Consider the radial Schrödinger equation

1) $$\left(-\frac{d^2}{dx^2}+V(x)\right)u = \lambda u \qquad u(0)=0$$

where V has compact support. Let $\phi(x,\lambda)$ denote the solution satisfying $\phi(0)=0$, $\phi'(0)=1$. (e.g. $\phi(x,\lambda)=\frac{\sin kx}{k}$ if $V=0$, where $k=\sqrt{\lambda}$).

Since V has compact support we have for [redacted] $x>>0$

2) $$\phi(x,k^2) = A(k)e^{-ikx} + B(k)e^{ikx}$$

provided $k\neq 0$, where A, B are holomorphic in k away from 0. Since V is assumed real, symmetry considerations show that $B(k)=\overline{A(k)}=A(-k)$ for k real, hence

$$B(k) = \overline{A(\bar{k})} = A(-k)$$

for all $k\neq 0$. (Note: If we assume only that V decays fast as $x\to+\infty$, then we can only expect the asymptotic formula

$$\phi(x,k^2) \sim A(k)e^{-ikx} + A(-k)e^{ikx}$$

to define $A(k)$ as a holomorphic function for $\operatorname{Im} k>0$ [redacted] $k\neq 0$.) From now on we [redacted] assume $\operatorname{Im}(k)\geq 0$ unless stated otherwise.

[redacted]

The goal now will be to understand the spectral measure for 1) in terms of the function $A(k)$. Point spectrum: $\phi(x,k^2)$ is square-integrable only when $\mathrm{Im}(k)>0$ and $A(k)=0$. In this case k^2 has to be real, so $k\in i\mathbb{R}_{>0}$. Thus the zeroes of $A(k)$ in the upper half-plane correspond to "bound[redacted]" states.

Let $f(x,k)$ be the solution of the DE 1) with $f(x,k)=e^{ikx}$ for $x>>0$. For $\mathrm{Im}(k)>0$, [redacted] $f(x,k)$ decays [redacted] as $x\longrightarrow+\infty$, so one [redacted] knows the Green's function is given by

$$G_{k^2}(x,x') = \frac{\phi(x_<,k^2)\,f(x_>,k)}{W(\phi(\ ,k^2),\, f(\ ,k))}$$

$$= \frac{\phi(x_<,k^2)\,f(x_>,k)}{2ik\,A(k)}$$

Emphasize: this formula holds for $\mathrm{Im}(k)>0$, so that both sides jump as we let k^2 cross $\mathbb{R}_{>0}$. So we get the formula

$$\frac{\phi(x_<,k^2)f(x_>,k)}{2ik\,A(k)} = \int \frac{\phi(x,\lambda)\,\phi(x',\lambda)}{k^2-\lambda}\,d\mu(\lambda)$$

where equality has to be understood [redacted] as kernels. I mean: [redacted] as distributions in two variables $(x,x')\in\mathbb{R}_{>0}\times\mathbb{R}_{>0}$ (at least $\mathbb{R}_{>0}\times\mathbb{R}_{>0}$).

Recall that if $h(\lambda)=\int\frac{\alpha(x)\,dx}{\lambda-x}$ is the Stieltjes transform of α, then for λ real

$$\lim_{\varepsilon \downarrow 0} \Big(h(\lambda+i\varepsilon) - h(\lambda-i\varepsilon) \Big) = \int \alpha(x)dx \left[\frac{1}{\lambda-x+i\varepsilon} - \frac{1}{\lambda-x-i\varepsilon} \right]$$

$$= \int \alpha(x)dx \, \frac{-2i\varepsilon}{(\lambda-x)^2+\varepsilon^2} = -2i\alpha(\lambda) \int \frac{\varepsilon \, dx}{\left(\frac{x^2}{\varepsilon^2}+1\right)\varepsilon^2} = -2\pi i \alpha(\lambda).$$

The jump in the Green's function as we cross the real axis, going upward, at $k^2>0$ is

$$\phi(x_<, k^2) \left(\frac{f(x_>, k)}{2ik\,A(k)} - \frac{f(x_>, -k)}{2i(-k)\,A(-k)} \right)$$

assuming $k>0$. Since $\phi(x,k^2) = A(k)f(x,k) + A(-k)f(x,k)$ this is

$$\frac{\phi(x_<, k^2)\,\phi(x_>, k^2)}{2ik\,A(k)A(-k)}$$

which is a nice smooth function of k for $k>0$. So from the Stieltjes inversion formula we get

$$-2\pi i \frac{d\mu}{d\lambda} = \frac{1}{2ik\,|A(k)|^2}$$

so

$$\boxed{d\mu(\lambda) = \frac{d\lambda}{4\pi k |A(k)|^2} = \frac{d\lambda}{4\pi\sqrt{\lambda}\,|A(\sqrt{\lambda})|^2} \qquad \lambda>0}$$

Here's how we can get the normalization constants for the bound states. Suppose $\lambda_0 = k_0^2 < 0$ corresponds to a bound state. Then (as $\operatorname{Im}(k_0) > 0$ by assumption)

$$\phi(x, k_0^2) = A(-k_0)f(x,k_0)$$

and $A(k_0) = 0$. Thus

$$G_{k^2}(x,x') = \frac{\phi(x,k_0^2)\,\phi(x',k_0^2)}{k^2-k_0^2}\,\lambda_0 \cdots + \text{ stuff reg. at } k_0$$

$$= \frac{\phi(x_<,k^2)\,f(x_>,k)}{2ik\,A(k)} = \frac{\phi(x_<,k^2)\,\phi(x_>,k^2)}{2ik_0\,A'(k_0)(k-k_0)A(-k_0)} + \cdots\cdots$$

so

$$\lambda_{k_0^2} = \frac{k_0+k_0}{2ik_0\,A'(k_0)\,A(-k_0)} \quad \text{or}$$

$$\boxed{\lambda_{k_0^2} = \frac{1}{i\,A'(k_0)\,A(-k_0)}}$$

So far we've only discussed the spectrum of the operator $-\frac{d^2}{dx^2}+V$; we have not discussed scattering. My hope is that if all of this is set up carefully, I will be able to appreciate the Hörmander idea that the wave equation gives the nice approach to the spectrum of this operator.

So we consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - V(x)u$$

on $0 \le x < \infty$ with the boundary condition $u(0;t)=0$. If $u(x;t)$ is a solution satisfying the bdry condition and decaying as $x \to +\infty$, then the energy at any t

$$E(u) = \int_0^\infty \left(|u_t|^2 + |u_x|^2 + V(x)|u|^2\right)dx$$

is constant, because

$$\frac{dE}{dt} = 2\,\mathrm{Re}\int_0^\infty \left(u_{tt}\bar{u}_t + u_{xt}\bar{u}_x + V(x)\,u_t\bar{u}\right)dx$$

$$= 2\,\mathrm{Re}\int_0^\infty u_t\left(\bar{u}_{tt} - \bar{u}_{xx} + V\bar{u}\right)dx = 0$$

where we have integrated by paths using that $u_t = 0$ when $x = 0$.

One assumes $V \geq 0$ so that E defines an inner product on the space of Cauchy data $(u, \dot{u})$ for the wave equation:

$$\|(u,\dot{u})\|^2 = \int_0^\infty \left(|\dot{u}|^2 + |u_x|^2 + V|u|^2\right)dx$$

where $u, \dot{u}$ are supposed to vanish for large x and also at 0. Time evolution preserves this pre-Hilbert space of C_0^∞ Cauchy data, so we get a Hilbert space with a 1-parameter unitary group of operators.

Note that $V \geq 0$ [redacted] implies there are no bound states, hence the operator $H = -\frac{d^2}{dx^2} + V$ + boundary condition is ≥ 0 and it has a positive square root. Somehow this Cauchy data Hilbert space should be related to this positive square root.

Another point is that on the Cauchy data [redacted] space one doesn't see any H_0, so that there has to be a mechanism [redacted] roughly equivalent to it. Lax + Phillips have incoming + outgoing subspaces instead of H_0.

For $H_0 = -\frac{d^2}{dx^2}$ one has

$$\phi(x,k) = \frac{\sin kx}{k} = \frac{e^{-ikx}}{-2ik} + \frac{e^{ikx}}{2ik} \qquad \text{so } A(k) = \frac{1}{-2ik}$$

hence

$$d\mu(k^2) = \frac{d(k^2)}{4\pi k}\,|2ik|^2 = \frac{k\,d(k^2)}{\pi} = \frac{2k^2\,dk}{\pi}$$

October 8, 1977:

In order to understand $H = -\frac{d^2}{dx^2} + V$ on $0 \leq x < \infty$ with the boundary condition $u(0) = 0$ via the wave equation
$$-\frac{\partial^2 u}{\partial t^2} = Hu$$
one introduces the space of Cauchy data $\begin{pmatrix} u \\ \dot{u} \end{pmatrix}$ with the norm
$$\begin{pmatrix} u \\ \dot{u} \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \|\dot{u}\|^2 + (Hu, u).$$
Here $u, \dot{u} \in C_c^\infty$ and they vanish at $x = 0$ and $x \gg 0$. We assume $V \geq 0$ so as to get an inner product. On the space of Cauchy data we have the time evolution described by
$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix} \begin{pmatrix} u \\ \dot{u} \end{pmatrix}.$$
One can easily verify that $\begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix}$ is skew-adjoint wrt the norm.

Suppose $V = 0$. Recall that $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ has the general solution (on $\mathbb{R}$)
$$u(x,t) = f(x-t) + g(x+t)$$
where $f(x-t)$ represents a wave travelling to the right and $g(x+t)$ one to the left. f, g can be determined from Cauchy data by the equations
$$u(x,0) = f(x) + g(x)$$
$$\dot{u}(x,0) = -f'(x) + g'(x)$$

This determines f, g up to an additive constant: $(f+c, g-c)$. If we want solutions vanishing at $x = 0$,
$$0 = f(-t) + g(t)$$

we have

$$u(x,t) = f(x-t) - f(-x-t).$$

Hence given $f \in C_0^\infty(\mathbb{R})$, this formula gives a solution of the wave equation on $0 \leq x < \infty$ satisfying the bdry condition. Conversely given Cauchy data $u, \dot{u}$ on $0 \leq x < \infty$, try to solve

$$u(x,0) = f(x) - f(-x) \qquad \Rightarrow \quad u_x(x,0) = f'(x) + f'(-x)$$
$$\dot{u}(x,0) = -f'(x) + f'(-x)$$

so we get

$$f'(x) = \frac{u_x(x,0) - \dot{u}(x,0)}{2} \qquad 0 \leq x < \infty$$

$$f'(-x) = \frac{u_x(x,0) + \dot{u}(x,0)}{2} \qquad \text{"}$$

(If f' is defined in this way, it is C^∞ provided $u, \dot{u}$ extend as odd C^∞ functions on the line, ie. their odd derivatives vanish at zero. So maybe I should think of this radial situation as the wave equation on the line but with u required to be odd.) You should check that the additive constant to be added to f can be chosen so that f has compact support. But since $\dot{u}$ is odd on the line:

$$\int_{-\infty}^{\infty} f'(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} u_x(x,0)\,dx = 0$$

So I see therefore that I can identify the Cauchy data space for the case $V=0$ with $C_0^\infty(\mathbb{R})$. The time evolution in this picture is given by $f(x) \mapsto f(x-t)$.

Consider a free trajectory

$$u_0(x,t) = f(x-t) - f(-x-t) = \int e^{-ikt}\left(e^{ikx} - e^{-ikx}\right)\hat{f}(k)\,dk$$

Remembering that $0 \le x < \infty$, one sees that as $t \to +\infty$ $f(-x-t)$ tends to zero in norm. Next consider the perturbed system. Recall the notation

$$\phi(x,k^2) = A(k)\,e^{-ikx} + \underset{\substack{\parallel \\ A(-k)}}{\overline{A(k)}}\,e^{+ikx}$$

and denote by $\psi^-(x,k)$ the solution asymptotic to e^{-ikx}, namely

$$\psi^-(x,k) = e^{-ikx} + \frac{A(-k)}{A(k)}\,e^{ikx}$$

and put $-S(k) = \dfrac{A(-k)}{A(k)}$. Now we get a honest trajectory

$$u(x,t) = \int e^{-ikt}\,\psi^-(x,k)\,\hat{f}(k)\,dk$$

which for $x \gg 0$ has the form

$$u(x,t) = \int e^{-ikt}\left(e^{-ikx} - S(k)\,e^{ikx}\right)\hat{f}(k)\,dk.$$

As $t \to -\infty$, this is asymptotic to

$$\int e^{-ik(t+x)}\,\hat{f}(k)\,dk = f(-x-t)$$

which is asymptotic to $-u_0(x,t)$. As $t \to +\infty$ it is asymptotic to

$$-\int e^{-ikt+ikx}\,S(k)\,\hat{f}(k)\,dk = -(Sf)(x-t)$$

Hence we see that S is the scattering operator

Jost function terminology:

Start with $\left(-\frac{d^2}{dx^2}+V\right)u = k^2 u$ on $0 \le x < \infty$ with the bdry condition $u(0)=0$. Define $\phi(x,k)$ to be the solution with $\phi(0,k)=0$, $\phi'(0,k)=1$. Let $f(x,k)$ be the solution agreeing with e^{ikx} for x large (assume $V(x)$ has compact support). The Jost function is the Wronskian.

$$\boxed{\mathcal{L}(k) = W(f(x,k), \phi(x,k))}$$

(Note: no problems with $k=0$ here)

and it is an entire function of k.

When $k=0$ we know $f(x,k)$ and $f(x,-k)$ are two linearly independent solutions so we have

$$\phi(x,k) = B(k)f(x,k) + A(k)f(x,-k)$$

where necessarily $B(k) = A(-k)$ as ϕ is an even fn. of k.

Calculate Wronskians:

$$\mathcal{L}(k) = W(f,\phi) = A(k)\,W(e^{ikx}, e^{-ikx}) = -2ik\,A(k)$$

Hence we have

$$\phi(x,k) = \frac{\mathcal{L}(-k)f(x,k) - \mathcal{L}(k)f(x,-k)}{2ik}$$

and this formula is valid in the limit ~~as~~ $k \to 0$.

Also

$$\mathcal{G}_{k^2}(x,x') = -\frac{\phi(x,k)\,f(x,k)}{\mathcal{L}(k)}$$

provided $\mathrm{Im}(k) > 0$ and [redacted] $\mathcal{L}(k) \neq 0$.

October 9, 1977

Summary: Consider

1) $$\left(-\frac{d^2}{dx^2}+V(x)\right)u = \lambda u \qquad \text{on} \qquad 0\leq x<\infty$$

[redacted] with a given real boundary condition at $x=0$, say $u(0)=0$ to fix the ideas. Denote by $\phi(x,\lambda)$ the solution satisfying the boundary condition normalized in some suitable way (e.g. [redacted] $\phi(0,\lambda)=0$, $\phi'(0,\lambda)=1$). Let $\Phi(x,\lambda)$ denote the solution satisfying a complementary bdry condition [redacted] such that $W(\phi,\Phi)=1$ (e.g. $\Phi(0,\lambda)=-1$, $\Phi'(0,\lambda)=0$.) Assuming that the limit point case at [redacted] $x\to+\infty$ prevails, we know that there exists an analytic function $m(\lambda)$ defined for $\operatorname{Im}\lambda\neq 0$ by the requirement that

2) $$v(x,\lambda) = m(\lambda)\phi(x,\lambda)+\Phi(x,\lambda)$$

is in $L^2(0,\infty)$. The Green's function is [redacted] given by

3) $$G_\lambda(x,x') = \phi(x_<,\lambda)\,v(x_>,\lambda)$$

since $W(\phi,v)=W(\phi,m\phi+\Phi)=W(\phi,\Phi)=1$. Hence we have

4) $$\phi(x_<,\lambda)\,v(x_>,\lambda) = \int\frac{\phi(x,\hat\lambda)\,\phi(x',\hat\lambda)}{\lambda-\hat\lambda}\,d\mu(\hat\lambda)$$

with equality as distributions.

(I could list more features of the Weyl theory - things that hold without hypotheses on V other than the limit point situation. Instead let's look at the scattering case).

Suppose V has compact support. Then there is a unique solution $f(x,k)$ of 1) where $\lambda = k^2$ such that $f(x,k) = e^{ikx}$ for $x \gg 0$. Define the Jost function to be

$$\mathcal{L}(k) = W(f(x,k), \phi(x,k^2)).$$

This is an entire function of k. For $\mathrm{Im}(k) > 0$, the solution $f(x,k)$ is square-integrable, hence we have

$$G_{k^2}(x,x') = \frac{\phi(x_<, k^2)\, f(x_>, k)}{-\mathcal{L}(k)}$$

provided $\mathrm{Im}(k) > 0$ and $\mathcal{L}(k) \neq 0$. Hence

$$v(x,\lambda) = -\frac{f(x,k)}{\mathcal{L}(k)}.$$

From the above formula for the Green's function one can see that the spectrum of the operator consists of $\{k^2 \mid k \in \mathbb{R}$ or $\mathrm{Im}\, k > 0$ and $\mathcal{L}(k) = 0\}$. Also G_λ is a meromorphic function off the cut $\lambda \geq 0$ with poles

~~I claim that the zeroes of~~ $\cancel{\mathcal{L}(k)}$ ~~for~~ $\cancel{\mathrm{Im}(k) > 0}$ ~~are all on~~ $\cancel{i\mathbb{R}_{\geq 0}}$ ~~and are all simple. To see this, suppose~~ $\cancel{\varphi, \tilde{\varphi}}$ ~~are chosen as in the example so that~~

$$\cancel{m(\lambda) = v'(0,\lambda) = -\frac{f'(0,k)}{\mathcal{L}(k)}} \quad \text{~~and~~} \quad \cancel{f(0,k) = \mathcal{L}(k)}.$$

~~From Weyl theory we know~~ $\cancel{\mathrm{Im}(\lambda) > 0 \Rightarrow \mathrm{Im}(m(\lambda)) > 0}$.

at the points $\lambda = k_0^2$ where $\mathcal{L}(k_0) = 0$ and $\mathrm{Im}(k_0) > 0$. These correspond to square-integrable eigenfunctions, so k_0 has to be purely imaginary. Also from Weyl we know that $\operatorname{Im}(\lambda) > 0 \Longrightarrow \operatorname{Im} m(\lambda) < 0$. It follows the poles of $v(x,\lambda)$, for $\lambda < 0$ are simples, and hence the zeroes of $\mathcal{L}$ in the upper half-plane are simple.

Consider next $k=0$. From

$$Hf = k^2 f$$
$$H\frac{df}{dk} = 2kf + k^2\frac{df}{dk}$$

one sees that $f(x,0)$, $\frac{df}{dk}(x,0)$ are solutions of $Hu=0$. They are independent since for $x \gg 0$, $f(x,0)=1$, $\frac{df}{dk}(x,0) = ix$. Hence if $\mathcal{L}(k) = f(0,k)$ vanishes at 0, then $\mathcal{L}'(k) = \frac{df}{dk}(0,0)$ has to be $\neq 0$.

Better proof uses the formula

$$\phi(x,k) = \frac{1}{2ik}\left(\mathcal{L}(-k)f(x,k) - \mathcal{L}(k)f(x,-k)\right)$$

which shows that if $k=0$ is a double zero of $\mathcal{L}$, then $\phi(x,0) \equiv 0$ which isn't the case.

Hence we see that the zeroes of $\mathcal{L}(k)$ for $\operatorname{Im} k \geq 0$ are all on $i\mathbb{R}_{\geq 0}$ and they are simple. ($\mathcal{L}(k) \neq 0$ for k real $\neq 0$ as e^{ikx}, e^{-ikx} are linearly independent.)

For scattering we are interested in the following solution of the DE & bdry condition

$$\psi(x,k) = f(x,-k) - \underbrace{\frac{\mathcal{L}(-k)}{\mathcal{L}(k)}}_{S(k)} f(x,k)$$

This is a meromorphic function of k, holomorphic for k real.

Consider

$$u(x,t) = \int_{-\infty}^{\infty} e^{-ikt}\, \psi(x,k)\, \alpha(k)\, dk \qquad \text{where} \quad \alpha \in C_0^\infty(\mathbb{R})$$

This is a solution of the wave equation

$$-\frac{\partial^2 u}{\partial t^2} = Hu \qquad u(0,t) = 0$$

The Riemann-Lebesgue lemma says that $u(x,t) \to 0$ as $t \to \pm\infty$ uniformly for x in a compact set. For x large we have

$$u(x,t) = \int e^{-ikt}\left(e^{-ikx} - S(k)e^{ikx}\right)\alpha(k)\,dk$$

$$= f(-x-t) - (Sf)(x-t)$$

where $f(x) = \int e^{-ikx}\alpha(k)\,dk$ and $(Sf)(x) = \int e^{ikx} S(k)\alpha(k)\,dk$.
Now if one considers the L^2 norm over $R \le x < \infty$, it is clear that $(Sf)(x-t)$ converges to zero as $t \to -\infty$. Hence we have

$$\lim_{t\to-\infty} \int_0^\infty |u(x,t) - f(-x-t)|^2\, dx = 0$$

and similarly

$$\lim_{t\to+\infty} \int_0^\infty |u(x,t) + Sf(x-t)|^2\, dx = 0$$

?

Question: Can one derive from the wave equation the fact that the wave operator:

$$f(x) = \int e^{-ikx}\alpha(k)\,dk \longmapsto u(x,0) = \int \psi(x,k)\, \alpha(k)\, dk$$

is an isometry?

October 10, 1977:

Consider the system

$$\left(\frac{d}{dx}-p\right)v_1 = -kv_2$$

$$\left(\frac{d}{dx}+p\right)v_2 = kv_1$$

on the line where p is real. It gives rise to a Schroedinger equation

$$\left(\frac{d}{dx}+p\right)\left(\frac{d}{dx}-p\right)v_1 = -k^2v_1$$

or

$$\left(-\frac{d^2}{dx^2}+V\right)v_1 = k^2v_1$$

where

(*) $$V = p^2+p'$$

Conversely suppose we are given V and wish to find p. Note that (*) is the Riccati equation belonging to the linear DE: $u''=Vu$ since

$$\left(\frac{u'}{u}\right)' = V - \left(\frac{u'}{u}\right)^2$$

so $p=\frac{u'}{u}$ satisfies (*). Hence the general solution of (*) is given by

$$p = \begin{pmatrix} u_1' & u_2' \\ u_1 & u_2 \end{pmatrix}(c)$$

c an arb. constant, u_1, u_2 (real) linear. ind. solns. of $u''=Vu$.

However the requirement that $p=\frac{u'}{u}$ be everywhere regular means that u never vanishes. This implies that any non-zero solution of $u''=Vu$ has at most one zero. Suppose to fix the ideas that V has compact support. Then solutions of $u'' = Vu$ are linear for $|x|$ large. ?

We've seen that given V there is a p with $p^2 + p' = V \iff$ there is a non-vanishing solution of $u'' = Vu$. In this case we have for $\varphi \in C_0^\infty(\mathbb{R})$

$$(H\varphi, \varphi) = -\left(\left(\frac{d}{dx}+p\right)\left(\frac{d}{dx}-p\right)\varphi, \varphi\right)$$

$$= -\left(\left(\frac{d}{dx}-p\right)\varphi, \left(-\frac{d}{dx}+p\right)\varphi\right) = \left\|\left(\frac{d}{dx}-p\right)\varphi\right\|^2 \geq 0$$

hence the spectrum of H is contained in $\mathbb{R}_{\geq 0}$.

Suppose V grows as $|x| \to +\infty$ so that the spectrum of H is discrete. For each λ there is an eigenfn. $\varphi^+(x,\lambda)$ which decays as $\lambda \to +\infty$ and φ^+ is unique up to a multiplicative constant. The j-th square integrable eigenfunction has $(j-1)$-zeroes. So if all the eigenvalues are ≥ 0 it is clear that $\varphi^+(x,0)$ is a non-vanishing solution of $u'' = Vu$.

So the moral seems to be that when the spectrum of $H = -\frac{d^2}{dx^2} + V(x)$ is contained in $\mathbb{R}_{\geq 0}$, then it is possible to factorize H into $-\left(\frac{d}{dx}+p\right)\left(\frac{d}{dx}-p\right)$

Consider $\left(-\frac{d^2}{dx^2}+V\right)u = \lambda u$ on $0 \le x < \infty$
with the boundary condition $u'(0) = hu(0)$ where $h \in \mathbb{R}$.
Denote by $\phi(x,\lambda)$ the solution with $\phi(0,\lambda)=1$, $\phi'(0,\lambda)=h$.
Assume $\phi(x,0) \neq 0$ all x, and put

$$p(x) = \frac{\phi'(x,0)}{\phi(x,0)}$$

Then

$$p^2 + p' = \frac{\phi'^2}{\phi^2} + \frac{\phi''}{\phi} - \frac{\phi'^2}{\phi^2} = V$$

so

$$\left(\frac{d}{dx}+p\right)\left(\frac{d}{dx}-p\right) = \frac{d^2}{dx^2} - V.$$

If u is C^∞ on $0 \le x < \infty$ and vanishes for $x >> 0$, then

$$\int_0^\infty \left[\left(-\frac{d^2}{dx^2}+V\right)u\right]\bar{u}\,dx = \int_0^\infty -\left[\left(\frac{d}{dx}+p\right)\left(\frac{d}{dx}-p\right)u\right]\bar{u}\,dx$$

$$= -\int_0^\infty \left(\frac{d}{dx}\left(\frac{d}{dx}-p\right)u\right)\bar{u}\,dx - \int_0^\infty \left[\left(\frac{d}{dx}-p\right)u\right]\overline{pu}\,dx$$

$$= \left[-\left[\left(\frac{d}{dx}-p\right)u\right]\bar{u}\right]_0^\infty + \int_0^\infty \left(\frac{d}{dx}-p\right)u\,\overline{\left(\frac{d}{dx}-p\right)u}\,dx$$

or

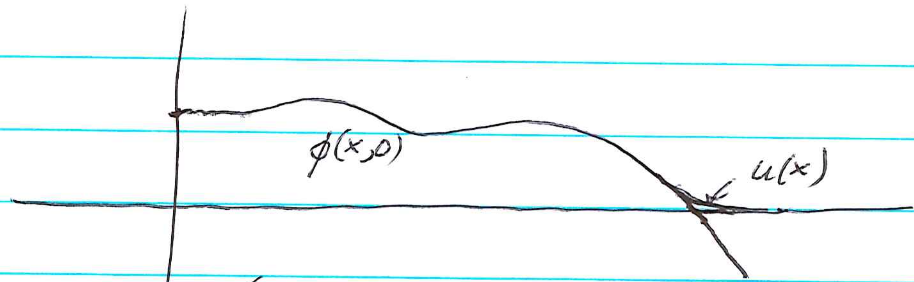
$$(Hu,u) = \left(u'(0)-p(0)u(0)\right)\overline{u(0)} + \left\|\left(\frac{d}{dx}-p\right)u\right\|^2$$

But $p(0)=h$, so if u satisfies the boundary condition we have

$$(Hu,u) \ge 0$$

and so $H \ge 0$. Conversely suppose $(Hu,u) \ge 0$ for all

$u \in C^\infty([0,\infty))$ vanishing for large x and such that $u'(0) = h\,u(0)$. Suppose $\phi(x,0)$ vanishes at $x=a$. Let $u(x) = \phi(x,0)$ for $x<a$ and 0 for $x>a$ with the corner at $x=a$ rounded out:



Then $Hu = \left(-\frac{d^2}{dx^2}+V\right)u$ will be zero except in a small nbd of a where it will have sign opposite to that of u, hence $(Hu,u) < 0$. This contradiction shows that $\phi(x,0)$ never vanishes.

Prop: Given $\left(-\frac{d^2}{dx^2}+V\right)u = \lambda u$ with bdry condition $u'(0) = h\,u(0)$. TFAE:

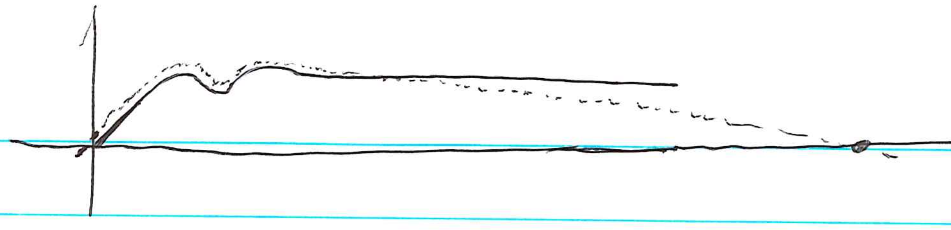
i) spectrum ≥ 0

ii) $V = p^2 + p'$ where $p(0) = h$.

There seems to be some delicacy concerned with the case $h=\infty$, i.e. the boundary condition $u(0)=0$. If we take $p(x) = \frac{\phi'(x,0)}{\phi(x,0)}$, then $p(x) \sim \frac{1}{x}$ as $x \to 0$. We still have the implication $V = p^2+p'$ with p regular on $0 \leq x < \infty$ $\Longrightarrow$ $(Hu,u) \geq 0$ for $u \in C_0^\infty([0,\infty))$ & $u(0)=0$, but it doesn't seem to be possible always to find a non-vanishing solution of $u'' = Vu$.

For example: suppose V has compact support in $(0,\infty)$ and that $u'' = Vu$ has a solution with $\phi = x$ near 0

and $\phi = 1$ near ∞ which doesn't vanish in between



It's then clear that the solution with $u(0)=\varepsilon$, $u'(0)=1$ has to have a zero at $x=-\varepsilon$ and at a point x_ε far out tending to $+\infty$ as $\varepsilon \downarrow 0$. Then any solution of the equation $u''=Vu$ must vanish between ε and x_ε, hence any solution $\neq \phi$ must vanish on $0 \le x < \infty$.

Scattering for the radial Dirac equation. Let us consider the equation

1) $$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

on $0 \le x < \infty$, p having compact support, with a boundary condition given at $x=0$, say

2) $$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(0) \text{ proportional to } \begin{pmatrix} e^{i\gamma} \\ e^{-i\gamma} \end{pmatrix} \qquad \gamma \text{ real}$$

Let $\phi(x,\lambda)$ denote the solution of the DE with the initial values $\begin{pmatrix} e^{i\gamma} \\ e^{-i\gamma} \end{pmatrix}$.

Rewrite 1) in the form

$$\underbrace{\begin{pmatrix} \frac{1}{i}\frac{d}{dx} & i\bar{p} \\ -ip & -\frac{1}{i}\frac{d}{dx} \end{pmatrix}}_{H} u = \lambda u$$

The operator H is (formally) self-adjoint; with the boundary condition 2) and the fact p has compact support, we should get a self-adjoint extension, and hence a unitary 1-parameter group whose trajectories are described by the wave equation

$$Hu = i\frac{\partial u}{\partial t}$$

For example $u(x,t) = e^{-i\lambda t}\phi(x,\lambda)$ satisfies this wave equation. More generally if $\alpha(\lambda) \in C_0^\infty(\mathbb{R})$

$$(3) \qquad u(x,t) = \int e^{-i\lambda t}\phi(x,\lambda)\alpha(\lambda)\,d\lambda$$

is a solution of the wave equation, and hence its L^2 norm over $0 \le x < \infty$ is independent of t.

Since p has compact support there are functions $A(\lambda)$, $B(\lambda)$ which are entire such that

$$4) \qquad \phi(x,\lambda) = \begin{pmatrix} A(\lambda)e^{i\lambda x} \\ B(\lambda)e^{-i\lambda x} \end{pmatrix} \qquad x >> 0$$

Recall the equation 1) is symmetric under $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} \overline{u}_2 \\ \overline{u}_1 \end{pmatrix}$ provided λ is real, hence for any solution $u_2 = \overline{u}_1$ provided this holds at some point. Hence

$$\lambda \text{ real} \implies \phi_2(x,\lambda) = \overline{\phi_1(x,\lambda)}$$

and so λ real $\implies$ $B(\lambda) = \overline{A(\lambda)}$. Combining 3) + 4)

$$u(x,t) = \int_{-\infty}^{\infty} e^{-i\lambda t}\begin{pmatrix} A(\lambda)e^{i\lambda x} \\ B(\lambda)e^{-i\lambda x} \end{pmatrix}\alpha(\lambda)\,d\lambda \qquad x >> 0$$

As $t \to +\infty$ the bottom component goes to zero in $L^2([R,\infty))$ because its of the form $g(x+t)$ with g rapidly decreasing. But from the Riemann Lebesgue lemma the same is true for $0 \leq x \leq R$ for $u(x,t)$. Hence we see that $\lim_{t\to+\infty} u_2(\cdot,t) = 0$ in L^2, and so

$$\|u(\cdot,t)\|^2 = \lim_{t\to+\infty} \|u_1(\cdot,t)\|^2 = \left\| \int e^{i\lambda(\cdot)} A(\lambda)\alpha(\lambda)\,d\lambda \right\|^2$$

$$= 2\pi \int |A(\lambda)\alpha(\lambda)|^2 d\lambda$$

Setting $t=0$ we get

$$\left\| \int \phi(x,\lambda)\,\alpha(\lambda)\,d\lambda \right\|^2 = \int |\alpha(\lambda)|^2\, 2\pi\, |A(\lambda)|^2 d\lambda$$

which shows that $d\mu(\lambda) = 2\pi |A(\lambda)|^2 d\lambda$ assuming that the system $\{\phi(x,\lambda),\ \lambda \in \mathbb{R}\}$ is complete.

? We can establish the completeness provided we use facts about the wave equation. ? Start with $u(x)$ satisfying the boundary condition and with compact support. Choose the solution with $u(x,0) = u(x)$ and forms its Fourier transform

$$u(x,t) = \int e^{-i\lambda t}\, \hat{u}(x,\lambda)\, d\lambda$$

Then it is easily seen that $H\hat{u} = \lambda \hat{u}$ and that $\hat{u}$ satisfies the boundary condition, hence $\hat{u}(x,\lambda) = \phi(x,\lambda)\alpha(\lambda)$ for some α. NOT CLEAR.

We've seen that

$$\int e^{-i\lambda t}\phi(x,\lambda)\alpha(\lambda)\,d\lambda \sim \int e^{-i\lambda t}\begin{pmatrix} e^{i\lambda x} \\ 0 \end{pmatrix} A(\lambda)\alpha(\lambda)\,d\lambda \qquad t\to+\infty$$

$$\int e^{-i\lambda t}\begin{pmatrix} 0 \\ e^{-i\lambda x} \end{pmatrix} B(\lambda)\alpha(\lambda)\,d\lambda \qquad t\to-\infty$$

and consequently [redacted] if we view solutions of the free equation in the form

$$\int e^{-i\lambda t}\begin{pmatrix} e^{i\lambda x} \\ e^{-i\lambda x}\end{pmatrix} g(\lambda)\,d\lambda$$

the scattering operator is $g(\lambda) \mapsto \frac{A(\lambda)}{B(\lambda)}\, g(\lambda)$ or

$$S(\lambda) = \text{[redacted]} \quad \frac{\overline{B(\lambda)}}{B(\lambda)}$$

where

$$B(\lambda) = \begin{vmatrix} e^{i\lambda x} & A(\lambda) e^{i\lambda x} \\ 0 & B(\lambda) e^{-i\lambda x} \end{vmatrix}$$

is the Wronskian of the solution decaying at $x = \infty$ for $\mathrm{Im}(\lambda) > 0$ with the solution $\phi(x, \lambda)$.

Let $f(z)$ be analytic for $|z| \leq 1$ (i.e. $|z| < 1+\varepsilon$), [redacted] say

$$f(z) = \sum_{n \geq 0} a_n z^n.$$

Then [redacted]

$$\overline{f\left(\frac{1}{\bar{z}}\right)} = \sum_{n \geq 0} \bar{a}_n z^{-n}$$

is analytic for $|z| \geq 1$ including $z = \infty$. Cauchy's formula yields for $|z| < 1$

$$f(z) = \frac{1}{2\pi i} \oint_{|\lambda|=1} \frac{f(\lambda)\, d\lambda}{\lambda - z}$$

$$[redacted] \quad \frac{1}{2\pi i} \oint_{|\lambda|=1} \frac{\overline{f\left(\frac{1}{\bar{\lambda}}\right)}\, d\lambda}{\lambda - z} = \frac{1}{2\pi i} \oint_{|\lambda|=R} \frac{\overline{f\left(\frac{1}{\bar{\lambda}}\right)}\, d\lambda}{\lambda - z} = \overline{f(0)}$$

Hence

$$f(z) = \overline{f(0)} + \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{f(\lambda) - \overline{f(\lambda)}}{\lambda - z}\, d\lambda$$

$$f(z) = \overline{f(0)} + \frac{1}{\pi} \int_{|\lambda|=1} \frac{\mathrm{Im}(f(\lambda))\, d\lambda}{\lambda - z}$$

~~Now suppose only that $f(z)$ is analytic for $|z|<1$, but assume $\mathrm{Im}\, f(z) \geq 0$ for $|z|<1$. Then for $r<1$ we have~~

$$f(rz) = \overline{f(0)} + \frac{1}{\pi} \int_{|\lambda|=1} [redacted] \frac{1}{\lambda - z} \cdot \mathrm{Im}\, f(r\lambda)\, d\lambda \; [redacted]$$

~~where $\mathrm{Im}\, f(r\lambda)\, d\theta = \mathrm{Im}\, f(\lambda) \frac{d\lambda}{i\lambda}$, $\lambda = e^{i\theta}$, is a measure on S^1.~~

$$f(z) = \overline{f(0)} + \frac{i}{\pi}\int_0^{2\pi} \frac{\operatorname{Im} f(e^{i\theta})}{e^{i\theta} - z}\, e^{i\theta}\, d\theta$$

$$= \overline{f(0)} + \frac{i}{\pi}\int_0^{2\pi} \frac{1}{1 - e^{-i\theta} z}\, \operatorname{Im} f(e^{i\theta})\, d\theta$$

which shows that

$$(*) \qquad a_n = 2i \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\left(\operatorname{Im} f(e^{i\theta})\, d\theta\right) \qquad n \geq 1$$

$$a_0 - \bar{a}_0 = 2i \cdot \frac{1}{2\pi}\int_0^{\infty} \operatorname{Im} f(e^{i\theta})\, d\theta \qquad n = 0$$

Next suppose only that f is analytic for $|z| < 1$ but that $\operatorname{Im} f(z) \geq 0$ for $|z| < 1$. Apply the above to $f(r \cdot z)$, where $0 < r < 1$. We get

$$f(rz) = \overline{f(0)} + \frac{2i}{2\pi}\int_0^{2\pi} \frac{1}{1 - e^{-i\theta} z}\, \operatorname{Im} f(re^{i\theta})\, d\theta$$

where $d\mu_r(\theta) = \operatorname{Im} f(re^{i\theta}) \frac{d\theta}{2\pi}$ is a positive measure on S^1 of mass $\operatorname{Im}(f(0))$. Now let $r \uparrow 1$ and use Helly selection principle to see that at least a subsequence of the $d\mu_r$ converges to a measure $d\mu$. Actually if one uses the formula

$$a_n r^n = 2i\, \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\left(\operatorname{Im} f(re^{i\theta})\, d\theta\right) \qquad n \geq 1$$

it is clear that $\int p(\theta)\, d\mu_r(\theta) \longrightarrow \int p(\theta)\, d\mu$ for every trig polynomial, hence $d\mu_r \to d\mu$ as $r \uparrow 1$. Consequently given any analytic fn. $f(z)$ in $|z| < 1$ with positive imaginary part, there is a unique measure $d\mu$

on S^1 such that

$$f(z) = \overline{f(0)} + 2i\int_0^{2\pi} \frac{1}{1-e^{-i\theta}z}\, d\mu(\theta)$$

[redacted]

$$\operatorname{Im} f(0) = \int_0^{2\pi} d\mu(\theta)$$

$$f(z) = \operatorname{Re} f(0) + i\int_0^{2\pi} \left(\frac{2}{1-e^{-i\theta}z} - 1\right) d\mu(\theta)$$

$$= \operatorname{Re} f(0) + i\int_0^{2\pi} \frac{1+e^{-i\theta}z}{1-e^{-i\theta}z}\, d\mu(\theta)$$

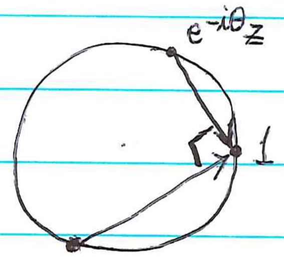
So we get

Herglotz formula: Any f holom. in $|z|<1$ with $\operatorname{Im} f(z) \geq 0$ can be uniquely represented

$$f(z) = \operatorname{Re} f(0) + i\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)$$

where $d\mu$ is a measure on S^1, and conversely any [redacted] measure on S^1 [redacted] corresponds to a holom. fn. in $|z|<1$ with positive imaginary part, unique up to a real constant.

Note $z \longmapsto \dfrac{1+e^{-i\theta}z}{1-e^{-i\theta}z}$ maps $|z|=1$ to $i\mathbb{R}$ and $z=0 \longmapsto 1$



hence it maps $|z|<1$ onto $\operatorname{Re} > 0$.

In this derivation the formula at the top of 469 should be changed to

$$\boxed{f(z) = \mathrm{Re}\, f(0) + \frac{i}{2\pi}\int_0^{2\pi} \frac{1+e^{-i\theta}z}{1-e^{-i\theta}z}\, \mathrm{Im} f(e^{i\theta})\, d\theta \quad \text{for } f \text{ analytic in } |z|\leq 1.}$$

What does this have to do with Bochner's thm. which describes $d\mu$ on S^1 in terms of pos. def. functions on $\widehat{S^1} = \mathbb{Z}$? Given $d\mu$ put

$$c_n = \int e^{+in\theta}\, d\mu(\theta).$$

Then the matrix $c_{j-k} = \int e^{+i(j-k)\theta}\, d\mu(\theta)$ is positive definite. Conversely given $\{c_n, n\in\mathbb{Z}\}$ such that $\{c_{j-k}\}_{j,k\in\mathbb{Z}}$ is positive definite we consider the Hilbert spaces V obtained by completing the set of finite support fns. $C_0(\mathbb{Z},\mathbb{C})$ on $\mathbb{Z}$ with the inner product defined by the matrix c_{j-k}. Thus you want

$$(e_j, e_k) = c_{j-k}$$

Shifting gives a unitary operator U such that e_0 is cyclic and so you get a measure $d\mu(\theta)$ such that

$$c_j = (e_j, e_0) = (U^j e_0, e_0) = \int e^{ij\theta}\, d\mu(\theta).$$

So given $d\mu(\theta)$ with the $\{c_n\}$ defined this way we want $f(z)$ analytic for $|z|<1$, $f(z) = \sum a_n z^n$ with

$$a_n = 2i\int e^{-in\theta}\, d\mu(\theta) \qquad n>0$$

$$a_0 = i\int d\mu(\theta) \qquad \text{assuming } f(0) \text{ purely imag.}$$

Hence $\quad f(z) = i c_0 + \sum_{n\geq 1} 2i\bar{c}_n z^n \quad$ and

$$\operatorname{Im}(f(z)) = \operatorname{Re}\Big(c_0 + 2\sum_{n\geq 1} \bar{c}_n z^n\Big)$$

$$= c_0 + \sum_{n\geq 1} (\bar{c}_n z^n + c_n \bar{z}^n)$$

So it seems that we get the following criterion for $\{c_n\}$ to be positive-definite: $\quad \bar{c}_n = c_{-n}$, [redacted] the sequence is bounded, and the harmonic function

$$c_0 + \sum_{n\geq 1} (c_{-n} z^n + c_n \bar{z}^n) \quad \text{is} > 0 \quad \text{for } |z|<1.$$

Note that if $\quad c_n = \int e^{+in\theta}\, d\mu$, then

$$c_0 + \sum_{n\geq 1} (c_{-n} z^n + c_n \bar{z}^n) = \int \Big(\sum_{n\geq 0} e^{-in\theta} z^n + \sum_{n\geq 1} e^{in\theta} \bar{z}^n \Big) d\mu$$

$$= \int \Big(\frac{1}{1-e^{-i\theta} z} + \frac{e^{i\theta}\bar{z}}{1-e^{i\theta}\bar{z}} \Big) d\mu$$

$$= \int \frac{1-|z|^2}{|1-e^{-i\theta} z|^2}\, d\mu \quad > 0 \quad \text{for } |z|<1$$

October 19, 1977

$$\int \frac{d\mu(x)}{\lambda - x} = \frac{1}{\lambda - b_1 -}\;\frac{a_1^2}{\lambda - b_2 -}\cdots\frac{a_{n-1}^2}{\lambda - b_n}$$

Suppose the measure is even and n is even.

$$\int \frac{d\mu(x)}{\lambda - x} = \int_0^\infty \left(\frac{1}{\lambda - x} + \frac{1}{\lambda + x}\right) d\mu(x) = \lambda \int_0^\infty \frac{2\,d\mu(x)}{\lambda^2 - x^2} = \lambda \int_0^\infty \frac{d\nu(y)}{\lambda^2 - y}$$

where $d\nu(x^2) = 2\,d\mu(x)$. Then

$$\int_0^\infty \frac{d\nu(y)}{\lambda^2 - y} = \frac{1}{\lambda}\left(\frac{1}{\lambda -}\;\frac{a_1^2}{\lambda -}\cdots\frac{a_{2n}^2}{\lambda}\right)$$

$$= \frac{1}{\lambda^2 -}\;\frac{a_1^2}{1 -}\;\frac{a_2^2}{\lambda^2 -}\cdots\frac{a_{2n}^2}{\lambda^2}$$

and I ought to be able to write this in the form of the J-fraction belonging to $d\nu$. Identity:

$$\frac{a}{1 +}\;\frac{b}{u} = a\left[\frac{u}{b+u}\right] = a\left[1 - \frac{b}{b+u}\right] = a - \frac{ab}{b+u}$$

$$\frac{a}{1 -}\;\frac{b}{u} = a - \frac{ab}{-b+u}\cdots$$

$$\frac{1}{\lambda^2 -}\;\frac{a_1^2}{1 -}\;\frac{a_2^2}{\lambda^2 -}\cdots = \frac{1}{\lambda^2 - a_1^2 -}\;\frac{a_1^2 a_2^2}{-a_2^2 + \lambda^2 - a_3^2 -}\cdots\frac{a_{2n-1}^2 a_{2n}^2}{\lambda^2 - a_{2n}^2}$$

$$= \frac{1}{\lambda^2 - a_1^2 -}\;\frac{a_1^2 a_2^2}{\lambda^2 - (a_2^2 + a_3^2) -}\;\frac{a_3^2 a_4^2}{\lambda^2 - (a_4^2 + a_5^2)}\cdots\frac{a_{2n-1}^2 a_{2n}^2}{\lambda^2 - a_{2n}^2}$$

If

$$L = aT + \quad + T^{-1}a$$

$$L^2 = aT(a)T^2 + a^2 + T^{-1}(a^2) + T^{-2}aT(a)$$

$$L^2_{\text{even}} = \begin{pmatrix} a_1^2 & a_1 a_2 & \\ a_1 a_2 & a_2^2 + a_3^2 & a_2 a_3 \\ & a_2 a_3 & a_4^2 + a_5^2 \\ & & \end{pmatrix}$$

First review scattering for Dirac system

$$\frac{d}{dx}u = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$$

on $0 \leq x < \infty$ with a boundary condition $u_1(0) = e^{i\alpha} u_2(0)$ given at 0 and with p of compact support. [redacted] $\phi(x,\lambda)$ denotes the solution with $\phi(0,\lambda) = \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix}$, $f^+(x,\lambda)$ denotes the solution of the D.E. with

$$f^+(x,\lambda) = \begin{pmatrix} e^{i\lambda x} \\ 0 \end{pmatrix} \qquad x \gg 0$$

and $f^-(x,\lambda)$ the solution with

$$f^-(x,\lambda) = \begin{pmatrix} 0 \\ e^{-i\lambda x} \end{pmatrix} \qquad x \gg 0$$

(Hence f^+ is the solution [redacted] representing an outgoing [redacted] wave for large x, and f^- represents an incoming wave). Define $A(\lambda)$, $B(\lambda)$ by

$$\phi(x,\lambda) = A(\lambda) f^-(x,\lambda) + B(\lambda) f^+(x,\lambda)$$

whence
$$A(\lambda) = W(f^+(x,\lambda), \phi(x,\lambda))$$
$$B(\lambda) = -W(f^-(x,\lambda), \phi(x,\lambda))$$
since $W(f^+, f^-) = 1$.

[redacted] Write the Dirac system in the form

$$\lambda u = \underbrace{\begin{pmatrix} \frac{1}{i}\frac{d}{dx} & i\bar{p} \\ -ip & -\frac{1}{i}\frac{d}{dx} \end{pmatrix}}_{H} u$$

and consider the one-parameter unitary group e^{-itH} belonging

to the self-adjoint operator H whose [redacted] trajectories
are solutions of the wave equation:

$$i\frac{\partial}{\partial t}u = Hu$$

[redacted] Clearly

$$u(x,t) = \int e^{-i\lambda t}\,\phi(x,\lambda)\,\alpha(\lambda)\,d\lambda$$

is a solution of the wave equation for any $\alpha \in C_0^\infty(\mathbb{R})$. For [redacted] a fixed x the Riemann-Lebesgue lemma says this goes to zero as $t \to +\infty$ or $-\infty$. The same holds [redacted] for L^2 norm taken [redacted] as x ranges over a compact set. For x large

$$u(x,t) = \int e^{-i\lambda t}\begin{pmatrix} B(\lambda)e^{i\lambda x} \\ A(\lambda)e^{-i\lambda x}\end{pmatrix}\alpha(\lambda)\,d\lambda = \begin{pmatrix} (\widehat{B\alpha})(x-t) \\ (\widehat{A\alpha})(-x-t)\end{pmatrix}$$

where $\hat{\alpha} = \int e^{i\lambda x}\alpha(\lambda)\,d\lambda$. As $t \to +\infty$ the L^2-norm of $(\widehat{A\alpha})(-x-t)$ over $[x_0,\infty)$ goes to zero; Similarly $\widehat{B\alpha}(x-t)$ goes to zero in $L^2[x_0,\infty)$ as $t \to -\infty$. Hence the [redacted] asymptotic limits of $u(x,t)$ are clear. If we compare H with the free operator H_0 having the solutions

$$\begin{pmatrix} \hat{\alpha}(x-t) \\ \hat{\alpha}(-x-t)\end{pmatrix}$$

i.e. $p=0$ with boundary condition $u_1(0) = u_2(0)$, then u is asymptotic at $t \to -\infty$ with the free solution belonging to $\widehat{A\alpha}$ and asymptotic [redacted] as $t \to +\infty$ to the free solution belonging to $\widehat{B\alpha}$. Hence the scattering operator is $\quad S\hat{\alpha} = \left(\widehat{\frac{B}{A}\alpha}\right) \quad$ or $\quad S(\lambda) = \dfrac{B(\lambda)}{A(\lambda)}$

Also from the invariance of L^2 norm under e^{-iHt} we see that

$$\left\| \int \phi(x,\lambda)\alpha(\lambda)\,d\lambda \right\|^2_{0\le x<\infty} = \|\widehat{A\alpha}\|^2_{-\infty<x<\infty} = 2\pi \int |A|^2|\alpha|^2\,d\lambda$$

Hence if $\alpha(\lambda) = \beta(\lambda)\dfrac{d\mu(\lambda)}{d\lambda}$ we find

$$\left\| \int \phi(x,\lambda)\beta(\lambda)\,d\mu(\lambda)\right\|^2_{[0,\infty)} = 2\pi \int |A|^2 |\beta|^2 \left(\frac{d\mu(\lambda)}{d\lambda}\right)^2 d\lambda$$

$$\int |\beta|^2\,d\mu(\lambda) \qquad\qquad 2\pi \int |A|^2 |\beta|^2 \frac{d\mu}{d\lambda}\,d\mu(\lambda)$$

Hence $2\pi |A|^2 \dfrac{d\mu}{d\lambda} = 1$ or

$$\boxed{\frac{d\mu}{d\lambda} = \frac{1}{2\pi|A(\lambda)|^2}}$$

Now I want to find a discrete analogue of the above, but I do not yet know what H should be. However I believe that any J-matrix L with $a_i = \frac{1}{2}$, $b_i = 0$ for large i and having no bound states, i.e. no eigenvalues outside of $[-1,1]$ should give rise to an H. This is by analogy with a S-L system $-u'' + Vu = k^2u$ which can be factored to get a Dirac system when no eigenvalues are < 0.

How does one proceed for $H = -\dfrac{d^2}{dx^2} + V$? $\phi(x,\lambda)$ denotes the eigenfunction for H satisfying the boundary condition at $x=0$ normalized. $f^+(x,k) \sim e^{ikx}$, $f^-(x,k) \sim e^{-ikx}$ where $k^2 = \lambda$. Put $L(k) = W(f^+,\phi)$ so that

$$G(x,x';\lambda) = -\frac{\phi(x,\lambda) f^+(x,k)}{\mathcal{L}(k)} \qquad \operatorname{Im}(k) > 0$$

Recall that $G(x,x';\lambda) = \int \frac{\phi(x,\hat\lambda)\phi(x';\hat\lambda)\, d\mu(\hat\lambda)}{\lambda - \hat\lambda}$ and that

$$\operatorname{Im} \int \frac{\alpha(\hat\lambda)\, d\hat\lambda}{x + i\varepsilon - \hat\lambda} = -\int \frac{\varepsilon\, \alpha(\hat\lambda)\, d\hat\lambda}{(x-\hat\lambda)^2 + \varepsilon^2} \longrightarrow -\pi\, \alpha(x) \qquad \text{as } \varepsilon \downarrow 0$$

Hence

$$\lim_{\varepsilon \downarrow 0} \operatorname{Im} G(x,x';\lambda+i\varepsilon) = -\pi\, \phi(x,\lambda)\phi(x';\lambda) \frac{d\mu(\lambda)}{d\lambda} \qquad \text{for } \lambda > 0$$

$$\parallel$$

$$-\phi(x,\lambda) \operatorname{Im}\left(\frac{f^+(x,k)}{\mathcal{L}(k)}\right) \qquad k = \sqrt{\lambda} > 0$$

But if

$$\phi(x,\lambda) = A(k) e^{-ikx} + A(-k) e^{ikx} \qquad x >> 0$$

$$= A(k) f^-(x,k) + A(-k) f^+(x,k) \qquad x \gg 0$$

then

$$\mathcal{L}(k) = W(f^+, \phi) = A(k)\, W(e^{ikx}, e^{-ikx}) = -2ik\, A(k)$$

or

$$\phi = \frac{\mathcal{L}(k)}{-2ik} f^- + \frac{\mathcal{L}(-k)}{2ik} f^+ \qquad \frac{\mathcal{L}(k)}{k} = -2iA(k)$$

$$k \frac{\phi(x,\lambda)}{|\mathcal{L}(k)|^2} = \left(-\frac{f^-(x,k)}{\mathcal{L}(-k)} + \frac{f^+(x,k)}{\mathcal{L}(k)} \right) \frac{1}{2i} = \operatorname{Im}\left(\frac{f^+(x,k)}{\mathcal{L}(k)}\right)$$

and so we get the formula

$$\frac{k}{|\mathcal{L}(k)|^2} = \pi \frac{d\mu}{d\lambda}$$

$$d\mu = \frac{k\, 2k\, dk}{\pi |\mathcal{L}(k)|^2} = \frac{2}{\pi} \frac{k^2}{|\mathcal{L}(k)|^2} dk = \frac{dk}{2\pi |A(k)|^2}$$

Next let's go to the discrete case

$$Lu = \lambda u \qquad L = aT + b + T^* a$$

where $a_n = \frac{1}{2}$ $b_n = 0$ for large n. So for large n any solution is a linear combination of the solutions z^n, z^{-n} where $\frac{1}{2}(z + z^{-1}) = \lambda$.

Here $-1 \le \lambda \le 1$ is the "cut"; off the cut there is a unique $|z| < 1$ such that $\frac{1}{2}(z + z^{-1}) = \lambda$. Can define eigenfunctions $f^{\pm}(z)$ of L by

$$f^{\pm}(z)_n = z^{\pm n} \qquad n \text{ large},$$

and a Jost function by

$$\mathcal{L}(z) = W(f^+, \phi).$$

If

$$\phi(\lambda) = A(z^{-1}) f^+(z) + A(z) f^-(z)$$

then

$$\mathcal{L}(z) = W(f^+, \phi) = A(z)\, W(f^+, f^-)$$

$$= A(z) \begin{vmatrix} z^{n+1} & z^{-n-1} \\ \frac{1}{2} z^n & \frac{1}{2} z^{-n} \end{vmatrix} = A(z) \frac{1}{2}(z - z^{-1}) = A(z)\, i \sin\theta$$

The solution ψ decaying at ∞ with $W(\phi, \psi) = 1$ is

$$\psi = -\frac{f^+(z)}{\mathcal{L}(z)} = \int \frac{\phi(\hat\lambda)\, d\mu(\hat\lambda)}{\lambda - \hat\lambda}$$

So by Stieltjes inversion $+\pi \phi(\lambda) \frac{d\mu(\lambda)}{d\lambda} = \operatorname{Im}\left(\frac{f^+(z)}{\mathcal{L}(z)}\right)$

$$\frac{\phi(\lambda)}{i \sin\theta\, |A(z)|^2} = \frac{f^+(z)}{i \sin\theta\, A(z)} + \frac{f^-(z)}{i \sin\theta\, A(z^{-1})} = 2i \operatorname{Im}\left(\frac{f^+}{\mathcal{L}}\right)$$

$$\pi \frac{d\mu(\lambda)}{d\lambda} = \frac{1}{-2\sin\theta\,|A(z)|^2}$$

Here $z = e^{i\theta}$ and we have to choose θ so that λ is on the top of the cut. Recall that z is approached from the inside of $|z|<1$. As we go around $|z|=1$ clockwise $|z|<1$ is on the left, hence $\lambda = \cos\theta$ must be on the bottom of the cut for $0<\theta<\pi$ and on the top for $\pi<\theta<2\pi$, hence I get that $\pi<\theta<2\pi$ is the top of the cut. So

$$d\mu(\lambda) = \frac{d\lambda}{2\pi\sqrt{1-\lambda^2}\,|A(z)|^2}$$

(For example, if all $a_i = \frac{1}{2}$, then $\phi_n = \frac{\sin n\theta}{\sin\theta} = \frac{z^n - z^{-n}}{2i\sin\theta}$

so $A(z) = -\frac{1}{2i\sin\theta}$ and $d\mu(\lambda) = \frac{d\lambda}{2\pi\sqrt{1-\lambda^2}}\, 4(\sqrt{1-\lambda^2})^2 = \frac{2}{\pi}\sqrt{1-\lambda^2}\, d\lambda$

which checks.)

In this example $d\mu(\lambda)$ is the image of a probability measure $d\nu$ on S^1 under the mapping $z \mapsto \frac{1}{2}(z+z^{-1})$, namely

$$d\nu(\theta) = \frac{d\theta}{4\pi\,|A(z)|^2}$$

and $d\nu(\theta)$ is invariant under $\theta \mapsto -\theta$ because $\overline{A(z)} = A(z^{-1})$, so $|A(z)|^2 = A(z)A(z^{-1})$ is symmetric.

October 26, 1977:

Let $d\nu(\theta)$ be a probability measure on S^1 such that in $\mathcal{H} = L^2(S^1, d\nu)$ the closed subspace D spanned by $1, z, z^2, \ldots$ is outgoing with respect to the unitary operator of multiplication by z. This means that $\bigcap_{n\geq 0} z^n D = 0$. If I choose a unit vector $\delta \in D$ orthogonal to zD, then we have an isomorphism

$$L^2\left(S^1, \frac{d\theta}{2\pi}\right) \longrightarrow \mathcal{H}$$

$$z^n \longmapsto z^n \delta$$

and $\delta_{ij} = (z^i\delta, z^j\delta) = \int z^{i-j} |\delta(z)|^2 d\nu(\theta)$, so

$$d\nu(\theta) = \frac{d\theta}{2\pi |\delta(z)|^2}$$

suppose $\delta(z) \in \mathbb{C}[z]$.

$$(z^i, \delta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} z^i \frac{d\theta}{\delta(z)} = \frac{1}{2\pi i}\oint \frac{z^i}{\delta(z)}\frac{dz}{z} = 0$$

for all $i \geq 1$ implies that δ has no zeroes in $|z| < 1$, hence also none in $|z| \leq 1$.

The goal here is to construct a nice orthonormal basis for $\mathcal{H}$ of a certain kind. $\delta(z)$ is orthogonal to $z^1, z^2, \ldots$ hence $z^{-n}\delta(z) \in \mathbb{C}[z^{-1}]$ for $n \geq \deg(\delta(z)) = d$, and $z^{-n}\delta(z)$ is $\perp$ to $z^{-n+1}, z^{-n+2}, \ldots, z^0, z^1, z^2, \ldots$. In particular $z^{-n}\delta(z)$ is $\perp$ to $z^n\overline{\delta(z)}$ for $n \geq d$. In fact $z^{-i}\delta(z) \perp z^j\overline{\delta(z)}$ for $i + j > d$.

$\delta(z)$

$0 \cdots\cdots\cdots d$

$j-d \cdots\cdots\cdots j$

$z^j \overline{\delta(z)}$

So what I want to do is to put $e_i = z^i \overline{\delta(z)}$ for $i \gg 0$ and $e_i = z^{-i}\delta(z)$ for $i \ll 0$. This will give an orthonormal basis outside of an interval of width d where I have to do something else.

Suppose $d = 1$. Then I have defined e_i for $i \neq 0$; hence if $\delta = \delta_0 + \delta_1 z$, then $e_{-1} = z^{-1}\delta = \delta_0 z^{-1} + \delta_1 \perp 0$ and $e_1 = z\overline{\delta(z)} = \bar{\delta}_0 z + \bar{\delta}_1$. So we get the required orthonormal basis by putting $e_0 = 1$.

Herglotz formulas:

$f(z)$ analytic for $|z| \leq 1$; if $|z|<1$

$$f(z) = \frac{1}{2\pi i}\int_{S^1} \frac{f(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int \frac{1}{1-(z/\zeta)} f(\zeta)\frac{d\zeta}{\zeta}$$

$$= \frac{1}{2\pi i}\int \sum z^n/\zeta^n \, f(\zeta)\frac{d\zeta}{\zeta} = \sum_{n\geq 0} a_n z^n$$

where

$$\frac{1}{2\pi i}\int_{S^1} \frac{f(\zeta)\,d\zeta}{\zeta^{n+1}} = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(e^{i\theta})\,d\theta = \begin{cases} a_n & n\geq 0 \\ 0 & n<0 \end{cases}$$

Conjugating

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\,\overline{f(e^{i\theta})}\,d\theta = \begin{cases} \bar{a}_{-n} & n \leq 0 \\ 0 & n>0 \end{cases}$$

so

$$(*) \qquad \frac{i}{\pi}\int_0^{2\pi} e^{-in\theta}\, \operatorname{Im} f(e^{i\theta})\,d\theta = \begin{cases} a_n & n>0 \\ a_0 - \bar{a}_0 & n=0 \\ -\bar{a}_{-n} & n<0 \end{cases}$$

Hence

$$f(z) = \frac{a_0+\bar{a}_0}{2} + \frac{a_0 - \bar{a}_0}{2} + \sum_{n\geq 1} a_n z^n$$

$$= \operatorname{Re} f(0) + \frac{i}{\pi}\int_0^{2\pi} \underbrace{\Big\{\frac{1}{2} + \sum_{n\geq 1} e^{-in\theta} z^n\Big\}}_{} \operatorname{Im} f(e^{i\theta})\,d\theta$$

$$\frac{1}{2} + \frac{e^{-i\theta}z}{1-e^{-i\theta}z} = \frac{1}{2}\,\frac{1+e^{-i\theta}z}{1-e^{-i\theta}z}$$

$$\boxed{f(z) = \operatorname{Re} f(0) + \frac{i}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \operatorname{Im} f(e^{i\theta})\,d\theta}$$

$$\operatorname{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z} = \operatorname{Re}\frac{(e^{i\theta}+z)(e^{-i\theta}-\bar{z})}{|e^{i\theta}-z|^2} = \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

$$\boxed{\operatorname{Im} f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\operatorname{Im} f(e^{i\theta})\,d\theta}$$

Poisson formula for harmonic functions.

If now $f(z)$ is analytic for $|z|<1$ and $\operatorname{Im} f(z) > 0$, then for $0<r<1$ we apply the above to $f(rz)$ getting

$$f(rz) = \operatorname{Re} f(0) + \frac{i}{\cancel{2\pi}}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu_r(\theta)$$

where $d\mu_r(\theta)$ is the measure $\operatorname{Im} f(re^{i\theta})\frac{d\theta}{2\pi}$. Also we have

$$2\frac{i}{\cancel{\pi}}\int_0^{2\pi} e^{-in\theta}\,d\mu_r(\theta) = \begin{cases} a_n r^{-n} & n>0 \\ a_0-\bar{a}_0 & n=0 \\ -\bar{a}_{-n} r^{-n} & n<0 \end{cases}$$

which shows that as $r\uparrow 1$ the Fourier coeffs. of $d\mu_r(\theta)$ converge. Since trigonometric polys are dense in $C(S^1)$ by Fejer, we see that for any $\varphi\in C(S^1)$

$$\lim_{r\uparrow 1}\int \varphi(\theta)\,d\mu_r(\theta) \quad \exists$$

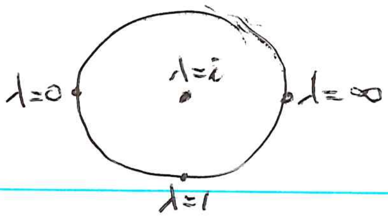
hence we get a positive linear fnl. on $C(S^1)$, hence a measure $d\mu(\theta) = \lim_{r\uparrow 1} d\mu_r(\theta)$. We have

$$f(z) = \operatorname{Re} f(0) + i\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)$$

where we think of $d\mu(\theta)$ as $\operatorname{Im}(f(e^{i\theta}))\frac{d\theta}{2\pi}$.

Now I want to translate from the disk $|z|<1$ to the upper half plane $\operatorname{Im}\lambda>0$. Use the mapping

$$z = \frac{\lambda - i}{\lambda + i}$$



$$\lambda = \frac{1}{i}\,\frac{z+1}{z-1}$$

Suppose $f(\lambda)$ is analytic on the closed upper half-plane including $\lambda=\infty$; then $f\left(\frac{1}{i}\frac{z+1}{z-1}\right)$ is analytic for $|z|\le 1$. Put $\zeta = e^{i\theta} = \frac{x-i}{x+i}$ so that as runs over $\mathbb{R}$, ζ runs over $S^1 - \{1\}$.

$$i\,\frac{\zeta+z}{\zeta-z} = i\,\frac{\frac{x-i}{x+i}+\frac{\lambda-i}{\lambda+i}}{\frac{x-i}{x+i}-\frac{\lambda-i}{\lambda+i}} = i\,\frac{(x\lambda+\cancel{ix}-\cancel{i\lambda}+1)+(x\lambda-\cancel{ix}+\cancel{i\lambda}+1)}{(\cancel{x\lambda}+ix-i\lambda+\cancel{1})-(\cancel{x\lambda}-ix+i\lambda+\cancel{1})}$$

$$= \frac{1+\lambda x}{x-\lambda} = \frac{1+\lambda x - x^2 + x^2}{x-\lambda} = \frac{1+x^2}{x-\lambda} - x$$

$$= \left(\frac{1}{x-\lambda} - \frac{x}{1+x^2}\right)(1+x^2)$$

Also $\quad d\theta = \dfrac{d\zeta}{i\zeta} = \dfrac{1}{i}\left(\dfrac{dx}{x-i} - \dfrac{dx}{x+i}\right) = \dfrac{2dx}{x^2+1}$

So we get the representation formula

$$f(\lambda) = \operatorname{Re} f(i) + \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{x-\lambda} - \frac{x}{1+x^2}\right)\operatorname{Im} f(x)\,dx$$

valid for f analytic at $\lambda=\infty$ and on $\operatorname{Im}\lambda \ge 0$.

$$\operatorname{Im} f(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Im}(\lambda)}{|x-\lambda|^2}\operatorname{Im} f(x)\,dx \qquad \text{Poisson formula.}$$

Somehow the goal of the bounded type theory is to extend the validity of these formulas.

Let's now consider $f(\lambda)$ analytic in $\operatorname{Im}\lambda > 0$ with positive imaginary part. Then by the Herglotz thm for $f\left(\frac{1}{i}\frac{z+1}{z-1}\right)$ we get a representation in terms of a measure on S^1:

$$f(\lambda) = f\left(\frac{1}{i}\,\frac{z+1}{z-1}\right) = \operatorname{Re} f(i) + \int_0^{2\pi} i\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)$$

Think of $\mu(\theta)$ as a monotone right-continuous function on $0 \le \theta \le 2\pi$ with $\mu(0) = 0$, and $p = \mu(2\pi) - \mu(2\pi-) =$ measure of $\{2\pi\}$. We can parameterize $0 < \theta < 2\pi$ using $x \in \mathbb{R}$ as above, so we get a bounded measure $d\mu(x)$ on $\mathbb{R}$ given by this isomorphism: $\mathbb{R} \xrightarrow{\sim} S^1 - \{1\}$. So

$$f(\lambda) = \operatorname{Re} f(i) + p\, i\,\frac{1+z}{1-z} + \int_{-\infty}^{\infty} \frac{1+\lambda x}{x-\lambda}\, d\mu(x)$$

Hence we get the representation for f

$$f(\lambda) = \operatorname{Re} f(i) + p\lambda + \int_{-\infty}^{\infty} \frac{1+\lambda x}{x-\lambda}\, d\mu(x)$$

where $p \ge 0$ and $d\mu(x)$ is a bdd measure on $\mathbb{R}$.

It's convenient to replace $d\mu(x)$ by $\frac{1}{\pi}\frac{d\nu(x)}{(1+x^2)}$, whence this becomes

$$\boxed{f(\lambda) = c + p\lambda + \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{1}{x-\lambda} - \frac{x}{1+x^2}\right) d\nu(x)}$$

and we have: Any $f(\lambda)$ holomorphic for $\operatorname{Im}(\lambda) > 0$ with positive imaginary part can be uniquely represented in the above form where $c \in \mathbb{R}$, $p \ge 0$, and $d\nu(x)$ is a measure on $\mathbb{R}$ such that $\int \frac{d\nu(x)}{1+x^2} < \infty$, and conversely.

Special Case: $p=0$ and $\displaystyle\int \frac{|x|\,d\nu(x)}{1+x^2} < \infty$, or equivalently $\displaystyle\int_{|x|>R} \frac{d\nu(x)}{|x|} < \infty$.

Then we can define a holomorphic function in the UHP

$$h(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\nu(x)}{x-\lambda}$$

having the same [redacted] imaginary part as f, namely

$$\operatorname{Im} f(\lambda) = \text{[redacted]}\ \frac{\operatorname{Im}\lambda}{\pi}\int_{-\infty}^{\infty} \frac{d\nu(x)}{|x-\lambda|^2}$$

So we know that $f(\lambda) - h(\lambda)$ is a real constant. In this special case we've fixed a canonical choice for the class of all f belonging to the measure $d\nu$.

Suppose we let [redacted] $\lambda = it$ where $t \longmapsto +\infty$. Then

$$\left|\frac{1}{x-it}\right| = \frac{1}{(x^2+t^2)^{1/2}} < \frac{1}{(x^2+1)^{1/2}} < C\,\frac{|x|}{x^2+1}$$

and $\displaystyle\int \frac{|x|}{x^2+1}\,d\nu(x) < \infty$. Also $\displaystyle\lim_{t\to\infty} \frac{1}{x-it} = 0$, so by the Lebesgue dominated convergence thm. $\displaystyle\lim_{t\to+\infty} h(it) = 0$. In fact h has to go to zero uniformly as $\lambda \to \infty$ in a vertical strip.

[redacted] Since $\displaystyle\operatorname{Im} f(\lambda) = p\operatorname{Im}\lambda + \frac{\operatorname{Im}\lambda}{\pi}\int \frac{d\nu(x)}{|\lambda-x|^2}$ it is clear that $\operatorname{Im} f(\lambda) \geq p\operatorname{Im}\lambda$, so p has to be zero if f approaches 0 in [redacted] vertical directions. [redacted]

~~Proof why f(λ) holom. for Im λ > 0 with positive imaginary part and such that lim f(it) = 0 can be uniquely~~

October 30, 1977

Scattering: Start with $U_0 =$ mult. by z on $L^2(S^1, \frac{d\theta}{2\pi})$ and let U be a perturbation of U_0, say $U = U_0 V$ where V is a unitary operator of compact support, i.e. equal to the identity on the orthogonal complement of $\langle z^i, |i| \leq N \rangle$. Then we have an outgoing subspace $\langle z^i, i > N \rangle$ and an incoming subspace $\langle z^i, i < -N \rangle$; (strictly speaking we have to work orthogonally to the bound states).

Take invariant viewpoint: We have a unitary operator U on a Hilbert space $\mathcal{H}$ with an outgoing subspace D^+ and an incoming subspace D^-; assume U has multiplicity 1, whence D^+/UD^+ is one-dimensional and also $D^-/U^{-1}D^-$. Suppose also that $D^+ \perp D^-$ and that $\mathcal{H}/D^+ + D^-$ is finite dimensional. Can I describe this set-up completely? On $\mathcal{H}/D^+ + D^-$ you get an operator given by U followed by projection which is a contraction operator, hence you get a polynomial with roots in the unit circle as an invariant.

Suppose $d\nu = \frac{d\theta}{2\pi|\delta|^2}$ is a measure on S^1 where $\delta(z) = \delta_0 + \delta_1 z + \cdots + \delta_d z^d$ is a poly of degree d having no roots for $|z| \leq 1$ and such that $d\nu$ is a probability measure. Put $\mathcal{H} = L^2(S^1, d\nu)$. Then clearly $(z^i\delta, z^j\delta) = \delta_{ij}$, hence if $D =$ closed space $\{z^n, n \geq 0\}$ we have δ is a unit vector spanning $D \ominus zD$. The problem is to construct a nice orthonormal basis $\{e_n\}$ for $\mathcal{H}$ which contains the elements $e_n = z^{-n}\delta(z)$ $n >> 0$

and $e_n = z^n \delta^*(z)$ for $n \ll 0$. Note that

$$\left(z^{-m}\delta(z), z^n\delta^*(z)\right) = \int z^{-m-n}\, \delta(z)\, \delta(z) \frac{d\theta}{|\delta(z)|^2\, 2\pi}$$

$$= \frac{1}{2\pi i}\int z^{-m-n-1} \frac{\delta(z)}{\delta^*(z)} dz = \frac{-1}{2\pi i}\int z^{-m-n+1} \frac{\delta(z)}{\delta^*(z)}\, d\left(\frac{1}{z}\right)$$

As $\delta^*(z)$ is analytic for $|z| \geq 1$, so if $m+n-1 \geq d$ or $m+n > d$, we get zero.

In the scattering situation we can construct part of the basis e_i as follows, namely, you choose e_i to be unit vectors in $D^+ \ominus zD^+$ and translate these around via U. This defines e_i for $i \gg 0$ up to reindexing and a constant of absolute value 1. Similarly e_i $i \ll 0$ is defined using D^-.

Interesting point: Let us consider a perturbation situation $U = U_0\Theta$ where U_0 is a shift (precisely a shift $U_0 e_i = e_{i+1}$ on an orthonormal basis) and Θ is a "compact support" unitary operator. Then the scattering operator S is a rational function of z whose poles are outside S^1 and whose zeroes are inside S^1. I think that also

$$S = \frac{\delta^*(z)}{\delta(z)}$$

where $\delta(z)$ is a polynomial in z whose roots are outside S^1. $\delta(z)$ should be determined up to ± 1 by S so in fact we should have a canonical choice for cyclic vector. [redacted] It is probably necessary to assume there are no bound states.

It seems now that I have to understand these Schur functions

October 31, 1977:

Suppose we return to a scattering J-matrix situation: L such that $a_i = \frac{1}{2}$ $b_i = 0$ for i large. We have

$$\phi(\lambda)_n = A(z) z^{-n} + A(z^{-1}) z^n \qquad n >> 0$$

where $\lambda = \frac{z+z^{-1}}{2}$. Here $A(z)$ [redacted] is a Laurent polynomial and $A(z) = 0$, $|z| < 1$, signifies a bound state - hence λ is real and outside the cut $[-1, 1]$. Note that for $|z| = 1$ [redacted] we have

$$\phi(\lambda)_n = \overline{A(z)}\, z^{+n} + \overline{A(z^{-1})}\, z^{-n}$$

so that $A(z^{-1}) = \overline{A(z)}$ or $A(\bar{z}) = \overline{A(z)}$ so that A is a Laurent poly (no) with real coefficients. Spectral measure is determined by looking at the decaying soln. $\psi(z)$ with [redacted] $W(\phi, \psi) = 1$. If $|z| < 1$, then

$$\psi(z)_n = c\, z^n \qquad n >> 0$$

so

$$W(\phi, \psi) = W(A(z) z^{-n}, c z^n) = A(z) c \begin{vmatrix} z^{-n-1} & z^{n+1} \\ \frac{1}{2} z^{-n} & \frac{1}{2} z^n \end{vmatrix}$$

$$1 = A(z) c\, \frac{1}{2}(z^{-1} - z)$$

and [redacted] $\psi(z) = \dfrac{-1}{(z - z^{-1})/2} \, \dfrac{z^n}{A(z)}$

$$\psi(z) = \int \frac{\phi(x)}{\lambda - x} \, d\mu(x)$$

Stieltjes inversion formula: If

$$F(\lambda) = \int \frac{f(x)dx}{\lambda - x} \qquad \text{where } f \text{ is real}$$

$$\lim_{\varepsilon \downarrow 0} \operatorname{Im}(F(a+i\varepsilon)) = -\pi f(a) \qquad \text{for } a \text{ real.}$$

So recall $\lambda = (z+z^{-1})/2$. Let $a \in (-1,1)$; we want $\lambda = a + i\varepsilon$ where $\varepsilon \downarrow 0$. Rather we want to take $\lim -\frac{1}{\pi}\psi(z)$ as $\varepsilon \downarrow 0$ where $\lambda = a+i\varepsilon = \frac{z+z^{-1}}{2}$ and $|z|<1$.

The point is that if $z = e^{i\theta}$ then as θ goes from 0 to π $\lambda = \cos\theta$ goes from 1 to -1 and points on the left of S^1 are points on the left for to path from 1 to -1, hence in the lower half plane. Hence we ~~want~~ $\pi < \theta < 2\pi$ and we want to take $\operatorname{Im} \psi(e^{i\theta})$. Let $d\mu(\lambda) = \alpha(\lambda)\, d\lambda$

$$\psi(z) = \frac{i}{\sin\theta} \, \frac{z^n}{A(z)}$$

$$\operatorname{Im} \psi(z) = \frac{1}{\sin\theta}\left(\frac{z^n}{A(z)} + \frac{z^{-n}}{\overline{A(z)}}\right)\frac{1}{2} = -\pi\left(A(z)z^{-n} + A(z^{-1})z^{n}\right)\alpha(\lambda)$$

$$\frac{1}{-2\pi \sin\theta} \, \frac{1}{|A(z)|^2} = \alpha(\lambda)$$

But $d\lambda = -\sin\theta \, d\theta$ so this is $d\mu(\lambda) = \alpha(\lambda)\,d\lambda = \dfrac{d\theta}{2\pi |A(z)|^2}$

<u>Example:</u> $\phi(\lambda)_n = \dfrac{z^n - z^{-n}}{z - z^{-1}}$ so $A(z) = \dfrac{-1}{z - z^{-1}}$

or $\phi(\lambda)_n = \dfrac{z^n + z^{-n}}{\sqrt{2}}$ $A(z) = \dfrac{1}{\sqrt{2}}$

In the above $A(z)$ is not quite a Laurent polynomial. $\phi(\lambda)$ is a Laurent polynomial in z, so

$$\mathcal{L}(z) = W(z^n, \phi(\lambda)) = A(z)\, W(z^n, z^{-n}) = A(z)\left(\frac{z - z^{-1}}{2}\right)$$

is a Laurent poly in z. One has $\overline{\mathcal{L}(z)} = \mathcal{L}(z^{-1}) = \mathcal{L}(\bar{z})$ for $|z| = 1$, hence $\mathcal{L}$ has real coefficients: $\mathcal{L}(z) \in \mathbb{R}[z, z^{-1}]$. ~~Note that~~ $\overline{A(z)} = A(\bar{z})$ means that the measure is symmetric under $z \to \bar{z}$ on S^1.

Let $z \to 0$ whence $\lambda \mapsto \infty$. Recall that

$$\phi(\lambda)_n = \frac{\lambda^{n-1}}{a_1 \cdots a_{n-1}} + \text{lower terms}$$

$$\phi(\lambda)_{n+1} = \frac{\lambda^n}{a_1 \cdots a_n} + \text{lower terms.}$$

also $\lambda = \dfrac{z + z^{-1}}{2} \sim \dfrac{z^{-1}}{2}$, hence

$$\mathcal{L}(z) \sim \begin{vmatrix} (2\lambda)^{-n-1} & \dfrac{\lambda^n}{a_1 \cdots a_n} \\ \frac{1}{2}(2\lambda)^{-n} & a_n \dfrac{\lambda^{n-1}}{a_1 \cdots a_{n-1}} \end{vmatrix} = -\frac{2^{-(n+1)}}{a_1 \cdots a_n}$$

So it seems that $\mathcal{L}(z) \in \mathbb{R}[z]$ with non-zero leading coefficient. Also we know that $\mathcal{L}(z) = 0$ for $|z| < 1$ corresponds to bound states, hence with no bound states $\mathcal{L}(z)$ has no roots inside S^1. So it seems that $L(z)$ should be the analogue of $\delta(z)$ hence the good measure on the circle is

$$\frac{d\theta}{2\pi\,|L(z)|^2}$$

Note that $L(\bar{z}) = \overline{L(z)}$ implies this measure is symmetric under conjugation. Also if $\delta(z)\overline{\delta(z)} = \delta(\bar{z})\overline{\delta(\bar{z})}$ for $|z| = 1$, then $\delta(z)\,\delta^*(z) = \delta(z^{-1})\,\delta^*(z^{-1})$, hence

$$\underbrace{\frac{\delta(z)}{\delta^*(z^{-1})}}_{\text{analytic for } |z|<1} = \underbrace{\frac{\delta(z^{-1})}{\delta^*(z)}}_{\text{analytic for } |z|>1}$$

must be a constant which one can suppose to be one if one multiplies δ by a constant of modulus one. Thus $\delta(z) = \delta^*(z^{-1}) = \overline{\delta(\bar{z})}$, so $\delta \in \mathbb{R}[z]$.

November 2, 1977:

Schur's analysis of $f(z)$ analytic for $|z|<1$ modulus ≤ 1 there. Let $c_0 = f(0)$. If $|c_0|=1$, then by maximum modulus $f(z) \equiv c_0$. Otherwise $|c_0|<1$, so we can find a linear fractional transformation carrying c_0 to 0. So if we define $g(z)$ by

$$f(z) = \frac{g(z)+c_0}{\bar{c}_0 g(z)+1} = \begin{pmatrix} 1 & c_0 \\ \bar{c}_0 & 1 \end{pmatrix} g(z)$$

we have $|g(z)|<1$ for $|z|<1$ and $g(0)=0$. By maximum modulus it follows that $\left|\frac{g(z)}{z}\right| \leq 1$ for $|z|<1$, so we have

$$f(z) = \begin{pmatrix} 1 & c_0 \\ \bar{c}_0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} f_1(z)$$

Now repeat the process. If it stops in n steps, then

$$f(z) = \begin{pmatrix} 1 & c_0 \\ \bar{c}_0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_1 \\ \bar{c}_1 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & c_n \\ \bar{c}_n & 1 \end{pmatrix} (0)$$

and f is a rational function of degree n. In the other case we get a sequence of Schur parameters $c_0, c_1, c_2, \ldots$ in the disk and a sequence of functions $f_n \ni$

$$f(z) = \begin{pmatrix} 1 & c_0 \\ \bar{c}_0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} (f_n(z))$$

Now it remains only to show that for $|z|<1$ the sequence of analytic functions

$$g_n(z) = \begin{pmatrix} 1 & c_0 \\ \bar{c}_0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & c_n \\ \bar{c}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} (t_n),$$

where t_n is an arbitrary point in the disk, converges to $f(z)$.

November 4, 1977.

Schur's analysis. Suppose $f(z)$ analytic in $|z|<1$ and of modulus $\leqslant 1$ there. Put $f(0)=a_0$. If $|a_0|=1$, f has to be constant and otherwise we can define $g_1(z)$ with similar properties so that

$$f(z) = \begin{pmatrix} 1 & a_0 \\ \bar{a}_0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}(g_1(z))$$

Provided f is not a rational function of z we can continue this process indefinitely to get

$$f(z) = \begin{pmatrix} 1 & a_0 \\ \bar{a}_0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_1 \\ \bar{a}_1 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & a_n \\ \bar{a}_n & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}(g_{n+1}(z))$$

where $|a_i|<1$ and g_{n+1} is also an endomorphism of the disk D: $|z|<1$. I claim that for $|z|<1$, the sequence of linear fractional transformations of D:

$$S_\lambda(0,n) = \begin{pmatrix} 1 & a_0 \\ \bar{a}_0 & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_n \\ \bar{a}_n & 1 \end{pmatrix}$$

converges to a constant map. This can be seen in two ways.

First one can sees that $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, λ fixed of modulus <1 shrinks non-Euclidean distances by a factor $\leqslant |\lambda|$. This is because the non-Euclidean distance is

$$ds = \frac{|dz|}{1-|z|^2}$$

Hence the image under $S_\lambda(0,n)$ of $|z|=1$ is a circle having non-Euclidean radius on the order of $|\lambda|^{n-1}$ as $n \to \infty$.

Secondly, put $S_\lambda(0,n) = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$. Then

$$S_\lambda(0,n)\, z_1 - S_\lambda(0,n)\, z_2 = \frac{A_n z_1 + B_n}{C_n z_1 + D_n} - \frac{A_n z_2 + B_n}{C_n z_2 + D_n}$$

$$= \frac{\det\left(S_\lambda(0,n)\right)(z_1 - z_2)}{(C_n z_1 + D_n)(C_n z_2 + D_n)}$$

and

$$\det S_\lambda(0,n) = \lambda^n.$$

If we take z_1, z_2 to be endomorphisms of D, then it follows from this formula that the endos. $S_\lambda(0,n) z_1(\lambda)$ and $S_\lambda(0,n) z_2(\lambda)$ have the same Taylor series in degrees $< n$. On the other hand if f is an endo of D, then Cauchy's formula shows that the coefficients of f are of modulus ≤ 1 consequently

$$S_\lambda(0,n)\, z_1(\lambda) = \underbrace{c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1}}_{\text{independent of } z_1(\lambda)} + c_n \lambda^n + \cdots$$

where

$$|c_n \lambda^n + \cdots| \leq |\lambda|^n + \cdots = \frac{|\lambda|^n}{1 - |\lambda|}$$

so it's clear that as $n \to \infty$ the endos $S_\lambda(0,n)$ converge to the constant map with value

$$\lim_{n\to\infty} S_\lambda(0,n)(0) = \lim_{n\to\infty} \frac{B_n}{D_n}$$

Next consider orthogonal polys. on S^1. Let ds

be a measure on S^1 and let $p_n(z)$ be the sequence of orthogonal polynomials, which are monic, obtained from the sequence $1, z, z^2, \cdots$ in the Hilbert space $L^2(S^1; d\nu)$. Thus p_n is the unique monic polynomial of degree n which is orthogonal to $1, z, \ldots, z^{n-1}$. If $f(z)$ is a function analytic on S^1, put $f^*(z) = \overline{f(z^*)}$, $z^* = (\bar{z})^{-1}$, for the conjugate function: $(a z^n)^* = \bar{a} z^{-n}$.

Then

$$(z^i, p_n) = \int_{S^1} z^i p_n^* \, d\nu = (p_n^*, z^{-i}) = (z^n p_n^*, z^{n-i})$$

so we see that $z^n p_n^*$ is the poly of degree 1 with constant term 1 which is orthogonal to $z, z^2, \ldots, z^n$.

$z p_n$ is monic of degree $n+1$ orth. to $z, \ldots, z^n$ so $z p_n - p_{n+1}$ is of degree n and orth. to $z, \ldots, z^n$. Thus we have

$$z p_n - p_{n+1} = -\alpha_n z^n p_n^*$$

for some constant α_n; in fact $\alpha_n = p_{n+1}(0)$. So we have the recursion relation

$$p_{n+1} = z p_n + \alpha_n (z^n p_n^*)$$

starting from $p_0 = 1$. If I put $q_n = z^n p_n^*$ then

$$p_{n+1}^* = z^{-1} p_n^* + \bar{\alpha}_n (z^{-n} p_n)$$

$$z^{n+1} p_{n+1}^* = z^n p_n^* + \bar{\alpha}_n z p_n$$

or

$$q_{n+1} = q_n + \bar{\alpha}_n z p_n$$

so we get

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} z & \alpha_n \\ \bar{\alpha}_n z & 1 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & \alpha_n \\ \bar{\alpha}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

~~Change α_n to α_{n+1}.~~

So the difference equation being studied is

$$u_{n+1} = \begin{pmatrix} 1 & \alpha_n \\ \bar{\alpha}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} u_n$$

I think it would be better to use

$$\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \bar{\alpha}_n \\ \alpha_n & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} q_n \\ p_n \end{pmatrix}$$

for then we have the inverted form

$$\frac{q_n}{p_n} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\bar{\alpha}_n \\ -\alpha_n & 1 \end{pmatrix} \frac{q_{n+1}}{p_{n+1}}$$

November 5, 1977

Review discrete ~~strings~~: We have a [redacted] flexible weightless thread [redacted] under unit tension carrying particles m_i [redacted] at the abscissa x_i. Put $l_i = x_{i+1} - x_i$ and let u_i = displacement of ith particle, v_i = slope of the ith ~~segment~~ (between x_i, x_{i+1}). Then the equation of motion is

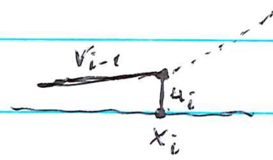
$$m_i \ddot{u}_i = \frac{u_{i+1}-u_i}{l_i} - \frac{u_i - u_{i-1}}{l_{i-1}} = v_i - v_{i-1}$$

$$v_i = \frac{u_{i+1}-u_i}{l_i}$$

A periodic solution: $u_i(t) = u_i e^{-i\omega t}$ leads to the equation

$$-\omega^2 m_i u_i = v_i - v_{i-1}$$

or $$\lambda m_i u_i = v_i - v_{i-1} \qquad \text{where } \lambda = -\omega^2.$$

Forward propagation: Given u_i, v_{i-1} 

to determine v_i, u_{i+1}. Gives equations:

$$v_i = v_{i-1} + \lambda m_i u_i$$

$$u_{i+1} = u_i + l_i v_i$$

or

$$\begin{pmatrix} v_i \\ u_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_i & 1 \end{pmatrix} \begin{pmatrix} v_i \\ u_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_i & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda m_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_{i-1} \\ u_i \end{pmatrix}$$

~~Backward propagation formula is~~

$$\cancel{\begin{pmatrix} v_{i-1} \\ u_i \end{pmatrix} = \begin{pmatrix} 1 & -\lambda m_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -l_i & 1 \end{pmatrix} \begin{pmatrix} v_i \\ u_{i+1} \end{pmatrix}}$$

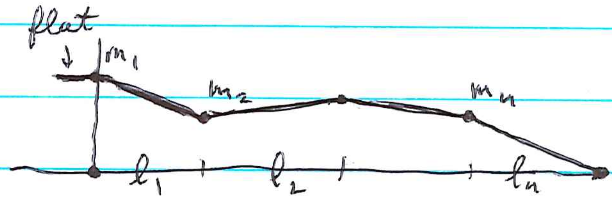
Backward propagation:

$$\left(\frac{-V_{i-1}}{u_i}\right) = \lambda m_i + \left(-\frac{V_i}{u_i}\right) \qquad \left(\frac{u_i}{-V_i}\right) = \left(\frac{u_{i+1}}{-V_i}\right) + \ell_i$$

hence

$$\left(\frac{u_i}{-V_{i-1}}\right) = \cfrac{1}{\lambda m_i + \cfrac{1}{\ell_i + \left(\cfrac{u_{i+1}}{-V_i}\right)}}$$

If we now require the m_i and ℓ_i to be defined for $i \geqslant 1$ (think of $x_1 = 0$), then define boundary conditions by $V_0 = 0$ and $u_{n+1} = 0$:



then the eigenvalues are given by the poles of

$$\frac{u_1}{-V_0} = \cfrac{1}{\lambda m_1 + \cfrac{1}{\ell_1 + \cdots \cfrac{1}{\lambda m_{n-1} + \cfrac{1}{\ell_n}}}}$$

Recall the problem: To start with a J-matrix

$$\begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & \ddots & \ddots \end{pmatrix}$$

having spectrum in $-1 \leq \lambda \leq 1$ and to construct a factorization of it involving some sort of discrete analogue of the Dirac-style DE

$$\frac{d}{dx} u = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u .$$

It seems now that I have almost a solution of this problem using orthogonal polys. on S^1.

Let $d\nu$ be a measure on S^1 and let $\phi_0, \phi_1, \ldots$ be the associated sequence of orthogonal polynomials. This means that ϕ_n is the unique poly of degree n having positive leading coefficient which is orthogonal to $1, z, \ldots, z^{n-1}$:

$$(z^i, \phi_n) = \int z^i \phi_n \, d\nu = 0 \qquad 0 \leq i < n,$$

and is such that $\|\phi_n\|^2 = 1$. Let $\phi_n^*(z) = \overline{\phi(z^*)}$ where $z^* = (\bar{z})^{-1}$. Then $z^n \phi_n^*$ is the unique poly of degree n with norm 1, positive constant term, orthogonal to $z, z^2, \ldots, z^n$. $z\phi_n$ is of degree $n+1$ orth to $z, \ldots, z^n$ with positive leading term, hence $\exists!\ k_n > 0$ with $z\phi_n - k_n\phi_{n+1}$ of degree $\leq n$. Clearly $z\phi_n - k_n \phi_{n+1}$ is orth. to $z, \ldots, z^n$ hence we have ! h_n $\ni$

$$z\phi_n = k_n \phi_{n+1} - h_n z^n \phi_n^*$$

As $(\phi_{n+1}, z^n \phi_n^*) = 0$, we have $1 = \|z\phi_n\|^2 = k_n^2 + |h_n|^2$ and

$k_n = \sqrt{1 - |h_n|^2}$

Also

$$z^{-1}\phi_n^* = k_n \phi_{n+1}^* - \bar{h}_n z^{-n}\phi_n$$

$$z^n\phi_n^* = k_n\left(z^{n+1}\phi_{n+1}^*\right) - \bar{h}_n z\phi_n$$

Thus

$$k_n\begin{pmatrix}\phi_{n+1}\\ z^{n+1}\phi_{n+1}^*\end{pmatrix} = \begin{pmatrix}1 & h_n\\ \bar{h}_n & 1\end{pmatrix}\begin{pmatrix}z\phi_n\\ z^n\phi_n^*\end{pmatrix}$$

or

$$\begin{pmatrix}\phi_{n+1}\\ z^{n+1}\phi_{n+1}^*\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{1-|h_n|^2}} & \frac{h_n}{\sqrt{1-|h_n|^2}}\\ \frac{\bar{h}_n}{\sqrt{1-|h_n|^2}} & \frac{1}{\sqrt{1-|h_n|^2}}\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\phi_n\\ z^n\phi_n^*\end{pmatrix}$$

I'd like to write these equations in a more symmetrical form. It seems to be necessary to separate into n even and n odd.

So start with a sequence $h_0, h_1, h_2, \ldots\ldots$ of complex numbers of modulus < 1 and consider the first order difference system

$$\begin{pmatrix}u_{2n+1}\\ v_{2n+1}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{1-|h_{2n}|^2}} & \frac{h_{2n}}{\sqrt{\ }}\\ \frac{\bar{h}_{2n}}{\sqrt{\ }} & \frac{1}{\sqrt{\ }}\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_{2n}\\ v_{2n}\end{pmatrix}$$

$$\begin{pmatrix}u_{2n+2}\\ v_{2n+2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{1-|h_{2n+1}|^2}} & \frac{h_{2n+1}}{\sqrt{\ }}\\ \frac{\bar{h}_{2n+1}}{\sqrt{\ }} & \frac{1}{\sqrt{\ }}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & z^{-1}\end{pmatrix}\begin{pmatrix}u_{2n+1}\\ v_{2n+1}\end{pmatrix}$$

Notation:

$$R(h) = \frac{1}{\sqrt{1-|h|^2}}\begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}$$

It might be better to write the system in the form

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = R(h_n)\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

as in the Lee-Yang business. [redacted] As long as we concentrate on even n, [redacted] or we work with ratio $\frac{u_n}{v_n}$ the $z^{1/2}$ will not appear.

So it's clear from the way things have been set up we have that the solution of the system starting with [redacted] $u_0 = v_0 = 1/(\int d\nu)^{1/2} = \phi_0 = \phi_0^*$ is just

$$\begin{pmatrix} z^{-n/2}\phi_n \\ z^{+n/2}\phi_n^* \end{pmatrix}$$

and in the situation where $h_n = 0$ for $n \gg 0$ we have $\phi_n = z^n \delta^*(z)$, $\phi_n^* = z^{-n}\delta(z)$ for $n \gg 0$ so that the above solution is

$$\begin{pmatrix} z^{n/2}\delta^*(z) \\ z^{-n/2}\delta(z) \end{pmatrix} \qquad n \gg 0.$$

November 6, 1977

Up to the $z^{1/2}$ problem, we have found a discrete Dirac-style system associated to the prob measure $d\nu$ namely

$$\begin{pmatrix} u \\ v \end{pmatrix}_{n+1} = R(h_n)\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_n \qquad n \geq 0$$

with the ϕ-solution

$$(\phi)_n = \begin{pmatrix} z^{-n/2}\phi_n \\ z^{n/2}\phi_n^* \end{pmatrix} \sim \begin{pmatrix} z^{n/2}\delta^* \\ z^{-n/2}\delta \end{pmatrix} \qquad n \gg 0$$

The problem now is to locate within this system a unitary operator & cyclic vector giving back the measure $d\nu$. The first idea is to use the ϕ-solution to map functions $f(z)$ on S^1 to vectors by

(1) $$f(z) \longmapsto \int \underline{\phi}(z) f(z)\, d\nu$$

In the case $d\nu = \frac{d\theta}{2\pi}$ we have

$$\underline{\phi}_n = \begin{pmatrix} z^{n/2} \\ z^{-n/2} \end{pmatrix}$$

So at least if we restrict n to be even, then we get an isomorphism between $L^2(S^1)$ and the space of square integrable vectors

$$u = \begin{pmatrix} u_0^1, & u_2^1, & u_4^1, & \cdots\cdots, & u_{2n}^1, & \cdots\cdots \\ u_0^2, & u_2^2, & u_4^2, & \cdots\cdots, & u_{2n}^2, & \cdots\cdots \end{pmatrix}$$

such that $u_0^1 = u_0^2$, given by sending f to

$$u_{2n}^1 = \int f(z) z^n dz$$

$$u_{2n}^2 = \int f(z) z^{-n} dz .$$

But for this to work in general is impossible, because take e_{2n}^1 to be the basic vector which is 1 in the indicated position. Then

$$(\underline{\phi}, e_{2n}^1) = z^{-n} \phi_{2n}$$

$$(\underline{\phi}, e_{2n}^2) = z^{n} \phi_{2n}^*$$

so

$$\left(\int \underline{\phi}(z) f(z) d\nu ,\ e_{2n}^1 \right) = \int z^{-n} \phi_{2n} f\, d\nu = (f, z^n \phi_{2n}^*)$$

hence if we have an L^2 isomorphism given by (1) above, it forces $z^n \phi_{2n}^* \longmapsto e_{2n}^1$. Similarly $z^{-n}\phi_{2n} \mapsto e_{2n}^2$ and hence $z^n \phi_{2n}^*$ and $z^{-n}\phi_{2n}$ would be orthogonal. But this clearly isn't the case in general.

So if I wish to locate a unitary operator here it will be necessary to use a non-standard inner product on the space spanned by e_{2n}^1, e_{2n}^2. Notice however that the poly $z^{+n}\phi_{2n}^*$ is orthogonal to $z^{+m}\phi_{2m}^*$ and $z^{-m}\phi_{2m}$ for $m \neq n$. Also because $z^n\phi_{2n}^*$ and $z^{-n}\phi_{2n}$ are both unit vectors (assuming $\int d\nu = 1$) we know that their sum and difference are orthogonal.

Suppose $d\nu$ is invariant under $\theta \leftrightarrow -\theta$ or $z \leftrightarrow z^{-1}$. Then

$$c_{-n} = \int e^{-in\theta}\, d\nu = \int e^{+in\theta}\, d\nu = c_n$$

so as $c_{-n} = \bar{c}_n$ in general we see c_n is real. Conversely if all c_n are real, then $d\nu$ is invariant under $\theta \leftrightarrow -\theta$. It follows that the ϕ_n have real coefficients in this case because they are obtained by orthonormalizing $1, z, z^2, \ldots$ and the inner products $(z^i, z^j) = c_{i-j}$ are all real. Consequently the numbers h_n defined by

$$z\phi_n = \sqrt{1-|h_n|^2}\,\phi_{n+1} - h_n z^n \phi_n^*$$

are real. Converse clear, for if the $\phi_n \in \mathbb{R}[z]$, then z^n is a real linear combination of the ϕ_i, so $(z^i, z^j) \in \mathbb{R}$.

November 9, 1977:

Paradox: Suppose a set of moments $c_0, c_1, c_2, \dots$ is given such that the associated Hankel matrix is positive-definite. Then I've seen that the associated J-matrix on $\ell^2_{[1,\infty)}$ gives a closed symmetric operator with equal deficiency indices, hence it has a self-adjoint extension A. Then if $A = \int \lambda \, dE_\lambda$, we get a measure $d\mu(\lambda) = d(E_\lambda e, e)$ such that

$$\int \lambda^i d\mu(\lambda) = (A^i e, e) = c_i$$

and such that

$$L^2(d\mu) \xrightarrow{\sim} \ell^2_{[1,\infty)}$$

$$p(\lambda) \longmapsto p(A)e$$

for any polynomial p. Hence it follows that $\mathbb{C}[\lambda]$, which goes into finite support vectors in ℓ^2, is dense in $L^2(d\mu)$. But we have for $f \in L^2(d\mu)$, that

$$\int f \, d\mu = (f, 1)$$

so $f \mapsto \int f \, d\mu$ is continuous on $L^2(d\mu)$. Since $\mathbb{C}[\lambda]$ is dense in $L^2(d\mu)$ it follows that $\int f \, d\mu$ is determined by its value on polynomials, i.e. by the moments. Thus the moments will determine $\int f \, d\mu$ for all $f \in C_0(\mathbb{R})$ which implies that $d\mu$ is unique. This contradicts the fact there are indeterminate moment problems.

Resolution of the paradox: The point is that given $f \in C_0(\mathbb{R})$ one approximates it by p in $L^2(d\mu)$ and another poly. q in $L^2(d\nu)$. No reason for p to equal q.