January 29, 1977

Ising inequalities.

Let X be a finite poset and μ a probability measure on X given by a function: $\mu(\{x\}) = U(x)$. One wants to prove correlation inequalities:

$$(*) \qquad \sum_{x\in X}' f(x)g(x)U(x) \geq \Big(\sum_{x\in X}' f(x)\,U(x)\Big)\Big(\sum_{x\in X}' g(x)\,U(x)\Big)$$

when f, g are monotone real-valued functions on X. (Note that one has equality when f, g are independent, so this means monotone functions tend to behave non-independently.)

Note that any monotone function f on X is a non-negative linear combination of characteristic functions of subsets closed under specialization (I call them open): Specifically suppose the range of f is

$\{a_0 < \cdots < a_n\}$. Then

$$f(x) = a_0 + \sum_{i=1}^{n} (a_i - a_{i-1})\, \chi_{\{x \mid f(x) \geq a_i\}}(x)$$

Therefore (1) is equivalent to

(2) $$\mu(A \cap B) \geq \mu(A)\,\mu(B)$$

if A, B are open subsets of the poset X. (Recall two subset A, B are independent if $\mu(A \cap B) = \mu(A)\mu(B)$; this is the same as χ_A and χ_B being independent).

Other versions of (1): Consider the space of real functions on X with mean: $\sum f(x)u(x) = 0$. On this space one has the inner product (assume $u(x)$ always >0).

$$(f, g) = \sum f(x)\,g(x)\,u(x) = E(fg).$$

Note that the monotone functions form a convex cone with non-empty interior. Condition (1) is equivalent to $(f, g) \geq 0$ if f, g are in this cone, i.e. the angle between two vectors in the cone is $\leq 90°$.

Observe that (2) holds if X is a chain because then either $B \subset A$ or $A \subset B$.

FKG theorem asserts (1) holds if X is a distributive lattice and u satisfies

(4) $$u(x \cup y)\,u(x \cap y) \geq u(x)\,u(y)$$

for example $u(x) = e^{-h(x)}$ with

$$h(x \cup y) + h(x \cap y) \leq h(x) + h(y).$$

<u>Thm.</u> (Simon's book p. 280)

Let $\mathbb{R}^n$ be given the product order, let $d\nu_1, \dots, d\nu_n$ be measures on $\mathbb{R}$ and $u(x_1, \dots, x_n)$ a strictly positive function $\ni$

(1) $$u(x \cup y)\, u(x \cap y) \geq u(x) u(y).$$

$$d\mu = u(x_1, \dots, x_n)\, d\nu_1(x_1) \cdots d\nu_n(x_n)$$

Put $\langle f \rangle = \int f\, d\mu \big/ \int d\mu$. If f, g are monotone then

(2) $$\langle fg \rangle \geq \langle f \rangle \langle g \rangle.$$

Assume the $d\nu_i$ have compact support; other cases can be handled by passing to the limit.

Proof by induction on n. If $n = 1$, (1) is trivial and (2) follows from

(3) $$\int \big(f(x) - f(y)\big)\big(g(x) - g(y)\big)\, d\mu(x)\, d\mu(y) \geq 0$$

$$\parallel$$

$$2\left(\int fg\, d\mu\right)\left(\int d\mu\right) - 2\left(\int f\, d\mu\right)\left(\int g\, d\mu\right)$$

and the fact that the integrand in (3) is always ≥ 0 when f, g are monotone.

Write $x \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ as (p, s) and

$$\int \big(f(x) - f(y)\big)\big(g(x) - g(y)\big)\, d\mu(x)\, d\nu(x) = \int \beta(s,t)\, d\nu_n(s)\, d\nu_n(t)$$

$$\beta(s,t) = \int \big(f(p,s) - f(q,t)\big)\big(g(p,s) - g(q,t)\big)\, u(p,s)\, u(q,t) \prod_1^{n-1} d\nu_i(p_i)\, d\nu_i(q_i).$$

It suffices to prove $\beta(s,t) \geq 0$ and since $\beta(s,t) = \beta(t,s)$

we can suppose $s \leq t$. Put

$$F(s) = \int f(p,s)\, u(p,s) \prod_1^{n-1} d\nu_i(p_i)$$

$$G(s) = \int g(p,s)\, u(p,s) \quad \text{"}$$

$$H(s) = \int f(p,s) g(p,s)\, u(p,s) \quad \text{"}$$

$$Z(s) = \int u(p,s) \quad \text{"}$$

Then

$$Z(s)Z(t)\beta(s,t) = Z(s)Z(t)\left[H(s)Z(t) + Z(s)H(t) - F(s)G(t) - F(t)G(s)\right]$$

$$= Z(s)^2\left[Z(t)H(t) - F(t)G(t)\right]$$

$$+ Z(t)^2\left[Z(s)H(s) - F(s)G(s)\right]$$

$$+ \left[Z(s)F(t) - Z(t)F(s)\right]\left[Z(s)G(t) - Z(t)G(s)\right]$$

Now

$$Z(t)H(t) \geq F(t)G(t)$$

namely apply induction to $f(\cdot,t)$ $g(\cdot,t)$ $u(\cdot,t)$ and $d\nu_1 \dots d\nu_{n-1}$. Similarly the second term is ≥ 0.

Next

$$p \longmapsto \frac{u(p,t)}{u(p,s)}$$

is increasing. Hence by induction

$$F(s)Z(t) = \int f_s(q) u_s(q) \prod d\nu_i(q_i) \int \frac{u_t}{u_s} u_s \prod d\nu_i$$

$$\leq \int u_s \int f_s u_t \leq \int u_s \int f_t u_t = Z(s)F(t)$$

because $f_s \le f_t$. Similarly $G(s)Z(t) \le G(t)Z(s)$, so the proof is complete.

Another version of the proof in the special case $X = S \times Y$ where $S = \{0,1\}$, and X is finite. Put

$$F(s) = \sum_{y \in Y}' f(s,y)\,\mu(s,y)$$

and define $G(s)$, $H(s)$, $Z(s)$ similarly. We want to prove:

$$\sum_S H(s) \sum_S Z(s) \ge \sum_S F(s) \sum_S G(s)$$

By the induction assumption $\frac{F(s)}{Z(s)}$ $\frac{G(s)}{Z(s)}$ are increasing. Note that if f, g are increasing and $s \le t$

$$(*) \qquad f(s)g(s) + f(t)g(t) \ge f(s)g(t) + f(t)g(s)$$

because the difference is $\big(f(s)-f(t)\big)\big(g(s)-g(t)\big) \ge 0$.

Thus

$$\big(H(0)+H(1)\big)\big(Z(0)+Z(1)\big) \ge \left(\frac{F(0)G(0)}{Z(0)} + \frac{F(1)G(1)}{Z(1)}\right)\big(Z(0)+Z(1)\big)$$

$$\ge F(0)G(0) + F(1)G(1) + \frac{F(0)}{Z(0)}\frac{G(0)}{Z(0)}Z(0)Z(1) + \frac{F(1)G(1)}{Z(1)Z(1)}Z(0)Z(1)$$

$$\overset{*}{\ge} F(0)G(0) + F(1)G(1) + \left(\frac{F(1)}{Z(1)}\frac{G(0)}{Z(0)} + \frac{F(0)}{Z(0)}\frac{G(1)}{Z(1)}\right)Z(0)Z(1)$$

$$= \big(F(0)+F(1)\big)\big(G(0)+G(1)\big)$$

<u>Better proof:</u> Assume $H(s)Z(s) \geq F(s)G(s)$ with $Z(s) > 0$ and that $\frac{F(s)}{Z(s)}, \frac{G(s)}{Z(s)}$ are increasing.

Then
$$\sum H(s) \sum Z(s) \geq \sum \frac{F(s)}{Z(s)} \frac{G(s)}{Z(s)} Z(s) \sum Z(s)$$
$$\geq \sum \frac{F(s)}{Z(s)} Z(s) \cdot \sum \frac{G(s)}{Z(s)} Z(s)$$
$$= \sum F(s) \sum G(s)$$
because of what we know for monotone functions on a chain.

Note the same formula will hold under the weaker assumption that $Z(s) \geq 0$ but that $Z(s) = 0 \implies F(s) = G(s) = H(s) = 0$; namely you delete these from S.

Generalize: Suppose I have a map of posets $p: X \longrightarrow S$ where S is a chain. If $s \leq t$ in S, then

$$F(t)Z(s) = \sum_{x \in X_t}' f(x)u(x) \sum_{y \in X_s}' u(y) \geq \sum_{x \in X_t} f(\varphi x)u(x) \sum_{y \in X_s} u(y)$$

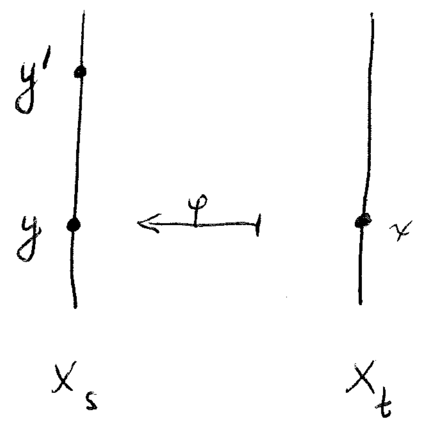
$$= \sum_{y \in X_s}' f(y) \frac{\sum'_{x \in \varphi^{-1}y} u(x)}{u(y)} u(y) \cdot \sum_{y \in X_s}' u(y)$$

$$\geq \sum_{y \in X_s}' f(y)u(y) \sum_{y \in X_s}' \sum_{x \in \varphi^{-1}y} u(x) = F(s)Z(t)$$

Here $\varphi : X_t \to X_s$ is some sort of pull-back map satisfying

a) $\varphi(x) \leq x$

b) $$y \leq y' \implies \frac{\sum'_{x \in \varphi^{-1}(y)} U(x)}{U(y)} \leq \frac{\sum_{x \in \varphi^{-1}(y')} U(x)}{U(y')}$$



Note b) holds if one has a map $x \mapsto x \underset{y}{\cup} y'$ embedding $\varphi^{-1}(y)$ in $\varphi^{-1}(y')$ and if

$$U(x \underset{y}{\cup} y')\, U(y) \geq U(y')\, U(x).$$

So now if $H(s) = \sum'_{x \in X_s} f(x) g(x) U(x)$, then assuming $H(s) Z(s) \geq F(s) G(s)$ on each fibre, we get

$$\sum_s' H(s) \sum_s' Z(s) \geq \sum' \frac{F(s)}{Z(s)} \frac{G(s)}{Z(s)} Z(s) \sum' Z(s)$$

$$\geq \sum' \frac{F(s)}{Z(s)} \cdot Z(s) \sum' \frac{G(s)}{Z(s)} Z(s)$$

$$= \sum' F(s) \sum' G(s) .$$

Next I want to allow U to be zero sometimes. To simplify return to the case $X = S \times Y$.

$s \leq t$

$$F(t)Z(s) = \sum_y f(t,y)\,U(t,y) \sum_y{}' U(s,y)$$

$$\geq \sum_y{}' f(s,y) \frac{U(t,y)}{U(s,y)} U(s,y) \cdot \sum_y{}' U(s,y)$$

Note that the function $y \mapsto \dfrac{U(t,y)}{U(s,y)}$ defined on the set of y such that $U(s,y) > 0$ is increasing:

$$U(s,y)\,U(t,y') \geq U(s,y')\,U(t,y)$$

$$\begin{tikzcd}
(s,y') \arrow[r, dash] \arrow[d, dash] & (t,y') \arrow[d, dash] \\
(s,y) \arrow[r, dash] & (t,y)
\end{tikzcd}$$

and $U(s,y),\ U(s,y') > 0$

$$\Longrightarrow \frac{U(t,y')}{U(s,y')} \geq \frac{U(t,y)}{U(s,y)}$$

But $\{y \mid U(s,y) > 0\}$ is a sublattice.

Suppose next that L is a finite distributive lattice. Let J be the set of irreducibles in L, so that L is isomorphic to the lattice of closed subsets of J. Let $U : L \longrightarrow \mathbb{R}_{\geq 0}$ satisfy $U(x \vee y)\,U(x \wedge y) \geq U(x)\,U(y)$, and let f, g be monotone functions on L. To prove

$$\sum_{x \in L}{}' f(x)g(x)U(x) \sum_{x \in L} U(x) \geq \sum_{x \in L} f(x)U(x) \sum_{x \in L} g(x)U(x)$$

I want to prove this by induction on card(L).

First note that we can always enlarge L to the lattice 2^J of all subsets of J. In effect we extend u to 2^J by zero outside of L. The inequality $u(x \cup y)\, u(x \cap y) \geq u(x)\, u(y)$ still holds because if either $x, y \notin L$ then the right side is zero. (Notice that the support of u is a sublattice of L). Next one can extend f from L to 2^J by defining $f(x) = f(\bar{x})$; $x \leq x' \Rightarrow \bar{x} \leq \bar{x}'$, etc. So it suffices to prove the theorem when $L = 2^J$, but where u is allowed to have the value zero.

Then I would try induction on card(J). So write $L = S \times Y$ and put

$$F(s) = \sum_{y \in Y} f(s,y)\, u(s,y) = \sum_{y \in L_s} f(s,y)\, u(s,y)$$

where $L_s = \{ y \in Y \mid u(s,y) > 0 \}$. Note L_s is a sublattice of $Y = 2^{J'}$, $J' = J -$ some pt. If $s \leq t$, then the function

$$\frac{u(t,y)}{u(s,y)}$$

defined on L_s is monotone, hence it can be extended to all of Y to be a monotone function (its value at y is the value at the smallest element of $L_s \geq y$). Thus we can argue

$$F(t)Z(s) = \sum_y f(t,y)\frac{u(t,y)}{u(s,y)}\, u(s,y) \sum \cdots \ ??$$

I seem to run into trouble if L_s doesn't contain the largest element of Y.

So let's try selecting an irreducible $p \in J$ and considering the map $L \to \{0,1\}$
$x \mapsto 0,1$ according as $p \nleq x$ or $p \leq x$. Assume p maximal. Then $L_0 = \{x \in L \mid p \notin x\}$ = all closed subsets of J not containing p. Clearly L_0 contains the largest element of Y in this case, and the same is true for L_1.

Try again this time using induction on card J. Suppose then L is the lattice of closed subsets of the finite poset J and that $U : L \to \mathbb{R}_{>0}$ satisfies $U(x \cup y)\,U(x \cap y) \geq U(x)\,U(y)$ and that f, g are monotone functions on L. Then pick a maximal element p of J. We then have

$$\sum_{x \in L} F(x)U(x) = \sum_{p \notin x} F(x)U(x) + \sum_{p \in x} F(x)U(x)$$

Put $L_0 = \{x \in L \mid p \notin x\}$ = closed subsets of $J - \{p\}$
and $L_1 = \{x \in L \mid p \in x\} \simeq$ closed subsets of $J - J_{\leq p}$.
Thus

$$\sum_{x \in L} F(x)U(x) = F(0) + F(1) \qquad F(\alpha) = \sum_{x \in L_\alpha} F(x)U(x)$$

As before put $Z(s) = \sum_{x \in L_s} u(x)$

Then

$$F(1) = \sum_{y \in L_1}' f(y)\, u(y)$$

$$= \sum_{x \in L_0}' f(x \cup \{p\}) \frac{u(x \cup \{p\})}{u(x)} u(x)$$

where by convention $f(x \cup \{p\}) = f(\overline{x \cup \{p\}})$ and $u(x \cup \{p\}) = 0$ if $x \cup \{p\}$ is not closed. Now if $x \leq x'$ and $x' \cup \{p\}$ is not closed, then $x \cup \{p\}$ is not closed (for $x' \cup \{p\} = (x \cup \{p\}) \cup x'$), hence we have

$$\frac{u(x \cup \{p\})}{u(x)} \leq \frac{u(x' \cup \{p\})}{u(x')}$$

in all cases. Thus

$$F(1)Z(0) = \sum_{x \in L_0} f(x \cup \{p\}) \frac{u(x \cup \{p\})}{u(x)} u(x) \sum_{x \in L_0} u(x)$$

$$\geq \sum_{x \in L_0}' f(x \cup \{p\})\, u(x) \sum_{x \in L_0} u(x \cup \{p\})$$

$$\geq F(0)\, Z(1)$$

So

[redacted]

$$\sum_s H(s) \sum_s Z(s) \geq \sum_s' \frac{F(s)}{Z(s)} \frac{G(s)}{Z(s)} Z(s) \sum_s Z(s)$$

$$\geq \sum' F(s) \sum G(s)$$

as before.

February 1, 1977

Let L be a finite distributive lattice. I want to understand the different kinds of functions $U : L \to \mathbb{R}_{>0}$ such that

(1) $$U(x \cup y)\, U(x \cap y) \geq U(x)\, U(y)$$

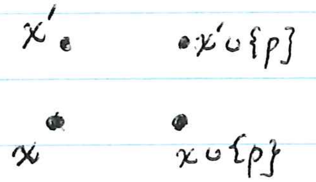
Put $H(x) = -\log U(x)$; then (1) becomes

(2) $$H(x \cup y) + H(x \cap y) \leq H(x) + H(y).$$

Call a function satisfying (2) semi-modular.

Begin by classifying semi-modular functions on $L \times \{0 \leq 1\}$. Think of L as the lattice of closed subsets of a poset J; then $L \times \{0 < 1\}$ is the lattice of closed subsets of $J \amalg \{p\}$. First note any semi-modular H on $L \times \{0 < 1\}$ gives semi-modular functions H_i on $L_i = L \times \{i\}$ for $i = 1, 2$ such that

(3) $$x \leq x' \in L_1 \implies H_1(x' \cup \{p\}) - H_1(x \cup \{p\}) \leq H_0(x') - H_0(x)$$



Conversely a pair H_i on L_i $i = 1, 2$ satisfying (3) gives an H on $L \times \{0 \leq 1\}$. To verify (3) one can suppose $x \in L_0$, $y \in L_1$ say $y = x_1 \cup \{p\}$. Then

$$H_0(x) + H_1(x_1 \cup \{p\}) - H_1(x \cup x_1 \cup \{p\}) - H_0(x \cap x_1)$$

$$\geq H_0(x \cup x_1) - H_0(x_1) - \left(H_1(x \cup x_1 \cup \{p\}) - H_1(x_1 \cup \{p\})\right)$$

≥ 0 by (3) applied to $x_1 \leq x \cup x_1$.

Suppose now that L is the lattice of closed sets of a poset $J = J' \cup \{p\}$ where p is maximal in J. Then

$$L = L_0 \cup L_1$$

where $L_0 =$ closed subsets of J' and $L_1 = \{y \in L \mid p \in y\}$. By sending $y \in L_1$ to $y - \{p\} \in L_0$ we get an isomorphism

$$L_1 \xrightarrow{\sim} (L_0)_{\geq w} \qquad w = J_{<p}$$

Any semi-modular H on L determines H_i on L_i such that

$$(4) \qquad w \leq x \leq x' \text{ in } L_0 \implies H_1(x' \cup \{p\}) - H_1(x \cup \{p\}) \leq H_0(x') - H_0(x)$$

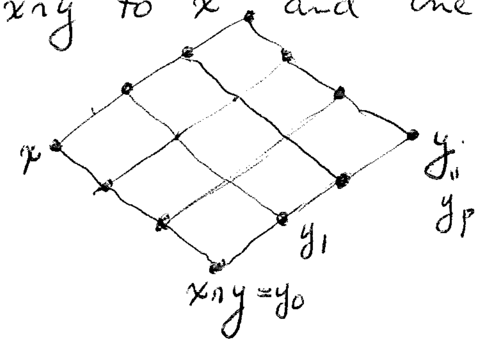
Conversely given H_i on L_i $i = 1, 2$ satisfying (4) I claim we get an H on L. To verify (2) one can suppose $x \in L_0$ and $y \in L_1$ say $y = x_1 \cup \{p\}$ with $w \leq x_1 \in L_0$. Then

$$H_0(x) + H_1(x_1 \cup \{p\}) - H_1(x \cup x_1 \cup \{p\}) - H_0(x \wedge x_1)$$

$$\geq H_0(x \cup x_1) - H_0(x_1) - (H_1(\underbrace{x \cup x_1}_{x' \text{ in } (4)} \cup \{p\}) - H_1(\underbrace{x_1}_{x \text{ in } (4)} \cup \{p\}))$$

$$\geq 0.$$

Simplification of (2): Suppose we choose a maximal chain from $x \cap y$ to x, and one from $x \cap y$ to y:



Then $$H(y) - H(x \cap y) = H(y_p) - H(y_0) = \sum_{i=1}^{p} H(y_i) - H(y_{i-1})$$

note $y_i \cap (x \cup y_{i-1}) = y_{i-1}$

$$\geqslant \sum_{i=1}^{p} H(x \cup y_i) - H(x \cup y_{i-1}) = H(x \cup y) - H(x)$$

Hence if we know (2) holds when y covers $x \cap y$, then it holds in general. Similarly we see that (2) when x, y cover $x \cap y$ imply it works in general.

Problem: What is the height of an element x of a finite distributive lattice L? Clearly the card of the number of irreducibles $\leq x$, i.e. the cardinality of x as a subset of J.

How to describe all semi-modular functions H on L: Construct them inductively with respect to height. First select $H(\emptyset)$, then $H(\{p\})$ for all $p \in J$. Then for each x of height 2 look at the two or one elements it covers; if $x = \{p, q\}$ with $\{p\}, \{q\}$ closed, then we have a condition

$$H(x) \leqslant H(\{p\}) + H(\{q\});$$

if on the other hand $x = \{p, q\}$ with $p < q$, then there is no condition on $H(x)$. In general $H(x)$ for x irreducible is completely arbitrary.

Note that the cone of semi-modular functions on L contains the subspace of "modular" functions, i.e. such that $H(x \cup y) + H(x \cap y) = H(x) + H(y)$. Such an H can be identified with a function on J and a constant

$$H(x) = \sum_{p \in x} f(p) + c \qquad c = H(\phi).$$

where $f(p) = H(J_{\leq p}) - H(J_{<p})$. So we can normalize things by requiring that $H(x) = 0$ if $x = \phi$ or if x is irreducible. For each reducible x one consider those elements it covers, i.e. $x - \{p\}$ for each p a generic point of x. For each pair p, q of generic points of x with $p \neq q$ one has a condition

$$H(x) \leq H(x - \{p\}) + H(x - \{q\}) - H(x - \{p, q\})$$

so the possible choices for $H(x)$ form a half-line $\mathbb{R}_{\leq a}$.