$+ \int_0^x \left(\frac{\partial^2 K}{\partial x^2} - q(x)K(x,y) - \frac{\partial^2 K}{\partial y^2}\right) f(y)\,dy$$

$$- K(x,0)f'(0) + K_y(x,0)f(0)$$

So provided K satisfies the wave equation

$$\frac{\partial^2 K}{\partial y^2} = \frac{\partial^2 K}{\partial x^2} - q(x)K \qquad \text{for } 0 \le y \le x$$

and $$q(x) = 2\frac{d}{dx}K(x,x)$$

and $$K(x,0) = 0$$

we have $LUf = UL_0 f$ for f such that $f(0)=0$.

Note the strange way q appears both in the wave equation and on the line $y=x$.

[redacted]

Let's consider a Dirac equation

$$Lu = \begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix} \frac{du}{dx} + \begin{pmatrix} 0 & i\bar{p} \\ -ip & 0 \end{pmatrix} u = \lambda u$$

on $0 \leq x < \infty$ with p smooth, and the boundary condition $u_1(0) = u_2(0)$. Denote by $\phi(x,\lambda)$ the solution starting out with $\binom{1}{1}$, and by $\phi^0(x,\lambda)$ the similar solution for the operator L_0 with $p = 0$. Thus

$$\phi^0(x,\lambda) = \begin{pmatrix} e^{i\lambda x} \\ e^{-i\lambda x} \end{pmatrix}.$$

For each $x_0 > 0$ denote by B_{x_0} the space of entire functions of the form

$$\int_0^{x_0} \alpha(x)^* \phi(x,\lambda)\, dx = \int_0^{x_0} \left\{ \overline{\alpha_1(x)}\, \phi_1(x,\lambda) + \overline{\alpha_2(x)}\, \phi_2(x,\lambda) \right\} dx$$

as α ranges over $L^2((0,x_0))^2$. For L_0 we get

$$\int_0^{x_0} \left(\bar{\alpha}_1(x) e^{i\lambda x} + \bar{\alpha}_2(x) e^{-i\lambda x} \right) dx = \int_{-x_0}^{x_0} \bar{\alpha}(x) e^{i\lambda x} dx$$

where $\alpha(x) = \begin{cases} \alpha_1(x) & x > 0 \\ \alpha_2(-x) & x < 0 \end{cases}$. Hence B_{x_0} consists of all Fourier transforms of square-integrable functions supported in $[-x_0, x_0]$. By Paley-Wiener this is the same as the space of entire functions of type $\leq x_0$ square-integrable on the real axis.

Assume known that B_x is the de Branges space based on $\phi_2(x,\lambda)$. (This involves the fact that $\phi(x,\lambda)$ is entire in λ + completeness of $\phi(\cdot,\lambda)$ on $(0,x)$.)

Conjecture: B_x is independent of L as a space of entire functions, i.e. it is the space of entire functions of type $\leq x$ square-integrable on the real axis.

Consider $u(x,t) = \frac{1}{2\pi}\int e^{-i\lambda t}\,\phi(x,\lambda)\,d\lambda$. This should make sense as a distribution. It should be a solution of the wave equation

$$i\frac{\partial u}{\partial t} = Lu$$

with the initial data

$$u(0,t) = \frac{1}{2\pi}\int e^{-i\lambda t}\begin{pmatrix}1\\1\end{pmatrix}d\lambda = \begin{pmatrix}\delta(t)\\ \delta(t)\end{pmatrix}$$

on the line $x=0$.

The theory of hyperbolic equations should tell us that for any initial data on the non-characteristic line $x=0$ there is a unique solution of the wave equation. From the theory of characteristics we should obtain that $u(x,t)=0$ for $|t|>x$ and that the singularities of $u(x,t)$ are located on the lines $x=|t|$.

In order to understand singularities along the characteristics you want to review WKB. Begin with the Schroedinger equation

$$\left(-\frac{d^2}{dx^2}+q\right)u = \lambda^2 u$$

as $\lambda^2 \longrightarrow +\infty$. The idea is that q becomes more and more negligible as $\lambda^2 \longrightarrow +\infty$.

Look for a solution $u = e^{A+iS}$ where A is a slowly varying amplitude and S is a rapidly-varying phase.

$$u' = e^{A+iS}(A'+iS')$$

$$u'' = e^{A+iS}\left[(A'+iS')^2 + A'' + iS''\right]$$

$$u'' + (\lambda^2 - q)u = 0 \iff (A'+iS')^2 + A'' + iS'' + \lambda^2 - q = 0$$

or

$$\begin{cases} (A')^2 - (S')^2 + A'' + \lambda^2 - q = 0 \\ 2A'S' + S'' = 0 \end{cases}$$

Now replace S by λS. Choose A so the second equation holds always:

$$A' = -\frac{1}{2}\frac{S''}{S'} = -\frac{1}{2}\frac{d}{dx}(\ln S) = \frac{d}{dx}\ln(S^{-1/2})$$

i.e. $u = S^{-1/2} e^{i\lambda S}$ up to a constant where

$$\lambda^2(1-(S')^2) = q + \left(\frac{d}{dx}\ln(S^{-1/2})\right)^2 - \frac{d^2}{dx^2}\left(\ln(S^{-1/2})\right)$$

This last formula can be used to grind out an asymptotic formula for S' of the form

$$S' = 1 + \frac{1}{\lambda^2}a_1(x) + \frac{1}{\lambda^4}a_2(x) + \cdots$$

i.e. $-2a_1(x) = q(x)$. As a check:

$$\frac{1}{\lambda}\sqrt{\lambda^2 - q} = \left(1 - \frac{q}{\lambda^2}\right)^{1/2} = 1 - \frac{1}{\lambda^2}\frac{q}{2} \cdots\cdots$$

But all of this is basic fanciness on something like

$$e^{i\lambda(x-x_0)}\left(1 + \frac{1}{\lambda^2}b_1(x) + \frac{1}{\lambda^4}b_2(x) + \cdots\right)$$

Go back to $\quad Lu = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \frac{du}{dx} + \begin{pmatrix} 0 & i\bar{p} \\ -ip & 0 \end{pmatrix} u = \lambda u.$

It should be the case that $\phi(x,\lambda)$ is entire of type $\leq x$ so that its Fourier tranform $\quad u(x,t) = \frac{1}{2\pi}\int e^{-i\lambda t}\phi(x,\lambda)\,d\lambda$ has support in [redacted] $-x \leq t \leq x$. What's more $u(x,t)$ should be smooth off $x = |t|$ and the singularity of $u(x,t)$ along the characteristics [redacted] should be linked to the asymptotic expansion of $\phi(x,\lambda)$ determined by WKB. To be more specific

$$\phi(x,\lambda) = \int_{-x}^{x} e^{i\lambda t}\, u(x,t)\,dt$$

$$e^{-i\lambda x}\phi(x,\lambda) = \int_{-x}^{x} e^{-i\lambda x + i\lambda t}\, u(x,t)\,dt \qquad \tau = x - t$$

$$= \int_0^{2x} e^{-i\lambda\tau}\, u(x, x-\tau)\,d\tau$$

Recall Watson's lemma:

$$\int_0^\infty e^{-st}\left(a_0 + \frac{a_1 t}{1!} + \frac{a_2 t^2}{2!} + \cdots\right) dt = \frac{a_0}{s} + \frac{a_1}{s^2} + \cdots$$

This is to be interpreted as saying that the Laplace tranform of $f(t)$ having the asymptotic expansion indicated on the left as $t \to 0$ has the asymptotic expansion on the right for s [redacted] $\to \infty$ in a sector $|\arg s| \leq \frac{\pi}{2} - \varepsilon$.

What do I know about the asymptotic expansion of $\phi(x,\lambda)$? Put

$$\phi(x,\lambda) = e^{i\lambda x}\left(a_0 + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \cdots\cdots\right)$$

and substitute into the [redacted] Dirac system.

$$\phi = e^{i\lambda x} v \qquad e^{-i\lambda x}\frac{d}{dx}\phi = \left(\frac{d}{dx} + i\lambda\right) v$$

$$\left(\frac{d}{dx}+i\lambda\right)v = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} v \qquad \frac{dv}{dx} = \begin{pmatrix} 0 & \bar{p} \\ p & -2i\lambda \end{pmatrix} v$$

Suppose $v = \begin{pmatrix} a_0 + a_1\lambda^{-1} + a_2\lambda^{-2} + \cdots \\ b_0 + b_1\lambda^{-1} + b_2\lambda^{-2} + \cdots \end{pmatrix}$. Then

$$\frac{dv_1}{dx} = \bar{p}\,v_2 \quad\Rightarrow\quad a_n' = \bar{p}\,b_n$$

$$\frac{dv_2}{dx} = p v_1 - 2i\lambda v_2 \quad\Rightarrow\quad b_n' = p a_n - 2i\,b_{n+1}$$

So $b_0 = 0 \Rightarrow a_0$ constant, say $a_0 = 1$

$$0 = p - 2i\,b_1 \quad\Rightarrow\quad b_1 = \frac{p}{2i} \quad\Rightarrow\quad a_1' = \frac{|p|^2}{2i}$$

~~So~~

$$\phi(x,\lambda) \sim e^{i\lambda x}\begin{pmatrix} 1 + \frac{1}{\lambda}\int \frac{|p|^2}{2i}\,dx + \cdots\cdots \\ + \frac{1}{\lambda}\frac{p}{2i} \quad + \cdots\cdots \end{pmatrix}$$

for more formulas see pg 41+42, April 27, '77

The only point which is confusing is how to evaluate the constant in a_1. Possibly this is determined by the boundary condition that $\phi(x,\lambda) = \binom{1}{1}$ at $x = 0$.

The above gives the asymptotic expansion for $\phi(x,\lambda)$ as $\lambda \to \infty$ in the sector $|\arg s| < \frac{\pi}{2} - \varepsilon$ where $s = i\lambda$, hence for $-\pi + \varepsilon < \arg\lambda < -\varepsilon$. We should also have

$$\phi(x,\lambda) \sim e^{-i\lambda x}\begin{pmatrix} -\frac{1}{\lambda}\frac{\bar{p}}{2i} \quad + \cdots\cdots \\ 1 - \frac{1}{\lambda}\int \frac{|p|^2}{2i}\,dx + \cdots\cdots \end{pmatrix}$$

for $\lambda \to \infty$ in $\varepsilon < \arg\lambda < \pi - \varepsilon$.

But we should be able to add these expressions to get an asymptotic expansion for λ real $\lambda \to \pm\infty$. In effect we should know that $u(x,t)$ is smooth for $|t| < x$

so by Riemann-Lebesgue the asymptotic behavior should be determined only by the behavior at $t=x$ and $t=-x$.

It seems reasonable that constants in a_1 should be rigged so that at $x=0$ we get the sum of the two asymptotic expansions should be zero. To simplify suppose $p(x)=0$ near $x=0$ so it's clear that we want $a_1 = \int \frac{|p|^2}{2i}\,dx$.

since

$$\phi(x,\lambda) \sim e^{i\lambda x}\begin{pmatrix} 1 + \frac{1}{\lambda}\int_0^x \frac{|p|^2}{2i} + \cdots \\ \frac{1}{\lambda}\frac{p}{2i} \end{pmatrix} + e^{-i\lambda x}\begin{pmatrix} -\frac{1}{\lambda}\frac{\bar{p}}{2i} \\ 1 - \frac{1}{\lambda}\int_0^x \frac{|p|^2}{2i} + \cdots \end{pmatrix}$$

is our asymptotic expansion, it seems reasonable to expect

$$u(x,t) = \begin{pmatrix} \delta(x-t) \\ \delta(-x-t) \end{pmatrix} + V(x,t)$$

where $V(x,t)$ is a nice function defined for $|t| \le x$ such that

missing a factor of i here because $s = i\lambda$

$$V(x,x) = \begin{pmatrix} \int_0^x \frac{|p|^2}{2i} \\ \frac{p}{2i} \end{pmatrix} \qquad V(x,-x) = \begin{pmatrix} -\frac{\bar{p}}{2i} \\ -\int_0^x \frac{|p|^2}{2i} \end{pmatrix}$$

drop i — drop $-i$

$$\int_{-x}^{x} V(x,t)\,e^{i\lambda t}\,dt = \int_0^x V(x,t)e^{i\lambda t} + V(x,-t)e^{-i\lambda t}\,dt$$

$$= \int_0^x K(x,t)\begin{pmatrix} e^{i\lambda t} \\ e^{-i\lambda t} \end{pmatrix} dt$$

where

$$K(x,t) = \begin{pmatrix} V_1(x,t) & V_1(x,-t) \\ V_2(x,t) & V_2(x,-t) \end{pmatrix}$$

satisfies

$$K(x,x) = \begin{pmatrix} \int_0^x \frac{|p|^2}{2i} & -\frac{\bar{p}}{2i} \\ \frac{p}{2i} & -\int_0^x \frac{|p|^2}{2i} \end{pmatrix}$$

should be: $\begin{pmatrix} \int_0^x \frac{|p|^2}{2}\,dx & \frac{\bar{p}}{2} \\ \frac{p}{2} & \int_0^x \frac{|p|^2}{2}\,dx \end{pmatrix}$

Therefore we ought to get by this procedure the formula

$$\phi(x,\lambda) = \begin{pmatrix} e^{i\lambda x} \\ e^{-i\lambda x} \end{pmatrix} + \int_0^x K(x,x') \begin{pmatrix} e^{i\lambda x'} \\ e^{-i\lambda x'} \end{pmatrix} dx'$$

and so we next want to show that the Volterra operator $I+K$ intertwines L and L_0.

So let's put

$$\mathcal{U} f(x) = f(x) + \int_0^x K(x,y) f(y)\,dy$$

and try to see what we need to get $L\mathcal{U} = \mathcal{U}L_0$.

$$A\frac{d}{dx}(\mathcal{U}f) = A\frac{df}{dx} + AK(x,x)f(x) + \int_0^x A\frac{\partial K}{\partial x}(x,y)f(y)\,dy$$

$$B\mathcal{U}f = (Bf)(x) + \int_0^x BK(x,y)f(y)\,dy$$

$$\therefore\ (L\mathcal{U}f)(x) = (Lf)(x) + AK(x,x)f(x) + \int_0^x L_x K(x,y) f(y)\,dy$$

$$(\mathcal{U}L_0 f)(x) = A\frac{df}{dx} + \int_0^x K(x,y)A\frac{df}{dy}(y)\,dy$$

$$= A\frac{df}{dx}(x) + \Big[K(x,y)Af(y)\Big]_{y=0}^{y=x} - \int_0^x \frac{\partial}{\partial y}K(x,y)A\,f(y)\,dy$$

$$= A\frac{df}{dx}(x) + K(x,x)Af(x) - K(x,0)Af(0) - \int_0^x \frac{\partial K}{\partial y}(x,y)A\,f(y)\,dy$$

For these to be equal it suffices that

1) $$L_x K(x,y) = -\frac{\partial K}{\partial y}(x,y)A$$

2) $$B(x) + AK(x,x) = K(x,x)A$$

3) $$K(x,0)Af(0) = 0$$

Now $L_x K(x,t) = L_x\big(v(x,t) \quad v(x,-t)\big) = \left(i\frac{\partial v}{\partial t}(x,t) \quad +i\frac{\partial v}{\partial t}(x,-t)\right)$

$$= \frac{\partial}{\partial t}\left(v(x,t)\, v(x,-t)\right)\begin{pmatrix} i & \\ & -i \end{pmatrix} = -\frac{\partial K}{\partial t}(x,t)A$$

proving 1).

$$AK(x,x) - K(x,x)A = \begin{pmatrix} 0 & \frac{1}{i}\left(-\frac{\bar{p}}{2i}\right) - \left(-\frac{\bar{p}}{2i}\right)i \\ i\frac{p}{2i} - \frac{1}{i}\frac{p}{2i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & +\bar{p} \\ p & 0 \end{pmatrix}$$

which doesn't work up to a factor of i.

$$\begin{pmatrix} \sigma & \tau \end{pmatrix}\begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \frac{1}{i}\left(\sigma f_1(0) - \tau f_2(0)\right)$$

is zero if $f_1(0) = f_2(0)$ and $\sigma = \tau$; thus 3) holds.

Corrections are

$$K(x,x) = \begin{pmatrix} \int_0^x \frac{|p|^2}{2} & \frac{\bar{p}}{2} \\ \frac{p}{2} & \int_0^x \frac{|p|^2}{2} \end{pmatrix}$$

hence

$$AK(x,x) - K(x,x)A = \begin{pmatrix} 0 & \frac{1}{i}\frac{\bar{p}}{2} - i\frac{\bar{p}}{2} \\ i\frac{p}{2} - \frac{1}{i}\frac{p}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\bar{p} \\ ip & 0 \end{pmatrix} = -B$$

proving 2). So it works.

December 19, 1977

Consider a Dirac system with p of compact support and vanishing near 0. Then for x large we have

$$\phi(x,\lambda) = \begin{pmatrix} B(\lambda)e^{i\lambda x} \\ A(\lambda)e^{-i\lambda x} \end{pmatrix} \qquad \text{where } B(\lambda) = A^{\#}(\lambda) = \overline{A(\bar{\lambda})}$$

Recall yesterday I saw that there was an asymptotic expansion

$$\phi(x,\lambda) \sim e^{i\lambda x} B(x,\lambda) + e^{-i\lambda x} A(x,\lambda)$$

where
$$B(x,\lambda) = \begin{pmatrix} 1 + \int_0^x \frac{|p|^2}{2i\lambda} + \cdots\cdots \\ 0 + \frac{p}{2i\lambda} + \cdots\cdots \end{pmatrix} \qquad A = B^{\#}$$

I also saw that the Fourier transform of $\phi(x,\lambda)$ has the form

$$u(x,t) = \begin{pmatrix} \delta(x-t) \\ \delta(-x-t) \end{pmatrix} + V(x,t)$$

where V is a smooth solution of the wave equation in $|t| \leq x$ with the boundary values

$$V(x,x) = \begin{pmatrix} \int_0^x \frac{|p|^2}{2} \\ \frac{p}{2} \end{pmatrix} \qquad V(x,-x) = \begin{pmatrix} \frac{\bar{p}}{2} \\ \int_0^x \frac{|p|^2}{2} \end{pmatrix}$$

For large x it is clear we have

$$u(x,t) = \begin{pmatrix} \hat{B}(x-t) \\ \hat{A}(-x-t) \end{pmatrix}$$

hence

$$V(x,t) = \begin{pmatrix} (B-1)^{\wedge}(x-t) \\ (A-1)^{\wedge}(-x-t) \end{pmatrix}.$$

We see that $(B-1)^{\wedge}(y)$ is supported in $0 \leq y \leq 2x_0$ $\qquad x_0 =$ range of p

December 20, 1977:

Review: I am considering the Dirac system

$$Lu = \begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix} \frac{du}{dx} + \begin{pmatrix} 0 & i\bar{p} \\ -ip & 0 \end{pmatrix} u = \lambda u \qquad \cdots$$

on $0 \le x < \infty$ where p has compact support. $\phi(x,\lambda)$ is the solution starting with $\phi(0,\lambda) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Put

$$u(x,t) = \frac{1}{2\pi} \int e^{-i\lambda t} \phi(x,\lambda)\, d\lambda \qquad \phi(x,\lambda) = \int e^{i\lambda t} u(x,t)\, dt$$

so that u is the solution of the wave equation $Lu = i\frac{\partial u}{\partial t}$ with the initial data

$$u(0,t) = \begin{pmatrix} \delta(t) \\ \delta(t) \end{pmatrix}$$

on the line $x = 0$. From the theory of hyperbolic DE's we should know that $u(x,t)$ is supported in $|t| \le x$ and that it is smooth for $|t| < x$. It should be that $\phi(x,\lambda)$ has an asymptotic expansion at $\lambda \to \pm\infty$ of the form

$$\phi(x,\lambda) \sim e^{ix\lambda}\left(B_0(x) + B_1(x)\frac{1}{\lambda} + B_2(x)\frac{1}{\lambda^2} + \cdots\right)$$

$$+ e^{-ix\lambda}\left(A_0(x) + A_1(x)\frac{1}{\lambda} + A_2(x)\frac{1}{\lambda^2} + \cdots\right)$$

where these coefficients satisfy certain ordinary DE's in x which can be ground out easily.

On the other hand for large x I have

$$\phi(x,\lambda) = \begin{pmatrix} B(\lambda) e^{i\lambda x} \\ A(\lambda) e^{-i\lambda x} \end{pmatrix}$$

so that I get an asymptotic expansion

$$\begin{pmatrix} B(\lambda) \\ 0 \end{pmatrix} \sim B_0(x) + B_1(x)\frac{1}{\lambda} + \cdots \qquad \text{as } \lambda \to \pm\infty$$

for any fixed x. For example, assuming that $p(0)$ is zero we know

$$B(x,\lambda) = \begin{pmatrix} 1 + \left(\frac{1}{2i}\int_0^x |p|^2\right)\frac{1}{\lambda} + \cdots \\ \frac{p}{2i}\frac{1}{\lambda} + \cdots \end{pmatrix}$$

and consequently we have

$$B(\lambda) \sim 1 + \left(\int_0^\infty \frac{|p|^2}{2}\right)\frac{1}{i\lambda} + \cdots \qquad \lambda \to \pm\infty$$

The coefficients involve various integrals of p and its derivatives. Notice that the asymptotic expansion for $B(x,\lambda)$ holds for $\operatorname{Im}\lambda \leq 0$ (where $e^{-ix\lambda}$ decays) so the above asymptotic expansion for $B(\lambda)$ should be valid in the [redacted] lower half-plane.

Algebraic formalism behind the Gelfand-Levitan equation. Let $\mathcal{H} = L^2((0,\infty))$ to fix the ideas and let $U = I + K$ be a Volterra operator on $\mathcal{H}$ which intertwines $L = -\frac{d^2}{dx^2} + q$, $L_0 = -\frac{d^2}{dx^2}$: $UL_0 = LU$.

$$\begin{tikzcd} (\mathcal{H}, L_0) \arrow[r, "U"] & (\mathcal{H}, L) \\ (L^2_+(\mathbb{R}), \lambda) \arrow[u, "T_0"] & \end{tikzcd}$$

T_0 is the unitary operator given by Fourier transform, but U is not unitary. Since $L_0U^* = U^*L$ one has that

$L_0(U^*U) = (U^*U)L_0$ so $T_0^* U^* U T_0$ [redacted] commutes with mult. by λ hence it is multiplication by a function $\rho(\lambda)$. $\rho(\lambda)$ is the spectral fn. for L because

$$\| U T_0 \alpha \|^2 = (T_0^* U^* U T_0 \alpha, \alpha) = \int |\alpha(\lambda)|^2 \rho(\lambda)\, d\lambda$$

and

$$(U T_0 \alpha) = \int \phi^0(x,\lambda)\alpha(\lambda)\, d\lambda + \int_0^x K(x,x') \int \phi^0(x',\lambda)\alpha(\lambda)\, d\lambda$$

$$= \int \phi(x,\lambda)\alpha(\lambda)\, d\lambda$$

So we get the equation

$$U^*U = T_0 \rho T_0^* = T_0 T_0^* + T_0(\rho - 1)T_0^*$$

$$= I + \Omega$$

or

$$(I+K^*)(I+K) = I + \Omega$$

On the surface this is a non-linear equation for K, however if $(I+K)^{-1} = I + \tilde{K}$, and recall that we only need to know $\tilde{K}(x,y)$ for $y \leq x$, then if the above is rewritten

$$I + K^* = (I+\Omega)(I+\tilde{K})$$

we get

$$0 = (\Omega + \tilde{K} + \Omega\tilde{K})(x,y) \qquad x > y$$

which is a linear equation for $\tilde{K}$.

Actually to do things [redacted] right, the spectral measure is really $\frac{d\lambda}{\rho(\lambda)}$, hence we want [redacted] $\Omega = T_0(\frac{1}{\rho} - 1)T_0^*$

or

$$(I+K^*)(I+K) = (I+\Omega)^{-1}$$

or

$$(I+K)(I+\Omega) = (I+K^*)^{-1} \qquad \text{hence}$$

[redacted] $K + \Omega + K\Omega = 0$ for $x > y$.

Linear algebra side: A positive-definite matrix, to find an upper-triangular matrix U with 1's on the diagonal so that

$$(AUe_i, Ue_j) = \lambda_i \delta_{ij}$$

or equivalently such that U^*AU = diagonal matrix D. Notice this implies that $AU = (U^*)^{-1}D$ is zero above the main diagonal which leads to the equations

$$0 = (AU)_{ik} = A_{ik} + \sum_{j=0}^{k-1} A_{ij} U_{jk} \qquad i<k$$

For k fixed, the matrix (A_{ij}) for $0 \le i,j < k$ is positive-definite, hence U_{jk} for $j<k$ can be found. This argument is nothing more than the Gram-Schmidt orthogonalization of $e_0, e_1, \dots$ with respect to (Ax,y).

Let's try to do the continuous version of orthogonal polys. on S^1. Suppose given a measure $d\mu(\lambda) = \rho(\lambda)d\lambda$ on $\mathbb{R}$ having properties to be specified later. For example I would want the sort of thing occurring in scattering for a Dirac system: $d\mu(\lambda) = \frac{d\lambda}{2\pi|A(\lambda)|^2}$ where $A(\lambda) \sim 1$. Now consider the Hilbert space $L^2(\mathbb{R}, d\mu)$ and the subspace "spanned by" $e^{ix\lambda}$, $x \ge 0$. I am going to try to find an "orthogonal basis" $\psi(x,\lambda)$ for this subspace of the form

$$\psi(x,\lambda) = e^{ix\lambda} + \int_0^x K(x,x') e^{ix'\lambda} dx'$$

We of course need the matrix of the inner product on the given basis $\{e^{ix\lambda}\}$

$$\int e^{ix\lambda} e^{-iy\lambda} \frac{d\lambda}{2\pi|A(\lambda)|^2} = \delta(x-y) + \Omega(x-y)$$

where $$\Omega(x-y) = \int e^{i(x-y)\lambda} \Big\{ \underbrace{\frac{1}{|A(\lambda)|^2}}_{W(\lambda) \text{ in Fadeev notation}} - 1 \Big\} \frac{d\lambda}{2\pi}$$

The Gelfand-Levitan equation should express the orthogonality:

$$0 = \int \psi(x,\lambda)\, e^{-iy\lambda}\, d\mu(\lambda) \qquad \text{for} \quad y < x$$

$$= \int \left(e^{ix\lambda} + \int K(x,x')\, e^{ix'\lambda}\, dx' \right) e^{-iy\lambda}\, d\mu(\lambda)$$

$$= \Omega(x-y) + \int_0^x K(x,x') \left\{ \delta(x'-y) + \Omega(x'-y) \right\} dx'$$

$$\boxed{0 = \Omega(x-y) + K(x,y) + \int_0^x K(x,x')\, \Omega(x'-y)\, dx' \qquad x > y}$$

This is the Gelfand-Levitan equation. For x fixed it is a Fredholm integral equation with hermitian symmetric kernel, so the Fredholm alternative says you can solve if the homogeneous equation
$$0 = f(y) + \int_0^x f(x')\, \Omega(x'-y)\, dx'$$
has no solutions, this following from positive-definiteness of the kernel $\delta(x-y) + \Omega(x-y)$

Let's return to the Dirac equation and the representation
$$\phi(x,\lambda) = \begin{pmatrix} e^{ix\lambda} \\ e^{-ix\lambda} \end{pmatrix} + \int_{-x}^{x} V(x,t)\, e^{i\lambda t}\, dt$$

This really consists of a single equation because the components are conjugate. Now $\phi_1(x,\lambda)$ should be orthogonal to $e^{i\lambda y}$ with respect to the spectral measure for all $|y| < x$

hence if we put

$$\int e^{i\lambda x} e^{-i\lambda y} d\mu(\lambda) = \int e^{i\lambda(x-y)} \frac{d\lambda}{2\pi}\left(1 + \frac{1}{|A|^2} - 1\right)$$

$$= \delta(x-y) + \Omega(x-y)$$

as above, we find that for $|y| < x$

$$0 = \int \phi(x,\lambda) e^{-i\lambda y} d\mu(\lambda) = \begin{pmatrix} \Omega(x-y) \\ \Omega(-x-y) \end{pmatrix} + \int_{-x}^{x} v(x,t)\,\Omega(t-y)\,dt + v(x,y)$$

Thus we get the Gelfand-Levitan equation

$$\boxed{x > y \qquad \Omega(x-y) + v_1(x,y) + \int_{-x}^{x} v_1(x\,\text{[redacted]},t)\,\Omega(t-y)\,dt = 0}$$

December 21, 1977:

Start with a measure $d\mu(\lambda)$ on the line and form its Fourier transform

$$\int e^{it\lambda}\, d\mu(\lambda)$$

We want this to have the form $\delta(t) + \Omega(t)$ where $\Omega(t)$ is smooth.

$$\Omega(t) = \int e^{it\lambda} \left\{ d\mu(\lambda) - \frac{d\lambda}{2\pi} \right\}$$

It would seem that smoothness of $\Omega(t)$ is related to $d\mu(\lambda)$ having an asymptotic expansion [redacted] as $\lambda \to \infty$.

Example: Suppose

$$d\mu(\lambda) = \sum_{n \in \mathbb{Z}} \delta(\lambda - n) \frac{d\lambda}{2\pi}$$

Then

$$\int e^{it\lambda}\, d\mu(\lambda) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{int}$$

$$= \sum_{m \in \mathbb{Z}} \delta(t - 2\pi m)$$

so $\Omega(t)$ is smooth for $|t| < 2\pi$. [redacted]

Let me first understand what happens for the scattering by a potential with compact support. Then for large x I have

$$\phi(x,\lambda) = \begin{pmatrix} B(\lambda) e^{ix\lambda} \\ A(\lambda) e^{-ix\lambda} \end{pmatrix} = \begin{pmatrix} e^{ix\lambda} \\ e^{-ix\lambda} \end{pmatrix} + \int_{-x}^{x} v(x,t) e^{i\lambda t}\, dt$$

so

$$B(\lambda) = 1 + \int_{-x}^{x} v_1(x,t) e^{i\lambda(t-x)}\, dt$$

$y = x - t$
$dy = -dt$

$$B(\lambda)-1 = \int_0^{2x} v_1(x,x-y)e^{-i\lambda y}\,dy$$

so $$v_1(x,x-y) = \int (B(\lambda)-1)e^{i\lambda y}\frac{d\lambda}{2\pi} = (\hat{B}-1)(y)$$

$$v_1(x,t) = (\hat{B}-1)(x-t) \qquad \text{for large } x.$$

Thus [redacted] $B-1$ is the Fourier transform of a function $v_1(x,x-y)$ (ind. of x for x large) supported in $y \geq 0$ smooth for $y \geq 0$ and zero for large y.

[redacted]

December 22, 1977

Recall the S^1 case: One starts with a measure $d\mu$ on S^1 i.e. a positive-definite (semi-[redacted] definite) function on $\mathbb{Z}$. [redacted] One then applies Gram-Schmidt to the sequence $1, z, z^2, \ldots$. Better to say one looks at the filtration $F_0 \subset F_1 \subset F_2 \subset \cdots$ inside $L^2(S^1, d\mu)$ where F_n is spanned by $1, z, \ldots, z^n$.

Finite case: If $d\mu$ is supported on n-points, then $F_n = L^2(S^1, d\mu)$.

Suppose from now on that $d\mu$ has infinite support so that $\dim F_n = n+1$. Put

$$D = \overline{\bigcup F_n} \quad \text{in } L^2(S^1, d\mu)$$

so that $\bigcup_n z^{-n}D$ is dense in $L^2(S^1, d\mu)$. We have

[redacted] $$\boxed{D = \mathbb{C}\cdot 1 + zD}$$

In effect zD is a closed subspace, hence $(\mathbb{C}\cdot 1 + zD)/zD$

being finite-dimensional in $L^2(S^1, d\mu)/zD$ is closed so $\mathbb{C}\cdot 1 + zD$ is closed. (Simpler to put $1 = A \oplus A_1$ with $A \perp (zD)$, $A_1 \in zD$; then $\mathbb{C}\cdot 1 + zD = \mathbb{C}A \oplus zD$ is obviously closed.) Since $F_n = \mathbb{C}\cdot 1 + zF_{n-1} \subset \mathbb{C}\cdot 1 + zD \subset D$ we conclude $D = \mathbb{C}\cdot 1 + zD$).

So there are two cases depending on whether $D = zD$ or $D > zD$. Szegö's alternative decides between the two. To understand this, let's suppose $D > zD$. Then we can form the closed subspace

$$D_\infty = \bigcap_{n \geq 0} z^n D$$

and break up the Hilbert space into

$$L^2(S^1, d\mu) = D_\infty \oplus D_\infty^\perp$$

invariant under multiplication by z, z^{-1}. Suppose we are in the

Scattering Case: $D_\infty = \bigcap z^n D = (0)$, and $D > zD$.

[redacted]

Let $A \in D$ be a unit vector perp. to zD. Then

$$(z^i A, z^j A) = (z^{i-j} A, A) = \delta_{ij}$$

so that we get an isomorphism

$$L^2\left(S^1, \frac{d\theta}{2\pi}\right) \xrightarrow{\sim} L^2(S^1, d\mu)$$

given by multiplication by A. It follows that

$$d\mu = \frac{1}{|A|^2} \frac{d\theta}{2\pi}$$

so that $d\mu$ is in particular absolutely continuous with respect to Lebesgue measure on S^1.

This is too hard. Let's begin by studying the algebraic scattering case, by which I mean that the sequence of orthogonal polynomials satisfy:

$$p_n = z p_{n-1}$$

for large n. Another way of putting this is to have

$$d\mu(\lambda) = \frac{1}{|A(z)|^2} \frac{d\theta}{2\pi}$$

where $A(z) \in \mathbb{C}[z]$ has its roots outside the disk. Here A is a unit vector in D perpendicular to zD.

Question: How do you go about finding A starting with $d\mu$?

~~In this algebraic case one has~~ $\cancel{\frac{d\mu}{d\lambda} = \rho(z) \frac{d\theta}{2\pi}}$ ~~where~~ $\cancel{\rho(z)}$ ~~is a rational function with positive values on~~ $\cancel{S^1}$. ~~Notice the rational function~~ $\cancel{\rho^*(z) = \overline{\rho(z^*)}}$ ~~coincides with~~ $\cancel{\rho}$, ~~consequently if~~ $\cancel{\rho(a) = 0}$, ~~then also~~ $\cancel{\rho(a^*) = 0}$. $\cancel{a^* = (\bar a)^{-1}}$

$$\cancel{\frac{z-a}{1-\bar a z}} \qquad \cancel{\frac{z-(\bar a)^{-1}}{1-a^{-1}z}}$$

Put $d\mu(z) = \rho(z) \frac{d\theta}{2\pi}$. Then $\rho(z) = \frac{1}{|A(z)|^2}$, or

$$\log|A(z)| = -\frac{1}{2}\log\rho(z) \qquad \text{for } |z| = 1$$

Because $A(z)$ doesn't vanish for $|z| \le 1$, $\log A(z)$ is a well-defined analytic function and

$$\operatorname{Re}\log A(z) = \log|A(z)|$$

hence $\log A(z)$ is an analytic function in $|z| \le 1$ whose real part is $-\frac{1}{2}\log \rho(z)$. So

$$\log A(z) = \text{imag constant} + \int \left(-\frac{1}{2}\log \rho(e^{i\theta})\right) \frac{e^{i\theta}+z}{e^{+i\theta}-z}\frac{d\theta}{2\pi}$$

or

$$A(z) = e^{ic} \exp\left(-\frac{1}{2}\int \log \rho(e^{i\theta}) \frac{e^{i\theta}+z}{e^{i\theta}-z}\frac{d\theta}{2\pi}\right)$$

So provided A is normalized so as to have positive constant term we get Szegö's formula

$$A(0) = \exp\left(-\frac{1}{2}\int \log(\rho(e^{i\theta}))\frac{d\theta}{2\pi}\right)$$

Interpretation: One is interested in the orthogonal projection of 1 perpendicular to zD, which is $\frac{A(z)}{A(0)}$. Thus the length squared of this projection is

$$\boxed{\frac{1}{|A(0)|^2} = \exp\left(\int \log(\rho(e^{i\theta}))\frac{d\theta}{2\pi}\right)}$$

Recall for positive quantities

$$\sqrt{ab} \le \frac{a+b}{2}$$

or

$$\exp\left(\frac{1}{2}\log a + \frac{1}{2}\log b\right) \le \frac{a+b}{2}$$

or more generally for $f(t) > 0$ and $\int d\mu = 1$

$$\exp\left(\int \log f(t)\,d\mu\right) \le \int f(t)\,d\mu$$

This checks with $\frac{1}{|A(0)|^2} \le \int 1\,d\mu$

Suppose given a Toeplitz form on $F_n\,\mathbb{C}[z]$, that is, a positive-definite Hermitian form $(\ )$ with (z^i, z^j) depending only on $i-j$. [redacted] Let $\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_n$ be the orthogonal system obtained from $1, z, \ldots, z^n$ and $p_i = \dfrac{\tilde{p}_i}{\|p_i\|}$ the associated orthonormal system. The Toeplitz determinant $\det(c_{ij})$ where $c_{ij} = (z^i, z^j)$ is the same as

$$D_n = \det(\tilde{p}_i, \tilde{p}_j) = \prod_{i=0}^{n} \|\tilde{p}_i\|^2$$

But recall the recursion relation:

$$z p_{i-1} = k_i p_i - h_i z^{i-1} p_{i-1}^* \qquad k_i = \sqrt{1-|h_i|^2}$$

If l_i = leading coefficient of p_i [redacted] we have

$$p_i = l_i \tilde{p}_i \qquad \text{so} \qquad l_i = \frac{1}{\|\tilde{p}_i\|}$$

Also $\quad l_{i-1} = k_i l_i \quad$ so

[redacted]

$$\frac{1}{l_i} = k_i \frac{1}{l_{i-1}} = k_i k_{i-1} \cdots k_1 \frac{1}{l_0}$$

$$= k_i \cdots k_1 \int d\mu$$

Now [redacted] l_n = constant term $A(0)$, so

$$\frac{1}{|A(0)|^2} = \frac{1}{l_n^2} = \|\tilde{p}_n\|^2 = \frac{D_n}{D_{n-1}}$$

This leads one ~~to~~ [redacted] suspect that in general

$$\lim_{n\to\infty} \frac{D_n}{D_{n-1}} = \exp\left(\int \log \rho(z) \frac{d\theta}{2\pi}\right)$$

Toeplitz forms, i.e. orthogonal polys. on S^1.

The following gadgets are equivalent

1) measures $d\nu$ on S^1

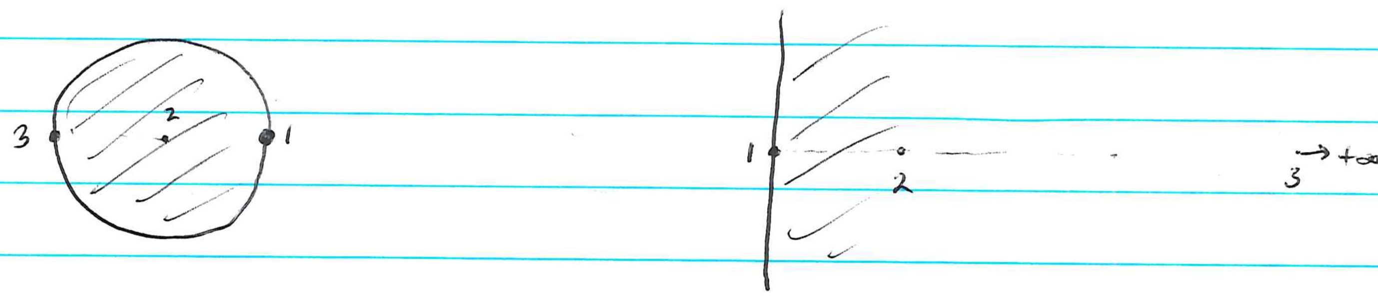
2) triples $(\mathcal{H}, U, e)$ up to isom, consisting of a unitary operator on a Hilbert space $\mathcal{H}$ plus cyclic vector e.

3) positive-definite functions $n \mapsto c_n$ on $\mathbb{Z}$

4) analytic functions $g(z)$ in $|z|<1$ with non-negative real part and real value at 0.

5) analytic functions $f(z)$ of modulus ≤ 1 in $|z|<1$ with [redacted] $-1 < f(0) \leq 1$.

Relation between 4) + 5) uses the conformal transformation of $|z|<1$ onto $\mathrm{Re}(w)>0$ which sends 1 to 0, -1 to ∞, 0 to 1



$$w = \frac{-z+1}{z+1} = \frac{1-z}{1+z}$$

~~Suppose $g(z)$ analytic in $|z|<1$ with $\mathrm{Re}\, g(z) \geq 0$ and $g(0) \geq 0$. Then $g(z) = \sum_{n \geq 0} a_n z^n$, $a_n = \oint g(\zeta) \frac{d\zeta}{2\pi i \zeta^{n+1}}$~~

Formulas: $\quad c_n = \int e^{+in\theta} d\nu$ $\quad$ and

(so $d\nu = \sum c_{-n} e^{in\theta} \frac{d\theta}{2\pi}$ in some sense)

$$g(z) = 2\Big(\frac{c_0}{2} + \sum_{n\geq 1}{}' c_{-n} z^n\Big) = \int \Big(\frac{1}{2} + \sum_{n\geq 1} \zeta^{-n} z^n\Big)\, d\nu(\zeta)$$

$$= \int \frac{1+\zeta^{-1}z}{1-\zeta^{-1}z}\, d\nu(\zeta)$$

Then $\quad \mathrm{Re}\, g(z) = c_0 + \sum_{n\geq 1} (c_{-n} z^n + c_n z^{-n}) \text{ ``=''}\ 2\pi \frac{d\nu}{d\theta}$, and

$$g(0) = \int d\nu = c_0 .$$

[redacted]

Next we bring in the Schur parameters & orthogonal polys. So restrict attention to probability measures.

Starting with the measure we construct the sequence of orthonormal polys. $1 = p_0, p_1, p_2, \ldots$ This goes up to p_d where $d+1 = \text{card support}(d\nu)$. We have recursion formulas for $n \leq d$

$$z p_{n-1} = k_n p_n - h_n z^{-n+1} p_{n-1}^*$$

with $\quad |h_n| < 1 \quad$ and $\quad k_n = \sqrt{1-|h_n|^2}$. These can be written

$$\begin{pmatrix} p_n \\ z^n p_n^* \end{pmatrix} = R(h_n) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ z^{n-1} p_{n-1}^* \end{pmatrix}$$

$$R(h_n) \leftrightarrow \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}$$

Let $n = d$. Then $p_d \in F_d \overset{L^2(d\nu)}{\mathbb{C}[z]}$ is orth. to $1, z, \ldots, z^{d-1}$ hence

zp_d is orth. to $z, \dots, z^d$ and since $F_d\,\mathbb{C}$[redacted]$[z] = L^2(d\nu)$ we have that it is proportional to $z^{-d}p_d^*$. Thus

$$zp_d = -h_{d+1}\, z^{-d} p_d^*$$

where $|h_{d+1}| = 1$. Consequently to $d\nu$ we have associated a sequence of Schur parameters

$$h_1, h_2, \dots\dots,$$ [redacted]

of modulus < 1 for $n \le d$ and with $|h_{d+1}| = 1$.

Recall Schur's theory: $f(z)$ analytic in $|z| < 1$ and of modulus ≤ 1 there. Put $\gamma_1 = f(0)$. If $|\gamma_1| = 1$ maximum modulus thm $\Rightarrow$ $f(z) = \gamma_1$ identically. If $|\gamma_1| < 1$, then

$$f(z) = \frac{g(z) + \gamma_1}{1 - \bar{\gamma}_1 g(z)} = \begin{pmatrix} 1 & \gamma_1 \\ \bar{\gamma}_1 & 1 \end{pmatrix}(g(z))$$

where $g(z)$ is analytic in $|z| < 1$ of modulus ≤ 1 and also $g(0) = 0$. Then $f_1(z) = \frac{g(z)}{z}$ is analytic in $|z| < 1$ of modulus ≤ 1 by maximum modulus thm, and

$$f(z) = \begin{pmatrix} 1 & \gamma_1 \\ \bar{\gamma}_1 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}(f_1(z))$$

Now repeat the process. One gets a sequence of numbers $\gamma_1, \gamma_2, \gamma_3, \dots$ all of modulus < 1 if the sequence is infinite, or all but the last have modulus < 1 and the last has modulus 1 when the sequence is finite.

I've seen that a similar sequence belongs to any probability measure $d\nu$ on S^1. What is the relation between f and $d\nu$?

$$\frac{z p_n}{z^n p_n^*} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is the image of the boundary condition $u_1(0) = u_2(0)$ at the $(n+1)$-th spot. Hence

$$f(z) = \begin{pmatrix} 1 & \gamma_1 \\ \bar{\gamma}_1 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & \gamma_n \\ \bar{\gamma}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} (\gamma_{n+1})$$

should be the image of the boundary condition $u_1(n+1) = \gamma_{n+1} u_2(n+1)$ at the 0-th spot. The relation between f and $d\nu$ is therefore clear in principle, although the formulas still have to be worked out.

Let $d\nu$ be a probability measure on S^1 with support of card $d+1$, let $p_0, \dots, p_d$ be the corresponding orthonormal polys and $h_1, h_2, \dots, h_d, h_{d+1}$ the Schur parameters given by

$$\begin{pmatrix} p_n \\ z^n p_n^* \end{pmatrix} = R(h_n)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ z^{n-1} p_{n-1}^* \end{pmatrix} \qquad n = 1, 2, \dots, d$$

$$z p_d + h_{d+1}(z^d p_d^*) \equiv 0 \quad \text{in} \quad L^2(d\nu)$$

The roots of $z p_d + h_{d+1}(z^d p_d^*)$ make up the support of $d\nu$. The condition that $z \in \operatorname{Supp}(d\nu)$ is hence that

$$z p_d + h_{d+1}(z^d p_d^*) = \begin{pmatrix} 1 & h_{d+1} \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(h_d) \cdots\cdots R(h_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

vanish. Taking the transpose this polynomial is also

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(\bar{h}_1) \cdots R(\bar{h}_d)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ h_{d+1} \end{pmatrix}$$

since $|h_{d+1}| = 1$ we have

$$\bar{h}_{d+1} \cdot z p_d + z^d p_d^* = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(\bar{h}_1) \cdots\cdots R(\bar{h}_d)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{h}_{d+1} \\ 1 \end{pmatrix}.$$

Let $f(z)$ denote the rational function with the Schur parameters $\bar{h}_1, \dots, \bar{h}_d, \bar{h}_{d+1}$ ie.

$$f(z) = R(\bar{h}_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots R(\bar{h}_d)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}(\bar{h}_{d+1})$$

Problem: Relate $f(z)$ to the measure $d\nu$ directly.

We have $\quad f(z) = \dfrac{N}{D} \quad$ where $\begin{pmatrix} N \\ D \end{pmatrix} = R(\bar{h}_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots R(\bar{h}_d)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{h}_{d+1} \\ 1 \end{pmatrix}$

hence

$$\bar{h}_{d+1}\, z p_d + z^d p_d^* = zN + D$$

and therefore the eigenvalues are the roots of

$$zf(z) + 1 = 0$$

Change zf into f so that now

$$f = \frac{N}{D} \qquad \begin{pmatrix} N \\ D \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(\bar{h}_1) \cdots\cdots \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{h}_{d+1} \\ 1 \end{pmatrix}$$

The eigenvalues are the roots of

$$f(z) = -1$$

Also

$$\begin{pmatrix} N^* \\ D^* \end{pmatrix} = \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} R(h_1) \cdots R(h_d)\begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h_{d+1} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} D^* \\ N^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} R(\bar{h}_1) \cdots R(\bar{h}_d)\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}\begin{pmatrix} 1 \\ h_{d+1} \end{pmatrix}$$

$$\bar{h}_{d+1} z^{d+1}\begin{pmatrix} D^* \\ N^* \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(\bar{h}_1) \cdots R(\bar{h}_d)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{h}_{d+1} \\ 1 \end{pmatrix}$$

so

$$\boxed{\bar{h}_{d+1} z^{d+1}\begin{pmatrix} D^* \\ N^* \end{pmatrix} = \begin{pmatrix} N \\ D \end{pmatrix}}$$

Also

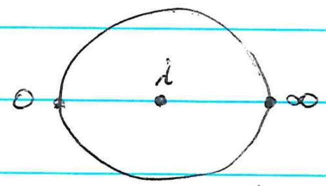
$$N + D = \bar{h}_{d+1}\, z p_d + z^d p_d^*$$

Look at the continuous case: Here one has the system
$$\frac{du}{dx} = \begin{pmatrix} i\lambda & h \\ \bar{h} & -i\lambda \end{pmatrix} u$$

on $0 \le x \le l$ starting with $\binom{1}{1}$ boundary condition at $x=0$. Let $S(\lambda) = S(0,l;\lambda)$ propagate initial values at $x=0$ to $x=l$, so that
$$\phi(l,\lambda) = S(\lambda)\binom{1}{1}.$$

Conditions for an eigenvalue are
$$\frac{\phi_1(l,\lambda)}{\phi_2(l,\lambda)} = e^{i\theta} \qquad \text{or}$$
$$\left(e^{-i\theta/2} \;\; -e^{i\theta/2}\right) S(\lambda) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

Recall that one way of handling this situation is the one used by de Branges. Suppose $e^{i\theta} = 1$ to simplify and let us transform $|z| < 1$ into $\operatorname{Im} \omega > 0$ by



$$\omega = \frac{1}{i} \frac{z+1}{z-1}$$

whence $$\frac{E^\#}{E} = \frac{\phi_1(l,\lambda)}{\phi_2(l,\lambda)}$$ which maps $\operatorname{Im} \lambda > 0$ into $|z| < 1$

goes into
$$\frac{1}{i} \frac{E^\# + E}{E^\# - E} = -\frac{A(\lambda)}{B(\lambda)}$$ where $E = A - iB$ in deB's notation.

Now we saw that because $-\frac{A(\lambda)}{B(\lambda)}$ has pos. imag. part for $\operatorname{Im}(\lambda) > 0$
$$-\frac{A(\lambda)}{B(\lambda)} = \sum p_n \left(\frac{1}{\lambda_n - \lambda} + \frac{1}{\lambda^2+1} \right) + \text{Real constant} + p\lambda$$

Key question: Because one is working on a finite interval it should be possible to sum

$$\sum{}' p_n\left(\frac{1}{\lambda_n - \lambda}\right)$$

à la Eisenstein. Does this give $-\dfrac{A(\lambda)}{B(\lambda)}$ on the nose?

Example: $l = \pi$, $h \equiv 0$. Eigenvalues given by

$$\frac{e^{i\lambda\pi}}{e^{-i\lambda\pi}} = 1 \qquad \text{or} \qquad \lambda \in \mathbb{Z}$$

$$\frac{A(\lambda)}{B(\lambda)} = \frac{\cos \pi\lambda}{\sin \pi\lambda}$$ is a meromorphic function with simple poles at integral points with residue $\frac{1}{\pi}$. In this case one knows

$$\pi \cot(\pi\lambda) = \sum_{n\in\mathbb{Z}} \frac{1}{\lambda - n} \qquad \text{(Eisenstein summ.)}$$

and this could be proved by contour integration.

Completeness of eigenfunctions for [redacted] a Sturm–Liouville problem on a finite interval.

$$L = -\frac{d^2}{dx^2} + q$$

Green's function is $G(x, x', \lambda) = \dfrac{\varphi(x_<)\,\psi(x_>)}{W(\varphi, \psi)}$. The identity used is

$$(\lambda - L)^{-1} f = (\lambda - L)^{-1}\left(\frac{(\lambda - L)f + Lf}{\lambda}\right) = \frac{1}{\lambda} f + \frac{1}{\lambda}(\lambda - L)^{-1} L f$$

Suppose then that f is smooth with compact support inside the interval. Then $Lf \in C_0^\infty$ also. But [redacted] it should be possible to see that $\forall x'$

$$G(\ , x', \lambda) \longrightarrow 0$$

as $|\lambda| \longrightarrow \infty$ provided one avoids getting too [redacted] close to the eigenvalues. The idea is that on a finite interval the potential should be negligible for large $|\lambda|$, i.e. φ, ψ should be asymptotic to trigonometric functions. Thus if one integrates around a large circular contour avoiding the eigenvalues one should have

$$\frac{1}{2\pi i}\oint (\lambda - L)^{-1} Lf \, \frac{d\lambda}{\lambda} \longrightarrow 0$$

and hence that

$$\frac{1}{2\pi i}\oint (\lambda - L)^{-1} f \, \frac{d\lambda}{\lambda} \longrightarrow \frac{1}{2\pi i}\oint f \, \frac{d\lambda}{\lambda} = f$$

But now one uses contour integration to write the former as a sum over the eigenvalues.

December 26, 1977

Recall the equivalence between

1) measures $d\nu$ on S^1
2) iso classes of triples $(\mathcal{H}, U, e)$, e cyclic vector for the unitary operator U on $\mathcal{H}$.
3) pos. def. functions $n \mapsto c_n$ on $\mathbb{Z}$
4) analytic fns. $g(z)$ in $|z|<1$ with $\operatorname{Re} g(z) \geq 0$ and $g(0)$ real.

Formulas: $\quad c_n = \int z^n \, d\nu$

$$g(z) = c_0 + 2\sum_{n\geq 1} c_{-n} z^n = \int \frac{1+\zeta^{-1}z}{1-\zeta^{-1}z} \, d\nu(\zeta)$$

$$d\nu(\zeta) = \operatorname{Re} g(\zeta) \cdot \frac{d\theta}{2\pi}$$ when g has nice boundary values.

I wanted to bring in the orthonormal polys and the Schur parameters. Assume $c_0 = \int d\nu = 1$, and that $\operatorname{Supp}(d\nu)$ has card $d+1$, so that one has $p_0, \ldots, p_d$ hence numbers $h_1, \ldots, h_d$ of modulus <1, and also an h_{d+1} of modulus 1 with

$$zp_d + h_{d+1} z^d p_d^* = 0 \quad \text{on } \operatorname{Supp}(d\nu).$$

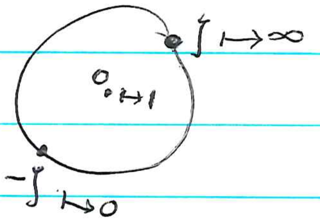
Consider the rational function

$$\tilde{g}(z) = \frac{-zp_d + h_{d+1} z^d p_d^*}{zp_d + h_{d+1} z^d p_d^*} = \frac{(-h_{d+1}) + \frac{zp_d}{z^d p_d^*}}{(-h_{d+1}) - \frac{zp_d}{z^d p_d^*}}$$

We have

$$\tilde{g}(0) = 1 \qquad \tilde{g}(\infty) = -1$$

$$\tilde{g}(z) = \frac{\zeta + f}{\zeta - f} \qquad \text{where } |f| < 1 \text{ in disk}$$



so $\operatorname{Re} \tilde{g}(z) \geq 0$ in disk.

The conjecture therefore is that

$$\tilde{g}(z) = g(z) \overset{\text{i.e.}}{=} \int \frac{1+\zeta^{-1}z}{1-\zeta^{-1}z}\, d\nu(\zeta)$$

There should be a corresponding conjecture for arbitrary probability measures $d\nu$, except that you want to use the "solution nice at $n \to \infty$" instead of the "ϕ-solution":

$$(1 \;\; h_{d+1})\begin{pmatrix} zp_d \\ z^d p_d^* \end{pmatrix} = (1+h_{d+1})\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(h_d) \cdots R(h_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Taking transpose this polynomial becomes

$$(1 \;\; 1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(\bar{h}_1) \cdots\cdots R(\bar{h}_d)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ h_{d+1} \end{pmatrix}$$

and multiplying by $\bar{h}_{d+1}$ we get

$$\bar{h}_{d+1} z p_d + z^d p_d^* = (1 \;\; 1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(\bar{h}_1) \cdots R(\bar{h}_d)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{h}_{d+1} \\ 1 \end{pmatrix}.$$

Better: put $S(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(h_d) \cdots R(h_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$.
so that

$$\begin{pmatrix} zp_d \\ z^d p_d^* \end{pmatrix} = S(z)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and so that the eigenvalues are roots of $S(z)(1) = -h_{d+1}$

Then this [redacted] condition can also be written

$$1 = S(z)^{-1}(-h_{d+1})$$

where $S(z)^{-1} = \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} R(-h_1) \cdots\cdots R(-h_d) \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix}$

Put $$\tilde{f}(z) = S(z)^{-1}(-h_{d+1})$$

so that

$$\tilde{f}(0) = \infty \qquad |\tilde{f}(z)| > 1 \quad \text{inside } S^1$$
$$\tilde{f}(\infty) = 0 \qquad = 1 \quad \text{on } \text{"}$$
$$< 1 \quad \text{outside } \text{"}$$

But this is too ugly. Instead you conjugate by $\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -h_{d+1} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ h_{d+1} \end{pmatrix}$$

$$S_1(z) = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} S(z)^{-1} \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} R(\bar{h}_1) \cdots\cdots R(\bar{h}_d) \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}$$

and put

$$f(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(\bar{h}_1) \cdots\cdots R(\bar{h}_d) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} (\bar{h}_{d+1}) = -\frac{1}{\tilde{f}}$$

so that now

$$f(0) = 0 \qquad |f(z)| < 1 \quad \text{when } |z| < 1$$
$$f(\infty) = \infty \qquad = 1 \qquad = 1$$
$$> 1 \qquad > 1$$

and so the eigenvalue condition is

$$f(z) = -1$$

Now we can transform the circle to right half plane using $-1 \mapsto \infty$, $1 \mapsto 0$, $0 \mapsto 1$

$$g(z) = \frac{1-f}{1+f}$$

Then this g satisfies the conditions for being representable in the form

$$g(z) = \int \frac{1+\zeta^{-1}z}{1-\zeta^{-1}z}\, d\nu(\zeta)$$

so what I want to prove is that this $d\nu$ coincides with the one we started with.

December 27, 1977

Consider a finite Toeplitz form $\quad T_d = \sum_{i,j=0}^{d} c_{i-j} u_i \bar{u}_j$ which is positive (>0).

Equivalently we have an inner product on $F_d\,\mathbb{C}[z]$ such that $(z^i, z^j) = c_{i-j}$ depends only on $i-j$.

Problem: Extend T to an inner product on $\mathbb{C}[z,z^{-1}]$.

Construct an orthogonal sequence $\tilde{p}_0, \ldots\ldots, \tilde{p}_d$ from $1, \ldots\ldots, z^d$. $\qquad \tilde{p}_i = \sum_{j\le i} a_{ij} z^j \qquad a_{ii} = 1$

$$(\tilde{p}_i, z^k) = \sum_{j\le i} a_{ij} c_{jk} = 0 \qquad k < i$$

Put $z^i \tilde{p}_i^* = \sum_{j \le i} \bar{a}_{ij} z^{i-j}$ and notice that for $0 \le k < i$

$$\left(z^i \tilde{p}_i^*, z^{i-k}\right) = \sum_{j\le i} \bar{a}_{ij} \underset{c_{k-j} = \bar{c}_{j-k}}{\underset{\shortparallel}{(z^{i-j}, z^{i-k})}} = 0$$

Consequently $z^i\tilde{p}_i^*$ is the unique poly of degree $\le i$ with constant term 1 which is orthogonal to $z, z^2, \ldots, z^i$. It follows therefore that

$$z\tilde{p}_{n-1} = \tilde{p}_n + (\text{const})\, z^{n-1}\tilde{p}_{n-1}^* \qquad \left(= \tilde{p}_n - h_n z^{n-1}\tilde{p}_{n-1}^*\right)$$

Hence if we form the orthonormal sequence $p_0, \ldots\ldots, p_d$ we have recursion formulas

$$z p_{n-1} = k_n p_n - h_n z^{n-1} p_{n-1}^* \qquad k_n = \sqrt{1-|h_n|^2}$$

i.e.

$$\begin{pmatrix} p_n \\ z^n p_n^* \end{pmatrix} = R(h_n)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots R(h_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ p_0^* \end{pmatrix}$$

It follows that $\left|\frac{p_n}{z^n p_n^*}\right| < 1$ for $|z|<1$, hence p_n has all its roots inside S^1.

Now consider the measure $d\nu = \frac{d\theta}{2\pi\,[redacted]\,|q|^2}$ where q [redacted] is a poly of degree d with its roots [redacted] inside S^1.

$$(z^i, q) = \int z^i \bar{q} \frac{d\theta}{2\pi q\bar{q}} = \int \frac{z^i}{q} \frac{dz}{2\pi i z}$$

Deform the contour to a big circle. If $i<d$, then $\frac{z^i}{q} \to 0$ and so $(z^i,q)=0$ for $i<d$. Therefore if $p_0, p_1, \ldots, p_d$ is the sequence of orthonormal polys with respect to $d\nu$ we have $p_d = q$ provided q has positive leading coefficients.

On the other hand if we start with q we can form the rational function $\frac{q}{z^d q^*}$ which has a unique Schur representation

$$\frac{q}{z^d q^*} = R(h_d)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots R(h_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}(\gamma) \qquad |\gamma| = 1.$$

Hence we see that if $p_0, \ldots, p_d$ is the [redacted] orthonormal sequence belonging to the finite Toeplitz form T_d, then these are also the first d orthonormal polys associated to the measure $\frac{d\theta}{2\pi |p_d|^2}$.

Actually it should be possible to start with the form $\sum c_{i-j} u_i \bar{u}_j$ and then to solve the equations for $\tilde{p}_d$:

$$(\tilde{p}_d, z^k) = \sum_{j \le i} a_{dj} c_{j-k} = 0 \qquad k = 0, \ldots, d-1$$

whence it should be easy to see that the measure $\dfrac{d\theta}{2\pi|\tilde{p}_d|^2}$ should have similar moments to the given c_n.

Put $\tilde{p}_d = \sum_{0\leq j\leq d}' \alpha_j z^j$ $\quad \alpha_d = 1$. Then

$$0 = (\tilde{p}_d, z^k) = \sum_{0\leq j\leq d}' \alpha_j c_{j-k} \qquad \text{for } 0\leq k < d$$

Multiply by z^k and sum

$$\sum_{k\in\mathbb{Z}}' \Big(\sum_{0\leq j\leq d}' \alpha_j c_{j-k}\Big) z^k = \sum_{0\leq j\leq d}' \alpha_j \Big(\sum_{k\in\mathbb{Z}}' c_{j-k} z^{k-j}\Big) z^j$$

$$= \tilde{p}_d(z) \cdot \sum_{n\in\mathbb{Z}}' c_{-n} z^n$$

This formal Laurent series has no terms involving $1, z, \ldots, z^{d-1}$.

Notice that the Toeplitz form gives c_i for $|i|\leq d$, however the equations

$$\sum_{0\leq j\leq d}' \alpha_j c_{j-k} = 0 \qquad 0\leq k < d$$

only use the c_i for $d < i \leq d$. Coeff. of z^k in

$$\sum_{0\leq j\leq d}' \alpha_j z^j \cdot \sum_{-k\ n<d}' c_n z^n \qquad \begin{matrix} j+n=k \\ j-k=-n \end{matrix}$$

is

$$\sum_{\substack{0\leq j\leq d \\ -d< j-k< d}}' \alpha_j c_{j-k}$$

?

If $k>0$

Let's suppose given a rational function $f(z)$ of modulus <1 in the disk, and form its Schur representation

$$f(z) = R(\gamma_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(\gamma_2)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots R(\gamma_n)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} f_{n+1}(z)$$

Problem: Does this process stop ie. $f_{n+1}(z) = \text{constant}$?

Write $f = \frac{N}{D}$ in lowest terms and look at the degrees. We remove factors of z from N, then when $f(0) = h$ we change f to

$$\frac{f-h}{-\bar{h}f+1} = \frac{N-hD}{D-\bar{h}N}$$

Once $\deg(N) < \deg(D)$ the degree of f doesn't change. In effect [redacted] if $\deg D = n$, then $N-hD$ has degree $\leq n$ so removing a factor of z we get a numerator of degree $<n$, whereas the denominator $D-\bar{h}N$ has degree n.

For example take $f =$ [redacted] $\frac{\varepsilon p}{q}$ where p, q are rel. prime, q is of degree $>$ than p and ε is sufficiently small.

Let q_n be a poly of degree n with all roots inside S^1. Put

$$f_n(z) = \frac{q_n}{z^n q_n^*} = \frac{c}{c^*} \prod_{i=1}^{n} \left(\frac{z-\lambda_i}{1-\bar{\lambda}_i z} \right)$$

so that $|f_n(z)| < 1, = 1, > 1$ according to $|z| < 1, = 1, > 1$

Let $h_n = f_n(0)$ and define $f_{n-1}(z)$ by

$$z f_{n-1}(z) = R(-h_n) f_n(z) = \frac{g_n - h_n z^n g_n^*}{-\bar{h}_n g_n + z^n g_n^*}$$

Define g_{n-1} by

$$z\, g_{n-1} = \frac{1}{k_n}\left(g_n - h_n z^n g_n^*\right) \qquad k_n = \sqrt{1-|h_n|^2}$$

Then
$$z f_{n-1} = \frac{z g_{n-1}}{z^n (z g_{n-1})^*} = \frac{z\, g_{n-1}}{z^{n-1} g_{n-1}^*}$$

so
$$f_{n-1} = \frac{g_{n-1}}{z^{n-1} g_{n-1}^*}$$

Because we know $|f_{n-1}(z)|<1$ for $|z|<1$, the roots of the denominator are outside S^1, hence g_{n-1} has its roots inside S^1. (Note that because g_n and $z^n g_n^*$ are rel. prime so are $z g_{n-1}$ and $z^{n-1} g_{n-1}^*$, as

$$\begin{pmatrix} g_n \\ z^n g_n^* \end{pmatrix} = R(h_n) \begin{pmatrix} z g_{n-1} \\ z^{n-1} g_{n-1}^* \end{pmatrix}$$

hence g_{n-1} and $z^{n-1} g_{n-1}^*$ are rel. prime)

So it's clear that we can repeat the process and we get a formula

$$\begin{pmatrix} g_n \\ z^n g_n^* \end{pmatrix} = R(h_n)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots R(h_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g_0 \\ g_0^* \end{pmatrix}$$

The process just described is the analogue of way of starting from a de B function E and building it up. So I want to understand the continuous version of the above Schur process.

December 28, 1977

Consider $Lu = \left(-\frac{d^2}{dx^2} + q\right)u = \lambda^2 u$ on $-\infty < x < \infty$
with the boundary condition $u(0)=0$ and $q(x) = 0$ for $x > a$.
Denote by $f(x,\lambda)$ the solution of $Lf = \lambda^2 f$ with

$$f(x,\lambda) = e^{i\lambda x} \qquad x > a$$

Then for each x, $f(x,\lambda)$ is an entire function of λ. As usual put

$$\phi(x,\lambda) = B(\lambda) f(x,\lambda) + A(\lambda) f(x,-\lambda)$$

so that

$$L(\lambda) = W(\phi, f) = A(\lambda)\, W(e^{-ix\lambda}, e^{ix\lambda}) = 2i\lambda A(\lambda)$$

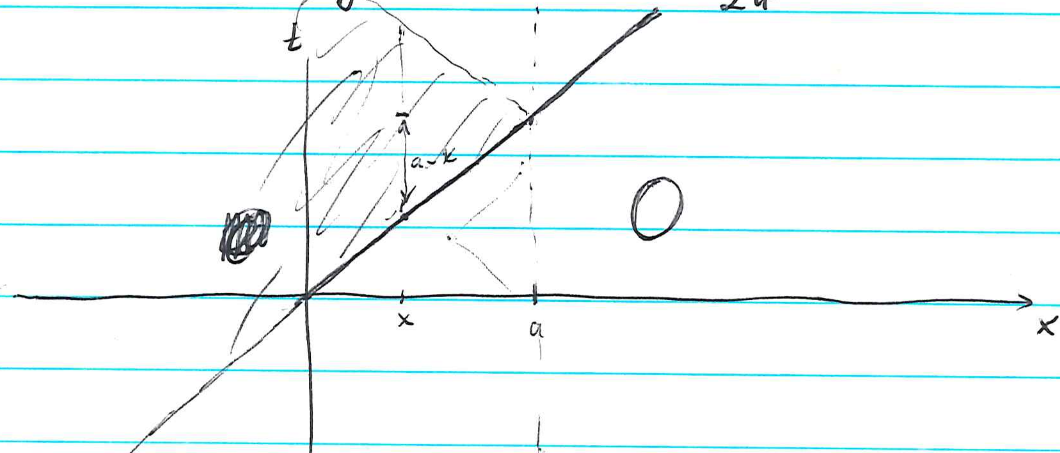
is an entire function of λ.

~~For Im $\lambda \neq 0$ we let $\psi(x,\lambda)$ denote the solution of $L\psi = \lambda^2\psi$ which decays as $x \to +\infty$,~~ [redacted] ~~suitably normalized. Thus~~

$$\cancel{\psi(x,\lambda) = \begin{cases} f(x,\lambda) & \text{Im}\,\lambda > 0 \\ f(x,-\lambda) & \text{Im}\,\lambda < 0 \end{cases}}$$

Take the Fourier transform of $f(x,\lambda)$:

$$V(x,t) = \int e^{-i\lambda t} f(x,\lambda) \frac{d\lambda}{2\pi} = \delta(x-t) \qquad x > a$$



From hyperbolic equation theory we see that $v(x,t)$ should be supported for $x < t < 2(a-x)+x = 2a-x$ with a δ singularity along $x=t$. So

$$f(x,\lambda) = e^{i\lambda x} + \int_x^{2a-x} v(x,t)\, e^{i\lambda t}\, dt \qquad x \leq a$$

and more generally when $q \to 0$ at $x \to \infty$ we expect to have

$$f(x,\lambda) = e^{i\lambda x} + \int_x^{\infty} v(x,t)\, e^{i\lambda t}\, dt$$

The problem is whether $v(x,t)$ satisfies an analogue of the Gelfand-Levitan equation. (I think this is the Marchenko equation, but what is its significance?)