December 3, 1977.

$\mathcal{H}_1 = \ell^2$ has the operator L

$\mathcal{H}_2 =$ Hilbert space of $t \mapsto u(t) \in \ell^2$ $\ni$ $Lu(t) = \frac{u(t+1)+u(t-1)}{2}$
with norm $E(u)$.

We have produced an isom

$$L^2(S^1, 4\pi|B|^2 d\theta) \xrightarrow{\sim} \mathcal{H}_2$$

$$\alpha \longmapsto u(n,t) = \int z^{-t}\phi(n,\lambda)\frac{\alpha}{\sin\theta}d\theta$$

Moreover α even $\Rightarrow \frac{\alpha}{\sin\theta}$ odd $\Rightarrow u(-t) = -u(t)$
and α odd $\Rightarrow \frac{\alpha}{\sin\theta}$ even $\Rightarrow u(t) = u(-t)$

Recall $\mathcal{H}_2$ is given the norm

$$E(u) = \|u(t)\|^2 - \tfrac{1}{2}(u(t+1), u(t-1)) - \tfrac{1}{2}(u(t-1), u(t+1))$$

$$= \|u(t)\|^2 + \|u(t+1)\|^2 - (u(t+1), Lu(t)) - (Lu(t), u(t+1))$$

$$= \|u(t)\|^2 + \|u(t+1)\|^2 - (Lu(t+1), u(t)) - (u(t), Lu(t+1))$$

$$= \|u(t+1)\|^2 - \tfrac{1}{2}(u(t+2), u(t)) - \tfrac{1}{2}(u(t), u(t+2)).$$

Let $\mathcal{H}_2^{ev}$ consist of u such that $u(-t) = u(t)$
and $\mathcal{H}_2^{odd}$ consist of u " " $u(-t) = -u(t)$. Since
$u(t) \mapsto u(-t)$ is a symmetry of $\mathcal{H}_2$ it follows that

$$\mathcal{H}_2 = \mathcal{H}_2^{ev} \oplus \mathcal{H}_2^{odd}$$

is an orthogonal direct sum.

Claim $\mathcal{H}_2^{odd} \xrightarrow{\sim} \mathcal{H}_1$ via $u(t) \longmapsto u(1)$. In effect for $u \in \mathcal{H}_2^{odd}$ we have $u(0) = 0$, so for any $v \in \mathcal{H}_1$ there is a unique $u(t)$ such that $u(0)=0$, $u(1) = v$. It follows that $\frac{u(1)+u(-1)}{2} = Lu(0) = 0$ so $u(-1) = -v$. Hence $u(-t) = -u(t)$ since both coincide for $t = 0, 1$. Moreover we have

$$
\begin{aligned}
E(u) &= \|u(0)\|^2 - \tfrac{1}{2}(u(1), u(-1)) - \tfrac{1}{2}(u(-1), u(1)) \\
&= \|u(1)\|^2 .
\end{aligned}
$$

Next let $u(t) \in \mathcal{H}_2^{ev}$.

$$\frac{u(1) - u(-1)}{2} = 0$$

$$\frac{u(1) + u(-1)}{2} = Lu(0)$$

$$\Longrightarrow \quad u(1) = Lu(0)$$

hence $\mathcal{H}_2^{ev} \longrightarrow \mathcal{H}_1$, $u(t) \longmapsto u(0)$ is injective. Given $v \in \mathcal{H}_1$ let $u(t) \in \mathcal{H}_2$ be given by

$$
\begin{aligned}
u(0) &= v \\
u(1) &= Lv
\end{aligned}
$$

Then

$$\frac{u(-1) + u(1)}{2} = Lv \Longrightarrow \frac{u(-1)}{2} = \frac{Lv}{2} \Longrightarrow u(-1) = Lv$$

and so $u(-t) = u(t)$ as they coincide for $t = 0, 1$. Thus the map $\mathcal{H}_2^{ev} \to \mathcal{H}_1$ is bijective. We have

$$
\begin{aligned}
E(u) &= \|u(0)\|^2 - \tfrac{1}{2}(u(1), u(-1)) - \tfrac{1}{2}(u(-1), u(1)) \\
&= \|u(0)\|^2 - \|u(1)\|^2 = \|u(0)\|^2 - \|Lu(0)\|^2
\end{aligned}
$$

Thus we see that $\mathcal{H}_2$ is the direct sum of $\mathcal{H}_1$ and $\mathcal{H}_1$ with the norm $((1-L^2)u, u)$. Hence in general $\mathcal{H}_2$ won't be complete unless we require $\|L\| < 1$

So we get

$$u(t) \longmapsto u(1) = \frac{u(1)-u(-1)}{2}$$

$$L^2(S^1, 4\pi|B|^2 d\theta)^{ev} \xrightarrow{\sim} \mathcal{H}_2^{odd} \xrightarrow{\quad\sim\quad} \mathcal{H}_1$$

$$\alpha \longmapsto \int z^{-t}\phi(n,\lambda)\frac{\alpha}{\sin\theta}\,d\theta \longmapsto \int \phi(n,\lambda)\frac{z^{-1}-z}{2}\frac{\alpha}{\sin\theta}\,d\theta$$

$$= -i\int \phi(n,\lambda)\,\alpha\,d\theta$$

So inside $L^2(S^1, 4\pi|B|^2 d\theta)^{ev}$ there has to be an orthonormal basis corresponding to e_n under the isom. $\alpha \longmapsto \int \phi(\bullet,\lambda)\,\alpha\,d\theta$.

$$\left(\int \phi(\cdot,\lambda)\,\alpha\,d\theta,\ e_n\right) = \int \phi(n,\lambda)\,\alpha\,d\theta$$

$$= \int \overline{\frac{\phi(n,\lambda)}{4\pi|B|^2}}\,\alpha\,4\pi|B|^2\,d\theta$$

hence the element corresp. to e_n is the even function

$$\frac{\phi(n,\lambda)}{4\pi|B|^2}$$

As a check we recall $\quad d\mu(\lambda) = \dfrac{d\lambda}{2\pi|B|^2\sqrt{1-\lambda^2}}$

and

$$\int_{S^1} \frac{\phi(n,\lambda)}{4\pi|B|^2}\,\frac{\phi(m,\lambda)}{4\pi|B|^2}\,4\pi|B|^2 d\theta = 2\int_{-1}^{1} \frac{\phi(n,\lambda)\phi(m,\lambda)}{4\pi|B|^2}\,\frac{d\lambda}{\sqrt{1-\lambda^2}} = \delta_{nm}$$

For n large we have

$$\frac{\phi(n,\lambda)}{4\pi|B|^2} = \frac{\overline{B}(z)z^{-n} + B(z)z^n}{4\pi B\overline{B}} = \frac{1}{4\pi}\left(\frac{z^{-n}}{B} + \frac{z^n}{\overline{B}}\right)$$

$$= \frac{1}{2\pi}\operatorname{Re}\left(\frac{z^n}{A(z)}\right) = \frac{1}{4\pi}\left\{\frac{z^n}{A(z)} + \frac{z^{-n}}{A(z^{-1})}\right\}$$

Let's now consider the other half of the Hilbert space which consists of odd α on S^1 with the norm from $4\pi|B|^2 d\theta$. If I multiply α by $\sin\theta$ I get an even function.

Go back to

$$\alpha \longmapsto \int z^{-t}\phi(n,\lambda)\,\alpha\, d\theta$$

$$E\left(\int z^{-t}\phi(n,\lambda)\,\alpha\, d\theta\right) = \int |\alpha|^2\, 4\pi|B|^2\sin^2\theta\, d\theta$$

or

$$E\left(\int z^{-t}\phi(n,\lambda)\,\frac{\alpha}{f}\, f d\theta\right) = \int \left|\frac{\alpha}{f}\right|^2 f\, 4\pi|B|^2\sin^2\theta\; f d\theta$$

so if we choose $f = \dfrac{1}{4\pi|B|^2\sin^2\theta}$, and put

$$d\nu = f\, d\theta$$

we have

$$E\left(\int z^{-t}\phi(n,\lambda)\,\alpha\, d\nu\right) = \int |\alpha|^2\, d\nu$$

Then we have

$$L^2(S^1, d\nu) \xrightarrow{\sim} \mathcal{H}_2$$

$$\alpha \longmapsto \int z^{-t}\phi(n,\lambda)\,\alpha\, d\nu$$

and

$$L^2(S^1, d\nu)^{odd} \xrightarrow{\sim} \mathcal{H}_2^{odd} \xrightarrow{\sim} \mathcal{H}_1$$

$$\alpha \longmapsto \int z^{-t}\phi(n,\lambda)\,\alpha\, d\nu \longmapsto \int \frac{z^{-1}-z}{2}\phi(n,\lambda)\,\alpha\, d\nu$$

$$u(t) \longmapsto \tilde{u}(1) = \frac{u(1)-u(-1)}{2}$$

hence we get

$$L^2(S^1, d\nu)^{odd} \xrightarrow{\sim} \mathcal{H}^1 = \ell^2$$

$$\alpha \longmapsto \int \phi_\lambda (\sin\theta\, \alpha)\, d\nu$$

Figure out what corresponds to e_n:

$$\left(\int \phi_\lambda (\alpha \sin\theta) d\nu,\ e_n \right) = \int \phi(n,\lambda)\, \alpha \sin\theta\, d\nu$$

and so we see e_n corresp. to $\phi(n,\lambda)\sin\theta$. Conclude that

$$\phi(n,\lambda)\sin\theta \quad \text{is an orth. basis for } L^2(S^1, d\nu)^{odd}.$$

<u>December 4, 1977</u>:

Let $d\nu$ be a measure on S^1 and $p_0, p_1, \ldots$ the sequence of poly obtained by orthonormalizing $1, z, z^2, \ldots$ Define h_n, k_n for $n \geq 0$ by

1) $$z p_n = k_n p_{n+1} - h_n z^n p_n^*$$

Then p_{n+1} orth to $z^n p_n^* \Rightarrow \quad 1 = |k_n|^2 + |h_n|^2$. Let $\ell_n =$ leading coefficient of p_n. Then

$$\ell_n = k_n \ell_{n+1} \qquad \text{so} \qquad k_n > 0$$

and $$k_n = \sqrt{1 - |h_n|^2} \qquad \text{and}$$

$$\ell_n = (k_{n-1} \cdots k_0)^{-1} \ell_0 = \left(\prod_{i=0}^{n-1} |1 - h_i|^2 \right)^{-1/2} \ell_0$$

Also put $z = 0$ in 1) gives

$$0 = k_n p_{n+1}(0) - h_n \ell_n$$

or $$p_{n+1}(0) = \frac{h_n l_n}{k_n} = h_n \quad l_{n+1}$$

We get a basis for $\mathbb{C}[z,z^{-1}]$ using

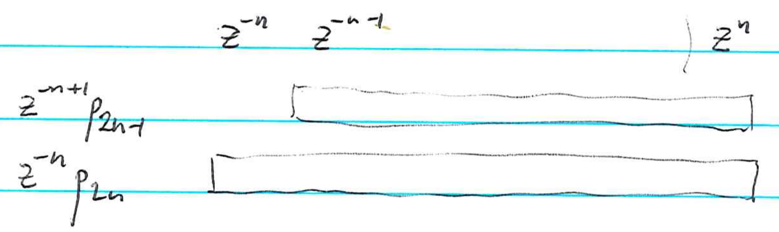
$$p_0, \quad \begin{matrix} z^{-1}p_2 & z^{-2}p_4 \\ z\, p_2^*, & z^2 p_4^* \end{matrix}$$

Better: Consider the filtration of $\mathbb{C}[z,z^{-1}]$ given by

$$F_n \,\mathbb{C}[z,z^{-1}] = \langle z^{-n}, \dots, z^n \rangle$$

so that
$$\begin{matrix} & F_0 < & F_1 < & F_2 < \cdots \\ \dim = & 1 & 3 & 5 & \cdots \end{matrix}$$

In $F_n \ominus F_{n-1}$ we find $z^{-n+1}p_{2n-1}$, $z^{-n}p_{2n}$
and their $*$'s.



Unfortunately $z^{-n}p_{2n}$, $z^n p_{2n}^*$ are not orthogonal.
In fact
$$(z^{-n}p_{2n}, z^n p_{2n}^*) = (p_{2n}, z^{2n} p_{2n}^*)$$
and
$$(p_n, \underset{\text{orth to } z, \dots, z^n}{z^n p_n^*}) = (p_n(0), z^n p_n^*) = \frac{p_n(0)}{(z^n p_n^*)(0)} (z^n p_n^*, z^n p_n^*)$$
$$= \frac{p_n(0)}{l_n} =$$

Check the formulas:

$$z p_n = k_n p_{n+1} - h_n z^n p_n^* \qquad \Rightarrow \quad k_n = \text{[redacted]} = \sqrt{1-|h_n|^2}$$

$$l_n = k_n l_{n+1} \qquad \text{so that}$$

$$l_{n+1} = \frac{1}{k_n} l_n = \frac{1}{k_n \cdots k_0} l_0 \ \text{[redacted]}$$

$$0 = k_n p_{n+1}(0) - h_n l_n$$

$$\therefore \quad p_{n+1}(0) = \frac{h_n l_n}{k_n} = h_n l_{n+1}$$

so that

$$\frac{p_{n+1}(0)}{l_{n+1}} = h_n$$

$$(p_n, z^n p_n^*) = p_n(0)(1, z^n p_n^*) = \frac{p_n(0)}{l_n}(z^n p_n^*, z^n p_n^*) = \frac{p_n(0)}{l_n}$$

$$= h_{n-1} \qquad \text{for } n \geq 1.$$

Suppose now that $d\nu$ is even: $d\nu(-\theta) = d\nu(\theta)$. Then $(z^i, z^j) \in \mathbb{R}$, so that the p_n as obtained by Gram-Schmidt are real polys, hence the h_n are real. Let

$$q_0, q_1, q_2, \ldots$$
$$r_1, r_2, \ldots$$

be the sequence of even (resp. odd) Laurent polys obtained by orthonormalizing the sequences

$$\text{[redacted]} \quad 1, \ z+z^{-1}, \ z^2+z^{-2}, \ldots$$
$$z-z^{-1}, \ z^2-z^{-2}, \ldots$$

In general if $F_n \mathbb{C}[z,z^{-1}]$ is the space of $\sum_{|i| \leq n} a_i z^i$, then it is the space $F_n \ominus F_{n-1}$ which I want a nice basis for. This space contains $z^{-n} p_{2n}$, $z^{-n+1} p_{2n-1}$ and their stars, and the problem seems to be to choose an orthonormal basis for this space. First we should try to find the recursion formulas for the $\{q_n\}$ and $\{r_n\}$ in terms of the number $\{h_n\}$

$z^{-n}p_{2n} + z^n p_{2n}^*$ is proportional to q_n and it has leading coefficient (of z^n) [redacted]

$$l_{2n} + p_{2n}(0) = l_{2n}(1 + h_{2n-1}) \qquad > 0$$

Also

$$\|z^{-n}p_{2n} + z^n p_{2n}^*\|^2 = 1 + 1 + 2\,\mathrm{Re}\,(z^{-n}p_{2n}, z^{+n}p_{2n}^*)$$

$$= 2(1 + h_{2n-1}) \qquad n \geq 1$$

so

$$q_n = \frac{1}{\sqrt{2(1+h_{2n-1})}}\left(z^{-n}p_{2n} + z^n p_{2n}^*\right) \qquad n \geq 1$$

has leading [redacted] term $\dfrac{l_{2n}}{\sqrt{2}}\sqrt{1+h_{2n-1}} \cdot z^n$

$z^{-n}p_{2n} - z^n p_{2n}^*$ is proportional to r_n and has the leading term [redacted]

$$\left(l_{2n} - p_{2n}(0)\right)z^n = l_{2n}(1 - h_{2n-1})\,z^n \qquad \text{coeff} > 0$$

[redacted] $$\|z^{-n}p_{2n} + z^n p_{2n}^*\| = 2(1 - h_{2n-1})$$

so

$$r_n = \frac{1}{\sqrt{2(1-h_{2n-1})}}\left(z^{-n}p_{2n} - z^n p_{2n}^*\right)$$

has the leading term

$$\frac{l_{2n}}{\sqrt{2}}\sqrt{1 - h_{2n-1}} \cdot z^n$$

Also

$$\|z^{-n+1}p_{2n-1} + z^{n-1}p^*_{2n-1}\|^2 = 1+1 + 2\mathrm{Re}\left(z^{-n+1}p_{2n-1}, z^{n-1}p^*_{2n-1}\right)$$

$$= 2 + 2\mathrm{Re}\left(z\,p_{2n-1}, z^{2n-1}p^*_{2n-1}\right)$$

$$= 2 + 2\mathrm{Re}\left(k_{2n-1}p_{2n} - h_{2n-1}z^{2n-1}p^*_{2n-1}, z^{2n-1}p^*_{2n-1}\right)$$

$$= 2(1 - h_{2n-1}) \qquad \text{(holds even if } h_i \text{ not real)}$$

$z^{-n+1}p_{2n-1} + z^{n-1}p^*_{2n-1}$ is proportional to g_n and has the leading term l_{2n-1}, hence

$$g_n = \frac{1}{\sqrt{2(1-h_{2n-1})}}\left(z^{-n+1}p_{2n-1} + z^{n-1}p^*_{2n-1}\right)$$

has leading term $\dfrac{l_{2n-1}}{\sqrt{2(1-h_{2n-1})}}\, z^n$

December 5, 1977:

Consider the Dirac system

$$\begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix} \frac{du}{dx} + \begin{pmatrix} 0 & i\bar{p} \\ -ip & 0 \end{pmatrix} u = \lambda u$$

on $0 \leq x < \infty$ with $\phi(0,\lambda) = \begin{pmatrix} \bar{\gamma} \\ \gamma \end{pmatrix}$ as usual, so that $\phi_2(x,\lambda)$ is a dB function for all $x > 0$. Question: Can we make sense out of this system when p is a δ-function?

Write the equation $A \frac{du}{dx} + Bu = \lambda u$ where $B = \beta \cdot \delta(x - x_0)$ where β is hermitian say to fix the ideas:

$$\beta = \begin{pmatrix} 0 & i\bar{b} \\ -ib & 0 \end{pmatrix} \qquad b \in \mathbb{C}.$$

Then [redacted] any solution should be of the form $\begin{pmatrix} e^{i\lambda x} c_1 \\ e^{-i\lambda x} c_2 \end{pmatrix}$ for $x < x_0$ and of a similar form for $x > x_0$. To see what goes on around [redacted] x_0 we integrate from x_0^- to x_0^+:

$$A\left(u(x_0^+) - u(x_0^-)\right) + \beta \int_{x_0^-}^{x_0^+} \delta(x - x_0)\, u(x)\, dx = \lambda \int_{x_0^-}^{x_0^+} u(x)\, dx$$

The last term should be zero, but there is some ambiguity about the first integral since u is not continuous at x_0. The obvious choice is the average:

$$A\left(u(x_0^+) - u(x_0^-)\right) + \beta\left(\frac{u(x_0^+) + u(x_0^-)}{2}\right) = 0$$

or

$$\left(A + \frac{\beta}{2}\right) u(x_0^+) = \left(A - \frac{\beta}{2}\right) u(x_0^-).$$

or

$$\left(I + \frac{A^{-1}\beta}{2}\right) u(x_0^+) = \left(I - \frac{A^{-1}\beta}{2}\right) u(x_0^-)$$

Now

$$-A^{-1}\beta = \begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix}\begin{pmatrix} 0 & i\bar{b} \\ -ib & 0 \end{pmatrix} = \begin{pmatrix} 0 & \bar{b} \\ b & 0 \end{pmatrix}$$

so

$$\left(I - \frac{A^{-1}\beta}{2}\right) = \begin{pmatrix} 1 & \frac{\bar{b}}{2} \\ \frac{b}{2} & 1 \end{pmatrix}. \qquad \left(I + \frac{A^{-1}\beta}{2}\right) = \begin{pmatrix} 1 & -\frac{\bar{b}}{2} \\ -\frac{b}{2} & 1 \end{pmatrix}$$

Except for the fact that $\frac{\bar{b}}{2}$ is not required to be of modulus <1 it is clear that

$$\left(I + \frac{A^{-1}\beta}{2}\right)^{-1}\left(I - \frac{A^{-1}\beta}{2}\right)$$

will give an element of $SU(1,1)$. Note that provided $\left|\frac{b}{2}\right| \neq 1$ this is well defined and it has determinant

$$\left(1 - \frac{|b|^2}{4}\right)^{-1}\left(1 - \frac{|b|^2}{4}\right) = 1$$

For $\left|\frac{\bar{b}}{2}\right| < 1$ it gives an element of $SU(1,1)$, hence probably also for $\left|\frac{\bar{b}}{2}\right| > 1$. Maybe in general?

$$\begin{pmatrix} 1 & -\frac{\bar{b}}{2} \\ -\frac{b}{2} & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & \frac{\bar{b}}{2} \\ \frac{b}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\bar{b}}{2} \\ \frac{b}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{\bar{b}}{2} \\ \frac{b}{2} & 1 \end{pmatrix}\frac{1}{1 - \frac{|b|^2}{4}}$$

$$= \begin{pmatrix} 1 + \frac{|b|^2}{4} & \bar{b} \\ b & 1 + \frac{|b|^2}{4} \end{pmatrix}\frac{1}{1 - \frac{|b|^2}{4}}$$

This definitely doesn't work for $|b| = 2$. This suggests that the averaging interpretation is probably not the good one.

I need to understand how to interpret

$$A\frac{du}{dx} + \beta\delta(x-x_0)u = \lambda u$$

simpler case

$$\frac{du}{dx} = \delta(x-x_0)u$$

or

$$du = (\delta(x-x_0)dx)u$$

$$\frac{du}{u} = \delta(x-x_0)dx$$

so u is right-continuous

$$\ln u = \begin{cases} 1 & x \geq x_0 \\ 0 & x < x_0 \end{cases} + c$$

or

$$u = \begin{cases} c & x < x_0 \\ c\text{[redacted]} & x \geq x_0 \end{cases}$$

Another interpretation: Choose a parameter t such that $\delta(x-x_0)dx$ is absolutely continuous with respect to dt. First, choose t such that $dt = \delta(x-x_0)dx$

i.e. $\begin{cases} t = 1 & \text{for } x \geq 0 \\ t = 0 & \text{for } x < 0 \end{cases}$

Then $\frac{du}{dt} = u$ so $u = ce^t$ as above.

2nd choice: Put $t = x$ for $x < x_0$ and $t = x+1$ for $x \geq x_0$. Then

$$dt = dx + \delta(x-x_0)dx$$

hence $\delta(x-x_0)dx = dt - dx = f\,dt$ $\quad f(t) = \begin{cases} 0 & \text{outside} \\ 1 & x_0 \leq t \leq x_0+1 \end{cases}$

Hence

$$\frac{du}{dt} = fu$$

so
$$\log(u) = \int_0^t f\,dt = \begin{cases} 0 & t \leq x_0 \\ t - x_0 & x_0 \leq t \leq x_0+1 \\ 1 & t \geq x_0+1 \end{cases} + C$$

Both of these choices agree with the idea of exponentiating the integral of $\delta(x-x_0)\,dx$.

So now return to
$$A\frac{du}{dx} + Bu = \lambda u$$

where $B = \beta\,\delta(x-x_0)$. The good interpretation of what happens as we pass through x_0 is now fairly clear. Let $dt = dx + \delta(x-x_0)\,dx$ and write the equation

$$A\frac{du}{dt} + \beta\left(\delta(x-x_0)\frac{dx}{dt}\right)u = \lambda\frac{dx}{dt}u$$

$$A\frac{du}{dt} + \beta f u = \lambda(1-f)u.$$

So we have $A\dfrac{du}{dt} = \lambda u$ on $t \leq x_0$. Then we have
$$A\frac{du}{dt} + \beta u = 0 \qquad \text{on} \quad x_0 \leq t \leq x_0+1$$
and back to $A\dfrac{du}{dx} = \lambda u$ on $x_0+1 \leq t$.

In the intermediate range $A^{-1}\beta$ is constant, hence we have
$$u(x_0^+) = e^{-A^{-1}\beta}\,u(x_0^-)$$
$$= e^{\begin{pmatrix} 0 & \bar{b} \\ b & 0 \end{pmatrix}}\,u(x_0^-)$$

and indeed
$$e^{\begin{pmatrix} 0 & \bar{b} \\ b & 0 \end{pmatrix}} = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\frac{1}{\sqrt{1-|h|^2}}$$

for some $|h| < 1$.

Recall that a map $g: S^1 \longrightarrow \mathbb{C}^*$ which is holomorphic determines a holomorphic line bundle over $\mathbb{P}_1$, [redacted] denote it $\mathcal{L}_g$, whose sections are pairs (f_0, f_∞) where f_0 is holom. in $|z| \leq 1$ and f_∞ is holom. in $|z| \geq 1$ such that

$$g f_\infty = f_0$$

For example if $g = z^n$, then we get the sections (z^i, z^{n-i}). Put $D_0 = \text{span } 1, z, \dots$ in $L^2(S^1)$, $D_\infty = \text{span } 1, z^{-1}, \dots\dots$ Then $\Gamma(\mathcal{L}_g)$ can be identified with $gD_\infty \cap D_0$, i.e. holom. fns. to on $|z| \leq 1$ when divided by g become holom. in $|z| \geq 1$. [redacted]

Suppose that g is in the form

$$g = \frac{z^n \delta^*(z)}{\delta(z)}$$

where δ is a poly of degree $\leq n$ having all its roots in $|z| > 1$. Now

$$gD_\infty \cap D_0 = \frac{z^n \delta^*(z)}{\delta(z)} D_\infty \cap D_0 \xleftarrow{\sim} z^n \delta^* D_\infty \cap \delta D_0$$

$$f_0 \longmapsto \delta f_0$$

and $\delta D_0 = D_0$, $\delta^* D_\infty = D_\infty$, hence $g D_\infty \cap D_0$ is $(n+1)$-dimensional with the basis $\frac{z^i}{\delta}$, $0 \leq i \leq n$.

Moreover if $g: S^1 \longrightarrow S^1$ we can introduce an inner product on $\Gamma(\mathcal{L}_g)$ by putting

$$\|(f_0, f_\infty)\|^2 = \int |f_0|^2 \frac{d\theta}{2\pi} = \int |f_\infty|^2 \frac{d\theta}{2\pi}$$

because $|g| = 1$. In the example $g = \frac{z^n \delta^*}{\delta}$ this amounts to using the inner product on polys of degree $\le n$

$$(z^i, z^j) = \int z^{i-j} \frac{d\theta}{2\pi|\delta|^2}.$$

So it is more or less clear that I have found some sort of discrete analogue of the de Branges spaces. For E take $z^{-n/2}\delta(z)$ for $n \ge \deg(\delta)$. Then $B(E)$ should consist of (say n even) all f holomorphic for $0<|z|<\infty$ such that

$$\frac{f(z)}{z^{-n/2}\delta(z)} \in D_0$$

and
$$\frac{f(z)}{z^{n/2}\delta^*(z)} \in D_\infty$$

with the norm $\displaystyle\int \left|\frac{f(z)}{z^{-n/2}\delta(z)}\right|^2 \frac{d\theta}{2\pi} = \int \frac{|f|^2 d\theta}{2\pi|\delta|^2}$

The first two conditions imply that f is a Laurent poly of degree $\le n/2$.

Lets shift so that B consists of f holomorphic for $0<|z|<\infty$ with

$$\frac{f}{\delta} \in D_0 \qquad \frac{f}{z^n \delta^*(z)} \in D_\infty$$

with the same norm. Then f has to be holomorphic at 0 and $\frac{f}{z^n}$ has to be holom. at ∞, hence f is a poly in z of degree $\le n$. Let's work out the point evaluator in B using Cauchy's formula:

$$\frac{1}{2\pi i}\int \frac{f(z)}{\delta(z)}\frac{dz}{z-a} = \begin{cases} \dfrac{f(a)}{\delta(a)} & |a|<1 \\ 0 & |a|>1 \end{cases}$$

$$(*)\qquad \frac{1}{2\pi i}\int \frac{f(z)}{z^n\delta^*(z)}\,\frac{dz}{z-a} = \begin{cases} -\dfrac{f(a)}{a^n\delta^*(a)} & |a|>1 \\ 0 & |a|<0 \end{cases}$$

So

$$\int_{S^1} f(z)\left\{\frac{\delta(a)}{\delta(z)} - \frac{a^n\delta^*(a)}{z^n\delta^*(z)}\right\}\frac{z}{z-a}\,\frac{dz}{2\pi i z} = f(a)$$

$$\|$$

$$\int_{S^1} f(z)\left\{\delta(a)\delta^*(z) - z^{-n}a^n\delta^*(a)\delta(z)\right\}\frac{1}{1-az^{-1}}\,d\theta$$

$$\overline{\frac{\overline{\delta(a)}\,\delta(z) - z^n\bar{a}^n\overline{\delta^*(a)}\,\delta^*(z)}{1-\bar{a}z}}$$

So

$$J_a(z) = \frac{\overline{\delta(a)}\,\delta(z) - z^n\delta^*(z)\cdot\overline{a^n\delta^*(a)}}{1-\bar{a}z} \qquad \text{not quite.}$$

The only problem is that this is a poly of degree $n-1$. The problem is that $(*)$ isn't valid unless $\deg f < n$ because $\frac{dz}{z-a}$ is singular at ∞

$$\frac{d\left(\frac{1}{w}\right)}{\frac{1}{w}-a} = \frac{-\frac{1}{w^2}\,dw}{\frac{1}{w}-a} = -\frac{dw}{w(1-aw)}$$

so you need to replace n by $n+1$ in $*$. Thus the good formula is

$$J_a(z) = \frac{\overline{\delta(a)}\,\delta(z) - z^{n+1}\delta^*(z)\,\overline{a^{n+1}\delta^*(a)}}{1-\bar{a}z} = \sum_{i=0}^{n} \overline{p_i(a)}\,p_i(z)$$

Let's consider a Dirac system

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$$

where p is a δ-function at $x = 1, 2, \ldots$. If $\phi(x,\lambda)$ is the solution starting with $\phi(0,\lambda) = \binom{1}{1}$, then we have

$$\phi(x,\lambda) = \begin{pmatrix} e^{i\lambda x} \\ e^{-i\lambda x} \end{pmatrix} \qquad 0 \leq x < 1$$

$$\phi(1+,\lambda) = R(h_1)\phi(1-,\lambda) \qquad R(h_1) = \exp\begin{pmatrix} 0 & \bar{b}_1 \\ b_1 & 0 \end{pmatrix}$$

etc. so that $\qquad b_1 = \int_{1-}^{1+} p\,dx$

$$\phi(n+,\lambda) = R(h_n)\begin{pmatrix} e^{i\lambda} & \\ & e^{-i\lambda} \end{pmatrix} \cdots\cdots R(h_1)\begin{pmatrix} e^{i\lambda} & \\ & e^{-i\lambda} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If we put $z = e^{2i\lambda}$, then this system parallels an orthogonal system of polys. on S^1. I'd like to see if there is a sensible way of incorporating $z^{1/2}$ and more generally z^r, $r \in \mathbb{Q}$ into the circle setup.

First consider the case where n is even and the circular de B pair is $\begin{pmatrix} z^m \delta^* \\ z^{-m}\delta \end{pmatrix}$, $m = \frac{n}{2}$
δ has roots outside S^1 and degree $\delta \leq 2m$. Thus

$$\begin{pmatrix} z^{n/2}\delta^* \\ z^{-n/2}\delta \end{pmatrix} = R(h_n)\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \cdots\cdots R(h_1)\begin{pmatrix} z^{1/2} & \\ & z^{-1/2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that $\phi_2(n+,\lambda) = e^{-in\lambda}\delta(e^{2i\lambda})$.

I propose to find the point evaluator for the space of Laurent polys. of degree $\leq m$ with the norm $\int \left|\frac{f}{z^{-m}\delta}\right|^2 \frac{d\theta}{2\pi}$. Start with Cauchy:

$$\frac{1}{2\pi i}\int \frac{f(z)}{z^{-m}\delta(z)}\frac{dz}{z-a} = \begin{cases} \dfrac{f(a)}{a^{-m}\delta(a)} & |a|<1 \\ 0 & |a|>1 \end{cases}$$

$$\frac{1}{2\pi i}\int \frac{f(z)}{z^{m+1}\delta^*(z)}\frac{dz}{z-a} = \begin{cases} -\dfrac{f(a)}{a^{m+1}\delta^*(a)} & |a|>1 \\ 0 & |a|<1 \end{cases}$$

$$f(a) = \int f(z)\left\{\frac{a^{-m}\delta(a)}{z^{-m}}\delta^*(z) - \frac{a^{m+1}\delta^*(a)}{z^{m+1}}\delta(z)\right\}\frac{z}{z-a}\,\frac{1\text{[redacted]}}{|\delta|^2}\,\frac{dz}{2\pi i z}$$

$$= \int f(z)\left\{a^{-m-\frac{1}{2}}\delta(a)\cdot \cancel{z}^{m+\frac{1}{2}}\delta^* - a^{m+\frac{1}{2}}\delta^*(a)\cdot z^{-m-\frac{1}{2}}\delta(z)\right\}\frac{z^{1/2}a^{1/2}}{z-a}\,\frac{d\theta}{2\pi|\delta|^2}$$

$$= \int f(z)\left\{\overline{a^{-m-\frac{1}{2}}\delta(a)}\ \text{[redacted]}\ z^{-m-\frac{1}{2}}\delta - \overline{a^{m+\frac{1}{2}}\delta^*(a)}\ z^{m+\frac{1}{2}}\delta^*\right\}\overline{\left(\frac{1}{z^{-1/2}\overline{a^{-1/2}} - z^{1/2}\overline{a^{1/2}}}\right)}$$

hence

$$J_a(z) = \frac{\begin{vmatrix} z^{m+1/2}\delta^*(z) & \overline{a^{-m-\frac{1}{2}}\delta(a)} \\ z^{-m-\frac{1}{2}}\delta(z) & \overline{a^{m+\frac{1}{2}}\delta^*(a)} \end{vmatrix}}{\begin{vmatrix} z^{1/2} & \overline{a^{-1/2}} \\ z^{-1/2} & \overline{a^{1/2}} \end{vmatrix}}$$

Now if I put $E^{\#}(\lambda) = z^{m+\frac{1}{2}}\delta^*(z) = z^{\frac{n+1}{2}}\delta^*(z) = e^{i(n+1)\lambda}\delta^*(e^{2i\lambda})$

$$E(\lambda) = e^{-i(n+1)\lambda}\delta^{\text{[redacted]}}(e^{2i\lambda}).$$

and $e^{2i\alpha} = a$. Then

$$\begin{vmatrix} z^{1/2} & \overline{a^{-1/2}} \\ z^{-1/2} & \overline{a^{1/2}} \end{vmatrix} = e^{i\lambda}e^{-i\bar{\alpha}} - e^{-i\lambda}e^{i\bar{\alpha}} = 2i\sin(\lambda-\bar{\alpha}).$$

So

$$J_a(z) = \frac{\begin{vmatrix} E^\#(\lambda) & \overline{E^\#(\alpha)} \\ E(\lambda) & \overline{E^\#(\alpha)} \end{vmatrix}}{2i\sin(\lambda-\bar{z})} = \frac{i}{2\sin(\lambda-\bar{z})} \begin{vmatrix} E(\lambda) & E(\bar{\alpha}) \\ E^\#(\lambda) & E^\#(\bar{\alpha}) \end{vmatrix}$$

Somehow I have roughly the same Hilbert space except I am maybe replacing L^2 norms with the mean

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T}$$

on the real axis. No. The above formula makes sense only if $\frac{E^\#(\lambda)}{E(\lambda)} = \frac{E^\#(\lambda')}{E(\lambda')}$ when $\lambda - \lambda' \in \pi\mathbb{Z}$

Let $d\nu$ be a prob. measure on S^1 and $p_0 = 1, p_1, \ldots$ the associated sequence of orthogonal polynomials. Let h_n $n \geq 1$ be defined by

$$z p_{n-1} = k_n p_n - h_n z^{n-1} p_{n-1}^* \qquad k_n = \sqrt{1-|h_n|^2}$$

and let l_n be the leading coefficient of p_n, so that

$$l_{n-1} = k_n l_n \qquad l_n = \frac{1}{k_n \cdots k_1}$$

Recall

$$(p_n, z^n p_n^*) = p_n(0)(1, z^n p_n^*) = \frac{p_n(0)}{l_n}(z^n p_n^*, z^n p_n^*)$$

$$= \frac{p_n(0)}{l_n} = \frac{k_n p_n(0)}{k_n l_n} = \frac{h_n l_{n-1}}{k_n l_n} = h_n$$

Starting with $d\nu$ I want to construct a nice orthonormal basis in $L^2(S^1, d\nu)$ which is adapted to the filtration $F_n \mathbb{C}[z, z^{-1}] = \langle z^{-n}, \ldots, z^n \rangle$. Let $W_n = F_n \mathbb{C}[z, z^{-1}] \ominus F_{n-1}\mathbb{C}[z, z^{-1}]$, so that W_n is 2 dimensional for $n \geq 1$, one-dim'l for $n = 0$. Suppose $n \geq 1$. Note that W_n is closed under the operation of conjugation $*$ which preserves norm. Hence W_n is the complexification of a 2 dim'l Euclidean space, so it has a real orth. basis e_1, e_2. Let $ae_1 + be_2$ be an element of W_n. It is orthogonal to its conjugate

$$(ae_1 + be_2, \bar{a}e_1 + \bar{b}e_2) = a^2 + b^2 = 0$$

iff $b = \pm ia$. Hence the vectors orthogonal to their conjugates form the union of the two lines spanned by $e_1 + ie_2, e_1 - ie_2$.

so we can find an orthonormal basis for W consisting of (ϕ, ϕ^*) which is unique up to multiplying ϕ by a scalar of modulus 1 and interchanging ϕ and ϕ^*. In fact we take:

$$\phi = \frac{1}{\sqrt{2}}(e_1 + i e_2)$$

W_n has the basis $\bar{z}^{-n} p_{2n}, \; z^n p_{2n}^*$ so we try to find an element orthogonal to its conjugate in the form $\bar{z}^{-n} p_{2n} + t z^{+n} p_{2n}^*$:

$$\left(\bar{z}^{-n} p_{2n} + t z^{+n} p_{2n}^*, \; z^{+n} p_{2n}^* + \bar{t}\, \bar{z}^{-n} p_{2n}\right)$$

$$= \left(\bar{z}^{-n} p_{2n}, z^{+n} p_{2n}^*\right) + t \|z^{+n} p_{2n}^*\|^2 + \|\bar{z}^{-n} p_{2n}\|^2 t + t^2 \left(z^{+n} p_{2n}^*, \bar{z}^{-n} p_{2n}\right)$$

$$= h_{2n} + t + t + t^2 \bar{h}_{2n} = 0$$

$$\therefore \; t = \frac{-1 \pm \sqrt{1 - |h_{2n}|^2}}{\bar{h}_{2n}} = \frac{-1 \pm k_{2n}}{\bar{h}_{2n}}$$

$$\|\bar{z}^{-n} p_{2n} + t z^{+n} p_{2n}^*\|^2 = \left(\bar{z}^{-n} p_{2n} + t z^{n} p_{2n}^*, \; \bar{z}^{-n} p_{2n} + t z^{n} p_{2n}^*\right)$$

$$= 1 + t \bar{h}_{2n} + \bar{t} h_{2n} + |t|^2 \qquad t = \frac{-1 - k_{2n}}{\bar{h}_{2n}}$$

$$= 1 - 2(1 + k_{2n}) + \frac{1 + 2k_{2n} + k_{2n}^2}{|h_{2n}|^2}$$

December 10, 1977.

On $SU(1,1)$. This is the subgroup of $SL_2(\mathbb{C})$ consisting of matrices of the form $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, i.e. fixed under flipping conjugation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

followed by complex conjugation. The Lie algebra consists of all $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ with $a + \bar{a} = 0$, i.e all

$$\begin{pmatrix} i\alpha & \beta \\ \bar{\beta} & -i\alpha \end{pmatrix}$$

with $\alpha \in \mathbb{R}$. Thus good maximal compact T in $G = SU(1,1)$ is the diagonal matrices. Let $G = SU(1,1)$ act on $|z| < 1$ in the obvious way, then T fixes 0 so we should have an isomorphism

$$\begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} \longmapsto \exp\begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix}$$

between $\mathbb{C}$ and $|z| < 1$.

We have

$$\exp\begin{pmatrix} 0 & it \\ -it & 0 \end{pmatrix} = \begin{pmatrix} \cos(it) & \sin(it) \\ -\sin(it) & \cos(it) \end{pmatrix} = \begin{pmatrix} \cosh t & i\sinh t \\ -i\sinh t & \cosh t \end{pmatrix}$$

$$\| $$

$$\exp\left\{ \begin{pmatrix} +i & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

hence

$$\exp\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

In general
$$\begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & |b| \\ |b| & 0 \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix}$$

where $\theta = \arg(b)$, so that
$$\exp\begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} = \begin{pmatrix} \cosh|b| & e^{i\theta}\sinh|b| \\ e^{-i\theta}\sinh|b| & \cosh|b| \end{pmatrix}$$

December 11, 1977.

Consider a 2-dim'l complex Hilbert space W with involution $*$ and suppose given $u \in W \ni u, u^*$ are independent and $\|u\| = 1$. Then I have seen that the set of vectors $au + bu^*$ in W orthogonal to their stars is the union of two lines. Hence I can find an orthonormal basis of the form (ϕ, ϕ^*) which is unique up to multiplying ϕ by an elt of S^1 and also interchanging ϕ, ϕ^*.
$$\phi = au + bu^*$$
$$\phi^* = \bar{b}u + \bar{a}u^*$$

It should be the case that $\det\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} = |a|^2 - |b|^2 \neq 0$ so that we can normalize our choices by requiring $|a| > |b|$. Work out formulas: Suppose $(u, u^*) = h$
$$(u + tu^*, u^* + \bar{t}u) = h + t + t + t^2\bar{h}$$
$$\Rightarrow \quad t = \frac{-1 \pm \sqrt{1 - |h|^2}}{\bar{h}}$$

Put $k = \sqrt{1 - |h|^2}$. Notice that $|h| = |(u, u^*)| \leq \|u\|\|u\|^* = 1$ with equality iff u, u^* are proportional, which is impossible so $|h| < 1$. We want $|t| < 1$ so we take

$$t = -\left(\frac{1-k}{\bar{h}}\right) = \frac{-h}{1+k}$$

Hence if $\psi = (1+k)u - hu^*$, then $(\psi, \psi^*) = 0$.

$$\|\psi\|^2 = (1+k)^2 - 2(1+k)h\bar{h} + h\bar{h}$$
$$= (1+k)\{1+k - 2(1-k^2) + 1-k\} = 2k^2(1+k)$$

Thus

$$\begin{pmatrix}\phi \\ \phi^*\end{pmatrix} = \underbrace{\frac{1}{\sqrt{2k^2(1+k)}}\begin{pmatrix}1+k & -h \\ -\bar{h} & 1+k\end{pmatrix}}_{\text{Denote this } T(h)}\begin{pmatrix}u \\ u^*\end{pmatrix}$$

$\det(T(h)) = \frac{1}{k}$

So now given $d\nu$ a prob. measure on S^1 let us define ϕ_n by

$$\begin{pmatrix}z^{-n/2}\phi_n \\ z^{n/2}\phi_n^*\end{pmatrix} = T(h_n)\begin{pmatrix}z^{-n/2}p_n \\ z^{n/2}p_n^*\end{pmatrix}$$

where $h_n = (z^{-n/2}p_n, z^{n/2}p_n^*) = (p_n, z^n p_n^*)$. Then [redacted] ϕ_n is a poly of degree n in z since

$$\begin{pmatrix}\phi_n \\ z^n\phi_n^*\end{pmatrix} = T(h_n)\begin{pmatrix}p_n \\ z^n p_n^*\end{pmatrix}.$$

We have the recursion formula

$$\begin{pmatrix}\phi_n\\ z^n\phi_n^*\end{pmatrix} = T(h_n)\begin{pmatrix}p_n\\ z^n p_n^*\end{pmatrix} = T(h_n)R(h_n)\begin{pmatrix}z&0\\0&1\end{pmatrix}\begin{pmatrix}p_{n-1}\\ z^{n-1}p_{n-1}^*\end{pmatrix}$$

$$= T(h_n)R(h_n)\begin{pmatrix}z&0\\0&1\end{pmatrix}T(h_{n-1})^{-1}\begin{pmatrix}\phi_{n-1}\\ z^{n-1}\phi_{n-1}^*\end{pmatrix}$$

Observation: Except for a scalar factor $T(h)$ is the negative square root of $R(h)$:

$$T(h)^2 = \frac{1}{2k^2(1+k)}\begin{pmatrix}\overbrace{(1+k)^2+|h|^2}^{2+2k} & -2(1+k)h\\ -2\bar h(1+k) & |h|^2+(1+k)^2\end{pmatrix} = \frac{1}{k^2}\begin{pmatrix}1&-h\\-\bar h&1\end{pmatrix}$$

$$= \frac{1}{k}R(h)^{-1} \qquad\qquad T(h)^2R(h) = \frac{1}{k}$$

Hence

$$\begin{pmatrix}z^{-n/2}\phi_n\\ z^{n/2}\phi_n^*\end{pmatrix} = \frac{1}{k_n}T(h_n)^{-1}\begin{pmatrix}z^{1/2}&0\\0&z^{-1/2}\end{pmatrix}T(h_{n-1})^{-1}\begin{pmatrix}z^{\frac{-n+1}{2}}\phi_n\\ z^{n-1/2}\phi_n^*\end{pmatrix}$$

which is not such a bad transition formula.

Question: Suppose given $h_1, \ldots, h_n$ whence you get a matrix

$$F(z) = R(h_n)\begin{pmatrix}z^{1/2}&0\\0&z^{-1/2}\end{pmatrix}\cdots\cdots R(h_1)\begin{pmatrix}z^{1/2}&0\\0&z^{-1/2}\end{pmatrix}$$

which for $|z|=1$ maps S^1 to S^1. Now fix boundary conditions at the ends, say $u_1 = u_2$ at 0 and $\frac{u_1}{u_2} = e^{i\theta}$. Can those z which are compatible with these boundary values:

$$e^{i\theta} = F(z)\cdot 1$$

be interpreted as eigenvalues of a unitary operator?

Here's how this is done for the Dirac D.E.

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$$

on $0 \leq x \leq \ell$ with boundary conditions $u_1 = u_2$ at $x=0$, $u_1 = e^{i\theta} u_2$ at $x = \ell$. The point is that we get a self-adjoint operator in the Hilbert space $L^2([0,\ell])^2$ defined by the differential operator

$$\begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & i\bar{p} \\ -ip & 0 \end{pmatrix}$$

together with the boundary conditions. Hence I get a 1-parameter unitary group in this Hilbert space.

Idea is to get a J-matrix picture for a unitary operator U plus cyclic unit vectors (= probability measure on S^1) in the finite-dimensional case.

Suppose $\mathcal{H}$ is an n-dimensional Hilbert space with a unitary operator U and cyclic vector e (say $\|e\| = 1$). By Gram-Schmidt we can construct an orthonormal basis $e_0, e_1, \ldots, e_{n-1}$ for $\mathcal{H}$ from $e, Ue, \ldots, U^{n-1}e$. I might as well suppose $\mathcal{H} = L^2(S^1, d\nu)$ where $d\nu$ has support of card n. Clearly $p_i(U) = e_i$ for $i = 0, \ldots, n-1$. Moreover we get $h_1, \ldots, h_{n-1}$ such that

$$p_i = k_i z p_{i-1} + h_i z^{i-1} p_{i-1}^* \qquad i = 1, \ldots, n-1$$

~~Now p_{n-1} is equivalent in $L^2(d\nu)$ to a poly of degree $\ldots$ which is clearly orthogonal to $z^{-1}, 1, \ldots, z^{n-3}$~~

Let f be the unique monic poly of degree n such that $f(U)=0$, i.e. $\det(z-U)$. Then

$$z\,p_{n-1}(z) = l_{n-1} f(z) + r(z)$$

when $l_{n-1}=$ leading coeff of p_{n-1} and $\deg r < n$. r is orthogonal to $z, \dots, z^{n-1}$, hence r must be a multiple of $z^{n-1}p^*_{n-1}$. Put $r = -h_n z^{n-1} p^*_{n-1}$. Then

$$z\,p_{n-1}(z) = -h_n z^{n-1} p^*_{n-1} \quad \text{in } L^2(S^1, d\nu)$$

so taking norms: $|h_n| = 1$. Also

$$z p_{n-1}(z) + h_n z^{n-1} p^*_{n-1} = l_{n-1} f(z)$$

is a formula for the relation. Hence from $(\mathcal{H}, U, e)$ ords we have managed to construct $h_1, h_2, \dots, h_{n-1}$ of modulus <1 and h_n of modulus 1. So we get

$$\begin{pmatrix} p_i \\ z^i p_i^* \end{pmatrix} = R(h_i)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{i-1} \\ z^{i-1}p^*_{i-1} \end{pmatrix} \qquad i = 1, \dots, n-1$$

and finally

$$\boxed{l_{n-1} f(z) = \begin{pmatrix} 1 & h_n \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(h_{n-1})\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots R(h_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

In other words we have shown that the Lee-Yang polynomial belonging to a linear graph is essentially the characteristic poly. of a unitary operator.