September 14, 1977:

Scattering: Consider first the J-matrix with all $a_i = \frac{1}{2}$ $b_i = 0$. Eigenvector equations are:

$$y_{j+1} - 2\lambda y_j + y_{j-1} = 0$$

Solutions: $y_j = c_1 \omega^j + c_2 \omega^{-j}$ where $\frac{\omega + \omega^{-1}}{2} = \lambda$ or $\cos\theta = \lambda$ if $\omega = e^{i\theta}$. (Assuming $\lambda \neq \pm 1$, otherwise one has linear terms in the solution.)

Solution u with $u_0 = 0$, $u_1 = 1$ is clearly

$$u(\lambda)_n = \frac{\sin(n\theta)}{\sin(\theta)}$$

If $\lambda \in [-1, 1]$, then ω and ω^{-1} are conjugate of norm 1 and if $\lambda \notin [-1, 1]$ then one of the roots ω, ω^{-1} is inside S^1 and the other is outside. Choose ω to be the root inside S^1 so that

$$\omega = \lambda - \sqrt{\lambda^2 - 1} \sim \frac{1}{2\lambda} \quad \text{as } \lambda \to \infty$$

where the square root is defined off the "cut" $[-1, 1]$ so as to be $\sim \lambda$. Maybe a better procedure to do everything in terms of θ and put $e^{i\theta} = \omega$ so that $\lambda = \cos\theta$ runs over the exterior of the cut as θ runs over the upper half-plane.

So for $\lambda \notin [-1, 1]$, i.e. $\operatorname{Im}(\theta) > 0$ we can define v to be the solution with $a_0 v_0 = 1$ and which decays at ∞: i.e.

$$v(\lambda)_j = 2e^{ij\theta} = 2(\omega)^j$$

How does one gets the spectral measures.

Since

$$v(\lambda)_1 = \int \frac{d\mu(\hat\lambda)}{\lambda - \hat\lambda}$$

one looks at the jump in v as one crosses the Real axis.

Suppose we have $f(\lambda) = \int \frac{\phi(\hat\lambda)\,d\hat\lambda}{\lambda - \hat\lambda}$ where ϕ is a nice function. Take

$$f(ib) - f(-ib) = \int \left(\frac{1}{ib-\lambda} + \frac{1}{+ib+\lambda} \right) \phi(\lambda)\,d\lambda$$

$$= \int \frac{-2ib}{\lambda^2 + b^2}\, \phi(\lambda)\,d\lambda$$

and let $b \downarrow 0$. Notice that

$$\int_{-\infty}^{\infty} \frac{b}{\lambda^2+b^2}\,d\lambda = \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2+1} = \Big[\arctan \lambda\Big]_{-\infty}^{\infty} = \pi$$

Hence $\frac{1}{\pi}\frac{b}{\lambda^2+b^2} \to \delta(\lambda)$ in the limit. Hence

$$\lim_{b \downarrow 0} f(ib) - f(-ib) = -2i\pi\,\phi(0)$$

or

$$\lim_{b \downarrow 0} \frac{1}{2\pi i}\left[f(ib) - f(-ib)\right] = -\phi(0)$$

$$\lim_{b \downarrow 0} \frac{1}{2\pi i}\left[f(a+ib) - f(a-ib)\right] = -\phi(a) \qquad a \in \mathbb{R}.$$

Check: $v(\lambda)_1 = 2e^{i\theta}$. Fix $-1 < \lambda < 1$. ~~There are two values of θ with $\cos(\theta) = \lambda$. The one between 0 and π describes the limiting value of $v(\lambda)_1$ on the top of the cut. Hence $v(\lambda+i0)_1 = 2e^{i\theta}$ and~~

[redacted]

Better: $v(\lambda)_1 = 2(\lambda - \sqrt{\lambda^2-1})$

$$v(\lambda+i0)_1 = 2(\lambda - i\sqrt{1-\lambda^2}) \qquad \sqrt{1-\lambda^2} > 0$$

$$v(\lambda-i0)_1 = 2(\lambda + i\sqrt{1-\lambda^2})$$

$$\therefore \ + \phi(\lambda) = \frac{2}{\pi}\sqrt{1-\lambda^2}$$

and so the spectral measure is $\boxed{d\mu(\lambda) = \frac{2}{\pi}\sqrt{1-\lambda^2}}$

Next let's consider a perturbation of the above, i.e. a J-matrix with $a_i = 1/2, b_i = 0$ for $i > n_0$. Define $u(\lambda)$ as before, and $v(\lambda)$ so that we have

$$v(\lambda) = \int u(\hat{\lambda}) \frac{d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}}$$

We must have

$$u(\lambda)_n = a(\lambda) e^{in\theta} + b(\lambda) e^{-in\theta}$$

for $n \gg n_0$, at least when $\lambda \neq \pm 1$. For λ real $u(\lambda)$ is real, hence $\overline{a(\lambda)} = b(\lambda)$.

This is confusing: Consider the equation $\omega + \omega^{-1} = 2\lambda$. As long as $\lambda \notin [-1,1]$, there is exactly one solution ω such that $|\omega| > 1$, so we get a conformal equivalence between $|\omega| > 1$ and $\mathbb{C} - [-1,1]$

September 17, 1977

Stieltjes transform: Let $d\mu(\lambda)$ be a measure on the line such that

(1) $$\int \frac{d\mu(\lambda)}{1+|\lambda|} < \infty$$

so that we get a holomorphic function

(2) $$f(\lambda) = \int \frac{d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}}$$

in the upper and lower half-planes. Actually the weaker condition (3) $\int \frac{d\mu(\lambda)}{1+\lambda^2} < \infty$ suffices for the existence of a holom. fn. $f(\lambda)$ in the UHP with negative imaginary part:

(4) $$\mathrm{Im}(f(\lambda)) = \int \mathrm{Im}\left(\frac{1}{\lambda-\hat{\lambda}}\right) d\mu(\hat{\lambda})$$

$$= (-\mathrm{Im}\,\lambda) \int \frac{d\mu(\hat{\lambda})}{|\lambda-\hat{\lambda}|^2}$$

and this holomorphic function f is unique up to a real constant.

The question is how to recover $d\mu$ from the function f. So start with (2). Consider first a discrete case

$$f(\lambda) = \sum_{i=1}^{n} \frac{r_i}{\lambda-\lambda_i} \qquad r_i > 0 \qquad \lambda_1 < \cdots < \lambda_n$$

or more simply $$f(\lambda) = \frac{1}{\lambda - a} \qquad a \in \mathbb{R}.$$

Look at $\mathrm{Im}(f(\lambda)) = -(\mathrm{Im}\,\lambda) \cdot \dfrac{1}{|\lambda-a|^2} = -\dfrac{\mathrm{Im}\,\lambda}{(\mathrm{Re}\,\lambda - a)^2 + (\mathrm{Im}\,\lambda)^2}$

Put $\lambda = \nu + i\varepsilon$ so that

$$-\operatorname{Im} f(\lambda) = + \frac{\varepsilon}{(\nu-a)^2+\varepsilon^2}$$

As a function of ν this approaches $\pi\,\delta(\nu-a)$ as $\varepsilon \downarrow 0$, since

$$\int_{-\infty}^{\infty} \frac{\varepsilon}{(\nu-a)^2+\varepsilon^2}\,d\nu = \left[\arctan\frac{\nu-a}{\varepsilon}\right]_{-\infty}^{+\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

So the good distribution formula is that for any $\varphi(\nu) \in C_0^\infty(\mathbb{R})$, as $\varepsilon \downarrow 0$

$$-\int \operatorname{Im} f(\nu+i\varepsilon)\,\varphi(\nu)\,d\nu \longrightarrow \pi\int \varphi(\nu)\,d\mu(\nu).$$

In fact you can write ~~[illegible]~~

$$-\int \operatorname{Im}(f(\nu+i\varepsilon))\,\varphi(\nu)\,d\nu = \int\left[\int \left(\frac{\varepsilon}{(\nu-\hat\lambda)^2+\varepsilon^2}\right) d\mu(\hat\lambda)\right]\varphi(\nu)\,d\nu$$

$$= \int\left(\int \frac{\varepsilon}{(\nu-\hat\lambda)^2+\varepsilon^2}\,\varphi(\nu)\,d\nu\right) d\mu(\hat\lambda)$$

$$\to \pi \int \varphi(\hat\lambda)\,d\mu(\hat\lambda).$$

Some technical stuff is required to let φ be the characteristic function of an interval.

The moral is this: $d\mu(\hat\lambda)$ is the limit in the sense of distributions of $-\frac{1}{\pi}\operatorname{Im} f(\hat\lambda+i\varepsilon)\,d\hat\lambda$ as $\varepsilon \downarrow 0$. Hence if $-\frac{1}{\pi}\operatorname{Im}\left(f(\hat\lambda+i\varepsilon)\right)$ does converge to a function $g(\hat\lambda)$ in a dominated way one has $d\mu(\hat\lambda) = g(\hat\lambda)\,d\hat\lambda$.

So let's consider a J-matrix with $a_i = \frac{1}{2}$ and $b_i = 0$ for i large. Define the solutions $u(\lambda)$, $\tilde{u}(\lambda)$ so as to get the solution matrix, if one writes the system in the matrix form:

$$\begin{pmatrix} y_n \\ a_{n-1} y_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{a_n} \\ -a_n & \frac{\lambda - b_n}{a_n} \end{pmatrix} \begin{pmatrix} y_{n+1} \\ a_n y_n \end{pmatrix}$$

so that

$$\begin{pmatrix} u_1 & \tilde{u}_1 \\ a_0 u_0 & a_0 \tilde{u}_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and so that $v = mu + \tilde{u}$, i.e. $v(\lambda)_1 = m(\lambda)$.

So $u, \tilde{u}$ are a basis for the solution nicely adapted to initial conditions. Clearly $u(\lambda)_n$ is a poly of degree $n-1$, and $\tilde{u}(\lambda)_n$ is a poly of degree $n-2$ in λ (for $n \geq 1$).

We also have solutions two independent solutions adapted to the behavior at ∞. Let $\psi(\theta)$ denote the solution such that

$$\psi(\theta)_n = e^{in\theta}$$

for large n, where θ satisfies: $\cos(\theta) = \lambda$. Provided $\lambda \neq \pm 1$, i.e. $\sin\theta \neq 0$, $\psi(\theta)$ and $\psi(-\theta)$ give two linearly independent solutions of the equation, hence we have

$$u(\lambda)_n = a(\theta) e^{in\theta} + b(\theta) e^{-in\theta}$$

for unique functions $a(\theta)$, $b(\theta)$ defined for $\sin(\theta) \neq 0$.

Suppose $\lambda \notin [-1, 1]$. [redacted]
In this case we can choose $e^{i\theta}$ uniquely so that $|e^{i\theta}| > 1$ and $\lambda = \cos\theta$. Then $\psi(\theta)_n$ blows up as $n \to \infty$ and $\psi(\theta)_n = e^{-in\theta}$ decays as $n \to \infty$. If [redacted] $\psi(\theta)_0 \neq 0$, [redacted] for example, if $\lambda \notin$ spectrum, then we have

$$v(\lambda) = c(\lambda)\,\psi(-\theta)$$

for some function $c(\lambda)$ defined in the complement of the spectrum. We can determine c using Wronskian:

$$1 = \begin{vmatrix} u(\lambda)_n & v(\lambda)_n \\ a_{n-1} u(\lambda)_{n-1} & a_{n-1} v(\lambda)_{n-1} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} a(\theta) e^{in\theta} & c(\lambda) e^{-in\theta} \\ a(\theta) e^{-i(n-1)\theta} & c(\lambda) e^{-i(n-1)\theta} \end{vmatrix}$$

$$= \frac{1}{2} a(\theta) c(\lambda) \left(e^{i\theta} - e^{-i\theta}\right) = a(\theta) c(\lambda)\, i \sin\theta$$

hence

$$c(\lambda) = \frac{1}{i a(\theta) \sin\theta}$$

Now

$$\frac{e^{-in\theta}}{i a(\theta) \sin\theta} = v(\lambda)_n = \int [redacted] \frac{u(\hat{\lambda})_n}{\lambda - \hat{\lambda}}\, d\hat{\mu}(\lambda)$$

for n large. I want to [redacted] apply the Stieltjes inversion formula to this. Thus I want to look at the imaginary part of what sits on the left as $\lambda = \nu + i\varepsilon$, and $\varepsilon \downarrow 0$.

From the equations:

$$u(\lambda)_n = a(\theta)e^{in\theta} + b(\theta)e^{-in\theta}$$

$$u(\lambda)_{n-1} = a(\theta)e^{i(n-1)\theta} + b(\theta)e^{-i(n-1)\theta}$$

we get

$$a(\theta) = \frac{\begin{vmatrix} u(\lambda)_n & e^{-in\theta} \\ u(\lambda)_{n-1} & e^{-in\theta+i\theta} \end{vmatrix}}{2i\sin\theta}$$

$$i\,a(\theta)\sin\theta = e^{-in\theta}\left(\frac{u(\lambda)_n e^{i\theta} - u(\lambda)_{n-1}}{2}\right)$$

so

$$\therefore \frac{e^{-in\theta}}{i\,a(\theta)\sin\theta} = v(\lambda)_n = \frac{2}{u(\lambda)_n e^{i\theta} - u(\lambda)_{n-1}} \qquad ?$$

This is clearly a rational function in $e^{i\theta}$. Hence except for poles, finite in number there is a definite limiting value as λ approaches the positive side of the real axis. Hence at these points $d\mu(\hat{\lambda})$ will be $\varphi(\hat{\lambda})\,d\hat{\lambda}$, where

$$-\frac{1}{\pi}\,\mathrm{Im}\left(\frac{e^{-in\theta}}{i\,a(\theta)\sin\theta}\right) = u(\lambda)_n\,\varphi(\lambda)$$

Maybe the good way to proceed is to assume $d\mu = \varphi(\lambda)\,d\lambda$. Then from

$$v(\lambda) = \int \frac{u(\hat{\lambda})\,d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}}$$

and by Stieljes inversion one gets the formula

$$\boxed{-\frac{1}{\pi}\,\mathrm{Im}\big(v(\lambda + i0^+)\big) = u(\lambda)\,\varphi(\lambda) \qquad \lambda \in \mathbb{R}}$$

For example in the constant coefficient case: all $a_i = \frac{1}{2}$ all $b_i = 0$, one has

$$u(\lambda)_n = \frac{\left(\lambda + \sqrt{\lambda^2-1}\right)^n - \left(\lambda - \sqrt{\lambda^2-1}\right)^n}{2\sqrt{\lambda^2-1}}$$

(where $\sqrt{\lambda^2-1}$ is defined off the cut to agree with λ far out), and

$$v(\lambda)_n = 2\left(\lambda - \sqrt{\lambda^2-1}\right)^n$$

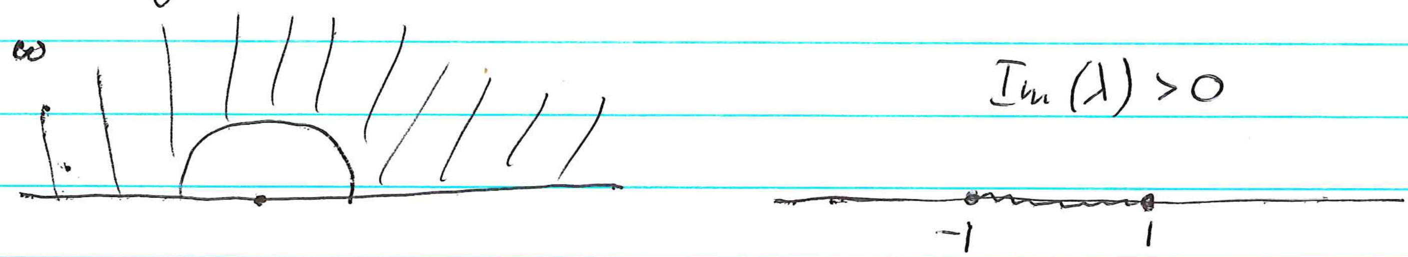
Notice that $v(\lambda)_n$ has a limiting value as λ approaches the real axes from the upper half-plane. The imaginary part of the limit is zero unless $\lambda \in [-1, 1]$. ~~If $-1 \le \lambda \le 1$, put $\lambda = \cos\theta$ with $0 \le \theta \le \pi$. Then if θ moves into the upper half-plane $\lambda = \cos\theta$ moves downward: $\theta = \theta_0 + i\varepsilon$, then~~

$$\cancel{\cos\theta = \frac{e^{-\varepsilon + i\theta_0} + e^{\varepsilon - i\theta_0}}{2} = \frac{(1-\varepsilon)e^{i\theta_0} + (1+\varepsilon)e^{-i\theta_0}}{2}}$$

$$\cancel{= \cos\theta_0 - \frac{\varepsilon}{2}\, 2i \sin\theta_0 .}$$

~~so that~~

Use conformal map



given by $\lambda = \dfrac{\omega + \omega^{-1}}{2}$ or $\omega = \lambda + \sqrt{\lambda^2-1}$

If $-1 \le \lambda \le 1$, then the corresponding point is $\omega = e^{i\theta}$ where $0 \le \theta \le \pi$ and $\lambda = \cos\theta$. We have

$$v(\lambda)_n = 2\omega^{-n} \qquad \omega^{-1} = \left(\lambda - \sqrt{\lambda^2 - 1}\right)$$

hence the limiting value for $v(\lambda)_n$ $\quad -1 \le \lambda \le 1$ is

$$v(\lambda)_n = 2e^{-in\theta}$$

hence

$$-\frac{1}{\pi}\,\mathrm{Im}\left[v(\lambda)_n\right] = \frac{2}{\pi}\sin(n\theta).$$

On the other hand $u(\lambda)_n = \dfrac{\sin(n\theta)}{\sin\theta}$ so we have

$$\frac{2}{\pi}\sin(n\theta) = \frac{\sin(n\theta)}{\sin\theta}\,\frac{d\mu}{d\lambda}$$

or

$$\frac{d\mu}{d\lambda} = \frac{2}{\pi}\sin\theta = \frac{2}{\pi}\sqrt{1-\lambda^2}$$

along the cut.

Next consider the perturbed system. One has

$$v(\lambda)_n = c(\lambda)\,\omega^{-n}$$

$$u(\lambda)_n = a(\lambda)\,\omega^{n} + b(\lambda)\,\omega^{-n}$$

for large n where the $a(\lambda)$, $b(\lambda)$ are holomorphic off the cut and have nice limiting values on both sides of the cut except possibly for $\lambda = \pm 1$. (See top of page 410).

To obtain $c(\lambda)$, let $\psi(\lambda)$ be the solution defined off the cut which agrees with ω^{-n} for large n. Then $v(\lambda) = c(\lambda)\psi(\lambda)$ where $c(\lambda)$ is chosen such that

$$a_0 v(\lambda)_0 = c(\lambda)\psi(\lambda)_0 = 1$$

$\psi(\lambda)$ is holomorphic in ω for $\omega \neq 0$; it has to be a Laurent polynomial in ω. It's roots for $|\omega|>1$ give all square integrable eigen solutions.

In effect, suppose one has a square-integrable eigenfunction. Up to a constant it agrees with $u(\lambda)$. Look at the asymptotic behavior. If $\lambda \in [-1,1]$ then $u(\lambda)_n$ is quasi-periodic in n hence can't be square-integrable unless it is zero. This is impossible so λ has to be outside the cut, and in this case $|\omega|>1$, so that $a(\lambda)=0$. (I don't see how Kac can claim that $\lambda=\pm 1$ can be part of the bound state spectrum.)

So we see that $c(\lambda)$ is holomorphic except on the cut where it has limiting values and except for poles at the bound-state eigenvalues.

By Wronskian calculation, we find

$$1 = \begin{vmatrix} u(\lambda)_n & v(\lambda)_n \\ a_{n-1}u(\lambda)_{n-1} & a_{n-1}v(\lambda)_{n-1} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \omega^n & \omega^{-n} \\ \omega^{n-1} & \omega^{-n+1} \end{vmatrix} a(\lambda)\, c(\lambda)$$

$$1 = \frac{1}{2}(\omega-\omega^{-1})\, a(\lambda)\, c(\lambda)$$

so

$$v(\lambda)_n = \frac{2\,\omega^{-n}}{a(\lambda)(\omega-\omega^{-1})} = \frac{e^{-in\theta}}{a(\lambda)\, i\sin\theta} \qquad \text{as } \lambda \to \cos\theta \text{ with } \mathrm{Im}(\lambda)>0$$

or

$$-\frac{1}{\pi}\,\mathrm{Im}\left[v(\lambda)_n\right] = \frac{1}{2\pi}\left[\frac{e^{+in\theta}}{a(\lambda)\,\sin\theta} - \frac{e^{-in\theta}}{a(\lambda)\,\sin\theta}\right]$$

$$= u(\lambda)_n \frac{d\mu}{d\lambda} = \left[a(\lambda)e^{in\theta} + b(\lambda)e^{-in\theta}\right]\frac{d\mu}{d\lambda}$$

Comparing coefficients we find

$$\frac{1}{2\pi}\frac{1}{\overline{a(\lambda)}\sin\theta} = a(\lambda)\frac{d\mu}{d\lambda}$$

or

$$\boxed{\frac{d\mu}{d\lambda} = \frac{1}{2\pi\,|a(\lambda)|^2 \sin\theta}}$$

As a check take unperturbed case:

$$u(\lambda)_n = \frac{\sin(n\theta)}{\sin(\theta)} = \frac{e^{in\theta}}{2i\sin\theta} - \frac{e^{-in\theta}}{2i\sin\theta}$$

$$a(\lambda) = \frac{1}{2i\sin\theta} = \frac{1}{2\sqrt{\lambda^2-1}}$$

hence

$$\frac{1}{2\pi\,|a(\lambda)|^2\sqrt{1-\lambda^2}} = \frac{2}{\pi}\sqrt{1-\lambda^2}$$

So what's the formula for the spectral measure?

September 18, 1977:

Start with a rational function $\rho(\lambda) = \dfrac{f_1(\lambda)}{f_0(\lambda)}$ where f_0, f_1 are relatively prime polynomials. If $\rho(\infty) = 0$, i.e. $\deg f_0 > \deg f_1$ we can perform Euclid's algorithm:

$$\begin{cases} f_0 = q_1 f_1 + f_2 & \deg(f_2) < \deg f_1 \\ f_1 = q_2 f_2 + f_3 & \\ & \\ f_{n-1} = q_n f_n & \end{cases}$$

and obtain a sequence of polys $q_1, \ldots, q_n$ of positive degree such that

$$\deg(f_i) = \deg(q_i) + \deg(f_{i+1}).$$

Note that f_n is a non-zero constant as f_0, f_1 are rel. prime. and f_n is the g.c.d. of f_0, f_1. Then

$$\frac{f_i}{f_{i-1}} = \frac{1}{q_i +} \; \frac{f_{i+1}}{f_i}$$

so

$$\frac{f_1}{f_0} = \frac{1}{q_1 +} \cdots \frac{1}{q_n}$$

If $\deg(f_0) \leq \deg(f_1)$ we get an expansion

$$\frac{f_1}{f_0} = q_0 + \frac{1}{q_1 +} \cdots \frac{1}{q_n}$$

Geometric significance: Given a rational function ρ there is a unique poly q_0 such that $(\rho - q_0)(\infty) = 0$, now repeat the process with $\dfrac{1}{\rho - q_0}$, etc.

So let's concentrate on the case $\rho(\infty)=0$.

$$\begin{pmatrix} f_i \\ f_{i-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & q_i \end{pmatrix}\begin{pmatrix} f_{i+1} \\ f_i \end{pmatrix}$$

$$\begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & q_n \end{pmatrix}\begin{pmatrix} 0 \\ \underset{\substack{\uparrow \\ \text{constant } \neq 0}}{f_n} \end{pmatrix}$$

Clear that $\frac{1}{q_1+} \cdots \frac{1}{q_n}$ always [redacted] represents a rational functions <u>vanishing</u> at ∞.

<u>Proposition:</u> Any rational function $\rho(\lambda) \neq 0$ with $\rho(\infty)=0$ has a unique CF expansion

$$\frac{f_1}{f_0} = \frac{1}{q_1+} \cdots \frac{1}{q_n}$$

where the q_i are polynomials of [redacted] degree >0. If f_0, f_1 are relatively prime, then

$$\deg(f_0) = \sum_{i=1}^{n} \deg(q_i)$$

The degree of a rational function ρ is the max of the degrees of the numerator and denominator when expressed in lowest terms. (The degree is defined as $[k(\lambda) : k(\rho)]$ = number of solns. of $\rho(\lambda) = a$ for any a if counted with multiplicity). We see that any rational function of degree n has a unique expansion

$$\rho = \frac{f_1}{f_0} = q_0 + \frac{1}{q_1 +} \cdots \frac{1}{q_n}$$

where $n = \sum_{i=0}^{n} \deg(q_i)$ and $\deg(q_i) > 0$ for $i = 1, \ldots, n$.

<u>Question:</u> Describe all rational functions of degree n such that $\rho = \frac{1}{q_1 +} \cdots \cdots \frac{1}{q_n}$

where $\deg(q_i) = 1$ for all i.

<u>September 19, 1977</u>:

Consider a rational fn. $\rho(\lambda) = \frac{1}{r_1\lambda + s_1 +} \cdots\cdots \frac{1}{r_n\lambda + s_n}$

where $r_1, \ldots, r_n$ are all $\neq 0$. We can perform an equivalence transformation

$$\rho(\lambda) = \left(\frac{1}{r_1}\right) \frac{1}{\lambda + \frac{s_1}{r_1} -} \; \frac{\left(-\frac{1}{r_1 r_2}\right)}{\lambda + \frac{s_2}{r_2} -} \; \frac{\left(-\frac{1}{r_2 r_3}\right)}{\lambda + \frac{s_3}{r_3} -} \cdots \frac{\left(-\frac{1}{r_{n-1} r_n}\right)}{\lambda + \frac{s_n}{r_n}}$$

and so get it into the form

$$(*) \qquad \rho(\lambda) = \frac{a_0}{\lambda - b_1 -} \; \frac{a_1}{\lambda - b_2 -} \cdots\cdots \frac{a_{n-1}}{\lambda - b_n}$$

On the other hand, if we take the equations

$$\psi_{i-1} = (\lambda - b_i)\psi_i - a_i \psi_{i+1} \qquad i = 1, \dots, n$$
$$\text{with } \psi_{n+1} = 0$$

we get

$$\frac{\psi_i}{\psi_{i-1}} = \frac{1}{\lambda - b_i - \left(a_i \frac{\psi_{i+1}}{\psi_i}\right)}$$

hence

$$\frac{\psi_1}{\psi_0} = \frac{1}{\lambda - b_1 -}\,\frac{a_1}{\lambda - b_2 -} \cdots \frac{a_{n-1}}{\lambda - b_n}.$$

So I see that provided $a_0 = 1$, $\rho(\lambda)$ is the rational fn. obtained in the usual way from the matrix:

$$(+) \qquad \begin{pmatrix} b_1 & a_1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & 1 & b_n \end{pmatrix}$$

A necessary condition that ρ be in the form $(*)$ with $a_0 = 1$ and all $a_1, \dots, a_{n-1} \neq 0$ is that

$$\rho(\infty) = 0 \qquad \lim_{\lambda \to \infty} \lambda\rho(\lambda) = 1$$

i.e. ρ has a simple pole at ∞ with residue -1.

Try to prove the converse, at least supposing that the poles of ρ are all simple. Thus we assume

$$\rho(\lambda) = \sum_{i=1}^{n} \frac{r_i}{\lambda - \lambda_i} \qquad \sum r_i = 1$$

where $\lambda_1, \dots, \lambda_n$ are distinct and each $r_i \neq 0$. From ρ we can construct a vector space $V = \mathbb{C}[\lambda]/(f_0(\lambda))$ where

$f_0(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$ together with an operator A = multiplication by λ and cyclic vector e = image of 1 and linear functional $e^* : V \to \mathbb{C}$ given by

$$(f, e^*) = \sum_{i=1}^{n} f(\lambda_i) r_i = \operatorname{res}\left(f(\lambda) \rho(\lambda) d\lambda\right).$$

Then $(e, e^*) = 1$ and e^* restricted to each eigenspace V_{λ_i} is $\neq 0$, hence e^* is cyclic in V^*.

Let's consider such quadruples (V, A, e, e^*). The triple (V, A, e) is determined by the set of eigenvalues of A counted with multiplicity, [redacted] call these $\{\lambda_1, \ldots, \lambda_n\}$ so that

$$\prod (\lambda - \lambda_i) = \det(\lambda - A),$$

because

$$V \xleftarrow{\sim} \mathbb{C}[\lambda] / \det(\lambda - A).$$

[redacted] Next I want to understand what it means for e^* to be cyclic in V^* — perhaps this means that one gets a duality in V. In any case given any e^* in V^* one can define a unique A-linear homomorphism

$$\begin{aligned} V &\longrightarrow V^* \\ e &\longmapsto e^* \end{aligned}$$

which is an isomorphism iff e^* is cyclic. The bilinear form belonging to this iso is

$$\begin{array}{ccccc} f(A)e \otimes g(A)e & \longmapsto & f(A)e \otimes g(A)e^* & \longmapsto & (f(A)e, g(A)e^*) \\ V \otimes V & \longrightarrow & V \otimes V^* & \xrightarrow[(\ )]{} & \mathbb{C} \end{array}$$

and

$$(f(A)e, g(A)e^*) = (g(A)f(A)e, e^*).$$

So I see that e^* cyclic $\Leftrightarrow$ the pairing $f(A)e \otimes g(A)e \mapsto (f(A)g(A)e, e^*)$ is non-degenerate.

The next thing to try is to construct an orthogonal basis out of the sequence $e, Ae, A^2e, \ldots\ldots$. Put $\phi_1 = e$, so that $(e,e) = (e, e^*) = 1$.

First [redacted] note the following about the case:

$$(*) \qquad A = \begin{pmatrix} b_1 & a_1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ 0 & & 1 & b_n \end{pmatrix} \qquad e = e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad e^* = e^t = (1, 0, \ldots, 0)$$

$F_iV = \operatorname{span}\{e, Ae, \ldots, A^{i-1}e\} = \operatorname{span}\{e_1, \ldots, e_i\}$

$F_iV^* = \operatorname{span}\{e^*, Ae^*, \ldots, A^{i-1}e^*\} = \operatorname{span}\{e_1^t, \ldots, e_i^t\}$

The point is that these two spaces are in duality. In other words it is necessary in order to get (V, A, e, e^*) into Jacobi form that the [redacted] quadratic form on V be non-degenerate when restricted to F_iV for $i = 1, \ldots, n$. Conversely if this condition is satisfied then we can construct by Gram-Schmidt an orthogonal basis out of $e, Ae, A^2e, \ldots, A^{n-1}e$ and hence get a Jacobi matrix.

The real point therefore seem to be whether you can find $\rho(\lambda) = \sum_{i=1}^{n} \frac{r_i}{\lambda - \lambda_i}$, $r_i \neq 0$, $\sum r_i = 1$ such that the bilinear form [redacted] $(f,g) = \sum_{i=1}^{n} r_i f(\lambda_i) g(\lambda_i)$ which is non-degenerate on polys. of degree $\leq n-1$ is degenerate on polys of degree $\leq i-1$ for some $1 \leq i < n$.

Clearly $\sum r_i \neq 0$ is necessary [redacted] and sufficient that [redacted] this quadratic form be non-degenerate on F_1; (F_i = polys. of degree $\leq i-1$). So what's the condition that this form be non-degenerate on [redacted] linear polys? We don't want there to be a linear poly $a\lambda + b \neq 0$ with

$$0 = (1, a\lambda+b) = a\left(\sum r_i\lambda_i\right) + b\left(\sum r_i\right)$$
$$0 = (\lambda, a\lambda+b) = a\left(\sum r_i\lambda_i^2\right) + b\left(\sum r_i\lambda_i\right)$$

Can assume $a = 1$. Then we can take $b = -\sum r_i\lambda_i = -c_1$ and so we want the norm of the linear polynomial $\lambda - c_1$ to be $\neq 0$. [redacted]

So [redacted] to construct a counterexample with $n=3$, simply construct a ρ with $\sum r_i = 1$, $\sum r_i\lambda_i = 0$ and $\sum r_i\lambda_i^2 = 0$.

New viewpoint: Consider the variety of all matrices A as on preceding page with $a_i \neq 0$. This is an affine variety, non-singular. The subset of A with given characteristic poly. is an affine subvariety, call it Z_{f_0} and we've constructed an injective map of Z_{f_0} into the space of [redacted] monic polys f_1 of degree $n-1$ which is an affine space of dimension $n-1$. It seems clear from the proceeding that one gets an isom. of Z_{f_0} with a Zariski open of $\mathbb{A}^{n-1}$ in this way, and since Z_{f_0} is [redacted] affine, Z_{f_0} should be the complement of a divisor in $\mathbb{A}^{n-1}$. This leads to

Prop: Let f_0 be a monic poly of deg. n, and let $f_1 = \lambda^{n-1} + \alpha_1 \lambda^{n-2} + \cdots + \alpha_{n-1}$ be a variable monic poly. of degree $n-1$. Then there exists a poly $\Phi(\alpha_1, \dots, \alpha_{n-1})$ whose non-vanishing is necessary and sufficient for $\frac{f_1}{f_0}$ to have a continued fraction expansion of the form

$$\frac{f_1}{f_0} = \frac{1}{q_1 +} \cdots \frac{1}{q_n}$$

where $\deg(q_i) = 1$.

It's even simpler: Note that if you want a collection of polys:

$$c_0, \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix}, \dots\dots, \begin{vmatrix} c_0 & \cdots & c_n \\ \vdots & & \vdots \\ c_n & \cdots & c_{2n} \end{vmatrix}$$

to be $\neq 0$ you require their product to be $\neq 0$.

September 24, 1977: after Newton: Scattering Theory of Waves and Particles

Time-dependent scattering:

Consider a quantum system described by a Hamiltonian which is a self-adjoint operator H acting on a Hilbert space $\mathcal{H}$ whose lines we think of as states of the system. Time evolution is given by the flow e^{-itH} on $\mathcal{H}$; a possible history of the system is described by a path in $\mathcal{H}$ of the form

1) $$\Psi(t) = e^{-itH}\psi$$

where $\psi = \Psi(0)$. Thus

$$i \frac{\partial \Psi}{\partial t} = H\Psi$$

Now suppose $H = H_0 + H'$ where H_0 represents the Hamiltonian in the absence of interaction, so that H is a perturbation of H_0. Given $\Psi(t)$ as in 1) we expect $\Psi(t)$ as $t \to -\infty$ to be asymptotic to a free state $\Psi_{in}(t) = e^{-itH_0}\Psi_{in}(0)$ evolving under H_0. Hence

$$e^{-itH}\Psi(0) = \Psi(t) \sim e^{-itH_0}\Psi_{in}(0) \qquad \text{as } t \to -\infty$$

or

$$\Psi_{in}(0) = \lim_{t\to-\infty} e^{itH_0} e^{-itH}\Psi(0)$$

Similarly we expect a state $\Psi_{out}(t)$ evolving under H_0 $\ni$

$$\Psi(t) \sim \Psi_{out}(t) \qquad \text{as } t \to +\infty$$

i.e.

$$\Psi_{out}(0) = \lim_{t\to+\infty} e^{itH_0} e^{-itH}\Psi(0)$$

Now $$\frac{d\Psi}{dt} = \frac{1}{i} H\Psi = \frac{1}{i}(H_0 + H')\Psi$$

or
$$\frac{d\Psi}{dt} - \frac{1}{i}H_0\Psi = \frac{1}{i}H'\Psi$$

$$\frac{d}{dt}\left(e^{iH_0 t}\Psi\right) = \frac{1}{i}\, e^{iH_0 t}H'\Psi$$

$$\left[e^{iH_0 t}\Psi\right]_{-\infty}^{t} = \frac{1}{i}\int_{-\infty}^{t} e^{iH_0 t'}H'\Psi(t')\,dt'$$

$$e^{iH_0 t}\Psi(t) - \Psi_{in}(0) = \frac{1}{i}\int_{-\infty}^{t} e^{iH_0 t'}H'\Psi(t')\,dt'$$

or
$$\boxed{\Psi(t) = \Psi_{in}(t) + \frac{1}{i}\int_{-\infty}^{t} e^{-iH_0(t-t')}H'\Psi(t')\,dt'}$$

similarly
$$\boxed{\Psi(t) = \Psi_{out}(t) - \frac{1}{i}\int_{t}^{\infty} e^{-iH_0(t-t')}H'\Psi(t')\,dt'}$$

September 25, 1977

In the above one must assume $\Psi(t)$ is such that it has asymptotic behaviors Ψ_{in} and Ψ_{out} – which is the case for the state vector of a scattering experiment.

One assumes that for the subspace of such Ψ, call it $\mathcal{H}_1$, one gets isomorphisms

$$\mathcal{H} \xrightarrow{\Omega^+} \mathcal{H}_1 \xleftarrow{\Omega^-} \mathcal{H}$$

$$\Psi_{in}(0) \longleftrightarrow \Psi(0) \longleftrightarrow \Psi_{out}(0).$$

The operators Ω^+, Ω^- are called the Møller wave operators and defined by

$$\Omega^+ = \lim_{t\to-\infty} e^{iHt}e^{-iH_0 t} \qquad \Omega^- = \lim_{t\to+\infty} e^{iHt}e^{-iH_0 t}$$

The orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}$ is the subspace of so-called bound states.

The scattering operator is defined by

$$S = (\Omega^-)^* \Omega^+$$

It is a unitary operator on $\mathcal{H}$ commuting with H_0 given by

$$S = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} e^{iH_0 t_2} e^{-iHt_2} e^{iHt_1} e^{-iH_0 t_1}$$

In other words given $\psi \in \mathcal{H}$, you propagate it backwards freely $e^{-iH_0 t}\psi$ to get $\Psi_{in}(t)$, then you change to $\Psi(t)$ the good solution asymptotic to $\Psi_{in}(t)$, then you propagate forward, i.e. let $t \to +\infty$ and find $\Psi_{out}(t) = e^{-itH_0} S\psi$.

The basic perturbation formula for S is obtained as follows: Recall

$$\frac{d}{dt}\left(e^{iH_0 t} e^{-iHt}\right) = e^{iH_0 t}(iH_0 - iH)e^{-iHt}$$

$$= e^{iH_0 t}\left(\frac{1}{i}H'\right)e^{-iH_0 t} \quad e^{iH_0 t} e^{-iHt}$$

So if $Q(t) = e^{iH_0 t} e^{-iHt} \Omega^+$ one has

$$\frac{d}{dt} Q(t) = e^{iH_0 t}\left(\frac{H'}{i}\right)e^{-iH_0 t} \; Q(t)$$

$$Q(-\infty) = I$$

so $Q(t)$ satisfies the integral equation

$$Q(t) = I + \int_{-\infty}^{t} e^{iH_0 \hat{t}} \frac{H'}{i} e^{-iH_0 \hat{t}} \; Q(\hat{t})\, d\hat{t}$$

which can be formally solved by iterating

$$Q(t) = I + \int_{-\infty}^{t} dt_1\, e^{iH_0 t_1} \frac{H'}{i} e^{-iH_0 t_1} + \int_{-\infty}^{t} dt_1\, e^{iH_0 t_1} \frac{H'}{i} e^{-iH_0 t_1} \int_{-\infty}^{t_1} e^{iHt_2} \frac{H'}{i} e^{-iHt_2}\, dt_2 + \dots$$

$$Q(t) = I + \int_{-\infty}^{t} dt_1\, e^{iH_0 t_1} \frac{H'}{i} e^{-iH_0 t_1} + \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\, e^{iH_0 t_1} \frac{H'}{i} e^{-iH_0 (t_1 - t_2)} \frac{H'}{i} e^{-iH_0 t_2} + \dots$$

Now $Q(+\infty) = S$ so we get

$$\boxed{S = I + \int_{-\infty}^{\infty} dt_1\, e^{iH_0 t_1} \frac{H'}{i} e^{-iH_0 t_1} + \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2\, e^{iH_0 t_1} \frac{H'}{i} e^{-iH_0 (t_1 - t_2)} \frac{H'}{i} e^{-iH_0 t_2} + \dots}$$

It aids our insight to interpret this as saying that the scattering is a superposition of terms which correspond to a ψ state propagating freely for all t, plus ψ propagating freely to t_1 interacting and then progating freely, plus ψ propagating freely to t_1, interacting, then to t_2 interacting, and then propaging freely, etc.

Time-Independent Scattering: Here one wants to describe the above situation directly in terms of H, H_0 and H' without using t. In other words we have $H = H_0 + H'$ acting on $\mathcal{H}$ and we want to describe the bound states and wave operators directly.

Method is to take Fourier tranform

$$\hat{\Psi}(E) = \int_{-\infty}^{\infty} dt\, e^{iEt}\, \Psi(t)$$

Do this to the equation

$$\Psi(t) = \Psi_{in}(t) + \int_{-\infty}^{t} e^{-iH_0(t-t^\wedge)}\frac{H'}{i}\Psi(t^\wedge)\,dt^\wedge$$

$$\hat{\Psi}(E) = \hat{\Psi}_{in}(E) + \int_{-\infty}^{\infty} dt\; e^{iEt}\int_{-\infty}^{t} d\hat{t}\; e^{-iH_0(t-\hat{t})}\frac{H'}{i}\Psi(\hat{t})$$

$$= \hat{\Psi}_{in}(E) + \int_{-\infty}^{\infty} d\hat{t}\int_{\hat{t}}^{\infty} dt\; e^{iEt}\, e^{-iH_0(t-\hat{t})}\frac{H'}{i}\Psi(\hat{t})$$

$$= \hat{\Psi}_{in}(E) + \int_{-\infty}^{\infty} d\hat{t}\int_{0}^{\infty} du\; e^{iE(\hat{t}+u)}\, e^{-iH_0u}\frac{H'}{i}\Psi(\hat{t})$$

$$= \hat{\Psi}_{in}(E) + \int_{-\infty}^{\infty} d\hat{t}\; e^{iE\hat{t}}\underbrace{\int_{0}^{\infty} e^{i(E-H_0)u}\,du}_{i(E-H_0)^{-1}}\;\frac{H'}{i}\Psi(\hat{t})$$

$$\boxed{\hat{\Psi}(E) = \hat{\Psi}_{in}(E) + (E-H_0)^{-1}H'\hat{\Psi}(E)}$$

Maybe it's useful to note that for the above calculation to have any meaning we want E to be in the upper half-plane so that

$$\int_0^\infty e^{i(E-H_0)u}\,du = i(E-H_0)^{-1}$$

makes sense. In other words in the box the resolvent is to be interpreted as obtained from the upper half plane.

Thus put

$$G(E) = (E-H_0)^{-1}$$

for E not in the spectrum of H_0. Then

$$G^{\pm}(E) = \lim_{\varepsilon \downarrow 0} G(E \pm i\varepsilon)$$

and we have

$$\boxed{\hat{\Psi}(E) = \hat{\Psi}_{in}(E) + G^{+}(E)H'\hat{\Psi}(E)}$$

so-called Lippmann-Schwinger equation.

It should be possible to express these things in terms of the spectral resolution of H, H_0, e.g. one has

$$H = \int E\,dP(E)$$

where $P(E)$ is the spectral resolution of H. Hence

$$\Psi(t) = e^{-itH}\psi = \int_{-\infty}^{\infty} e^{-itE}\,dP(E)\psi = \int_{-\infty}^{\infty} dE\, e^{-itE}\,\frac{dP(E)\psi}{dE}$$

so that

$$\frac{dP(E)\psi}{dE} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itE}\,\Psi(t)\,dt = \frac{1}{2\pi}\hat{\Psi}(E).$$

$\therefore$

September 26, 1977:

What you want to understand is the relation between time-dependent scattering which is based upon the unitary groups e^{-itH} and e^{-itH_0} and time-independent scattering which concerns the actual Schroedinger equation

$$-\frac{d^2u}{dr^2} + V(r)u = k^2u \qquad\qquad k^2 = E.$$

and its spectrum. At the same time I want to work in the Hörmander emphasis on the wave equation giving the good approach to the spectrum.

Look at the above Schroedinger equation with the boundary condition $u(0)=0$ so as to get a self-adjoint problem as well as a spectrum of multiplicity 1. Then what is $\Psi(t) = e^{-itH}\psi$? Here $\psi \in L^2(0,\infty)$, so ψ can be expanded in terms of eigenfunctions for H:

$$\psi = \int f(E)\,\varphi(E)\,d\mu(E)$$

(More specifically

$$\psi(r) = \int_{-\infty}^{\infty} f(k)\,\varphi(r,k)\,d\mu(k)$$

where $f \in L^2(d\mu)$ is an even function of k. But let's avoid working with k until we have to.). Thus we have an isomorphism

$$L^2(\mathbb{R}, d\mu) \xrightarrow{\sim} L^2(0,\infty)$$

$$f \longmapsto \int f(E)\,\varphi(E)\,d\mu(E)$$

so that

$$H = \int E\,\varphi(E)\,d\mu(E)$$

is the spectral resolution of H. So

$$\Psi(t) = e^{-itH}\psi = \int e^{-itE} f(E)\,\varphi(E)\,d\mu(E)$$

and so if

$$\Psi(t) = \frac{1}{2\pi}\int e^{-itE}\,\hat{\Psi}(E)\,dE$$

we have

$$\frac{1}{2\pi}\hat{\Psi}(E) = f(E)\,\varphi(E)\,\frac{d\mu}{dE}$$

The problem: Suppose $H = -\frac{d^2}{dr^2} + V(r)$ on $0 \leq r < \infty$ with the boundary condition $u(0) = 0$ so as to get a self-adjoint problem. The spectrum for $H_0 = -\frac{d^2}{dr^2}$ consists of $E \geq 0$. The solution of $H_0\varphi = E\varphi$ with initial value $\varphi(0) = 0$, $\varphi'(0) = 1$ is

$$\varphi_0(E, r) = \frac{\sin \sqrt{E}\, r}{\sqrt{E}}$$

Let $\varphi(E, r)$ denote the solution of $H\varphi = E\varphi$ with some initial values. Presumably $\Omega^+\varphi_0 = \varphi$ as these are the unique eigenfunctions. But then also $\Omega^-\varphi_0 = \varphi$ which is non-sense. So we expect $\Omega^+\varphi_0 = c^+(E)\varphi$ and similarly $\Omega^-\varphi_0 = c^-(E)\varphi$, whence

$$S\varphi_0 = \frac{c^+(E)}{c^-(E)}\varphi_0$$

So the problem is to explain in what sense $\Omega^+\varphi_0(E)$ and $\varphi(E)$ are proportional.

Now these eigenfunctions are not in $L^2(0,\infty)$, hence we really should work with a "wave packet"

$$\psi_0 = \int f(E)\, \varphi_0(E)\, dE$$

One has $\Omega^+\psi_0 = \int f(E)\, \Omega^+\varphi_0(E)\, dE$ and on the other hand $\Omega^+\psi_0 = \psi$ should admit an expansion in terms of the $\varphi(E)$:

$$\psi = \int g(E)\, \varphi(E)\, dE$$

So our problem is to find g so that

$$e^{-itH_0}\psi_0 = \int e^{-itE} f(E)\varphi_0(E)\, dE \sim e^{-itH}\psi = \int e^{-itE} g(E)\varphi(E)\, dE$$

as $t \longrightarrow -\infty$.

It's clear I am missing some ingredient which interprets this asymptotic condition.

In any case let's notice that there exists a measure $d\mu_0(E)$ on $[0,\infty)$ such that

$$f(E) \longmapsto \int_0^\infty f(E)\,\varphi_0(E)\,\frac{d\mu_0}{dE}\cdot dE$$

is an isomorphism of $L^2(d\mu_0)$ with $L^2(0,\infty)$. To find it note that

$$\int_0^\infty f(x)\sin kx\,dx = \int_0^\infty f(x)\,\frac{e^{ikx}-e^{-ikx}}{2i}\,dx = \frac{1}{2i}\int_{-\infty}^\infty f(x)\,e^{ikx}\,dx$$

if $f(-x) = f(x)$, so

$$\frac{2i}{\sqrt{2\pi}}\int_0^\infty f(x)\sin kx\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{ikx}\,dx$$

hence by Plancherel

$$\int_{-\infty}^\infty \left|\frac{2i}{\sqrt{2\pi}}\int_0^\infty f(x)\sin kx\,dx\right|^2 dk = \int_{-\infty}^\infty |f(x)|^2\,dx$$

$$\int_0^\infty |f(r)|^2\,dr = \frac{2}{\pi}\int_0^\infty \left|\int_0^\infty f(r)\,\frac{\sin kr}{k}\,dr\right|^2 k^2\,dk$$

$$= \int_0^\infty \left|\int_0^\infty f(r)\,\frac{\sin\sqrt{E}r}{\sqrt{E}}\,dr\right|^2 \frac{\sqrt{E}\,dE}{\pi}$$

so $\quad d\mu_0(E) = \dfrac{\sqrt{E}\,dE}{\pi}$.

Next since Ω^+ intertwines H_0 and H it should be the case that

$$\Omega^+\varphi_0(E)\frac{d\mu_0}{dE} = c^+(E)\,\varphi(E)\,\frac{d\mu}{dE}$$

where $d\mu$ is the spectral measure measure for H on $\mathcal{H}_{scatt} = \Omega^{+\alpha}\mathcal{H}$ and where $|c^+(E)| = 1$.

Again consider $H = -\frac{d^2}{dr^2} + V(r)$ on $[0,\infty)$ with bdry condition $u(0) = 0$. To simplify suppose V has compact support, so that we know what solutions look like for large r. For example, there is a solution

$$f^+(k,r) \sim e^{ikr}$$

To obtain it, write the Schroedinger equation in the form

$$\left(\frac{d^2}{dr^2} + k^2\right) u = V(r)\, u.$$

The variation-of-constants formula gives

$$f^+(k,r) = e^{ikr} - \int_r^\infty \frac{\sin k(r-\hat{r})}{k} V(\hat{r}) f^+(k,\hat{r})\, d\hat{r}$$

Question: What have these solutions to do with the wave operators?

Situation: You have specific self-adjoint operators $H = -\frac{d^2}{dr^2} + V(r)$, $H_0 = -\frac{d^2}{dr^2}$ on a Hilbert space $\mathcal{H} = L^2(0,\infty)$ (boundary condition $u(0)=0$ necessary to define these operators), and so consequently you should have wave operators $\Omega^\pm$, etc. In particular to each $\psi \in \mathcal{H}$ you should have another element $\Omega^+\psi$ such that

$$e^{-itH}\,\Omega^+\psi \sim e^{-itH_0}\,\psi \quad \text{as } t \to -\infty$$

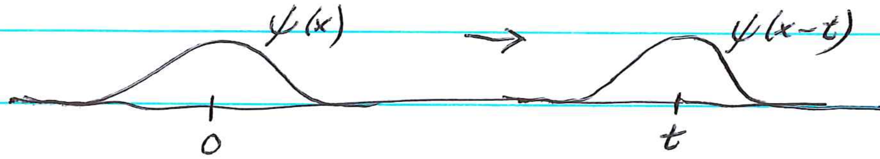
On the other hand this has to relate to the solutions of the differential equations, where I can see a definite asymptotic behavior. What's the missing ingredient?

Suppose we consider instead [redacted]

$$H = \frac{1}{i}\frac{d}{dx} + V(x) \qquad \text{on } (-\infty, \infty)$$

with $\mathcal{H} = L^2(-\infty, \infty)$. Here $H_0 = \frac{1}{i}\frac{d}{dx}$ and

$$\left(e^{-iH_0 t}\psi\right)(x) = e^{-t\frac{d}{dx}}\psi(x) = \psi(x$$ [redacted] $$-t)$$



[redacted] So the group $e^{-iH_0 t}$ is easily visualized. Next we can solve the [redacted] equation

[redacted] $$H\psi = \lambda\psi$$

because it's a linear first order DE.

$$\frac{1}{i}\frac{d\psi}{dx} + V\psi = \lambda\psi$$

$$\frac{1}{\psi}\frac{d\psi}{dx} = i(\lambda - V)$$ [redacted]

[redacted] $$\ln\psi = \int^x i(\lambda - V)$$

or $$\psi(x) = \psi(0)\, e^{\int_0^x [i\lambda - V(\hat{x})]d\hat{x}}$$

$$= \psi(0)\, e^{i\lambda x - i\int_0^x V(\hat{x})d\hat{x}}$$

[redacted] It's more straightforward to note that H is conjugate to H_0. Let $\sigma(x) = \int_{-\infty}^{x} V(\hat{x})d\hat{x}$. Then

$$e^{-i\sigma}\frac{1}{i}\frac{d}{dx}e^{i\sigma} = \frac{1}{i}\frac{d}{dx}+\sigma' = \frac{1}{i}\frac{d}{dx}+V = H$$

Hence given $\psi(x)\in\mathcal{H}$, one has

$$H\left(e^{-i\sigma}\psi\right) = e^{-i\sigma}H_0\psi$$

so that

$$e^{-itH}\left(e^{-i\sigma}\psi\right) = e^{-i\sigma}e^{-itH_0}\psi.$$

If I put

$$\Psi(t,x) = \left[e^{-itH}\left(e^{-i\sigma}\psi\right)\right](x) = e^{-i\sigma(x)}\psi(x-t)$$

Then

$$i\frac{\partial\Psi}{\partial t} = e^{-i\sigma(x)}\frac{1}{i}\frac{d\psi}{dx}(x-t)$$

$$H\Psi = \left(\frac{1}{i}\frac{d}{dx}+V(x)\right)\left(e^{-i\sigma(x)}\psi(x-t)\right)$$

$$= e^{-i\sigma(x)}\frac{1}{i}\frac{d}{dx}\psi(x-t)$$

so $i\dfrac{\partial\Psi}{\partial t} = H\Psi$. On the other hand

$$\Psi_0(t) = e^{-itH_0}\psi = \psi(x-t)$$

Compare $\Psi_0(t)$ and $\Psi(t)$ as $t\to-\infty$.

$$e^{itH_0}\left(\Psi_0(t)-\Psi(t)\right) = \psi(x) - e^{-i\sigma(x+t)}\psi(x)$$

As $t\to-\infty$, we know that $\sigma(x+t)\longrightarrow 0$ so that the above difference $\Psi_0-\Psi\to 0$ in norm, so one can conclude that

$$\Omega^+\psi = e^{-i\sigma}\psi$$

A similar argument shows that

$$\Omega^{-}\psi = e^{+i\int_x^{\infty} V(\hat{x})\,d\hat{x}}\,\psi(x)$$

Hence the scattering matrix is

$$S = (\Omega^{-})^{-1}\Omega^{+} = e^{-i\int_x^{\infty}}\, e^{-i\int_{-\infty}^{x}}$$

$$\boxed{S = e^{-i\int_{-\infty}^{\infty} V(\hat{x})\,d\hat{x}}}$$

October 1, 1977: Scattering:

Consider $H = H_0 + H' = -\frac{d^2}{dx^2} + V(x)$

where V decays fast as $|x| \to \infty$, say of compact support if necessary. What is the Green's function for $H_0 = -\frac{d^2}{dx^2}$. Put $k^2 = E$ and note that $\operatorname{Im} k > 0$ is mapped bijectively onto $\mathbb{C} - \mathbb{R}_{\geq 0}$. The solution of

$$(E - H_0)\, G(x,x') = \left(k^2 + \frac{d^2}{dx^2}\right) G(x,x') = \delta(x - x')$$

which decays as $|x| \to \infty$ is

$$G(x,x') = \begin{cases} a e^{ikx} & x > x' \\ b e^{-ikx} & x < x' \end{cases}$$

since $\operatorname{Im} k > 0$. So continuity requires

$$G(x,x') = \begin{cases} c e^{ik(x-x')} & x \geq x' \\ c\, e^{ik(x'-x)} & x \leq x' \end{cases}$$

$$\therefore \quad \frac{dG}{dx}(x'_+, x') = cik$$

$$\frac{dG}{dx}(x'_-, x') = -cik$$

But

$$\frac{dG}{dx}(x'_+, x') - \frac{dG}{dx}(x'_-, x') = \int_{x'_-}^{x'_+} \left(k^2 + \frac{d^2}{dx^2}\right) G(x,x')\, dx = \underbrace{\int_{x'_-}^{x'_+} \delta(x - x')\, dx}_{=1}$$

so that $c\,2ik = 1$ or $c = 2ik$, hence the free G-fn is

$$\boxed{G(x,x') = \frac{e^{ik(x_> - x_<)}}{2ik}}$$

Suppose $\varphi_0(x)$ is a rapidly-decreasing function. Then we can expand it into the eigenfunctions we have for $H_0 = -\frac{d^2}{dx^2}$, namely

$$\varphi_0(x) = \int_{-\infty}^{\infty} e^{ikx}\,\hat{\varphi}_0(k)\,dk$$

where $\hat{\varphi}_0(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\varphi_0(x)\,dx$. If φ_0 decays fast enough this is even an analytic function of k.

Suppose the potential $V(x)$ decays fast enough so that we know that we can find a unique solutions $\psi^+(x,k)$

$$H\psi = -\frac{d^2\psi}{dx^2} + V(x)\psi = k^2\psi$$

with asymptotic behavior $\psi^+(x,k) \sim e^{ikx}$ as $x \to +\infty$. Here $\operatorname{Im} k \geq 0$ for this to be meaningful (otherwise e^{-ikx} is the decaying solutions). $\psi^+(x,k)$ should be analytic for $\operatorname{Im} k \geq 0$ and should be obtainable by solving the Volterra equation

$$\psi^+(x,k) = e^{ikx} - \int_x^{\infty} \frac{\sin k(x-x')}{k} V(x')\psi^+(x',k)\,dx'.$$

Question: I can form

$$\varphi(x) = \int_{-\infty}^{\infty} \psi^+(x,k)\,\hat{\varphi}_0(k)\,dk$$

where $\hat{\varphi}_0$ is as above. Is this $\Omega^+\varphi_0$? In other words, is

$$e^{-itH}\varphi \sim e^{-itH_0}\varphi_0 \quad \text{as } t \to -\infty?$$

$$e^{-itH_0}\varphi_0 = \int_{-\infty}^{\infty} e^{-itk^2} e^{ikx} \hat{\varphi}_0(k)\,dk$$

$$e^{-itH}\varphi = \int_{-\infty}^{\infty} e^{-itk^2} \psi^+(x,k)\, \hat{\varphi}_0(k)\,dk$$

But $\psi^+(x,k) = e^{-ikx}$ for $x > x_0$ if $V(x)$ has compact support, hence $e^{-itH_0}\varphi_0$ and $e^{-itH}\varphi$ will agree for $x > x_0$ and I can't see why they would be asymptotic as $t \to -\infty$ (?).

October 2, 1977:

Continue with scattering for $H = -\frac{d^2}{dx^2} + V(x)$ on the line. Suppose to simplify that V is smooth with compact support. I want to determine the wave operator Ω^+ which intertwines H and $H_0 = -\frac{d^2}{dx^2}$. It should be the case that that Ω^+ carries the eigenfunction e^{ikx}, $k \in \mathbb{R}$, into an eigenfunction $\psi^+(x,k)$ for the operator H, and that $\psi^+(x,k)$ satisfies the Lipmann-Schwinger equation

$$\psi^+(x,k) = e^{ikx} + \int_{-\infty}^{\infty} \frac{e^{ik|x-x'|}}{2ik} V(x')\psi^+(x',k)\,dx'$$

Note that this equation determines the asymptotic behavior of ψ^+: For $x > \mathrm{Supp}(V)$ we have

$$\psi^+(x,k) = e^{ikx} + e^{ikx} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-ikx'}}{2ik} V(x')\psi^+(x',k)\,dx'}_{\text{function of } k}$$

so $\psi^+(x,k)$ is proportional to e^{ikx} for $x >> 0$.
Similarly for $x <$ Supp(V) we have

$$\psi^+(x,k) = e^{ikx} + e^{-ikx} \underbrace{\int_{-\infty}^{\infty} \frac{e^{ikx'}}{2ik} V(x')\psi^+(x';k)\, dk}_{\text{function of } k}$$

Maybe we have to understand the LS equations:

$$\psi = \psi_0 + (E - H_0)^{-1} V\psi$$

To begin with lets work in L^2 and suppose $E \notin \mathbb{R}_{\geq 0}$. Then $(E-H_0)^{-1}$ is a bounded operator and this equation is equivalent to

$$(E-H_0)\psi = (E-H_0)\psi_0 + V\psi$$

or

$$\boxed{(E-H)\psi = (E-H_0)\psi_0}$$

So for each ψ_0 (in the domain of H_0) there is a unique solution ψ provided E is not in the spectrum of H.

I have seen that if $\psi^+(k,x)$ is the solution of Lipmann-Schwinger with $\psi_0 = e^{ikx}$, then $\psi^+(k,x)$ is a solution of $H\psi = E\psi$ such that

$$\psi^+ \sim c_1 e^{ikx} \qquad x \to +\infty$$

$$\psi^+ \sim e^{ikx} + c_2 e^{-ikx} \qquad x \to -\infty$$

But this pins ψ^+ down, because we recall there is a unique solution f^+ of $H\psi = E\psi$ with $f^+ \sim e^{ikx}$ as

$x \to +\infty$ for $\mathrm{Im}(k) \geq 0$. If we [redacted] continue f^+ toward $-\infty$ it has an expansion

$$f^+(x,k) = a(k)e^{ikx} + b(k)e^{-ikx} \qquad x \to -\infty$$

where [redacted]

$$\overline{f^+(x,k)} = f^+(x,-k) \quad \text{[redacted]}$$

$$\overline{f^+(x,k)} = \overline{a(k)}\,e^{-ikx} + \overline{b(k)}\,e^{+ikx}, \qquad f^+(x,-k) = a(-k)e^{-ikx} + b(-k)e^{+ikx}$$

so $\overline{a(k)} = a(-k)$ and $\overline{b(k)} = b(-k)$. [redacted] Assuming $\psi^+(x,k)\ \exists$ we see that $a(k) \neq 0$ and

$$\psi^+(x,k) = \frac{f^+(x,k)}{a(k)}$$

<u>Summary</u>: Assume V has compact support. Denote by $f^+(x,k)$ the unique solution of

$$\left(-\frac{d^2}{dx^2} + V\right)u = k^2 u$$

such that $f^+(x,k) = e^{ikx}$ for $x \gg 0$. (f^+ is entire in k, but if one [redacted] weakens V compact support f^+ is only well-defined for $\mathrm{Im}\, k \geq 0$, which corresponds to the "physical sheet" for $E = k^2$). Let $a(k)$, $b(k)$ be defined [redacted] by

$$f^+(x,k) = a(k)e^{ikx} + b(k)e^{-ikx} \qquad \text{for } x \ll 0$$

(These are defined for $k \neq 0$.) [redacted] If $\psi^+(x,k)$ is a solution of the L-S equation

$$\psi^+ = e^{ikx} + \int_{-\infty}^{\infty} \frac{e^{ik|x-x'|}}{2ik} V(x')\psi^+(x')\,dx'$$

then $a(k) \neq 0$ and $\psi^+(x,k) = \dfrac{f^+(x,k)}{a(k)}$. Conversely $a(k) \neq 0 \Rightarrow \psi^+ = \dfrac{f^+}{a}$ is a solution (necessarily unique) of the L-S equation.

To see the last point we note that for θ of compact support

$$\cancel{e^{ikx} + \int_{-\infty}^{\infty} \frac{e^{ik|x-x'|}}{2ik} V(x') \frac{f^+(x',k)}{a(k)}\, dx'} \qquad (k \neq 0)$$

$$g(x) = \int_{-\infty}^{\infty} \frac{e^{ik|x-x'|}}{2ik}\, \theta(x')\,dx'$$

is the unique solution of

$$(*) \qquad \left(\frac{d^2}{dx^2} + k^2\right) g(x) = \theta(x)$$

such that

$$g'(b) = ik\, g(b)$$
$$g'(a) = -ik\, g(a)$$

where $\mathrm{Supp}(\theta) \subset (a,b)$. In other words for any $k \neq 0$ and compactly supported θ, there is a unique solution of $(*)$ such that g is proportional to e^{ikx} for x large and e^{-ikx} for $x \ll 0$. (g is a so-called outgoing solution.) But

$$\frac{f^+(x,k)}{a(k)} - e^{ikx} = \begin{cases} \left(\dfrac{1}{a(k)} - 1\right) e^{ikx} & x \gg 0 \\[2ex] \dfrac{b(k)}{a(k)}\, e^{-ikx} & x \ll 0 \end{cases}$$

and

$$\left(\frac{d^2}{dx^2} + k^2\right)\left(\frac{f^+}{a} - e^{ikx}\right) = V\left(\frac{f^+}{a}\right)$$

hence by the uniqueness $\dfrac{f^+}{a}$ has to satisfy LS equation.

Next point is to notice that if $a(k)=0$ with $\mathrm{Im}(k)>0$, then $f^+(x,k)$ is square integrable and one has a bound state. Since H is self-adjoint this forces k^2 to be real, hence $k \in i\mathbb{R}_{>0}$.

Question: Is it possible for $a(k)=0$ with k real $\neq 0$?

October 3, 1977:

Given a solution f of the Schroedinger equation

$$\left(-\frac{d^2}{dx^2}+V(x)\right)f = k^2 f$$

with V of compact support we know

$$f = \begin{cases} a e^{ikx} + b e^{-ikx} & x \gg 0 \\ c e^{ikx} + d e^{-ikx} & x \ll 0 \end{cases}$$

for certain constants a, b, c, d. (In fact there is a scattering matrix defined as follows:

$$\begin{pmatrix} a \\ b \end{pmatrix} = S(k)\begin{pmatrix} c \\ d \end{pmatrix}$$

if the solution asymptotic to $ce^{ikx}+de^{-ikx}$ for $x \ll 0$ becomes asymptotic to $ae^{ikx}+be^{-ikx}$ for $x \gg 0$.)

Given f we can determine f_0 by LS equation

$$f = f_0 + \int \frac{e^{ik|x-x'|}}{2ik} V(x') f(x')\, dx'$$

Then $\left(\frac{d^2}{dx^2}+k^2\right) f = \left(\frac{d^2}{dx^2}+k^2\right) f_0 + Vf \Rightarrow$

$\left(\frac{d^2}{dx^2}+k^2\right) f_0 = 0$

and $f - f_0 = \begin{cases} c_1 e^{ikx} & x \gg 0 \\ c_2 e^{-ikx} & x \ll 0 \end{cases}$

so clearly $f_0 = ce^{ikx} + be^{-ikx}$

October 4, 1977

Consider $\left(-\frac{d^2}{dx^2} + V\right)\psi = \sigma^2\psi$ on the line where $V(x)$ has support in $(-a,a)$. Start

$$\underset{x \ll 0}{e^{i\sigma x}} \longleftrightarrow \underset{x \gg 0}{A(\sigma)e^{i\sigma x} + B(\sigma)e^{-i\sigma x}}.$$

$$e^{-i\sigma x} \longleftrightarrow \overline{A(\sigma)}\, e^{-i\sigma x} + \overline{B(\sigma)}e^{i\sigma x}$$

This is for σ real, since

$$W(e^{i\sigma x}, e^{-i\sigma x}) = W(e^{i\sigma x}, e^{-i\sigma x}) \begin{vmatrix} A(\sigma) & \overline{B(\sigma)} \\ B(\sigma) & \overline{A(\sigma)} \end{vmatrix}$$

we conclude that $|A(\sigma)|^2 - |B(\sigma)|^2 = 1$ for σ real. In particular for σ real we have $|A(\sigma)| \geq 1$.

Let $\psi(x,\sigma)$ denote the solution with

$$\psi(x,\sigma) = \begin{cases} e^{i\sigma x} + R(\sigma)e^{-i\sigma x} & x \gg 0 \\ T(\sigma)e^{i\sigma x} & x \ll 0 \end{cases}$$

Interpretation: $e^{-i\sigma x}$ is a standing wave in $x \gg 0$ which encounters the obstacle. Part of the wave $T(\sigma)e^{i\sigma x}$ gets transmitted through the obstacle, and part of the wave $R(\sigma)e^{-i\sigma x}$ gets reflected:

<u>Def</u>: $T(\sigma) = \dfrac{1}{A(\sigma)}$ <u>transmission</u> coefficient

$R(\sigma) = \dfrac{B(\sigma)}{A(\sigma)}$ <u>reflection</u> coefficient

Note that

$$|T(\sigma)|^2 + |R(\sigma)|^2 = \frac{1}{|A|^2} + \frac{|B|^2}{|A|^2} = 1$$

Relation to Lipmann-Schwinger: I have to interpret σ as $-k$ so that $\mathrm{Im}(\sigma) < 0$ corresponds to the physical sheet. A zero of $A(\sigma)$ with $\mathrm{Im}(\sigma) < 0$ has to represent a bound state since one then has

$$\underset{x \ll 0}{e^{-ikx}} \quad \longleftarrow\cdots\cdots\longrightarrow \quad \underset{x \gg 0}{B(\sigma)e^{ik\sigma}}$$

Hence the zeroes of $A(\sigma)$ for $\mathrm{Im}(\sigma) < 0$ are on $i\mathbb{R}_-$, since $\sigma^2 = k^2$ has to be real. However the zeroes of A in the upper half plane represent something interesting called scattered states.

<u>Example</u>: $V(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

October 5, 1977:

A free particle in quantum mechanics is described by the Schroedinger equation $\frac{d\Psi}{dt} = -iH_0\Psi$ where

$$H_0 = -\frac{d^2}{dx^2}$$

Digress: Classically $H = \frac{p^2}{2m} + V(q)$ and Hamilton's eqns. are

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} = -\frac{\partial H}{\partial q} = -\frac{dV}{dq} \end{cases}$$

To get quantum mechanics in the coordinate representation, a state is a function $\psi(x)$, q is the operator of multiplying by x and $p = \frac{\hbar}{i}\frac{d}{dx}$ so that

$$[p, q] = \frac{\hbar}{i}$$

Then

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

$\hbar$ meas. in $\frac{gr\,cm^2}{sec}$

and Schroedinger's equation is

energy meas in $gr\frac{cm^2}{sec^2}$

$$i\hbar\frac{d\Psi}{dt} = H\Psi$$

whose solutions are

$$\Psi(t) = e^{-\frac{i}{\hbar}Ht}\psi \qquad \psi = \Psi(0)$$

Let's arrange the units so that $\hbar = 1$, $m = \frac{1}{2}$ so that $H_0 = -\frac{d^2}{dx^2}$ for a free particle. A state is a L^2 function on the line of norm 1, call it $\psi(x)$. The momentum representation is the Fourier transform. Thus if

$$\psi(x) = \int e^{ikx}\,\hat{\psi}(k)\,dk$$

we see that $\frac{1}{i}\frac{d}{dx} \leftrightarrow$ mult. by k. We have

$$e^{-iH_0 t}\psi = \int e^{ikx} e^{-ik^2 t}\,\hat{\psi}(k)\,dk$$

It's pretty complicated to see what happens as $t \to +\infty$. If $\hat{\psi}(k) = 0$ near $k=0$, then the Riemann-Lebesgue lemma says that for x fixed this integral goes to zero, so the probability of finding the particle in a bdd region goes to zero.

I should be able to completely describe the time evolution of Gaussian states. For example suppose we have a particle concentrated at the origin: $\hat{\psi}(k) \equiv 1$. Then

$$\int e^{ikx} e^{-ik^2 t}\,dk = \frac{\sqrt{\pi}}{\sqrt{it}}\, e^{\frac{(ix)^2}{4it}} = \frac{\sqrt{\pi}}{\sqrt{it}}\, e^{-\frac{x^2}{4it}}$$

assuming $t > 0$. Here I have used the formula

$$\int e^{-ax^2+bx}\,dx = \int e^{-a\left(x-\frac{b}{2a}\right)^2 + \frac{b^2}{4a}}\,dx = e^{\frac{b^2}{4a}} \int e^{-x^2}\,\frac{dx}{\sqrt{a}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{a}}\, e^{\frac{b^2}{4a}}$$

valid for $\mathrm{Re}(a) > 0$, maybe also for $\mathrm{Re}(a) \geq 0$.

Take $\hat{\psi}(k) = e^{-ak^2+bk}$, $\mathrm{Re}(a) > 0$. Then

$$e^{-itH}\psi = \int e^{ikx} e^{-itk^2}\,\hat{\psi}(k)\,dk = \int e^{-(a+it)k^2 + (b+ix)k}\,dk$$

$$= \frac{\sqrt{\pi}}{\sqrt{a+it}}\, e^{\frac{(b+ix)^2}{4(a+it)}}$$

For example suppose $b = 0$ and that a is real. Then

$$\mathrm{Re}\left(-\frac{x^2}{4(a+it)}\right) = -\frac{x^2}{4}\frac{a}{\sqrt{a^2+t^2}}$$

so as time increases the probability of finding the particle spreads out over space.

So lets return to $H = -\frac{d^2}{dx^2} + V$ where V has compact support. I have seen that for k real $\neq 0$ there is a unique solution $\psi^+(x,k)$ of Schroedinger eqn. with

$$\psi^+(x,k) = T(k)e^{ikx} \qquad x \gg 0$$

$$\psi^+(x,k) = e^{ikx} + R(k)e^{-ikx} \qquad x \ll 0$$

and that $\psi^+(x,k)$ is the unique solution of the L-S integral equation

$$\psi^+(x,k) = e^{ikx} + \int_{-\infty}^{\infty} \frac{e^{ik|x-x'|}}{2ik} V(x')\psi^+(x',k)\,dk$$

Take a free state

$$\varphi_0(x) = \int e^{ikx}\rho(k)\,dk$$

where ρ is C_0^∞ with support in a small nbd. of $k_0 \neq 0$. Thus φ_0 represents a [redacted] wave having momentum [redacted] nearly k_0. It was my idea that the state $\Omega^+\varphi_0 = \varphi$ should be given by

$$\varphi(x) = \int \psi^+(x,k)\rho(k)\,dk$$

If so then I should be able to show that the difference of $\Phi(t) = e^{-itH}\varphi = \int e^{-itk^2}\psi^+(x,k)\rho(k)\,dk$

$$\Phi_0(t) = e^{-itH_0}\varphi_0 = \int e^{-itk^2} e^{ikx}\rho(k)\,dk$$

tends to zero in norm as $t \to -\infty$

The first point is to notice that in any bounded region $|x| \leq R$ the wave functions $e^{-itH}\varphi$ and $e^{-itH_0}\varphi_0$ tend to zero by the Riemann-Lebesgue lemma as either $t \to \pm\infty$. This assumes perhaps that ρ vanishes near 0. Next look at the interval $[R, \infty)$. If $k_0 > 0$, then the particle should be moving to the right hence the wave function $\Psi_0(t)$ should tend to zero in this set as $t \to -\infty$. The same is true for $\Psi(t)$ since it agrees with $T(k)\Psi_0(t)$ in (R, ∞). Now look in $(-\infty, -R]$ where

$$\Psi(t) - \Psi_0(t) = \int e^{-itk^2} R(k) e^{-ikx} \rho(k)\,dk$$

The last integral is the wave function of a particle moving to the left with momentum $\sim k_0$ hence it decays as $t \to -\infty$ in $(-\infty, -R]$. So this all works if $k_0 > 0$.

If $k_0 < 0$ it doesn't work and so we have the wrong $\psi^+(x, k)$ for $k < 0$.