November 11, 1977:

Let U be a unitary operator on a Hilbert space $\mathcal{H}$, let D^+, D^- denote closed subspaces with the following properties: UD^+ is of codimension 1 in D^+ and $\bigcap_{n\geq 0} U^n D^+ = 0$; similarly $U^{-1}D^-$ is of cod. 1 in D^- and $\bigcap_{n\geq 0} U^{-n}D^- = 0$. Finally $D^+ \perp D^-$ and $D^+ \oplus D^-$ is of finite codimension in $\mathcal{H}$.

Let e_0 be a unit vector in D^- perpendicular to $U^{-1}D^-$, and put $e_{-n} = U^{-n}e_0$ for $n \geq 0$. It is clear that $e_0, e_{-1}, e_{-2}, \ldots$ is an orthonormal basis for D^-. Let $d = \dim(\mathcal{H}/D^+ \oplus D^-)$ and choose $e_1, \ldots, e_d$ to be an orthonormal basis for the orthogonal complement of $D^+ \oplus D^-$. Finally choose e_{d+1} to be a unit vector in D^+ perpendicular to UD^+, put $e_n = U^{n-d-1}e_{d+1}$ for $n \geq d+1$, so that D^+ has the orthonormal basis $e_{d+1}, e_{d+2}, \ldots$. Let U_0 be the shift operator: $U_0(e_n) = e_{n+1}$ for all n, and let $U_0^{-1}U = \Theta$. Since $Ue_n = e_{n+1}$ for $n \geq d+1$ and $n \leq -1$, it follows that Θ fixes e_n for n outside $[0, d]$, hence Θ is essentially a unitary operator on the space spanned by $e_0, e_1, \ldots, e_d$.

Let $W = \mathcal{H} \ominus (D^+ \oplus D^-)$. This is spanned by $e_1, \ldots, e_d$ which are not canonical, although the choice of e_0, e_{d+1} is unique up to scalars. I can identify W with the quotient $(D^-)^\perp / D^+$ which carries an induced operator from U, or with $(D^+)^\perp / D^-$ which carries an operator induced from U^{-1}. The induced operators are contraction operators. In fact let $i : W \to \mathcal{H}$ be the inclusion. Then i^*Ui and $i^*U^{-1}i$ are the induced operators. They are clearly adjoint and of norm < 1

if there are no bound states.

Assume from now on that there are no bound states. It should be the case that $(D^-)^\perp/D^+$ is cyclic wrt $\mathcal{U}$. Let p be the characteristic poly of $\mathcal{U}$ on $(D^-)^\perp/D^+$, so that for $w \in (D^-)^\perp$ we have $p(\mathcal{U})w \in D^+$.

Consider $p(\mathcal{U})e_0$ and its trajectory under $\mathcal{U}$: For $n \leq d$, $\mathcal{U}^n p(\mathcal{U})e_0 \in D^-$ and for $n \geq 1$

$$\mathcal{U}^n p(\mathcal{U})e_0 = p(\mathcal{U})\mathcal{U}^n e_0 \in D^+$$

hence the trajectory of $p(\mathcal{U})e_0$ starts in D^- and ends in D^+.

Work out the scattering operator using this trajectory. We have

$$\mathcal{U}^n p(\mathcal{U})e_0 = \mathcal{U}_0^n p(\mathcal{U}_0)e_0 \qquad n \leq d$$

$$\begin{aligned} \mathcal{U}^n p(\mathcal{U})e_0 &= \mathcal{U}^{n-1} p(\mathcal{U})\mathcal{U}e_0 \\ &= \mathcal{U}_0^{n-1} p(\mathcal{U})\mathcal{U}e_0 \qquad n \geq 1 \end{aligned}$$

hence

$$S(\mathcal{U}_0)p(\mathcal{U}_0)e_0 = \mathcal{U}_0^{-1} p(\mathcal{U})\mathcal{U}e_0$$

so if $p(\mathcal{U})\mathcal{U}e_0 = \mathcal{U}_0 q(\mathcal{U}_0)e_0$, then

$$S(\mathcal{U}_0) = \frac{q(\mathcal{U}_0)}{p(\mathcal{U}_0)}$$

<u>Example</u>:

$$\begin{aligned} \Theta e_0 &= ae_0 + be_1 \\ \Theta e_1 &= -\bar{b}e_0 + \bar{a}e_1 \end{aligned} \qquad d = 1$$

$$\begin{aligned} \mathcal{U}e_0 &= ae_1 + be_2 \\ \mathcal{U}e_1 &= -\bar{b}e_1 + \bar{a}e_2 \end{aligned}$$

$p(\lambda) = \lambda + \bar{b}$. $\qquad p(\mathcal{U})e_0 = \bar{b}e_0 + ae_1 + be_2$

$$\mathcal{U}p(\mathcal{U})e_0 = \bar{b}(ae_1+be_2)+a(-\bar{b}e_1+\bar{a}e_2)+be_3$$
$$= e_2+be_3 = \mathcal{U}_0\underbrace{(\mathcal{U}_0+b\mathcal{U}_0^2)}_{g(\mathcal{U}_0)}e_0$$

and $$S(\mathcal{U}_0) = \frac{\mathcal{U}_0+b\mathcal{U}_0^2}{\mathcal{U}_0+\bar{b}} = \frac{1+b\mathcal{U}_0}{1+\bar{b}\,\mathcal{U}_0^{-1}}$$

[redacted]

Question: Is $p(\mathcal{U})e_0$ an essentially unique cyclic vector for $\mathcal{U}$ [redacted] in some sense?

November 12, 1977:

Start with a [redacted] measure $d\nu$ on S^1 and construct the orthogonal polys $\phi_0, \phi_1, \ldots$ satisfying

$$\begin{pmatrix}\phi_{n+1}\\ z^{n+1}\phi_{n+1}^*\end{pmatrix} = R(h_n)\begin{pmatrix} z & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\phi_n\\ z^n\phi_n^*\end{pmatrix}$$

Assume $z\phi_n = \phi_{n+1}$ for $n \geq d$, [redacted] i.e. $h_d = h_{d+1} = \cdots = 0$. Then $z^n\phi_n^* = \delta$ [redacted] for $n \geq d$, where δ is the unique poly with positive constant term orthogonal to z, z^2, z^3, etc. We have

$$|\delta|^2 d\nu = \frac{d\theta}{2\pi}$$

so that $d\nu = \dfrac{d\theta}{2\pi|\delta|^2}$ assuming $d\nu$ absolutely cont. w.r.t. $d\theta$.

I want to set up the associated scattering: Start with the Dirac-style system

$$\begin{pmatrix} z^{-(n+1)/2}\phi_{n+1} \\ z^{+(n+1)/2}\phi^*_{n+1}\end{pmatrix} = R(h_n)\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2}\end{pmatrix}\begin{pmatrix} z^{-n/2}\phi_n \\ z^{n/2}\phi^*_n\end{pmatrix}$$

For $n \geq d$ we get

$$\begin{pmatrix} z^{-n/2}\phi_n \\ z^{+n/2}\phi^*_n\end{pmatrix} = \begin{pmatrix} z^{+n/2}\delta^* \\ z^{-n/2}\delta\end{pmatrix}$$

and these are orthogonal at least for n large. These should be used for the basis I want for $\mathcal{H}$. Put

$$\begin{cases} e_n = z^n\delta^* = \phi_n & n > d \\ e_n = z^{+n}\delta & n \leq 0 \end{cases}$$

These are orthogonal because

$$(z^n\delta^*, z^{-m}\delta) = \int z^{n+m}(\delta^*)^2\frac{d\theta}{2\pi|\delta|^2} = \int z^{n+m}\frac{\delta^*}{\delta}\frac{d\theta}{2\pi}$$
$$= 0$$

if $n+m > d$, because δ has no zeroes inside S^1, $\frac{dz}{2\pi i z} = \frac{d\theta}{2\pi}$, and $z^d\delta^* \in \mathbb{C}[z]$.

We can calculate the scattering matrix. The incoming spectral representation

$$L^2(S^1, \tfrac{d\theta}{2\pi}) \xrightarrow{\cdot\delta} L^2(S^1, d\mu)$$

sends z^n to e_n for $n \leq 0$, hence is given by multiplying by δ. The outgoing one sends z^n to e_n for $n > d$, hence is multiplication by δ^*. Thus

$$S = \frac{\delta}{\delta^*},$$

[redacted] so S has poles inside S^1 and zeroes outside S^1, (plus possible zeroes at $z=0$).

For the example before $S = \dfrac{1+bU_0}{1+\bar{b}U_0^{-1}} = \dfrac{1+bz}{1+\bar{b}z^{-1}}$.

Note that the basic cyclic vector 1 in $L^2(S^1; d\mu)$ has the incoming representative $\frac{1}{\delta}$ and the outgoing representative $\frac{1}{\delta^*}$. So the idea of finding the good cyclic vector using a trajectory starting in D^- & ending in D^+ is no good.

Suppose we consider carefully the example where $\delta(z) = 1 + bz$ with $|b| < 1$. Then we have

$$e_0 = \delta = 1 + bz$$

$$e_2 = z^2\delta^* = z^2(1+\bar{b}z^{-1}) = \bar{b}z + z^2$$

and the only possible choice for e_1 is a multiple of z.

$$\int \frac{d\theta}{\delta^2 \, 2\pi} = \frac{1}{2\pi i}\int \frac{1}{(1+bz)(1+\bar{b}z^{-1})}\,\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int \frac{1}{(1+bz)(z+\bar{b})}\,dz = \frac{1}{1-|b|^2}$$

hence

$e_1 =$ [redacted] $\bar{a}z$ where $|a|^2 + |b|^2 = 1$.

Then

$Ue_0 =$ [redacted] $z + bz^2 = a\dfrac{e_1}{\bar{a}} + b\left(e_2 - \bar{b}\dfrac{e_1}{\bar{a}}\right)$

$$= \frac{1-b\bar{b}}{\bar{a}}e_1 + be_2 = ae_1 + be_2$$

and

$$Ue_1 = \bar{a}z^2 = \bar{a}\left(e_2 - \bar{b}\frac{e_1}{\bar{a}}\right) = -\bar{b}e_1 + \bar{a}e_2.$$

November 13, 1977

Recall: If G is a group a positive-definite function φ on G is one such that for any $g_1, \cdots, g_n \in G$ the form $\sum_{i,j} \varphi(g_j^{-1} g_i)\, x_i \bar{x}_j$ is ≥ 0. In other words we get a (possibly-degenerate) inner product on $C_0(G)$ by

$$(f_1, f_2) = \int \varphi(g_2^{-1} g_1)\, f_1(g_1) \overline{f_2(g_2)}\, dg_1\, dg_2$$

and hence by completion a unitary representation ρ with cyclic vector e such that

$$\varphi(g) = (\rho(g)e, e).$$

Next let $G = \mathbb{Z}$. φ is then a sequence $\{c_n, n \in \mathbb{Z}\}$ such that the Toeplitz matrix (c_{i-j}) is ≥ 0. According to Riesz-Sz.Nagy (Appendix) if one has a contraction operator T on a Hilbert space W, there is a canonical way to embed W in a Hilbert space $\mathcal{H}$ with a unitary operator U such that $T^n = i^* U^n i$, $\forall n \geq 0$, where $i : W \hookrightarrow \mathcal{H}$ is the inclusion.

Consider the case $\dim(W) = 1$, where $T = \lambda$, $|\lambda| \leq 1$. (If $|\lambda| = 1$, then T is already unitary.) If $W = \langle e \rangle_\lambda$, $\|e\| = 1$, we have

$$(U^n e, e) = (i^* U^n i e, e) = (T^n e, e) = \lambda^n \qquad n \geq 0$$

and

$$(U^{-n} e, e) = (e, U^n e) = \bar{\lambda}^n \qquad n \geq 0.$$

so the function $c_n = \begin{cases} \lambda^n & n \geq 0 \\ \bar{\lambda}^{-n} & n \leq 0 \end{cases}$ should be positive-def. If we want

$$c_n = \int z^n \, g(z) \frac{dz}{2\pi i z}$$

then

$$g(z) = \sum_{n\geq 0}{}' \lambda^n z^{-n} + \sum_{n\geq 1}{}' \bar{\lambda}^n z^n$$

$$= \frac{1}{1-\lambda z^{-1}} + \frac{\bar{\lambda} z}{1-\bar{\lambda} z} = \frac{1-|\lambda|^2}{|1-z^{-1}\lambda|^2} > 0$$

is the Poisson kernel. Thus $\{c_n\}$ is the set of moments of $g(z)\,d\theta$ which is a measure.

More generally suppose $W, T, \mathcal{H}, U$ as above and let $e \in W$. Then it is the case that

$$c_n = \begin{cases} (T^n e, e) & n \geq 0 \\ ((T^*)^{-n} e, e) & n \leq 0 \end{cases}$$

is positive-definite, because $(T^n e, e) = (U^n e, e)$ for $n \geq 0$, etc. But I ought to be able to see this directly using that T is a contraction operator on W. Suppose $\|T\| < 1$ to simplify. Then

$$g(z) = \sum_{n\in\mathbb{Z}}{}' c_n z^n = \sum_{n\geq 0} (T^n e, e) z^{-n} + \sum_{n>0}{}' (T^{*n} e, e) z^n$$

$$= \Big(\underbrace{\Big(\sum_{n\geq 0} z^{-n} T^n + \sum_{n\geq 1}{}' (z T^*)^n \Big)}_{(1-z^{-1}T)^{-1} + (1-zT^*)^{-1} zT^*} e, \, e \Big)$$

Convergence for $|z|=1$ as $\|T\|<1$.

$$= (1-zT^*)^{-1} \Big[\underbrace{zT^*(1-z^{-1}T) + 1 - zT^*}_{1-T^*T} \Big] (1-z^{-1}T)^{-1}$$

so

$$g(z) = \big((1-T^*T)(1-z^{-1}T)^{-1} e, \, (1-z^{-1}T)^{-1} e \big) > 0$$

for $|z| = 1$.

In general given the contraction operator T on W, I can construct $\mathcal{H}, U$ by starting with the vector space of finite formal sums $\sum U^m w_m$ with $w_m \in W$ and introduce the inner product

$$\left(\sum U^m w_m, \sum U^n v_n\right) = \sum_{m,n} \left((i^* U^{m-n} i) w_m, v_n\right)$$

One has only to show that this is ≥ 0 when $v_n = w_n$. This can be done by writing the above as an integral over S^1.

$$\int \sum_{m,n,p} \left(z^p i^* U^{-p} i \, z^m w_m, z^n v_n\right) d\theta$$

$$= \int \Bigg(\underbrace{\Big(\sum_p z^p i^* U^{-p} i\Big)}_{\sum_{p\geq 0} z^{-p} T^p + \sum_{p\geq 1} z^p (T^*)^p \;=\; (1-zT^*)^{-1}(1-T^*T)(1-z^{-1}T)^{-1}} \cdot \Big(\sum_m z^m w_m\Big), \sum z^n v_n \Bigg) d\theta$$

so $$\left(\sum U^m w_m, \sum U^n v_n\right) = \int \left((1-T^*T)(1-z^{-1}T)^{-1} \sum w_m z^m, (1-z^{-1}T)^{-1} \sum v_n z^n\right) d\theta$$

which will be ≥ 0.

November 20, 1977:

Relate de Branges theory with 2 order systems.

Consider first the general system

$$Lu = \left(A\frac{d}{dx} + B\right)u = \lambda C u$$

where $L = L^*$, $C \geqslant 0$. If S is the solution matrix for $Lu = 0$, then changing variables $u \mapsto Su$ transforms this to a system where $B = 0$, and hence A is constant (since $L = L^* \Leftrightarrow A^* = -A$ and $\frac{dA}{dx} = B - B^*$). Changing variables $u \mapsto Su$ with S constant changes A to S^*AS, so we can suppose (provided $\det A > 0$)

$$A = \begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix} \qquad Lu = A\frac{du}{dx} = \lambda C u$$

Now if I want the solution matrix $S(x,\lambda)$ to be of determinant 1, then I want $tr(A^{-1}C) = 0$, and hence that C is of the form

$$C = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix} \qquad \alpha \text{ real}$$

Green's formula:

$$\frac{d}{dx} v^*Au = \left(\frac{dv}{dx}\right)^* Au + v^*A\frac{du}{dx}$$

$$= v^*(Lu) - (Lv)^*u$$

Suppose given a boundary values condition at $x = 0$

which kills $u^*Au = \frac{1}{i}(|u_1|^2 - |u_2|^2)$, and let $\phi(x,\lambda)$ be the solution of $Lu = \lambda Cu$ satisfying a normalized version of it:

$$\phi(0,\lambda) = \begin{pmatrix} e^{i\gamma} \\ e^{-i\gamma} \end{pmatrix} \qquad \gamma \text{ real fixed.}$$

Now the system

$$\begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix} \frac{du}{dx} = \lambda \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix} u$$

and the boundary condition admit the symmetry $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{u}_2 \\ \bar{u}_1 \end{pmatrix}$ for λ real, hence we have

$$\begin{pmatrix} \overline{\phi_2(x,\lambda)} \\ \overline{\phi_1(x,\lambda)} \end{pmatrix} = \begin{pmatrix} \phi_1(x,\lambda) \\ \phi_2(x,\lambda) \end{pmatrix}$$

for λ real, so in general

$$\phi_1(x,\lambda) = \overline{\phi_2(x,\bar{\lambda})} = \phi_2^{\#}(x,\lambda).$$

Next Green's formula gives for a solution of $Lu = \lambda Cu$

$$\frac{d}{dx}\left(\frac{1}{i}(|u_1|^2 - |u_2|^2)\right) = u^*Lu - (Lu)^*u$$
$$= (\lambda - \bar{\lambda})\, u^*Cu$$

Assume $C \geq 0$ and $\operatorname{Im}\lambda \geq 0$ we get

$$\frac{d}{dx}\left(|u_1|^2 - |u_2|^2\right) = -2(\operatorname{Im}\lambda)\, u^*Cu \leq 0.$$

Hence taking $u = \phi(x,\lambda)$ and integrating from 0 to b

$$\left(-|\phi_1|^2 + |\phi_2|^2\right)_b = 2\operatorname{Im}(\lambda) \int_0^b \phi(x,\lambda)^* C\, \phi(x,\lambda)\, dx$$

This will be >0 if $C>0$ at some point in $0 \leq x \leq b$, hence $\phi_2(b,\lambda)$ is a de Branges function.

[redacted]

Given $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ on $[0,b]$ we can associate to it the entire function $\hat{u} : \lambda \longmapsto \int_0^b \phi(x,\bar{\lambda})^* C u \, dx$

Thus
$$\hat{u}(\lambda) = (Cu, \phi_{\bar{\lambda}})$$

so if we want $u \longmapsto \hat{u}$ to be an isomorphism between $(L^2[0,b])^2$ and a [redacted] Hilbert space of entire functions; we [redacted] should have $\widehat{\phi_{\bar{z}}}$ = point evaluator at z. But

$$\widehat{\phi_{\bar{z}}}(\lambda) = \int_0^b \phi(x,\bar{\lambda})^* C \phi(x,\bar{z}) \, dx = (C\phi_{\bar{z}}, \phi_{\bar{\lambda}})$$

$$\begin{aligned}
(\lambda - \bar{z})\widehat{\phi_{\bar{z}}}(\lambda) &= (C\phi_{\bar{z}}, \bar{\lambda}\phi_{\bar{\lambda}}) - (C\bar{z}\phi_{\bar{z}}, \phi_{\bar{\lambda}}) \\
&= (\phi_{\bar{z}}, L\phi_{\bar{\lambda}}) - (L\phi_{\bar{z}}, \phi_{\bar{\lambda}}) \\
&= -\int_0^b [redacted] \frac{d}{dx}(\phi_{\bar{\lambda}}^* A \phi_{\bar{z}}) \\
&= -(\phi_{\bar{\lambda}}^* A \phi_{\bar{z}})(b) = i\Big(\overline{\phi_1(b,\bar{\lambda})}\,\phi_1(b,\bar{z}) \\
&\qquad - \overline{\phi_2(b,\bar{\lambda})}\,\phi_2(b,\bar{z})\Big) \\
&= i \begin{vmatrix} \phi_2(b,\lambda) & \phi_2(b,\bar{z}) \\ \phi_2^{\#}(b,\lambda) & \phi_2^{\#}(b,\bar{z}) \end{vmatrix}
\end{aligned}$$

[redacted]

Because C is assumed ≥ 0 if $\int_0^b u^* C u \, dx = 0$, then $u^* C u = 0$, so $Cu = 0$; so if also $A \frac{du}{dx} = \lambda C u$ we see that u is constant. Thus $\int \phi_\lambda^* C \phi_\lambda \, dx > 0$ unless the null-space of C contains the initial values for all x.

November 23, 1977:

Recall for the DE.

$$\underbrace{\begin{pmatrix} \frac{1}{i} & 0 \\ 0 & i \end{pmatrix}}_{A} \frac{du}{dx} = \lambda \underbrace{\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix}}_{C} u$$

we define $\phi(x,\lambda)$ to be the solution with initial value

$$\phi(0,\lambda) = \begin{pmatrix} e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix}$$

Then from the symmetry $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{u}_2 \\ \bar{u}_1 \end{pmatrix}$, $\lambda \mapsto \bar{\lambda}$ we get

$$\phi_1(x,\lambda) = \overline{\phi_2(x,\bar{\lambda})} = \phi_2^{\#}(x,\lambda)$$

and from Green's formula we get

$$\left(|\phi_1|^2 - |\phi_2|^2 \right)(b) = -\operatorname{Im}\lambda \int_0^b \phi^* C \phi \, dx$$

showing $\phi_2(b,\lambda)$ is a de Branges function for $b > 0$ provided $C > 0$ so that the integral is > 0.

Next consider the Hilbert space $\mathcal{H} = L^2([0,b], C\,dx)$ consisting of $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ on $[0,b]$ with $\|u\|^2 = \int_0^b u^* C u \, dx < \infty$

Then in $\mathcal{H}$ we have

$$(\lambda-\bar z)\left(\phi_{\bar z},\phi_{\bar\lambda}\right) = \int_0^b \left\{ -\phi_{\bar\lambda}^*\left(\bar z C\phi_{\bar z}\right) + \left(\bar\lambda C\phi_{\bar\lambda}\right)^*\phi_{\bar z}\right\} dx$$

$$= -\int_0^b \left\{ \phi_{\bar\lambda}^* L\phi_{\bar z} - \left(L\phi_{\bar\lambda}\right)^*\phi_{\bar z}\right\} dx$$

$$= -\left(\phi_{\bar\lambda}^* A\,\phi_{\bar z}\right)(b)$$

$$= i\left\{ \overline{\phi_1(b,\bar\lambda)}\,\phi_1(b,\bar z) - \overline{\phi_2(b,\bar\lambda)}\,\phi_2(b,\bar z)\right\}$$

$$= i\begin{vmatrix} \phi_2(b,\lambda) & \phi_2(b,\bar z) \\ \phi_2^\#(b,\lambda) & \phi_2^\#(b,\bar z)\end{vmatrix}$$

Thus if $E(\lambda)=\sqrt{2}\,\phi_2(b,\lambda)$, we have

$$\left(\phi_{\bar z},\phi_{\bar\lambda}\right) = \left(J_z, J_\lambda\right)$$

where J_z = point-evaluator at z in $B(E)$.

Because $B(E)$ is generated by the $\{J_z,\ z\in\mathbb{C}\}$ we get an isometry

$$\begin{aligned} B(E) &\hookrightarrow L^2([0,b], C\,dx) \\ J_z &\longmapsto \phi_{\bar z} \end{aligned}$$

whose adjoint is the map

$$u \longmapsto \hat u(z) = \left(u,\phi_{\bar z}\right)$$

In fact this ~~[illegible]~~ isometry is an isomorphism because one knows the $\phi_{\bar z}$ are dense. Here's why.

Fix a self-adjoint boundary condition at $x = b$:

$$u_1(b) = e^{i\theta'} u_2(b)$$

Then the eigenvalues [redacted] are those λ such that ϕ_λ satisfies this boundary condition:

$$\phi_1(b,\lambda) = e^{i\theta'} \phi_2(b,\lambda)$$

or
$$\frac{E^{\#}(\lambda)}{E(\lambda)} = e^{i\theta'}$$

or $\lambda \in \mathbb{R}$ and $(*)$ $\quad -2\arg(E(\lambda)) \equiv \theta' \pmod{2\pi\mathbb{Z}}$. The set of ϕ_λ for these λ forms an orthogonal basis for $\mathcal{H}$, by the known theory of eigenfunction expansions. So we see in this example that [redacted] if $\{\lambda_n\}$ is the set of real solutions of $(*)$, then $\{ J_{\lambda_n} / \|J_{\lambda_n}\| \}$ is an orthonormal basis for $B(E)$.

In general suppose E is a de Branges function, and $B = B(E)$. If z, λ are two real points where $\frac{E^{\#}(\lambda)}{E(\lambda)} = \frac{E^{\#}(z)}{E(z)}$, then,

$$J_z(\lambda) = \frac{i}{2(\lambda - \bar{z})} \begin{vmatrix} E(\lambda) & E(\bar{z}) \\ E^{\#}(\lambda) & E^{\#}(z) \end{vmatrix} = 0$$

so $(J_z, J_\lambda) = 0$. Let $\{\lambda_n\}$ run over those λ such that $E^{\#}(\lambda) = E(\lambda)$. [redacted] Then the corresponding family $J_{\lambda_n}/\|J_{\lambda_n}\|$ is orthonormal and the question is whether it is complete.

Possible approach is to use the fact that $aE + bE^{\#}$ is a de Branges function giving rise to the same space B, provided $|a|^2 - |b|^2 = 1$. Let $a \to +\infty$ and $b = -\sqrt{a^2-1}$

$$= -a\left(1 - \frac{1}{2a^2}\right) = -a + \frac{1}{2a} \cdots\cdots$$

Then $\displaystyle \|f\|^2 = \int_{\mathbb{R}} |f(\lambda)|^2 \frac{d\lambda}{|aE + bE^\#|^2 \pi}$

$$aE + bE^\# = a\left(E - E^\#\right) + \frac{1}{2a} E^\# \ldots$$

Now as $a \nearrow \infty$ one has that $\dfrac{1}{|aE+bE^\#|^2} \longrightarrow 0$ at those λ not in $\{\lambda_n\}$, and it goes to ∞ around points in $\{\lambda_n\}$. It should be the case that we get δ functions at λ_n.

Now $\displaystyle J_z(z) = \frac{i}{2}\left\{E'(z)\overline{E(z)} - \overline{E'(z)}\, E(z)\right\}$ $\quad z \in \mathbb{R}$

$$= -|E(z)|^2 \,\mathrm{Im}\left\{\frac{E'(z)}{E(z)}\right\}$$

$\overline{E(\lambda_n)} = E(\lambda_n)$

$$|(aE + bE^\#)(\lambda)|^2 = \left|a(E(\lambda) - \overline{E(\lambda)}) + \frac{1}{2a}\overline{E(\lambda)}\right|^2$$

$$\sim \left|a\, 2i\, \mathrm{Im}\, E'(\lambda_n)(\lambda - \lambda_n) + \frac{1}{2a}\overline{E(\lambda_n)}\right|^2 \qquad \lambda \sim \lambda_n$$

$$= (2a)^2\left(\mathrm{Im}\, E'(\lambda_n)\right)^2(\lambda - \lambda_n)^2 + \frac{1}{(2a)^2} E(\lambda_n)^2$$

So $\displaystyle \frac{d\lambda}{|aE+bE^\#|^2} \sim \frac{d\lambda}{(2a)^2(\mathrm{Im}\, E'(\lambda_n))^2(\lambda-\lambda_n)^2 + \frac{1}{(2a)^2} E(\lambda_n)^2}$

and as $a \longrightarrow \infty$

$$\beta^2 \frac{\beta}{\alpha} \quad \int_{-\infty}^{\infty} \frac{\frac{\alpha}{\beta}\, dx}{\alpha^2 x^2/\beta^2 + \beta^2/\beta^2} = \frac{1}{\alpha\beta} \arctan \frac{\alpha x}{\beta}\Big]_{-\infty}^{\infty} = \frac{\pi}{\alpha\beta}$$

so as $a \to \infty$ $\displaystyle \frac{d\lambda}{|aE+bE^\#|^2 \pi} \longrightarrow \frac{\delta(\lambda - \lambda_n)}{(\mathrm{Im}\, E'(\lambda_n))E(\lambda_n)} = \frac{\delta(\lambda - \lambda_n)}{J_{\lambda_n}(\lambda_n)}.$

The only problem with this approach is what happens for large λ.

Consider de Branges method which uses Riesz-Herglotz representation. One has the function $\frac{E^\#(\lambda)}{E(\lambda)}$ which maps the upper half-plane into the disk. Then one forms the corresponding map to the upper half-plane taking 1 to ∞

$$\frac{1}{i}\,\frac{E^\# + E}{E^\# - E} = -\frac{A}{B} \qquad \begin{aligned} E^\# &= A + iB \\ E &= A - iB \end{aligned}$$

By Riesz-Herglotz $\exists$ a measure $d\mu$ on $\mathbb{R}$ with $\int \frac{d\mu}{1+x^2} < \infty$ and $p \geq 0$ and $c \in \mathbb{R}$ $\ni$

$$-\frac{A(\lambda)}{B(\lambda)} = c + p\lambda + \int_{-\infty}^{\infty} \left\{ \frac{1}{x-\lambda} - \frac{x}{x^2+1} \right\} d\mu$$

Because $-\frac{A}{B}$ is analytic for $\lambda \neq \lambda_n$ on the real line it follows that $d\mu$ is supported at these points. (This uses the Stieltjes inversion formula). So we get

$$-\frac{A(\lambda)}{B(\lambda)} + \frac{A(\bar z)}{B(\bar z)} = p(\lambda - \bar z) + \sum_n \left\{ \frac{1}{\lambda_n - \lambda} - \frac{1}{\lambda_n - \bar z} \right\} p_n$$

where p_n = residue of $\frac{A(\lambda)}{B(\lambda)}$ at $\lambda = \lambda_n$ which is $\frac{A(\lambda_n)}{B'(\lambda_n)}$.

So $$J_z(\lambda) = \frac{-1}{\lambda - \bar z} \begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z) \end{vmatrix} = pB(\lambda)B(\bar z) + \sum_n \frac{B(\lambda)B(\bar z)}{(\lambda_n - \lambda)(\lambda_n - \bar z)}\, p_n$$

Now $$J_{\lambda_n}(\lambda) = \frac{+1}{\lambda - \lambda_n} A(\lambda_n) B(\lambda) = E(\lambda_n)\,\frac{B(\lambda)}{\lambda - \lambda_n}$$ and

$$p_n = \frac{A(\lambda_n)}{B'(\lambda_n)} = \frac{E(\lambda_n)}{- \operatorname{Im} E'(\lambda_n)} = \frac{E(\lambda_n)^2}{\|J_{\lambda_n}\|^2}$$

so

$$J_z(\lambda) = p B(\lambda) B(\bar z) + \sum_n \frac{J_{\lambda_n}(\bar z)\, J_{\lambda_n}(\lambda)}{\|J_{\lambda_n}\|^2}$$

and $\overline{J_{\lambda_n}(z)} = (J_z, J_{\lambda_n}) = (J_{\lambda_n}^{\#}, J_z^{\#}) = (J_{\lambda_n}, J_{\bar z}) = J_{\lambda_n}(\bar z)$

so

$$J_{\lambda_n}(\bar z) = J_z(\lambda_n).$$

If $p = 0$, then we have

$$J_z(\lambda) = \sum_n \frac{J_z(\lambda_n)\, J_{\lambda_n}(\lambda)}{\|J_{\lambda_n}\|^2}$$

which implies the $\dfrac{J_{\lambda_n}}{\|J_{\lambda_n}\|}$ are complete. In general Bessel's inequality says the sum converges in B, so when $p \neq 0$ we see that $B(\lambda) = \frac{1}{2i}(E^{\#} - E) \in B$. Clearly this is orthogonal to the J_{λ_n}.

November 27, 1977.

Converting an SL system to string form:
Start with

$$Lu = -u'' + Vu = \lambda^2 u$$

on $0 \le x < \infty$ with boundary condition $u'(0) = hu(0)$. Suppose that the spectrum is ≥ 0, or more precisely that $(Lu, u) \geq 0$ for $u \in C_0^\infty([0,\infty))$ satisfying the boundary conditions. Then I saw before that if $Ly = 0$, $y(0) = 1$, $y'(0) = h$ then $y > 0$ for all $x \geq 0$. Hence I can put

$$p = \frac{y'}{y} = \frac{d}{dx}(\log y)$$

and I have $\qquad p' = \frac{y''}{y} - \left(\frac{y'}{y}\right)^2 = V - p^2$, so I can factor L

$$-Lu = \left(\frac{d}{dx} + p\right)\left(\frac{d}{dx} - p\right)u.$$

This let's me replace L by the system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{dw}{dx} + \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix} w = \lambda w$$

where $u = w_1$. Then I can convert this to deB form using the solution matrix for $\lambda = 0$

$$S = \begin{pmatrix} e^{\int p} & 0 \\ 0 & e^{-\int p} \end{pmatrix}$$

getting

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{dv}{dx} = \lambda \begin{pmatrix} e^{2\int p} & 0 \\ 0 & e^{-2\int p} \end{pmatrix} v$$

But $e^{\int p} = y$ so we should go directly as follows.

Given $Lu = \left(-\frac{d^2}{dx^2} + V\right) u = \lambda^2 u$ & $u'(0) = hu(0)$

we [redacted] [redacted] define y by $Ly = 0$, $y(0) = 1$, $y'(0) = h$.
Then provided the spectrum of L is ≥ 0 we know $y > 0$ for $x \geq 0$ and we can change variables. Define v_1, v_2 by

$$u = y v_1$$

$$\lambda v_2 = -y^2 \frac{dv_1}{dx} \qquad v_1' = -\lambda y^{-2} v_2$$

Then

$$u' = y'v_1 + y v_1' = y'v_1 + y(-\lambda y^{-2} v_2)$$

$$= y'v_1 - \lambda y^{-1} v_2$$

$$u'' = y''v_1 + \cancel{y'(-\lambda y^{-2} v_2)} + \cancel{\lambda y^{-2} y' v_2} - \lambda y^{-1} v_2'$$

$$\parallel$$

$$Vu - \lambda^2 u = \frac{y''}{y} y v_1 - \lambda^2 y v_1$$

$$\therefore \quad v_2' = \lambda y^2 v_1$$

so you get

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dv}{dx} = \lambda \begin{pmatrix} y^2 & 0 \\ 0 & y^{-2} \end{pmatrix} v$$

Here are some versions of scattering analysis: By this I mean [redacted] finding a 1-parameter unitary group to [redacted] understand an operator: If I start with [redacted] $Lu = -u'' + Vu$, then one works with the wave equation

$$-u_{tt} = Lu$$

whose solutions are $u(x,t) = \int e^{-i\lambda t} \phi(x,\lambda)$ [redacted] $\alpha(\lambda)\, d\lambda$

One makes a Hilbert space out of solutions of the wave equation using the energy norm:

$$E(u) = \int_0^\infty |u_t|^2 \, dx + \int_0^\infty (Lu)\bar{u} \, dx$$

Check this is time-invariant:

$$\begin{aligned} \frac{d}{dt} E(u) &= (u_t, u_{tt}) + (u_{tt}, u_t) + (Lu_t, u) + (Lu, u_t) \\ &= (u_t, -Lu) + (-Lu, u_t) + (u_t, Lu) + (Lu, u_t) \\ &= 0 \end{aligned}$$

I was going to mimic this in the discrete case. Suppose I start with a J-matrix L = far out to $L_0 = \frac{1}{2}(T + T^{-1})$. Then

$$\phi(n, \lambda) = A(z) z^{-n} + B(z) z^{n} \qquad n >> 0$$

where $\frac{1}{2}(z + z^{-1}) = \lambda$. I consider transforms

$$u(n, t) = \int_{S^1} z^{-t} \phi(n, \lambda) \alpha(z) \, d\theta$$

for $\alpha \in C^\infty(S^1)$. This satisfy the "discrete wave equation"

$$\frac{1}{2}\big(u(n, t+1) + u(n, t-1)\big) = (Lu)(n, t).$$

<u>Question:</u> Let L be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with spectrum in $-1 \le \lambda \le 1$. Consider the vector space V of solutions $\mathbb{Z} \ni t \longmapsto u(t) \in \mathcal{H}$ of

$$\frac{1}{2}\big(u(t+1) + u(t-1)\big) = Lu(t).$$

There is an evident action of $\mathbb{Z}$ on V: $u(t) \mapsto u(t+1)$. Does V

have a natural Hilbert space structure such that this translation is unitary? Try the energy norm

$$E(u(\cdot)) = |u(0)|^2 + |u(1)|^2 - (Lu(0), u(1)) - (Lu(1), u(0))$$
$$= |u(0) - Lu(1)|^2 + ((1-L^2)u(1), u(1))$$
$$= |u(1) - Lu(0)|^2 + ((1-L^2)u(0), u(0))$$

Let's put

(1) $$E(u)(t) = |u(t) - Lu(t+1)|^2 + ((1-L^2)u(t+1), u(t+1))$$

Then by the above algebraic manipulation we have

(2) $$E(u)(t) = |u(t+1) - Lu(t)|^2 + ((1-L^2)u(t), u(t))$$

But if $Lu(t) = \frac{1}{2}\{u(t+1) + u(t-1)\}$

then

$$u(t+1) - Lu(t) = \frac{u(t+1) - u(t-1)}{2}$$

So (1) becomes

$$E(u)(t) = \left|\frac{u(t+2) - u(t)}{2}\right|^2 + ((1-L^2)u(t+1), u(t+1))$$

and (2) becomes

$$E(u)(t) = \left|\frac{u(t+1) - u(t-1)}{2}\right|^2 + ((1-L^2)u(t), u(t))$$

so it's clear now that $E(u)(t) = E(u)(t+1)$.

More direct proof:

$$E(u)(t) = |u(t)|^2 + |u(t+1)|^2 - (Lu(t), u(t+1)) - (u(t+1), Lu(t))$$
$$= |u(t)|^2 + |u(t+1)|^2 - \tfrac{1}{2}(u(t-1) + u(t+1), u(t+1)) - \tfrac{1}{2}(u(t+1), u(t-1) + u(t+1))$$
$$= |u(t)|^2 - \tfrac{1}{2}(u(t-1), u(t+1)) - \tfrac{1}{2}(u(t+1), u(t-1))$$

Observe the upper formula is centered at $t+\frac{1}{2}$ and the lower is centered at t; so we can revise the latter to center at $t-\frac{1}{2}$:

$$= |u(t)|^2 + |u(t-1)|^2 - \tfrac{1}{2}\big(u(t-1), u(t+1)-u(t-1)\big) - \tfrac{1}{2}\big(u(t+1)+u(t-1), u(t-1)\big)$$

$$= |u(t)|^2 + |u(t-1)|^2 - \big(u(t-1), Lu(t)\big) - \big(Lu(t), u(t-1)\big)$$

$$= E(u)(t-1).$$

Consider now the case $L = L_0 = \frac{1}{2}(T+T^{-1})$ for which we have

$$\phi(n,\lambda) = \frac{\sin n\theta}{\sin\theta} = \frac{z^n - z^{-n}}{z - z^{-1}} \qquad n \geq 1$$

I can define

$$u(n,t) = \int z^{-t}\, \phi(n,\lambda)\, \alpha(z)\, d\theta$$

$$= \int z^{-t}\, \frac{z^n - z^{-n}}{z-z^{-1}}\, \alpha(z)\, d\theta$$

$$= f(n-t) - f(-n-t) \qquad f = \widehat{\frac{\alpha}{z-z^{-1}}}$$

So $\quad E(u) = |u(t)|^2 - \mathrm{Re}\,\big(u(t-1), u(t+1)\big)$

should be independent of t.

$$|u(t)|^2 = \sum_{n=1}^{\infty} |f(n-t) - f(-n-t)|^2$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}} |f(n-t) - f(-n-t)|^2$$

$$\big(u(t-1), u(t+1)\big) = \sum_{n\geq 1}' \big(f(n-t+1) - f(-n-t+1)\big)\overline{\big(f(n-t-1) - f(-n-t-1)\big)} = \frac{1}{2}\sum_n$$

$$|u(t)|^2 = \frac{1}{2}\Big\{\sum_n |f(n-t)|^2 - 2\,\mathrm{Re}\, f(n-t)\overline{f(-n-t)} + |f(-n-t)|^2\Big\}$$

$$= \|f\|^2 - \mathrm{Re}\big(f(n-2t), f(-n)\big)$$

$$2\,\mathrm{Re}\big(u(t-1), u(t+1)\big) = \mathrm{Re}\big(f(?+2), f\big) + \mathrm{Re}\big(f(?+2), f\big)$$
$$- \mathrm{Re}\big(f(n-2t), f(-n)\big) - \mathrm{Re}\big(f(+n-2t), f(n)\big)$$

so
$$|u(t)|^2 - \mathrm{Re}\big(u(t-1), u(t+1)\big) = \|f\|^2 - \mathrm{Re}\big(f(n+2), f(n)\big)$$

Now if you use Plancherel [redacted]

$$\|f\|^2 \Big(= \sum_{n\in\mathbb{Z}} |f(n)|^2\Big) = 2\pi \int \left|\frac{\alpha(z)}{z-z^{-1}}\right|^2 d\theta$$

so
$$|u(t)|^2 - \mathrm{Re}\big(u(t-1), u(t+1)\big) = 2\pi \int \Big(1 - \frac{1}{2}z^2 - \frac{1}{2}z^{-2}\Big)\left|\frac{\alpha(z)}{z-z^{-1}}\right|^2 d\theta$$

$$1 - \frac{1}{2}z^2 - \frac{1}{2}z^{-2} = -\frac{1}{2}\big(z^2 - 2 + z^{-2}\big) = -\frac{1}{2}\big(z-z^{-1}\big)^2$$
$$= +\frac{1}{2}\big(z-z^{-1}\big)\overline{\big(z-z^{-1}\big)}$$

So
$$E(u) = \pi \int |\alpha|^2 d\theta \qquad \text{amazing.}$$

Consider $\quad Lu = \Big(-\Big(\dfrac{d^2}{dx^2}\Big) + V\Big)u = \lambda^2 u \quad$ in the finite range case

$$\phi(x,\lambda) = A(\lambda)e^{-i\lambda x} + A(-\lambda)e^{i\lambda x}$$

Then
$$u(x,t) = \int e^{-i\lambda t}\,\phi(x,\lambda)\,\alpha(\lambda)\,d\lambda \qquad \alpha \in C_0^\infty(\mathbb{R})$$

is a solution of $\quad u_{tt} = -Lu$, hence

$$E(u) = \|u_t\|^2 + (Lu, u) = \|u_t\|^2 + \|u_x\|^2 + (Vu, u)$$

is independent of t. As $t \to \pm\infty$, $u(x,t) \to 0$ for x fixed, uniformly on compact sets. For $x >> 0$

$$u(x,t) = \hat{A\alpha}(-x-t) + \hat{B\alpha}(x-t)$$

where the second term goes to zero globally as $t \to -\infty$. Thus we can compute $E(u)$ from the first term and we find

$$E(u) = \lim_{t\to-\infty} \left\|\frac{\partial}{\partial t} \hat{A\alpha}(-x-t)\right\|^2 + \left\|\frac{\partial}{\partial x} \hat{A\alpha}(-x-t)\right\|^2$$

$$= 2\|\hat{A\alpha}'\|^2 = 4\pi \int |A(\lambda)\alpha(\lambda)\lambda|^2 d\lambda$$

Now if α is odd, we know $u(x,0) \equiv 0$ hence

$$E(u) = \|u_t(0)\|^2 = \left\|\int (-i\lambda)\phi(x,\lambda)\alpha(\lambda)\,d\lambda\right\|^2$$

$$= \left\|\int \phi(x,\lambda)\,\lambda\alpha(\lambda)\,d\lambda\right\|^2$$

Hence we get

$$\left\|\int_0^\infty \phi(x,\lambda)\,\alpha(\lambda)\,d\lambda^2\right\|^2 = 4\pi\int_0^\infty |\alpha(\lambda)|^2\,|A(\lambda)|^2\lambda\;d\lambda^2$$

so $$d\mu(\lambda) = \frac{d\lambda^2}{4\pi\lambda\,|A(\lambda)|^2} \qquad \left(= \frac{d\lambda}{2\pi\,|A(\lambda)|^2}\right)$$

which agrees with the calculations on page 476.

L J-matrix $= \frac{1}{2}(T+T^{-1})$ far out

$$\phi(n,\lambda) = A(z)z^{-n} + B(z)z^{n} \qquad n \gg 0$$

$$u(n,t) = \int z^{-t}\phi(n,\lambda)\,\alpha(z)\,d\theta$$

$$= \widehat{A\alpha}(-n-t) + \widehat{B\alpha}(n-t) \qquad n \gg 0$$

So $\quad u(n,t) \sim \widehat{B\alpha}(n-t) \quad$ as $t \to +\infty$.

Problem: Calculate $E(u)$.

$$E(u) = \|u(t)\|^2 - \tfrac{1}{2}(u(t-1),u(t+1)) - \tfrac{1}{2}(u(t+1),u(t-1))$$

$$\sim \|\widehat{B\alpha}\|^2 - \tfrac{1}{2}(T\widehat{B\alpha}, T^{-1}\widehat{B\alpha}) - \tfrac{1}{2}(T^{-1}\widehat{B\alpha}, T\widehat{B\alpha})$$

$$= 2\pi\int\left(1 - \tfrac{1}{2}z^2 - \tfrac{1}{2}z^{-2}\right)|B\alpha|^2\,d\theta$$

$$= \int \pi\,|z - z^{-1}|^2\,|B\alpha|^2\,d\theta$$

Digression: If $L = \dfrac{T+T^{-1}}{2}$ where T is unitary on $\mathcal{H}$, then

$$\tfrac{1}{2}\left\{\|Tu(t) - u(t+1)\|^2 + \|T^{-1}u(t) - u(t+1)\|^2\right\}$$

$$= \|u(t)\|^2 + \|u(t+1)\|^2 - (Lu(t),u(t+1)) - (u(t+1),Lu(t))$$

$$= E(u)$$

so that for a solution $u(n,t) = \widehat{B\alpha}(n-t)$ for which $Tu(t) = u(t+1)$, one has $E(u) = \frac{1}{2}\|u(t-1)+u(t+1)\|^2$.

Next use the formula

$$E(u) = \| u(t+1) - Lu(t) \|^2 + \|u(t)\|^2 - \|Lu(t)\|^2$$

to conclude

$$E(u) = \|u(1)\|^2 \qquad \text{if} \qquad u(0) = 0.$$

For example, if $\alpha(z) = -\alpha(z^{-1})$, then

$$u(n,0) = \int \phi(n,\lambda)\, \alpha(z)\, d\theta = 0$$

Hence we get the formula

$$\left\| \int z^{-1} \phi(\cdot,\lambda)\, \alpha(z)\, d\theta \right\|^2 = \int \pi \left| (z - z^{-1}) B(z)\, \alpha(z) \right|^2 d\theta$$

for $\alpha(z)$ an odd function. But

$$u(1) = \int z^{-1} \phi(n,\lambda)\, \alpha(z)\, d\theta = \int z\, \phi(n,\lambda)\, \alpha(z^{-1})\, d\theta$$

$$= -\int z\, \phi(n,\lambda)\, \alpha(z)\, d\theta \qquad = -u(-1)$$

so that
$$u(1) = \int \left(\frac{z^{-1} - z}{2} \right) \phi(n,\lambda)\, \alpha(z)\, d\theta$$

So a better formula is

$$\left\| \int \frac{z - z^{-1}}{2}\, \phi(\cdot,\lambda)\, \alpha(z)\, d\theta \right\|^2 = 4\pi \int \left| \left(\frac{z - z^{-1}}{2}\right) B(z)\, \alpha(z) \right|^2 d\theta$$

or
$$\left\| \int \phi_\lambda\, \beta(z)\, d\theta \right\|^2 = 4\pi \int |\beta(z)|^2\, |B(z)|^2\, d\theta$$

for any β such that $\beta(z) = \beta(z^{-1})$. [redacted]

$$\int_0^{2\pi} \phi_\lambda\, \beta(\lambda)\, d\theta = 2 \int_0^{\pi} \phi_\lambda\, \beta(\lambda). \text{[redacted]} \sin\theta \ \frac{d\theta}{\sin\theta}$$

Better: suppose $\psi(\lambda)$ given on $-1 \leq \lambda \leq 1$.

$$\left\| \int_{-1}^{1} \phi_\lambda \, \psi(\lambda) \, d\lambda \right\|^2 = \left\| \int_{0}^{\pi} \phi_\lambda \, \underbrace{\psi(\lambda)}_{\alpha(z)} \, \underbrace{\sin\theta}_{\frac{z-z^{-1}}{2i}} \, d\theta \right\|^2$$

define α to be odd on $\pi \leq \theta \leq 2\pi$.

$$= \left\| \frac{1}{2} \int_{S^1} \phi_\lambda \, \alpha(z) \, \frac{z-z^{-1}}{2} \, d\theta \right\|^2$$

$$= \pi \int_{S^1} \left| \frac{z-z^{-1}}{2} B(z) \alpha(z) \right|^2 d\theta$$

$$= 2\pi \int_0^\pi \sin^2\theta \, |B(z)|^2 \, |\psi(\lambda)|^2 \, d\theta$$

$$= 2\pi \int_{-1}^{1} |\psi(\lambda)|^2 \, |B(z)|^2 \sqrt{1-\lambda^2} \, d\lambda$$

Hence the spectral measure is

$$d\mu(\lambda) = \frac{d\lambda}{2\pi \, |B(\lambda)|^2 \sqrt{1-\lambda^2}}$$

For example, when $L = L_0$ so that $\phi(n,\lambda) = \frac{z^n - z^{-n}}{z - z^{-1}}$, then $B(z) = \frac{1}{z-z^{-1}} = \pm \frac{1}{2i\sqrt{1-\lambda^2}}$, so $d\mu(\lambda) = \frac{2}{\pi}\sqrt{1-\lambda^2}\, d\lambda$

(Agrees with p. 478).