August 28, 1977

Recall that an entire function E is a de Branges fn. if $\operatorname{Im}(z)>0 \implies |E(\lambda)| > |E(\bar\lambda)|$ and that one gets a Hilbert space
$$B(E) = \Big\{ f \text{ entire} \;\Big|\; \|f\|^2 = \int_{\mathbb{R}} \Big|\frac{f}{E}\Big|^2 dx < \infty$$
$$\Big|\frac{f}{E}(\lambda)\Big| \le \frac{c(f)}{\sqrt{\operatorname{Im}\lambda}} \quad \text{for } \operatorname{Im}\lambda > 0$$
$$\Big|\frac{f}{E^\#}(\lambda)\Big| \le \frac{c(f)}{\sqrt{-\operatorname{Im}\lambda}} \quad \text{for } \operatorname{Im}\lambda < 0 \Big\}.$$

Suppose that $h(\lambda)$ is an entire function such that $h^\# = h$ (i.e. $h(\mathbb{R}) \subset \mathbb{R}$) which doesn't vanish for nonreal λ. Then hE is also a de Branges function and moreover $B(hE)$ is isomorphic to $B(E)$ i.e.
$$f \longmapsto hf \qquad B(E) \xrightarrow{\sim} B(hE).$$

Clear that the map is well-defined and is isometric, so one has to show it is onto. However if $g \in B(hE)$, then we know that $\frac{g}{hE}$ is analytic for $\operatorname{Re}(\lambda) \ge 0$, hence $\frac{g}{h}$ is analytic for $\operatorname{Re}(\lambda) \ge 0$ and hence $\frac{g}{h}$ is entire.

Consequently if E has a real zero one can remove it in some sense. If one is interested in classifying de Branges spaces, then one has to allow for the preceding modification.

Example: Suppose $B(E)$ is one-dimensional. Then every $f \in B(E)$ is a constant times J_i. Also since

$J_i^\# = J_{-i}$ one has $J_{+i}^\# = c\, J_i$ whence $J_i = \bar{c}\, J_i^\# = \bar{c} c J_i$ so $|c| = 1$. Hence every element of $B(E)$ is a constant times $h = e^{-i\alpha} J_i$, where α is chosen so that

$$h^\# = e^{-i\alpha} J_i^\# = e^{-i\alpha} c\, J_i = e^{-i\alpha} c e^{-i\alpha} h = h$$

i.e. $e^{2i\alpha} = c$. Notice that because $J_\omega \neq 0$ for ω nonreal, h has only real zeros.

Because one can take $E(\lambda) = (1 - i\gamma)\, J_i(\lambda)\, \|J_i\|^{-1}$ if one wants, it is clear that h divides E, hence one gets an isometry between $B(E)$ and $B((1-i\gamma)a)$ for some $a > 0$.

Suppose that $E(z)$ is a polynomial. It's roots λ_i have to satisfy $\mathrm{Im}(\lambda_i) \leq 0$, and one might as well suppose there are no real roots. Say

$$E(z) = \prod_{i=1}^{n} (z - \lambda_i) \qquad \mathrm{Im}(\lambda_i) < 0.$$

Then $$\frac{E^\#(z)}{E(z)} = \prod \frac{z - \bar{\lambda}_i}{z - \lambda_i} = \prod \left(1 - \frac{\bar{\lambda}_i}{z}\right)\left(1 + \frac{\lambda_i}{z} + O\left(\frac{1}{z^2}\right)\right)$$

$$\left|\frac{E^\#(z)}{E(z)}\right| = \prod \left(1 - \frac{\bar{\lambda}_i}{z}\right)\left(1 + \frac{\lambda_i}{z} + O\left(\frac{1}{z^2}\right)\right)\left(1 - \frac{\lambda_i}{\bar{z}}\right)\left(1 + \frac{\bar{\lambda}_i}{\bar{z}} + O\left(\frac{1}{\bar{z}^2}\right)\right)$$

Let $f(z) \in B(E)$. Then

$$|f(z)|^2 \leq \text{Const}\ \frac{|E(z)|^2 - |E(\bar{z})|^2}{\operatorname{Im} z} \leq \text{Const}\ \begin{matrix}\text{polynomial in}\\ \operatorname{Re}(z),\ \operatorname{Im}(z) \text{ of}\\ \text{degree} \leq 2n-1\end{matrix}$$

$$\leq C|z|^{2n-1} \qquad |z| \text{ large}$$

hence $f(z)$ is a polynomial of degree $\leq n-1$. Consequently $B(E)$ consists of all polynomials of degree $\leq n-1$.

To conclude the description of $B(E)$ we have to describe the inner product, or what amounts to the same thing, we have to describe the orthonormal basis of $B(E)$ one gets by orthonormalizing the sequence

$$1, z, \ldots, z^{n-1}.$$

Notice from the definition of inner product that we have

$$(zf, g) = (f, zg)$$

provided zf, zg are defined, i.e. $\deg(f), \deg(g) < n-1$. Let $\phi_0, \ldots, \phi_{n-1}$ be the orthonormal sequence of polys. One has for $i, j < n-1$

$$(z\phi_i, \phi_j) = (\phi_i, z\phi_j) = \begin{cases} 0 & \text{if } j < i-1 \end{cases}$$

hence $z\phi_i = a_i\phi_{i+1} + b_i\phi_i + c_i\phi_{i-1}$. Moreover

$$c_i = (z\phi_i, \phi_{i-1}) = (\phi_i, z\phi_{i-1}) = a_{i-1}$$

because in the orthonormalizing process ϕ_i has positive leading coefficient. So we get a Jacobi matrix if we extend the domain of multiplication by z by putting

$$z\phi_{n-1} = b_{n-1}\phi_{n-1} + a_{n-2}\phi_{n-2}$$

where b_{n-1} is an arbitrary real number.

<u>August 29, 1977:</u>

Again suppose E is a polynomial of degree n. From the formula

$$J_z(\lambda) = \frac{i}{\lambda - \bar{z}}\left[\overline{E(z)}\,E(\lambda) - \overline{E^{\#}(z)}\,E^{\#}(\lambda)\right]$$

one sees that J_z is a poly of degree $\leq n-1$. (Convention: $\|f\|^2 = \int |f|^2 \frac{dx}{2\pi|E|^2}$). If E has real zeroes these are also zeroes of J_z, hence of each $f \in B(E)$. Thus $B(E)$ consists of polys f of degree $\leq n-1$ such that f/E has no poles on $\mathbb{R}$, and also one can see that any such polynomial is in $B(E)$.

Question: How unique is E?

Suppose E has no real zeroes from now on and that it is of degree n. I have seen that $B = B(E)$ can be described in terms of the orthonormal sequence of polynomials one constructs by applying Gram-Schmidt to the sequence $1, z, \dots, z^{n-1}$. This gives a sequence of polys

$$\phi_0, \phi_1, \dots, \phi_{n-1}$$

satisfying

$$\phi_0 = \frac{1}{\|1\|}$$

$$z\phi_i = a_i\phi_{i+1} + b_i\phi_i + a_{i-1}\phi_{i-1}$$

where $a_0, \dots, a_{n-2} > 0$ and $b_0, \dots, b_{n-2} \in \mathbb{R}$. Hence $B(E)$ depends on n positive nos. and $(n-1)$ real numbers.

The reality condition that $f \mapsto f^-(z) = f(-z)$ be an isometry of B forces the b_i to be zero. In effect this isometry leaves invariant the filtration by degrees. If F_p = polys of degree $\le p$, then ϕ_p is the unique unit vector of F_p perpendicular to F_{p-1} such that $\phi_p - cz^p \in F_{p-1}$ for some $c > 0$. Hence $(-1)^p \phi_p^- \perp F_{p-1}$ and

$$(-1)^p \phi_p^- - cz^p \in F_{p-1}$$

so that by uniqueness $(-1)^p \phi_p^- = \phi_p$, i.e. ϕ_p is even or odd according to the parity of p. Hence the b_i are all zero.

So $B(E)$ depends $2n-1$ real parameters. The possible polys. E of degree n with roots in the lower half plane are described by $2(n+1)$ real parameters. We can eliminate one by noting that $B(E)$ doesn't change if E is replaced by ωE, $|\omega| = 1$, hence we can assume the leading coefficient of E is > 0. But this still gives $2n+1$ real parameters. However one can choose E of the form

$$\text{const } (1 - i\lambda) J_i(\lambda)$$

hence one can suppose $E(-i) = 0$, so in this way one ought to get $2n-1$ parameters.

Conjecture: There is a unique choice for E such that $E(-i) = 0$ and such that E has positive leading coefficient.

Consider the real case: $E^\#(\lambda) = E(-\lambda)$. If we put

$$E(\lambda) = A(\lambda) - iB(\lambda) \qquad A^{\#}(\lambda) = A(\lambda)\,,\ B^{\#} = B.$$

then $E^{\#}(\lambda) = A(\lambda) + iB(\lambda)$, so $E^{\#}(\lambda) = E(-\lambda) \Longleftrightarrow$ A even, B odd. Compute

$$J_z(\lambda) = \frac{i}{\lambda - \bar z}\begin{vmatrix} E(\lambda) & E(\bar z) \\ E^{\#}(\lambda) & E^{\#}(\bar z)\end{vmatrix} = \frac{i}{\lambda - \bar z}\begin{vmatrix} A(\lambda) - iB(\lambda) & A(\bar z) - iB(\bar z) \\ A(\lambda) + iB(\lambda) & A(\bar z) + iB(\bar z)\end{vmatrix}$$

$$= \frac{i}{\lambda - \bar z}\left[\begin{array}{l} i\,A(\lambda)B(\bar z) - i\,A(\bar z)B(\lambda) + A(\lambda)A(\bar z) + B(\lambda)B(\bar z) \\ + i\,A(\lambda)B(\bar z) - i\,A(\bar z)B(\lambda) - A(\lambda)A(\bar z) - B(\lambda)B(\bar z)\end{array}\right]$$

$$= \frac{-2}{\lambda - \bar z}\begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z)\end{vmatrix}$$

Suppose $\tilde E$ is another function giving the same Hilbert space, i.e.

$$A(\lambda)B(\bar z) - B(\lambda)A(\bar z) = \tilde A(\lambda)\tilde B(\bar z) - \tilde B(\lambda)\tilde A(\bar z)$$

and suppose both $E, \tilde E$ satisfy the reality condition. Then taking even & odd parts as fns of λ we have

$$A(\lambda)B(\bar z) = \tilde A(\lambda)\tilde B(\bar z)$$

$$B(\lambda)A(\bar z) = \tilde B(\lambda)\tilde A(\bar z)$$

It follows that $A(\lambda) = k\tilde A(\lambda)$ $\quad k \in \mathbb{R}^*$ and then

$$B(\lambda) = \frac{1}{k}\tilde B(\lambda).$$

Note that since $|E(\lambda)|^2 = |A(\lambda) - iB(\lambda)|^2 > |E^{\#}(\lambda)|^2 = |A(\lambda) + iB(\lambda)|^2$ for $\mathrm{Im}(\lambda) > 0$ one necessarily has that A, B have only real zeroes.

Let ϕ_i be a sequence of orthonormal polys. Then in the Hilbert space generated by $\phi_0, \dots, \phi_{n-1}$ one has the point evaluator

$$J_z(\lambda) = \sum_{i=0}^{n-1} \overline{\phi_i(z)}\,\phi_i(\lambda)$$

In effect $\quad J_z = \sum \phi_i (J_z, \phi_i) = \sum \phi_i \overline{(\phi_i, J_z)}$

$$= \sum \phi_i \cdot \overline{\phi_i(z)}$$

Any E giving rise to these point evaluators satisfies

(*) $\quad J_z(\lambda) = \dfrac{i}{\lambda - \bar z} \begin{vmatrix} E(\lambda) & E(\bar z) \\ E^\#(\lambda) & E^\#(\bar z) \end{vmatrix}$ $\qquad \overline{\phi_i(z)} = \phi_i(\bar z)$

so let's calculate $(\lambda - \bar z) J_z(\lambda)$.

$$= (\lambda - \bar z) \sum_{i=0}^{n-1} \phi_i(\lambda)\overline{\phi_i(z)} = \sum_0^{n-1} \Big(a_i \phi_{i+1}(\lambda) + \cancel{b_i \phi_i(\lambda)} + a_{i-1}\phi_{i-1}(\lambda)\Big)\phi_i(\bar z)$$
$$- \phi_i(\lambda)\Big(a_i \phi_{i+1}(\bar z) + \cancel{b_i \phi_i(\bar z)} + a_{i-1}\phi_{i-1}(\bar z)\Big)$$

$$= a_n \Big[\phi_n(\lambda)\phi_{n-1}(\bar z) - \phi_{n-1}(\lambda)\phi_n(\bar z)\Big]$$

Now if $E(\lambda) = A(\lambda) - iB(\lambda)$ with $A^\# = A$, $B^\# = B$, then

$$\begin{vmatrix} E(\lambda) & E(\bar z) \\ E^\#(\lambda) & E^\#(\bar z) \end{vmatrix} = \begin{vmatrix} A(\lambda) - iB(\lambda) & A(\bar z) - iB(\bar z) \\ A(\lambda) + iB(\lambda) & A(\bar z) + iB(\bar z) \end{vmatrix} = \begin{vmatrix} 1 & -i \\ 1 & +i \end{vmatrix} \begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z) \end{vmatrix}$$

$$= 2i \begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z) \end{vmatrix}$$

so that $\quad (\lambda - \bar z) J_z(\lambda) = -2 \begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z) \end{vmatrix} = \begin{vmatrix} a_n\phi_n(\lambda) & a_n\phi_n(\bar z) \\ \phi_{n-1}(\lambda) & \phi_{n-1}(\bar z) \end{vmatrix}$

Consequently we are interested in all solutions $A(\lambda), B(\lambda)$ of the equation

$$-2\begin{vmatrix} A(\lambda) & A(\bar{z}) \\ B(\lambda) & B(\bar{z}) \end{vmatrix} = \begin{vmatrix} a_n\phi_n(\lambda) & a_n\phi_n(\bar{z}) \\ \phi_{n-1}(\lambda) & \phi_{n-1}(\bar{z}) \end{vmatrix}$$

where A, B are polys. of degree $\leq n$ with real coefficients. In effect if I have a solution then putting $E(\lambda) = A(\lambda) - iB(\lambda)$, one gets the right formula for $J_z(\lambda)$ and in particular

$$J_z(z) = \frac{|E(z)|^2 - |E^\#(z)|^2}{2\,\mathrm{Im}(z)} > 0$$

showing $E(z)$ is a de Branges function.

~~If A is to be even and B is to be odd, then~~ Note

$$\begin{vmatrix} a_n\phi_n(\lambda) & a_n\phi_n(\bar{z}) \\ \phi_{n-1}(\lambda) & \phi_{n-1}(\bar{z}) \end{vmatrix} = \begin{vmatrix} \lambda\phi_{n-1}(\lambda) - a_{n-1}\phi_{n-2}(\lambda) & \bar{z}\phi_{n-1}(\bar{z}) - a_{n-1}\phi_{n-2}(\bar{z}) \\ \phi_{n-1}(\lambda) & \phi_{n-1}(\bar{z}) \end{vmatrix}$$

does not depend on b_n.

It is clear that if A is to be even and B is to be odd, then (for n even)

$$-2A(\lambda)B(\bar{z}) = a_n\phi_n(\lambda)\cdot\phi_{n-1}(\bar{z})$$

$$-2B(\lambda)A(\bar{z}) = \phi_{n-1}(\lambda)\ a_n\phi_n(\bar{z})$$

so that

$$A(\lambda) = k\frac{a_n\phi_n(\lambda)}{(-2)}$$

$$B(\lambda) = k^{-1}\phi_{n-1}(\lambda)$$

for a real constant $k \neq 0$.

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, then $\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$

satisfies

$$(*) \qquad \begin{vmatrix} \tilde{A}(\lambda) & \tilde{A}(\bar{z}) \\ \tilde{B}(\lambda) & \tilde{B}(\bar{z}) \end{vmatrix} = \begin{vmatrix} A(\lambda) & A(\bar{z}) \\ B(\lambda) & B(\bar{z}) \end{vmatrix}$$

so we have many solutions to the problem of finding an $E = A - iB$ describing the given Hilbert space. Suppose that we require the leading coefficient of E to be positive. Then A has degree n, say $A(\lambda) = \alpha_n \lambda^n$ + lower terms $\alpha_n > 0$, and B has degree $< n$. If we have an $\tilde{E} = \tilde{A} - i\tilde{B}$ of the same sort satisfying $(*)$, then by comparing coefficients of λ^n on both sides we find

$$\tilde{\alpha}_n \tilde{B}(\bar{z}) = \alpha_n B(\bar{z}).$$

To simplify suppose $\tilde{\alpha}_n = \alpha_n$. Then $\tilde{B} = B$, so $(*)$ gives

$$\tilde{A}(\lambda) B(\bar{z}) - B(\lambda) \tilde{A}(\bar{z}) = A(\lambda) B(\bar{z}) - B(\lambda) A(\bar{z})$$

$$[\tilde{A}(\lambda) - A(\lambda)] B(\bar{z}) = B(\lambda) [\tilde{A}(\bar{z}) - A(\bar{z})]$$

so $\tilde{A} = A + cB$, $\quad c$ real.

Instead suppose we require $E(\lambda_0) = A(\lambda_0) - iB(\lambda_0) = 0$ where λ_0 is a fixed point in the lower half-plane. Then

$$J_{\bar{\lambda}_0}(\lambda) = \frac{i}{\lambda - \lambda_0} \begin{vmatrix} E(\lambda) & E(\lambda_0) \\ E^{\#}(\lambda) & E^{\#}(\lambda_0) \end{vmatrix}$$

$$= \frac{i}{\lambda - \lambda_0} E^{\#}(\lambda_0) E(\lambda)$$

$E^{\#}(\lambda_0) = A(\lambda_0) + iB(\lambda_0)$
$\quad = 2A(\lambda_0)$

$$= \frac{i}{\lambda - \lambda_0} \tilde{E}^{\#}(\lambda_0) \tilde{E}(\lambda)$$

Hence E is determined up to a constant multiple by the condition $E(\lambda_0)=0$. In fact putting $\lambda = \bar{\lambda}_0$ in

$$E^{\#}(\lambda_0)E(\lambda) = \tilde{E}^{\#}(\lambda_0)\tilde{E}(\lambda)$$

one gets $\quad |E^{\#}(\lambda_0)|^2 = \overline{E(\bar{\lambda}_0)}\,E(\bar{\lambda}_0) = E^{\#}(\lambda_0)E(\bar{\lambda}_0) = \cdots = |\tilde{E}^{\#}(\lambda_0)|^2$.

Hence E is determined up to a constant multiple of absolute value 1 by the condition $E(\lambda_0)=0$.

<u>August 30, 1977</u>: Recall:

<u>Prop</u>: If E is a poly of degree n with roots in the lower half plane, then $B(E)$ consists of all polys. of degree $\leq n-1$.

Conversely let $B(E)$ be a de Branges space consisting of all polys of degree $\leq n-1$. Better: Let E be a de Branges function such that $B(E)$ consists of all polys. of degree $\leq n-1$.

$$J_z(\lambda) = \frac{i}{\lambda - \bar{z}} \begin{vmatrix} E(\lambda) & E(\bar{z}) \\ E^{\#}(\lambda) & E^{\#}(\bar{z}) \end{vmatrix}$$

If the functions $E, E^{\#}$ were linearly dependent this determinant would vanish. Hence one can find two values of z such that $\begin{vmatrix} E(\bar{z}_1) & E(\bar{z}_2) \\ E^{\#}(\bar{z}_1) & E^{\#}(\bar{z}_2) \end{vmatrix} \neq 0$. It follows that $E(\lambda)$ is a linear combination of $(\lambda - \bar{z}_1)J_{z_1}(\lambda)$ and $(\lambda - \bar{z}_2)J_{z_2}(\lambda)$, hence $E(\lambda)$ is a poly of degree $\leq n$. In fact $E(\lambda)$ must have degree n and have no real roots.

<u>Prop</u>: $B(E)$ consists of all polys of deg $< n \Rightarrow E$ poly of deg n with roots in $\operatorname{Im}\lambda < 0$.

$$J_z(\lambda) = \frac{i}{\lambda - \bar z}\left[E^\#(\bar z)\,E(\lambda) - E(\bar z)\,E^\#(\lambda)\right]$$

Take $z = ia$ and $z = -ia$ $a > 0$.

$$(a-i\lambda)\,J_{ia}(\lambda) = \overline{E(+ia)}\,E(\lambda) - E(-ia)\,E^\#(\lambda)$$

$$(a+i\lambda)\,J_{-ia}(\lambda) = -\overline{E(-ia)}\,E(\lambda) + E(ia)\,E^\#(\lambda)$$

so

$$\boxed{\begin{pmatrix} E(\lambda) \\ E^\#(\lambda)\end{pmatrix} = \frac{1}{|E(ia)|^2 - |E(-ia)|^2}\begin{pmatrix} E(ia) & E(-ia) \\ \overline{E(-ia)} & \overline{E(ia)}\end{pmatrix}\begin{pmatrix}(a-i\lambda)J_{ia}(\lambda) \\ (a+i\lambda)\,J_{-ia}(\lambda)\end{pmatrix}}$$

Now I recall that matrices $\begin{pmatrix}\alpha & \beta \\ \bar\beta & \bar\alpha\end{pmatrix}$ with $|\alpha|^2 - |\beta|^2 = 1$ form the conjugate subgroup to $SL_2(\mathbb{R})$ when one transforms: ~~between~~ $E = A - iB$ $E^\# = A + iB$. ~~and~~ Consequently except for the positive constant

$$\frac{1}{\sqrt{|E(ia)|^2 - |E(-ia)|^2}}\,,$$

the posible choices for E form an orbit under $SL_2(\mathbb{R})$.

Actually you should notice that

$$J_{ia}(ia) = \frac{i}{ia+ia}\left[|E(ia)|^2 - |E(-ia)|^2\right] = \frac{|E(ia)|^2 - |E(-ia)|^2}{2a}$$

so that

$$\begin{pmatrix} E(\lambda) \\ E^\#(\lambda)\end{pmatrix} = \frac{1}{\sqrt{|E(ia)|^2 - |E(-ia)|^2}}\begin{pmatrix} E(ia) & E(-ia) \\ \overline{E(-ia)} & \overline{E(ia)}\end{pmatrix}\begin{pmatrix}\dfrac{(a-i\lambda)\,J_{ia}(\lambda)}{\sqrt{2a\,J_{ia}(ia)}} \\[2ex] \dfrac{(a+i\lambda)\,J_{-ia}(\lambda)}{\sqrt{2a\,J_{ia}(ia)}}\end{pmatrix}$$

hence the possible choices for $E(\lambda)$ form an orbit under $SL_2(\mathbb{R})$.

Notice that

$$J_z^{\#}(\lambda) = \left[\frac{i}{\lambda-\bar{z}}\begin{vmatrix} E(\lambda) & E(\bar{z}) \\ E^{\#}(\lambda) & \overline{E(\bar{z})}\end{vmatrix}\right]^{\#} = \frac{-i}{\lambda - z}\begin{vmatrix} E^{\#}(\lambda) & \overline{E(\bar{z})} \\ E(\lambda) & E(z)\end{vmatrix}$$

$$= \frac{i}{\lambda - z}\begin{vmatrix} E(\lambda) & E(z) \\ E^{\#}(\lambda) & E^{\#}(z)\end{vmatrix} = J_{\bar{z}}(\lambda)$$

hence a typical $\tilde{E}(\lambda)$ is of the form

$$\tilde{E}(\lambda) = c_1 E(\lambda) + c_2 E^{\#}(\lambda) \qquad \text{where } |c_1|^2 - |c_2|^2 = 1$$

Question: Is it always the case that $\tilde{E}(\lambda)$ has zeroes in the lower half-plane? No, because of the example $E(\lambda) = e^{-i\lambda t}$, $t>0$. However any of the choices

$$c_1 e^{-i\lambda t} + c_2 e^{+i\lambda t} \qquad \text{with } c_1, c_2 \neq 0.$$

do have zeroes in fact infinitely many since this function is periodic of period $\frac{2\pi}{t}$. Zeroes are given by

$$e^{2i\lambda t} = -\frac{c_1}{c_2} \qquad \text{hence form a coset } \lambda_0 + \frac{\pi}{t}\mathbb{Z}.$$

We put down the following for reference

Prop: If E and $\tilde{E}$ de Branges functions, then $B(E)=B(\tilde{E})$ norms and all iff $\tilde{E}(\lambda) = c_1 E(\lambda) + c_2 E^{\#}(\lambda)$ for some $c_1, c_2 \in \mathbb{C}$ such that $|c_1|^2 - |c_2|^2 = 1$. If $a>0$, then

$$\frac{(a-i\lambda)\, J_{ia}(\lambda)}{\sqrt{2a\, J_{ia}(ia)}}$$

is a de Branges function giving $B(E)$ which vanishes at $\lambda = -ia$, and any $\tilde{E}$ with this property differs from it by a multiple of modulus 1.

So [redacted] at this point we somewhat understand the arbitrariness of $E(\lambda)$. We still have not decided if there is a particularly nice choice for E. One possibility would be to require A to be of degree n, B of degree $n-1$. Better: Take $B = \phi_{n-1}$ and take $A = \dfrac{a_n \phi_n}{(-2)}$.

Possibility: Start with a measure $d\mu(\lambda)$ having finite moments, [redacted] and let $\phi_i(\lambda)$ be the resulting sequence of orthonormal polys. We get a nested sequence of de Brange spaces of finite dim.

$$0 \subset [redacted]\ B_1 \subset B_2 \subset \cdots\cdots \subset B_n \subset \cdots\cdots \qquad \text{inside } L^2(d\mu)$$

where B_n consists of polys of degree $\leq n-1$. Suppose there is a reasonable way to choose E_n so that $B(E_n) = B_n$. Then it should be possible to set up some sort of recursion formula:

$$\begin{pmatrix} E_n \\ E_n^{\#} \end{pmatrix} = A_n \begin{pmatrix} E_{n-1} \\ E_{n-1}^{\#} \end{pmatrix}$$

where A_n is a linear matrix function of z.

Start with the case where $d\mu$ is even. First review the formulas:

$$_nJ_z(\lambda) = \sum_{i=0}^{n-1} \phi_i(\bar{z})\phi_i(\lambda) = \frac{i}{\lambda - \bar{z}} \begin{vmatrix} E_n(\lambda) & E_n(\bar{z}) \\ E_n^{\#}(\lambda) & E_n^{\#}(\bar{z}) \end{vmatrix}$$

$$= \frac{i}{\lambda - \bar{z}} \begin{vmatrix} A_n(\lambda) - i B_n(\lambda) & \\ A_n(\lambda) + i B_n(\lambda) & \end{vmatrix}$$

$$= \frac{i}{\lambda - \bar{z}} \begin{vmatrix} 1 & -i \\ 1 & +i \end{vmatrix} \begin{vmatrix} A_n(\lambda) & A_n(\bar{z}) \\ B_n(\lambda) & B_n(\bar{z}) \end{vmatrix} = \frac{-2}{\lambda - \bar{z}} \begin{vmatrix} A_n(\lambda) & A_n(\bar{z}) \\ B_n(\lambda) & B_n(\bar{z}) \end{vmatrix}$$

$$(\lambda - \bar{z})\, {}_nJ_z(\lambda) = \begin{vmatrix} a_n \phi_n(\lambda) & a_n \phi_n(\bar{z}) \\ \phi_{n-1}(\lambda) & \phi_{n-1}(\bar{z}) \end{vmatrix}$$

Absorb $\sqrt{2}$ into A_n, B_n. In the real case one wants A even and B odd. Thus the obvious choice is

$$\left. \begin{aligned} \sqrt{2} A_n &= a_n \phi_n(\lambda) \\ +\sqrt{2} B_n &= -\phi_{n-1}(\lambda) \end{aligned} \right\} \quad n \text{ even}$$

and

$$\left. \begin{aligned} \sqrt{2} A_n &= \phi_{n-1}(\lambda) \\ \sqrt{2} B_n &= a_n \phi_n(\lambda) \end{aligned} \right\} \quad n \text{ odd}$$

Since $a_n \phi_n(\lambda) + a_{n-1} \phi_{n-2}(\lambda) = \lambda \phi_{n-1}$, one gets

$$\begin{pmatrix} \sqrt{2} A_n \\ \sqrt{2} B_n \end{pmatrix} = \begin{pmatrix} a_n \phi_n(\lambda) \\ -\phi_{n-1}(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda \phi_{n-1} - a_{n-1} \phi_{n-2} \\ -\phi_{n-1}(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} -a_{n-1} & \frac{\lambda}{a_{n-1}} \\ 0 & -\frac{1}{a_{n-1}} \end{pmatrix} \begin{pmatrix} \phi_{n-2} \\ a_{n-1} \phi_{n-1} \end{pmatrix} = \begin{pmatrix} -a_{n-1} & \frac{\lambda}{a_{n-1}} \\ 0 & -\frac{1}{a_{n-1}} \end{pmatrix} \begin{pmatrix} \sqrt{2} A_{n-1} \\ \sqrt{2} B_{n-1} \end{pmatrix}$$

for n even and

$$\begin{pmatrix} \sqrt{2}A_n \\ \sqrt{2}B_n \end{pmatrix} = \begin{pmatrix} \phi_{n-1} \\ a_n\phi_n \end{pmatrix} = \begin{pmatrix} \phi_{n-1} \\ \lambda\phi_{n-1} - a_{n-1}\phi_{n-2} \end{pmatrix} = \begin{pmatrix} \frac{1}{a_{n-1}} & 0 \\ \frac{\lambda}{a_{n-1}} & a_{n-1} \end{pmatrix}\begin{pmatrix} a_{n-1}\phi_{n-1} \\ -\phi_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a_{n-1}} & 0 \\ \frac{\lambda}{a_{n-1}} & a_{n-1} \end{pmatrix}\begin{pmatrix} \sqrt{2}A_{n-1} \\ \sqrt{2}B_{n-1} \end{pmatrix}$$

for n odd. $\blacksquare$

There should be a simpler way to write these formulas. Take a string with masses $m_0, m_1, m_2, \ldots$ with separation α_i between m_i and m_{i+1}.

$$-\lambda^2 m_i y_i = \frac{y_{i+1} - y_i}{\alpha_i} - \frac{y_i - y_{i-1}}{\alpha_{i-1}}$$

Put $A_i = y_i$ and $B_i = -\frac{1}{\lambda}\frac{y_{i+1} - y_i}{\alpha_i}$. Then we get the equations

$$B_i - B_{i-1} = \lambda m_i A_i \qquad\qquad A_0 = 1, \ B_{-1} = 0$$

$$A_{i+1} - A_i = -\lambda \alpha_i B_i$$

$A_i(\lambda)$ is an even poly of degree $2i$, $B_i(\lambda)$ is an odd poly of degree $2i+1$.

August 31, 1977:

Suppose E is a poly of degree n with all roots satisfying $\mathrm{Im}(\lambda) < 0$, and $\underline{B} = B(E)$ is the associated de Branges space. Let $\phi_0, \ldots, \phi_{n-1}$ be the orthonormal sequence of polys in B such that $\phi_i = c_i z^i +$ lower terms, and $c_i > 0$. Put $\phi_n = \lambda\phi_{n-1} - a_{n-1}\phi_{n-2}$. I've seen that

$$\begin{vmatrix} \phi_n(\lambda) & \phi_n(\bar{z}) \\ \phi_{n-1}(\lambda) & \phi_{n-1}(\bar{z}) \end{vmatrix} = (-2)\begin{vmatrix} A(\lambda) & A(\bar{z}) \\ B(\lambda) & B(\bar{z}) \end{vmatrix}$$

and that there is a matrix α in $SL_2(\mathbb{R})$ such that

$$(-\tfrac{1}{2})\begin{pmatrix} \phi_n(\lambda) \\ \phi_{n-1}(\lambda) \end{pmatrix} = \begin{pmatrix} a & b \\ u & v \end{pmatrix}\begin{pmatrix} A(\lambda) \\ B(\lambda) \end{pmatrix}$$

Thus there exists $(u,v) \neq 0$ such that $uA + vB$ is a poly of degree $n-1$ and if we require $\|uA + vB\| = 1$, this determines (u,v) up to a complex scalar of modulus 1. But u,v are real so this determines (u,v) up to ± 1.

Suppose that $E_n = A_n - iB_n$ gives rise to $\underline{B} = \sum_{i=0}^{n-1} \mathbb{C}\phi_i$ and that $E_{n-1} = A_{n-1} - iB_{n-1}$ gives rise to $\sum_{i=0}^{n-2} \mathbb{C}\phi_i$. Then we have

$$(-2)\begin{vmatrix} A_n(\lambda) & A_n(\bar{z}) \\ B_n(\lambda) & B_n(\bar{z}) \end{vmatrix} = \begin{vmatrix} (\lambda\phi_{n-1} - a_{n-1}\phi_{n-2})(\lambda) & (\lambda\phi_{n-1} - a_{n-1}\phi_{n-2})(\bar{z}) \\ \phi_{n-1}(\lambda) & \phi_{n-1}(\bar{z}) \end{vmatrix}$$

and

$$(-1)\begin{vmatrix} A_{n-1}(\lambda) & A_{n-1}(\bar{z}) \\ B_{n-1}(\lambda) & B_{n-1}(\bar{z}) \end{vmatrix} = \begin{vmatrix} a_{n-1}\phi_{n-1}(\lambda) & a_{n-1}\phi_{n-1}(\bar{z}) \\ \phi_{n-2}(\lambda) & \phi_{n-2}(\bar{z}) \end{vmatrix}$$

We've seen this implies

$$\begin{pmatrix} A_n(\lambda) \\ B_n(\lambda) \end{pmatrix} = \underbrace{\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}}_{\text{some elt of } SL_2(\mathbb{R})} \begin{pmatrix} \frac{1}{2}\phi_{n-1}(\lambda) \\ \lambda\phi_{n-1}(\lambda) - a_{n-1}\phi_{n-2}(\lambda) \end{pmatrix}$$

$$\begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} \frac{1}{2}\phi_{n-1} \\ -a_{n-1}\phi_{n-2} \end{pmatrix}$$

So

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} \frac{1}{2}\phi_{n-1} \\ \lambda\phi_{n-1} - a_{n-1}\phi_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2\lambda & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\phi_{n-1} \\ -a_{n-1}\phi_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2\lambda & 1 \end{pmatrix} \begin{pmatrix} \beta \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}$$

where the matrices $\overbrace{\text{are}}^{\alpha,\beta}$ in $SL_2(\mathbb{R})$.

Question: How uniquely are α, β determined given E_n, E_{n-1}?

So consider the equation

α^{-1}

$$\# \quad \begin{pmatrix} u & v \\ s & t \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2\lambda & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}$$

This implies
$$uA_n + vB_n = aA_{n-1} + bB_{n-1}.$$
Since the latter is of degree $(n-1)$, this determines (u,v) up to a non-zero real scalar and (a,b) is determined up to the same scalar. It is clear that α^{-1} and β can be premultiplied by any matrix
$$\begin{pmatrix} f & 0 \\ g & f \end{pmatrix} \qquad f \neq 0$$
except $f = \pm 1$ to be in SL_2

without affecting (#). Note:
$$\begin{pmatrix} 1 & 0 \\ -2\lambda & b \end{pmatrix}\begin{pmatrix} f & 0 \\ g & h \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2\lambda & 1 \end{pmatrix} = \begin{pmatrix} f & 0 \\ -2\lambda f + g & h \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2\lambda & 1 \end{pmatrix} = \begin{pmatrix} f & 0 \\ -2\lambda f + g + 2h\lambda & h \end{pmatrix}$$

$$uA_n + vB_n \;\#\; aA_{n-1} + bB_{n-1}$$
$$sA_n + tB_n = 2\lambda\left(aA_{n-1} + bB_{n-1}\right) + cA_{n-1} + dB_{n-1}$$
$$(s - 2\lambda u)A_n + (t - 2\lambda v)B_n = \underbrace{cA_{n-1} + dB_{n-1}}_{\deg\ n-1}$$

Let $A_n = a_n t^n + \cdots$, $B_n = b_n t^n + \cdots$; then $ua_n + vb_n = 0$, so $a_n = -kv$, $b_n = +ku$ and $sa_n + tb_n = k(-sv + tu) = k$ $= 2(a a_{n-1} + b b_{n-1})$ where $a_{n-1}t^{n-1} + \cdots = A_{n-1}$ etc. Trying to change (u,v) by a scalar to $(\gamma u, \gamma v)$ changes k to $k\gamma^{-1}$ and (a,b) to $(\gamma a, \gamma b)$, which doesn't work unless $\gamma^{-1} = \gamma$ i.e. $\gamma = \pm 1$. So it appears that once E_{n-1}, E_n are fixed so are α, β except for matrices of the form $\pm\begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}$.

de Branges seems to choose (A_{n-1}, B_{n-1}) so that $\alpha\beta = 1$, whence:

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} t & -v \\ -s & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2\lambda & 1 \end{pmatrix} \begin{pmatrix} u & v \\ s & t \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} t-2\lambda v & -v \\ -s+2\lambda u & u \end{pmatrix} \begin{pmatrix} u & v \\ s & t \end{pmatrix}$$

$$= \begin{pmatrix} tu - sv - 2uv\lambda & tv - 2\lambda v^2 - vt \\ -su + 2\lambda u^2 + us & -sv + 2\lambda uv + ut \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 2uv\lambda & -2v^2\lambda \\ 2u^2\lambda & 1 + 2uv\lambda \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}$$

Review: Def. De Brange fn. = entire function E such that $\operatorname{Im} \lambda > 0 \Rightarrow |E(\lambda)| > |E(\bar\lambda)|$.

De Brange space based on E, $B(E)$ = all entire functions with

1) $$\int_{\mathbb{R}} |f(x)|^2 \frac{dx}{\pi |E(x)|^2} < \infty$$

2a) $\exists C, R \ni$ $$\left|\frac{f(\lambda)}{E(\lambda)}\right| < \frac{C}{(\operatorname{Im} \lambda)^{1/2}}$$ for $\operatorname{Im} \lambda > 0$, $|\lambda| \geq R$

2b) $\exists C, R$ $$\left|\frac{f(\lambda)}{E^{\#}(\lambda)}\right| < \frac{C}{(-\operatorname{Im} \lambda)^{1/2}}$$ for $\operatorname{Im} \lambda < 0$, $|\lambda| \geq R$.

where $E^{\#}(\lambda) = \overline{E(\bar\lambda)}$.

$B(E)$ is a vector space over $\mathbb{C}$ equipped with the inner product

$$(f,g) = \int_{\mathbb{R}} f(x)\overline{g(x)} \frac{dx}{\pi |E(x)|^2}$$

We will see below that it is a Hilbert space.

Prop: $$J_z(\lambda) = \frac{i}{2(\lambda-\bar{z})} \begin{vmatrix} E(\lambda) & E(\bar{z}) \\ E^{\#}(\lambda) & E^{\#}(\bar{z}) \end{vmatrix} \in B(E) \quad \text{and}$$

$$(f, J_z) = f(z), \qquad \forall f \in B$$

Proof: Let $f \in B$. Note that $\frac{f}{E}$ analytic for $\operatorname{Im}\lambda \geq 0$. Since as $r \to \infty$

$$\left| \int_0^{\pi} \frac{f(re^{i\theta})}{E(re^{i\theta})} \frac{ire^{i\theta}\,d\theta}{re^{i\theta} - z} \right| \leq C \int_0^{\pi} \frac{C}{(r\sin\theta)^{1/2}}\, d\theta = O\left(\frac{1}{r^{1/2}}\right)$$

Cauchy's formula gives ~~Note that E(λ)~~

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(\lambda)}{E(\lambda)} \frac{dz}{\lambda - z} = \begin{cases} 0 & \operatorname{Im} z < 0 \\ \frac{f(z)}{E(z)} & \operatorname{Im} z > 0 \end{cases}$$

Similarly $$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(\lambda)}{E^{\#}(\lambda)} \frac{dz}{\lambda - z} = \begin{cases} 0 & \operatorname{Im} z > 0 \\ -\frac{f(z)}{E^{\#}(z)} & \operatorname{Im} z < 0 \end{cases}$$

So $$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} f(\lambda) \left(\frac{E(z)}{E(\lambda)} - \frac{E^{\#}(z)}{E^{\#}(\lambda)} \right) \frac{d\lambda}{\lambda - z}$$ for $\operatorname{Im} z \neq 0$, and even for $\operatorname{Im} z = 0$.

$$= \int f(\lambda) \frac{1}{2i(\lambda - z)} \begin{vmatrix} \overline{E(\lambda)} & E^{\#}(z) \\ E(\lambda) & E(z) \end{vmatrix} \frac{d\lambda}{\pi |E(\lambda)|^2} = \left(f, \frac{i}{2(\lambda - \bar{z})} \begin{vmatrix} E(\lambda) & E(\bar{z}) \\ E^{\#}(\lambda) & E^{\#}(\bar{z}) \end{vmatrix} \right)$$

$$= (f, J_z)$$

It remains to show that $J_z \in B$.

$$\frac{J_z(\lambda)}{E(\lambda)} = \frac{i}{2(\lambda-\bar z)}\left(E^{\#}(\bar z) - \frac{E^{\#}(\lambda)}{E(\lambda)}E(\bar z)\right)$$

bdd in a closed UHP.

$$\left|\frac{J_z(\lambda)}{E(\lambda)}\right| = O\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda \to \pm\infty.$$

$\therefore \ \|J_z\|^2 < \infty$. Also argueing as above

$$\frac{J_z(\omega)}{E_z(\omega)} = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{J_z(\lambda)}{E(\lambda)}\frac{d\lambda}{\lambda-\omega} \qquad \text{if } Im(\omega) > 0$$

Schwarz $\Rightarrow$ $$\left|\frac{J_z(\omega)}{E_z(\omega)}\right|^2 \leq \left\|\frac{J_z}{E}\right\|^2 \int_{\mathbb{R}} \frac{d\lambda}{|\lambda-\omega|^2 4\pi} \leq \frac{C}{|Im\,\omega|}$$

because $\omega = a+bi$ $$\int \frac{d\lambda}{|\lambda-\omega|^2} = \int\frac{d\lambda}{|\lambda-bi|^2} = \frac{1}{b}\int\frac{d\lambda}{(\lambda-i)^2} = \frac{1}{b}\int\frac{d\lambda}{1+\lambda^2} = \frac{\pi}{b}$$

Similarly for 2b, so $J_z \in B$.

Cor: $$|f(\lambda)|^2 \leq \|f\|^2\|J_\lambda\|^2 = \|f\|^2(J_\lambda, \lambda) = \|f\|^2\frac{|E(\lambda)|^2 - |E(\bar\lambda)|^2}{4\,Im(\lambda)}$$

which improves 2).

Prop. 2: $B(E)$ is a Hilbert space

~~Proof: If f_n is a Cauchy sequence in $B(E)$, then the preceding corollary shows f_n converges uniformly on compact sets to a function f, + f is entire. But f_n converges in $L^2(\mathbb{R}, \frac{d\lambda}{\pi|E(\lambda)|^2})$ to a fn. g by~~

Proof: Let f_n be a Cauchy sequence in $B(E)$. Then f_n is a Cauchy sequence in $L^2\left(\mathbb{R}, \frac{d\lambda}{\pi |E(\lambda)|^2}\right)$, hence it converges to an element f of the latter. We have $(f, J_\lambda) = \lim_n (f_n, J_\lambda) = \lim_n f_n(\lambda)$, and the sequence f_n converges uniformly on compact sets to an entire fn. Thus $g(\lambda) = (f, J_\lambda)$ is an entire function of λ satisfying

$$|g(\lambda)|^2 \leq \|f\|^2 \frac{|E(\lambda)|^2 - |E(\bar\lambda)|^2}{4 \operatorname{Im} \lambda}$$

The only thing to check is that $g(\lambda)|\mathbb{R} = f$ almost everywhere. The point is that if $f_n \to f$ in the mean then the same is true for f_n restricted to a finite interval.

Return to

$$J_z(\lambda) = \frac{i}{2(\lambda - \bar z)} \begin{vmatrix} E(\lambda) & E(\bar z) \\ E^\#(\lambda) & E^\#(\bar z) \end{vmatrix}$$

Compute

$$J_z^\#(\lambda) = \overline{J_z(\bar\lambda)} = \frac{-i}{2(\lambda - z)} \begin{vmatrix} E^\#(\lambda) & E^\#(z) \\ E(\lambda) & E(z) \end{vmatrix} = J_{\bar z}(\lambda)$$

$$-2i(\lambda - \bar z) J_z(\lambda) = \overline{E(z)}\, E(\lambda) - E(\bar z)\, E^\#(\lambda)$$

$$+2i(\lambda - z) J_{\bar z}(\lambda) = -\overline{E(\bar z)}\, E(\lambda) + E(z)\, E^\#(\lambda)$$

$$4 \operatorname{Im}(z)\, J_z(z) = |E(z)|^2 - |E(\bar z)|^2$$

$$2(\operatorname{Im} z)^{1/2} \|J_z\| = \sqrt{|E(z)|^2 - |E(\bar z)|^2} \qquad \operatorname{Im}(z) > 0$$

$$\begin{pmatrix} E(\lambda) \\ E^\#(\lambda) \end{pmatrix} = \frac{1}{|E(z)|^2 - |E(\bar z)|^2} \begin{pmatrix} E(z) & E(\bar z) \\ \overline{E(\bar z)} & \overline{E(z)} \end{pmatrix} \begin{pmatrix} -2i(\lambda - \bar z) J_z(\lambda) \\ 2i(\lambda - z) J_{\bar z}(\lambda) \end{pmatrix}.$$

$$\begin{pmatrix} E(\lambda) \\ E^{\#}(\lambda) \end{pmatrix} = \underbrace{\frac{1}{\sqrt{|E(z)|^2 - |E(\bar z)|^2}} \begin{pmatrix} E(z) & E(\bar z) \\ \overline{E(\bar z)} & \overline{E(z)} \end{pmatrix}}_{} \begin{pmatrix} \dfrac{-2i(\lambda - \bar z) J_z(\lambda)}{\sqrt{|E(z)|^2 - |E(\bar z)|^2}} \\ \dfrac{2i(\lambda - z) J_{\bar z}(\lambda)}{\sqrt{|E(z)|^2 - |E(\bar z)|^2}} \end{pmatrix}$$

of the form $\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ where $|\alpha|^2 - |\beta|^2 = 1$.

Final formula:

$$\begin{pmatrix} E(\lambda) \\ E^{\#}(\lambda) \end{pmatrix} = \frac{1}{(|E(z)|^2 - |E(\bar z)|^2)^{1/2}} \begin{pmatrix} E(z) & E(\bar z) \\ \overline{E(\bar z)} & \overline{E(z)} \end{pmatrix} \begin{pmatrix} \dfrac{-i(\lambda - \bar z) J_z(\lambda)}{(\operatorname{Im} z)^{1/2} \|J_z\|} \\ \dfrac{i(\lambda - z) J_{\bar z}(\lambda)}{(\operatorname{Im} z)^{1/2} \|J_z\|} \end{pmatrix}$$

This shows that the different E's giving rise to the same de Branges space are all conjugate under the group of $\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ of determinant 1.

Prop: $B(\tilde E) = B(E)$ same functions and same norms

$\iff$ $\tilde E(\lambda) = \alpha E(\lambda) + \beta E^{\#}(\lambda)$ with $|\alpha|^2 - |\beta|^2 = 1$.

Introduce decomposition

$$E(\lambda) = A(\lambda) - iB(\lambda) \qquad E^{\#}(\lambda) = A(\lambda) + iB(\lambda)$$

where $A^{\#} = A$, $B^{\#} = B$. Then

$$J_z(\lambda) = \frac{i}{2(\lambda - \bar z)} \begin{vmatrix} A(\lambda) - iB(\lambda) & A(\bar z) - iB(\bar z) \\ A(\lambda) + iB(\lambda) & A(\bar z) + iB(\bar z) \end{vmatrix}$$

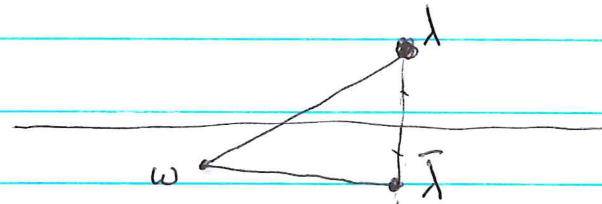
$$= \frac{i}{2(\lambda - \bar z)} \begin{vmatrix} 1 & -i \\ 1 & +i \end{vmatrix} \begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z) \end{vmatrix} = -\frac{1}{\lambda - \bar z} \begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z) \end{vmatrix}$$

Prop: $B(\tilde{E}) = B(E)$ with norms $\iff$ $\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$.

~~de Branges' characterization: Let B be a Hilbert space of entire functions.~~

September 1, 1977:

Clearly $\lambda - \omega$ is a de Branges function when $\mathrm{Im}(\omega) < 0$:



hence any polynomial $E(\lambda)$ having its roots in the lower half-plane is a de Branges function. If E has degree n, then any polynomial f of degree $< n$ is in $B(E)$ since

$$\frac{f(\lambda)}{E(\lambda)} = O\left(\frac{1}{\lambda}\right)$$

in the closed upper half-plane. Moreover from

$$J_z(\lambda) = \frac{i}{2(\lambda - \bar{z})} \begin{vmatrix} E(\lambda) & E(\bar{z}) \\ E^\#(\lambda) & E^\#(\bar{z}) \end{vmatrix}$$

one sees that J_z is a poly of degree $< n$ for each z. Since the linear combinations of J_z are dense in $B(E)$, we see $B(E)$ consists of all polys. of degree $< n$.

Apply Gram-Schmidt to the sequence $1, z, \ldots, z^{n-1}$

to get an orthonormal basis $\phi_0, \phi_1, \dots, \phi_{n-1}$ for $B(E)$ such that ϕ_i is a poly of degree i with positive leading coefficient. If $F_i B(E)$ is the subspace of polys of degree $\le i$, ϕ_i is the unique element of norm 1 with $\phi_i \equiv c_i z^i + F_{i-1} B(E)$ and $c_i > 0$. Since $\phi_i^\#$ also has this property, it follows that $\phi_i^\# = \phi_i$, that is, ϕ_i has real coefficients.

Since

$$(zf, g) = (f, zg)$$

for $\deg(f), \deg(g) < n-1$, we have

$$(z\phi_i, \phi_j) = (\phi_i, z\phi_j) = 0$$

for $i < n-1$, $j < i-1$. Hence we have for $i < n-1$

$$z\phi_i = a_i \phi_{i+1} + b_i \phi_i + a_{i-1} \phi_{i-1}$$

where $b_i = (z\phi_i, \phi_i)$, $a_i = (z\phi_i, \phi_{i+1})$; note that $(z\phi_i, \phi_{i-1}) = (\phi_i, z\phi_{i-1}) = (z\phi_{i-1}, \phi_i) = a_{i-1}$ and also that the numbers a_i, b_i are real with $a_i > 0$ because ϕ_i has real coefficients and positive leading coefficient.

One has

$$J_z(\lambda) = \sum_{i=0}^{n-1} (J_z, \phi_i)\phi_i(\lambda) = \sum_{i=0}^{n-1} \overline{\phi_i(z)}\, \phi_i(\lambda)$$

$$= \sum_{i=0}^{n-1} \phi_i(\bar{z})\phi_i(\lambda).$$

$$-\begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z) \end{vmatrix} = (\lambda-\bar z)\sum_{i=0}^{n-1} \phi_i(\bar z)\phi_i(\lambda)$$

$$= \lambda\phi_{n-1}(\lambda)\phi_{n-1}(\bar z) - \bar z\phi_{n-1}(\bar z)\phi_{n-1}(\lambda)$$
$$+ \sum_{i=0}^{n-2} (a_i\phi_{i+1} + \cancel{b_i\phi_i} + a_{i-1}\phi_{i-1})(\lambda)\,\phi_i(\bar z) - (a_i\phi_{i+1} + \cancel{b_i\phi_i} + a_{i-1}\phi_{i-1})(\bar z)\,\phi_i(\lambda)$$

$$= \big(\lambda\phi_{n-1}(\lambda) - a_{n-2}\phi_{n-2}(\lambda)\big)\phi_{n-1}(\bar z) - \big(\bar z\phi_{n-1}(\bar z) - a_{n-2}\phi_{n-2}(\bar z)\big)\phi_{n-1}(\lambda)$$

$$= -\begin{vmatrix} \phi_{n-1}(\lambda) & \phi_{n-1}(\bar z) \\ \lambda\phi_{n-1}(\lambda) - a_{n-2}\phi_{n-2}(\lambda) & \bar z\phi_{n-1}(\bar z) - a_{n-2}\phi_{n-2}(\bar z) \end{vmatrix}$$

Perhaps it is simplest to introduce a_n, ϕ_n, b_n so that

$$\lambda\phi_{n-1}(\lambda) = a_{n-1}\phi_n(\lambda) + b_{n-1}\phi_{n-1}(\lambda) + a_{n-2}\phi_{n-2}(\lambda)$$

and so we get

$$\begin{vmatrix} A(\lambda) & A(\bar z) \\ B(\lambda) & B(\bar z) \end{vmatrix} = \begin{vmatrix} \phi_{n-1}(\lambda) & \phi_{n-1}(\bar z) \\ a_{n-1}\phi_n(\lambda) & a_{n-1}\phi_n(\bar z) \end{vmatrix}.$$

We know that the only solutions of this equation are given by

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \phi_{n-1} \\ a_{n-1}\phi_n \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$.

I want to understand how one obtains $B(E)$ from the subspace of polys. of smaller degree. Suppose then that we are given $E_{n-1} = A_{n-1} - iB_{n-1}$ such that

$$F_{n-2}\, B(E) = B(E_{n-1})$$

i.e. such that

$$\begin{vmatrix} A_{n-1}(\lambda) & A_{n-1}(\bar z) \\ B_{n-1}(\lambda) & B_{n-1}(\bar z) \end{vmatrix} = \begin{vmatrix} \phi_{n-2}(\lambda) & \phi_{n-2}(\bar z) \\ a_{n-2}\phi_{n-1}(\lambda) & a_{n-2}\phi_{n-1}(\bar z) \end{vmatrix}$$

So we have

$$\begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix} = \beta \begin{pmatrix} \phi_{n-1} \\ -a_{n-2}\phi_{n-2} \end{pmatrix} \quad \text{for some } \beta \in SL_2(\mathbb{R})$$

Thus

$$\begin{pmatrix} A \\ B \end{pmatrix} = \alpha \begin{pmatrix} \phi_{n-1} \\ a_{n-1}\phi_n \end{pmatrix} = \alpha \begin{pmatrix} \phi_{n-1} \\ \lambda\phi_{n-1} - a_{n-2}\phi_{n-2} \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \phi_{n-1} \\ -a_{n-2}\phi_{n-2} \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \beta \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}$$

For some matrices $\alpha, \beta \in SL_2(\mathbb{R})$. Changing either E or E_{n-1} changes α, β.

deBranges considers requiring $\alpha = \beta^{-1}$ to be a natural requirement in the choice of E_{n-1}.

If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\alpha \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \alpha^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} +d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ d\lambda - c & -b\lambda + a \end{pmatrix}$$

$$= \begin{pmatrix} ad + bd\lambda - bc & -\cancel{ab} - b^2\lambda + \cancel{ab} \\ \cancel{cd} + d^2\lambda - \cancel{cd} & -bc - bd\lambda + ad \end{pmatrix}$$

Thus we get

$$\begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}1+bd\lambda & -b^2\lambda\\ d^2\lambda & 1-bd\lambda\end{pmatrix}\begin{pmatrix}A_{n-1}\\B_{n-1}\end{pmatrix}$$

General setup: suppose one chooses E_i so that $F_{i-1}B(E) = B(E_i)$, so E_i is of degree i, and put $E_n = E$. Then we get linear matrix functions $M_i(\lambda)$ of λ such that

$$\begin{pmatrix}A_i\\B_i\end{pmatrix} = M_i \begin{pmatrix}A_{i-1}\\B_{i-1}\end{pmatrix}$$

and such that M_i is of the form

$$M_i = \alpha_i \begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}\beta_i$$

with $\alpha_i, \beta_i \in SL_2(\mathbb{R})$. Note that M_i is unique for $i>2$, because if one has $(M_i - \tilde{M}_i)\begin{pmatrix}A_{i-1}\\B_{i-1}\end{pmatrix} = 0$ with $M_i - \tilde{M}_i \neq 0$ and linear, then one would have

$$l_1 A_{i-1} + l_2 B_{i-1} = 0$$

with l_1, l_2 linear polys, not both zero. Assuming $i-1>1$, this implies that A_{i-1}, B_{i-1} have a common divisor of positive degree which has to be real and hence E_{i-1} can't have its roots in the lower half-plane.

de Branges' normalization consists in requiring that $M_i(0) = 1$, or equivalently that $\alpha_i = \beta_i^{-1}$. This requires looking at different systems than

$$\frac{du}{dx} = \begin{pmatrix}i\lambda & \bar{p}\\ p & -i\lambda\end{pmatrix}u$$

where the solution for $\lambda = 0$ is not evident.

September 2, 1977

Start $d\mu$ having finite moments, whence you get a sequence of orthonormal polynomials $\phi_0, \phi_1, \dots$ satisfying

$$\lambda\phi_n = a_n\phi_{n+1} + b_n\phi_n + a_{n-1}\phi_{n-1}$$

where $n \geq 1$

Let F_n be the space of polys of degree $< n$. It has point evaluators

$$_nJ_z(\lambda) = \sum_{i=0}^{n-1} \phi_i(\bar{z})\phi_i(\lambda)$$

and

$$(\lambda - \bar{z})\,_nJ_z(\lambda) = \sum_0^{n-1} \lambda\phi_i(\lambda)\phi_i(\bar{z}) - \bar{z}\phi_i(\bar{z})\phi_i(\lambda)$$

$$= \sum_0^{n-1} (a_i\phi_{i+1} + \cancel{b_i\phi_i} + a_{i-1}\phi_{i-1})(\lambda)\,\phi_i(\bar{z})$$
$$- (a_i\phi_{i+1} + \cancel{b_i\phi_i} + a_{i-1}\phi_{i-1})(\bar{z})\,\phi_i(\lambda)$$

$$= a_{n-1}\left[\phi_n(\lambda)\phi_{n-1}(\bar{z}) - \phi_n(\bar{z})\phi_{n-1}(\lambda)\right]$$

Let $E_n = A_n - iB_n$ be a de Branges poly of degree n such that $B(E_n) = F_n$. Then we've seen that

$$_nJ_z(\lambda) = \frac{i}{2(\lambda - \bar{z})}\begin{vmatrix} E_n(\lambda) & E_n(\bar{z}) \\ E_n^\#(\lambda) & E_n^\#(\bar{z}) \end{vmatrix} = \frac{i}{2(\lambda - \bar{z})}\left|\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}\begin{pmatrix} A_n(\lambda) & A_n(\bar{z}) \\ B_n(\lambda) & B_n(\bar{z}) \end{pmatrix}\right|$$

$$= \frac{-1}{\lambda - \bar{z}}\begin{vmatrix} A_n(\lambda) & A_n(\bar{z}) \\ B_n(\lambda) & B_n(\bar{z}) \end{vmatrix}$$

So we get the following requirement for E_n

$$\begin{vmatrix} A_n(\lambda) & A_n(\bar{z}) \\ B_n(\lambda) & B_n(\bar{z}) \end{vmatrix} = \begin{vmatrix} \phi_{n-1}(\lambda) & \phi_{n-1}(\bar{z}) \\ a_{n-1}\phi_n(\lambda) & a_{n-1}\phi_n(\bar{z}) \end{vmatrix}$$

Since ϕ_{n-1} and $a_{n-1}\phi_n$ are linearly independent, it is easily seen that this implies $\exists!\ \Theta_n \in SL_2(\mathbb{R})$ such that

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \Theta_n \begin{pmatrix} \phi_{n-1} \\ a_{n-1}\phi_n \end{pmatrix}.$$

and conversely for any such Θ_n the A_n, B_n defined in this way constitute a de Branges function for F_n.

So

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \Theta_{n+1} \begin{pmatrix} \phi_n \\ \lambda\phi_n - b_n\phi_n - a_{n-1}\phi_{n-1} \end{pmatrix} = \Theta_{n+1} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \phi_n \\ -b_n\phi_n - a_{n-1}\phi_{n-1} \end{pmatrix}$$

$$= \Theta_{n+1} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{a_{n-1}} \\ -a_{n-1} & -\frac{b_n}{a_{n-1}} \end{pmatrix} \begin{pmatrix} \phi_{n-1} \\ a_{n-1}\phi_n \end{pmatrix}$$

$$= \Theta_{n+1} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{a_{n-1}} \\ -a_{n-1} & -\frac{b_n}{a_{n-1}} \end{pmatrix} \Theta_n^{-1} \begin{pmatrix} A_n \\ B_n \end{pmatrix}.$$

i.e.
$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = M_{n+1,n}(\lambda) \begin{pmatrix} A_n \\ B_n \end{pmatrix}$$

where
$$M_{n+1,n}(\lambda) = \Theta_{n+1} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{a_{n-1}} \\ -a_{n-1} & -\frac{b_n}{a_{n-1}} \end{pmatrix} \Theta_n^{-1}$$

is a linear matrix function of λ.

You notice from this formula that if Θ_n is given, there is a unique choice for Θ_{n+1} such that $M_{n+1,n}(0) = I$, and conversely given Θ_{n+1}, there is a unique Θ_n such that $M_{n+1,n}(0) = I$. Consequently once I pick E_1 then there is a unique way of choosing E_2, E_3, etc. so that one has $M_{n+1,n}(0) = I$ for all $n \geq 1$. As

E_1 itself is unique up to an element of $SL_2(\mathbb{R})$, one gets [redacted] a coherent choice of de Branges functions up to an element of $SL_2(\mathbb{R})$.

The next thing to understand is de Branges' version of this result. He consider a de Branges space $B(E)$ in which multiplication by λ is not densely defined.

First consider the partially-defined operator $f \mapsto \lambda f$ in $B(E)$. Its graph $\{(f,g) \in B(E)^2 \mid \lambda f = g\}$ is closed, since if $f_n \to f$ and $g_n \to g$ and $\lambda f_n = g_n$, then because convergence in $B(E)$ implies uniform converge on compact sets, we have $\lambda f = g$. The range consists of all g in B such that $g(0)=0$. In effect certainly $\left|\frac{g(\lambda)}{\lambda}\right| < |g(\lambda)|$ far out. Similar results hold for the operators $\lambda - c$.

Let $\underset{\sim}{B}$ be a de Branges space and suppose $S \in \underset{\sim}{B} + \lambda \underset{\sim}{B}$, say $S = f_1 + \lambda f_2$ with $f_1, f_2 \in B$. Then if $F \in \underset{\sim}{B}$

$$\frac{F(\lambda)\,S(z) - S(\lambda)\,F(z)}{\lambda - z} = \frac{F(\lambda) f_1(z) + F(\lambda) z f_2(z) - f_1(\lambda) F(z) \underbrace{-\lambda}_{-\lambda = z-\lambda-z} f_2(\lambda) F(z)}{\lambda - z}$$

$$= \left(\frac{F(\lambda) f_1(z) - f_1(\lambda) F(z)}{\lambda - z}\right) + z\left(\frac{F(\lambda) f_2(z) - f_2(\lambda) F(z)}{\lambda - z}\right) - f_2(\lambda) F(z)$$

$\in \underset{\sim}{B}$

for any choice of z. Conversely suppose S is an entire fn. such that $\exists z \in \mathbb{C}$ such that

$$\forall F \in \underset{\sim}{B} \qquad \frac{F(\lambda) S(z) - S(\lambda) F(z)}{\lambda - z} \in \underset{\sim}{B}$$

Then choose F so that $F(z) \neq 0$ and put

$$H(\lambda) = \frac{F(\lambda)S(z) - S(\lambda)F(z)}{\lambda - z}$$

whence $S(\lambda) = F(z)^{-1}\left[F(\lambda)S(z) - (\lambda - z)H(\lambda)\right] \in \underline{B} + \lambda \underline{B}$

Suppose ~~Let's go back to $L^2(d\mu)$ and the sequence E_n.~~ the sequence of E_n is normalized in the deBranges fashion so that $M_{n+1,n}(0) = I$, i.e.

$$\Theta_{n+1} = \Theta_n \begin{pmatrix} 0 & \frac{1}{a_{n-1}} \\ -a_{n-1} & -\frac{b_n}{a_{n-1}} \end{pmatrix}^{-1} = \Theta_n \begin{pmatrix} -\frac{b_n}{a_{n-1}} & -\frac{1}{a_{n-1}} \\ a_{n-1} & 0 \end{pmatrix}.$$

Then

$$M_{n+1,n} = \Theta_{n+1}\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\Theta_{n+1}^{-1}$$

so that if $\Theta_{n+1}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$M_{n+1,n}(\lambda) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d-b\lambda & -b \\ -c+a\lambda & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} ad - ab\lambda - bc & bd - b^2\lambda - bd \\ -ac + a^2\lambda + ac & -bc + ab\lambda + ad \end{pmatrix}$$

$$= \begin{pmatrix} 1 - ab\lambda & -b^2\lambda \\ a^2\lambda & 1 + ab\lambda \end{pmatrix}$$

In view of the importance of a, b we should regard Θ_{n+1}^{-1} as important, and

$$\Theta_{n+1}^{-1} = \begin{pmatrix} 0 & \frac{1}{a_{n-1}} \\ -a_{n-1} & -\frac{b_n}{a_{n-1}} \end{pmatrix} \Theta_n^{-1}$$

I now want to investigate the choices involved with Θ_1. We have

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \Theta_1 \begin{pmatrix} \phi_0 \\ a_0\phi_1 \end{pmatrix} = \Theta_1 \begin{pmatrix} \phi_0 \\ \lambda\phi_0 - b_0\phi_0 \end{pmatrix}$$

$$= \Theta_1 \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \phi_0 \\ -b_0\phi_0 \end{pmatrix}$$

The problem here is how to choose $M_{1,0}$ and $\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$.

In the case where $d\mu$ is even all the $b_n = 0$.

Regard ϕ_{-1} as zero, but regard a_{-1} as not yet determined. Then

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \Theta_1 \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 0 & a_{-1} \\ -a_{-1} & -\frac{b_0}{a_{-1}} \end{pmatrix} \begin{pmatrix} \phi_{-1}^{\text{“}0\text{”}} \\ a_{-1}\phi_0 \end{pmatrix}$$

Seems unclear how to proceed. ?

So let us instead regard as basic the fundamental recursion relation, and the measure $d\mu$ results only when one has selected out specific boundary conditions.

Transformation of a general system to the de Branges form: Start with

$$Lu = \left(A\frac{d}{dx} + B\right)u = \lambda C u$$

where $L = L^*$ i.e. $A\frac{d}{dx} + B = -\frac{d}{dx}A^* + B^* = -A^*\frac{d}{dx} - \frac{dA^*}{dx} + B^*$

or $A^* = -A \qquad \frac{dA}{dx} = B - B^*.$

Let S be the solution matrix with $\lambda = 0$, i.e.

$$A\frac{dS}{dx} + BS = 0 \qquad S(0) = I$$

Put $u = Sv$

$$A\left(\frac{dS}{dx}v + S\frac{dv}{dx}\right) + BSv = \lambda CSv$$

$$\boxed{S^*AS\frac{dv}{dx} = \lambda S^*CSv}$$

Note that S^*AS is constant:

$$\frac{d}{dx}(S^*AS) = \left(\frac{dS}{dx}\right)^*AS + S^*\frac{dA}{dx}S + S^*A\frac{dS}{dx}$$

$$= \left(-A\frac{dS}{dx}\right)^*S + S^*(B-B^*)S + S^*(-BS)$$

$$= (BS)^*S - S^*B^*S = 0$$

and skew-adjoint. Notice that if we have

$$\mathrm{tr}(A^{-1}B) = \mathrm{tr}(A^{-1}C) = 0$$

which is equivalent to having all $S(x,\lambda)$ of determinant 1, then also $\mathrm{tr}\left((S^*AS)^{-1}(S^*CS)\right) = \mathrm{tr}(A^{-1}C) = 0.$

So by the preceding transformation we can suppose $B=0$. By changing $L-\lambda C$ by a constant unitary matrix S i.e. into $S^*(L-\lambda C)S$ we can suppose A diagonal with purely imaginary eigenvalues and if we then conjugate with a diagonal matrix D, rather we replace $L-\lambda C$ by $D^*(L-\lambda C)D$, we can arrange that the eigenvalues of A be $\pm i$. So I might as well assume that

$$A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{1}{i} & 0 \\ 0 & -\frac{1}{i} \end{pmatrix}$$

Now let's consider changing the independent variable - this only changes C by a positive scalar function.

Is it possible to make C, which is a positive-definite matrix, into a real matrix. Suppose $A=\begin{pmatrix} \frac{1}{i} & 0 \\ 0 & -\frac{1}{i} \end{pmatrix}$. Since $tr(A^{-1}C) = i c_{11} + (-i)c_{22} = 0$ it follows that C has equal diagonal entries. Let write the equation

$$A\frac{du}{dx} = (\lambda c I + \lambda C_1)u \qquad c = c_{11} = c_{22}$$

and introduce the unitary operator

$$S = \begin{pmatrix} e^{if} & 0 \\ 0 & e^{-if} \end{pmatrix} \qquad f \text{ real}$$

where f is to be determined. Put

$$u = Sv.$$

$$S^*AS\frac{dv}{dx} + S^*A\frac{dS}{dx}v = \lambda S^*CSv$$

Now $S^*=S^{-1}$ and S commutes with A so $S^*AS = I$.
Also

$$S^*A\frac{dS}{dx} = \begin{pmatrix} e^{-i\lambda f}\frac{1}{i}\frac{d}{dx}e^{i\lambda f} & 0 \\ 0 & -e^{i\lambda f}\frac{1}{i}\frac{d}{dx}e^{-i\lambda f} \end{pmatrix} = \lambda \begin{pmatrix} \frac{df}{dx} & 0 \\ 0 & \frac{df}{dx} \end{pmatrix}$$

$$S^*CS = \begin{pmatrix} c_{11} & c_{12}e^{-2i\lambda f} \\ c_{21}e^{2i\lambda f} & c_{22} \end{pmatrix}$$

Unfortunately <u>S depends on λ.</u>

Here's how to proceed: Start with a general self-adjoint system

$$A\frac{du}{dx} + Bu = \lambda Cu$$

One thing that remains invariant under

$$A, C \longmapsto S^*AS, S^*CS$$

is the spectrum of $A^{-1}C$. Recall that $C^{-1}A$ is skew-adjoint with respect to the inner product defined by C:

$$(CC^{-1}Au, v) = -(u, Av) = -(Cu, C^{-1}Av).$$

Hence $C^{-1}A$ has purely imaginary eigenvalues. Since we are assuming $tr(A^{-1}C)=0$ and dealing with 2×2 matrices, it follows the eigenvalues of $A^{-1}C$ are purely imaginary and add to zero. Moreover by adjusting the independent variable I can suppose the eigenvalues of $A^{-1}C$ are always $\pm i$. So the next thing is to choose S so that

$$S^*CS = I$$
$$S^*AS = \begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix}.$$

S is unique up to right multiplication by a diagonal unitary matrix. So we can suppose that $A = \begin{pmatrix} \frac{1}{i} & 0 \\ 0 & -\frac{1}{i} \end{pmatrix}$ and $C = I$.

In addition by conjugating by a diagonal matrix

$$S = \begin{pmatrix} e^{if} & \\ & e^{-if} \end{pmatrix} \qquad f \text{ real}$$

we get

$$A\frac{d}{dx} + S^{-1}A\frac{dS}{dx} + S^{-1}BS$$

$$S^{-1}A\frac{dS}{dx} = \begin{pmatrix} e^{-if}\frac{1}{i}\frac{de^{if}}{dx} & 0 \\ 0 & e^{+if}\left(-\frac{1}{i}\right)\frac{de^{-if}}{dx} \end{pmatrix} = \begin{pmatrix} \frac{df}{dx} & 0 \\ 0 & \frac{df}{dx} \end{pmatrix}$$

and

$$S^{-1}BS = \begin{pmatrix} b_{11} & e^{-2if}b_{12} \\ e^{+2if}b_{21} & b_{22} \end{pmatrix}$$

Therefore there are two ways to proceed.

1) Because $tr(A^{-1}B) = 0$, $b_{11} = b_{22}$. Hence we can choose f smoothly so that $-\frac{df}{dx} = b_{11} = b_{12}$. Then the new equation will be in the form

$$\frac{1}{i}\begin{pmatrix} \frac{d}{dx} & -\bar{P} \\ P & -\frac{d}{dx} \end{pmatrix} u = \lambda u$$

2) We could try to choose f continously so that $S^{-1}BS$ is a real matrix, but there might be trouble when b_{12} becomes zero.

We will suppose 2) can be done, so that B is a real matrix. Note B is real symmetric and since $tr(A^{-1}B) = 0$ its diagonal entries are equal. Thus

$$B = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

where a, b are real. Now let

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}$$

Then $$U^*\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -i & -1 \\ i & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$U^*\begin{pmatrix} a & b \\ \bar b & a \end{pmatrix} U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} a & b \\ \bar b & a \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} a+b & (-a+b)i \\ \bar b + a & (-\bar b + a)i \end{pmatrix}$$

$$= \begin{pmatrix} a+\mathrm{Re}(b) & -\mathrm{Im}(b) \\ -\mathrm{Im}(b) & a-\mathrm{Re}(b) \end{pmatrix}$$

hence you end up with the system

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{du}{dx} + \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} u = \lambda u$$

which has <u>real</u> coefficients.

Final step is to let S be the solution matrix belonging to this equation with $\lambda = 0$. If we replace u by Su as on page 381 we get

$$S^* A S \frac{dv}{dx} = \lambda (S^* S) v$$

with S^*AS constant, so $S^*AS = A$ since $S(0) = 1$. So we end up finally with a system

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{dv}{dx} = \lambda C v$$

where C is a positive definite real matrix.

Simpler proof goes as follows: First choose S so that $S^*CS = I$ and $S^*AS = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; hence can suppose

$C = I$ and $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Now $B = B^*$ and

$$\operatorname{tr}(A^{-1}B) = \operatorname{tr}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = b_{21} - b_{12} = 0$$

But also $b_{21} = \bar{b}_{12}$, hence $b_{12} = b_{21} \in \mathbb{R}$ and so B is real. Rest is the same. Notice that there is no problem with picking an argument in this way.

Refinements: Look at the initial choice of S. We start with A, C varying smoothly, hence the eigenvalues of $C^{-1}A$ vary smoothly, as well as the eigenspaces. So by adjusting x we make eigenvalues $= \pm i$.

Better: Any positive definite matrix C has a unique positive square root $C^{1/2}$ which depends real analytically on C, because $\exp$: hermitian matrices $\to$ positive definite matrices is a real-analytic isomorphism. Hence taking $S = C^{-1/2}$ one can arrange $C = I$. Then the eigenspaces of A vary smoothly and we can select smoothly orthogonal unit eigenvectors, hence we can find a unitary S with $S^*AS = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with S varying smoothly. Consequently any smooth system can be transformed smoothly to a de Branges style system:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{du}{dx} = \lambda C u$$

where C is a real positive definite matrix of determinant 1 which is a smooth function of x

$$C = \begin{pmatrix} \alpha' & \beta' \\ \beta' & \gamma' \end{pmatrix}$$ de Brange notation

September 3, 1977

Start with (formally) self-adjoint system

$$A\frac{du}{dx} + Bu = \lambda C u \qquad C \geq 0$$

A supposed non-singular at each x. Self-adjointness means

$$A^* = -A \qquad \frac{dA}{dx} = B - B^*$$

~~Let $S(x,\lambda)$ be the solution matrix with $S(0,\lambda) = I$.~~

$$\cancel{\frac{dS}{dx} = A^{-1}(-B+\lambda C)S}$$

~~$\frac{d}{dx}\log(\det S) =$ [redacted] $\mathrm{tr}(-A^{-1}B + \lambda A^{-1}C)$~~

Let $S = S(x,0)$ be the solution matrix for $\lambda = 0$:

$$A\frac{dS}{dx} + B = 0.$$

Then putting $u = Sv$ we get the equation

$$S^*AS\frac{dv}{dx} = \lambda S^*CSv$$

so we can suppose $B = 0$, whence A is constant.

Look at solution matrix $S(x,\lambda)$. $A\frac{dS}{dx} = \lambda CS$

$$\frac{d}{dx}\log(\det S) = \text{[redacted]}\ \mathrm{tr}\left(\frac{dS}{dx}S^{-1}\right)$$

$$= \lambda\,\mathrm{tr}(A^{-1}C)$$

One has $\overline{\mathrm{tr}(A^{-1}C)} = \mathrm{tr}(C^*(A^{-1})^*) = \mathrm{tr}(C(-A)^{-1}) = -\mathrm{tr}(A^{-1}C)$

so $\mathrm{tr}(A^{-1}C)$ is purely imaginary. Hence if we change

S to $e^{-\frac{1}{2}\lambda\int^x tr(A^{-1}C)} S$ we get a [redacted] system whose [redacted] solution matrix is unimodular.

Notice that multiplication by

$$f = e^{+\frac{1}{2}\lambda\int^x tr(A^{-1}C)}$$

is a unitary transformation depending on λ, however, it doesn't seem to affect the eigenvalues of the operator.

[redacted]

Be careful: Put $u = fv$ in $A\frac{du}{dx}$ [redacted] $= \lambda C u$

[redacted]

$$\frac{1}{f}\left[A\frac{d}{dx} - \lambda C\right] f v = A\frac{dv}{dx} + \left(A\frac{1}{f}\frac{df}{dx} - \lambda C\right) v = 0$$

hence $\lambda A^{-1}C$ gets changed into

$$\text{[redacted]} \quad \lambda A^{-1}C - \frac{1}{f}\frac{df}{dx}$$

so if f is as above we have $\lambda\left(A^{-1}C - \frac{1}{2} tr A^{-1}C\right)$ which will have trace zero. So C gets replaced by

$$\tilde{C} = C - \frac{1}{2}\left(tr(A^{-1}C)\right) A$$

Question: Is $\tilde{C} > 0$? So say $A = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ so that $tr(A^{-1}C) = 0$ means the diagonal entries of C are equal. If

$$C = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \gamma \end{pmatrix} \qquad \alpha > 0, \gamma > 0 \qquad \alpha\gamma - |\beta|^2 > 0$$

$$\tilde{C} = C - \frac{1}{2} tr(A^{-1}C) = \begin{pmatrix} \frac{\alpha+\gamma}{2} & \beta \\ \bar{\beta} & \frac{\alpha+\gamma}{2} \end{pmatrix}$$

[redacted]

$$\left(\frac{\alpha+\gamma}{2}\right)^2 - |\beta|^2 = \left(\frac{\alpha-\gamma}{2}\right)^2 + \alpha\gamma - |\beta|^2 > 0$$

so $\tilde{C}$ is > 0.

Observe that the above argument would not have worked if $A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. So it is necessary to suppose that the eigenvalues of A are on opposite sides of 0. In this case we can find a constant matrix T such that $T^* A T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

So we reach the system

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{du}{dx} = \lambda C u$$

where

$$\operatorname{tr} \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} C = c_{12} - c_{21} = 0.$$

But C hermitian $\Longrightarrow$ $c_{12} = \bar{c}_{21}$, hence $c_{12} = c_{21}$ is <u>real</u>. Thus C is a real positive definite matrix.

Put

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and let $S(x,\lambda)$ be the solution matrix. Note that

$$\frac{d}{dx}\frac{dS}{d\lambda}(x,\lambda)\Big|_{\lambda=0} = \frac{d}{d\lambda}\left(\lambda J^{-1} C S(x,\lambda)\right)_{\lambda=0} = J^{-1} C S(x,0)$$

$$= J^{-1} C(x)$$

so that

$$\frac{dS}{d\lambda}(x,0) = \int_0^x J^{-1} C(x)\, dx$$

September 4, 1977

We've seen that if E is a de Branges function, then the de Branges functions equivalent to E are of the form

$$\tilde{E} = c_1 E + c_2 E^{\#}$$

where $|c_1|^2 - |c_2|^2 = 1$. We always assume that E has no real zeroes unless stated otherwise. Thus $E(0) \neq 0$ and so by taking $|c_1| = 1$, $c_2 = 0$ we can replace E by an equivalent one such that $E(0) > 0$. Next since $c_1 + c_2$ with $c_1 = \sqrt{1 + c_2^2}$ assumes all positive values as c_2 runs thru $\mathbb{R}$ we see that we can replace E by an equivalent de Branges function such that $E(0) = 1$. This is a convenient normalization to make. It is equivalent to $A(0) = 1$ $B(0) = 0$. Since a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{R})$ which fixes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

it follows that B is uniquely determined but A can be changed to $A + bB$ with any $b \in \mathbb{R}$. $\therefore$ We have

<u>Prop</u>: Any de Branges function is equivalent to one such that $E(0) = 1$, i.e. such that $A(0) = 1$, $B(0) = 0$. B is uniquely determined, and A is determined up to adding a real constant multiple of B.

Let's return to $L^2(d\mu)$, where $d\mu$ has finite moments. I've seen that if I fix a de Branges function

E_n giving the subspace of polys. of degree $< n$ in $L^2(d\mu)$, then there is a canonical way to construct de Branges functions E_i for $i = 1, 2, \ldots$ giving the subspace of polys. of degree $< i$. One gets linear matrix functions

$$(1) \qquad M_{i+1,i}(\lambda) = \theta_{i+1} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \theta_{i+1}^{-1}$$

such that

$$(2) \qquad \begin{pmatrix} A_{i+1} \\ B_{i+1} \end{pmatrix} = M_{i+1,i}(\lambda) \begin{pmatrix} A_i \\ B_i \end{pmatrix}$$

Before I didn't see how to get E_0. However one has

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \theta_1 \begin{pmatrix} \phi_0 \\ a_0 \phi_1 \end{pmatrix} = \theta_1 \begin{pmatrix} \phi_0 \\ (\lambda - b_0)\phi_0 \end{pmatrix} = \theta_1 \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \phi_0 \\ -b_0 \phi_0 \end{pmatrix}$$

where θ_1 is determined by $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$. If I want (1) (2) above to hold for $i = 0$, and for A_0, B_0 to be constants, then clearly

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} A_1(0) \\ B_1(0) \end{pmatrix} = \theta_1 \begin{pmatrix} \phi_0 \\ -b_0 \phi_0 \end{pmatrix}$$

so that if $M_{i+1,i}(\lambda)$ is defined by (1) for $i=0$ I do have

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \theta_1 \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \theta_1^{-1} \theta_1 \begin{pmatrix} \phi_0 \\ -b_0 \phi_0 \end{pmatrix} = M_{1,0}(\lambda) \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}.$$

Summary: Given a de Branges poly E_n of degree n there is a canonical way of associating to it a sequence of matrices

$$M_{i+1,i}(\lambda) = I + \begin{pmatrix} -\beta_i & -\gamma_i \\ +\alpha_i & \beta_i \end{pmatrix} \lambda$$

$i = 0, \ldots, n-1$ such that $\alpha_i, \gamma_i \geqslant 0$ and $\beta_i^2 = \alpha_i \gamma_i$, and such that

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = M_{n,n-1} \cdots \cdots M_{1,0} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$$

where $$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} A_n(0) \\ B_n(0) \end{pmatrix}$$

Check: $2n$ parameters required to describe E_n, because of the n roots of E_n in the lower half plane. $2n$ parameters α_i, γ_i to describe the matrices $M_{i+1,i}$.

~~Notice that it ought to be the case that the zeroes of A_n, B_n interlace each other. If true we know that $B_n'(0) \neq 0$.~~ Better:

$$\begin{pmatrix} A_n'(0) \\ B_n'(0) \end{pmatrix} = \begin{pmatrix} -\sum \beta_i & -\sum \gamma_i \\ +\sum \alpha_i & \sum \beta_i \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$$

suppose that we require $A_n(0) = A_0 = 1$ and $B_n(0) = B_0 = 0$. Then

$$\begin{pmatrix} A_n'(0) \\ B_n'(0) \end{pmatrix} = \begin{pmatrix} \sum \beta_i \\ +\sum \alpha_i \end{pmatrix}$$

Since $\alpha_i \geqslant 0$, one has $B_n'(0) > 0$ unless all $\alpha_i = 0$. But if all α_i are zero then we have also $\beta_i = 0$ so

$$M_{i+1,i}(\lambda) = \begin{pmatrix} 1 & -\gamma_i\lambda \\ 0 & 1 \end{pmatrix}$$

and so $$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} 1 & -\sum_1^n \gamma_i\lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which is impossible for $n \geq 1$. Thus $B_n'(0) < 0$ and so we see there is a unique choice for A_n such that $A_n'(0) = 0$.

Another proof that $B_n'(0) > 0$ is as follows: We know that for increasing real λ, $\arg(\lambda - \omega)$ is decreasing for $\operatorname{Im}(\omega) < 0$. Hence $\arg E_n(\lambda)$ is decreasing as λ increases along $\mathbb{R}$, so

$$\frac{d}{d\lambda} \arg E_n(\lambda) = \frac{d}{d\lambda} \operatorname{Im} \log E_n(\lambda) = \operatorname{Im} \frac{E_n'(\lambda)}{E_n(\lambda)} < 0$$

hence if $E_n(0) = 1$, we have $\operatorname{Im} E_n'(0) = -B_n'(0) < 0$.

However the condition $A_n'(0) = 0$ will not be inherited by A_{n-1} etc. Nevertheless we can always arrange for $A_1'(0) = 0$, i.e. $A_1(\lambda) = 1$.

Prop: Any de Branges function is equivalent to a unique de Branges function with $E(0) = 1$ and $A'(0) = 1$

Proof: We first arrange that $E(0) = 1$ ~~From the formula~~

$$J_z(\lambda) = \frac{-1}{\lambda - \bar{z}} \begin{vmatrix} A(\lambda) & A(\bar{z}) \\ B(\lambda) & B(\bar{z}) \end{vmatrix}$$

we get

$$J_0(\lambda) = \frac{+1}{\lambda} B(\lambda)$$

hence letting $\lambda \searrow 0$ one gets $B'(\lambda) = J_0(0) = \|J_0\|^2 > 0$. so one adjusts A by a real multiple of B to get $A'(0) = 0$.

<u>September 5, 1977:</u>

There doesn't seem to be any advantage in normalizing all (A_n, B_n) by requiring $A_n(0) = 1$, $B_n(0) = A_n'(0) = 0$.

Notice that if you use the formulas $E = A - iB$ and

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + (\ )\lambda & (\ )\lambda \\ (\ )\lambda & 1 - (\ )\lambda \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}$$

or

$$J\, d\begin{pmatrix} A \\ B \end{pmatrix} = \lambda\, dm \begin{pmatrix} A \\ B \end{pmatrix} = \lambda d \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

Then $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ doesn't work because it gives

$$\frac{d}{dx}\begin{pmatrix} A \\ B \end{pmatrix} = \lambda \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ \beta' & \gamma' \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \lambda \begin{pmatrix} +\beta' & +\gamma' \\ -\alpha' & -\beta' \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

hence $\frac{d}{dx}\left(\frac{dB}{d\lambda}\Big|_{\lambda=0} \right) = -\alpha'$ would be <u>negative</u>. So we want

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the formula

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 1-\beta_n\lambda & -\gamma_n\lambda \\ \alpha & 1+\beta_n\lambda \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}$$

or in continuous form

$$d\begin{pmatrix} A \\ B \end{pmatrix} = \lambda \begin{pmatrix} -d\beta & -d\gamma \\ d\alpha & d\beta \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\begin{pmatrix} A \\ B \end{pmatrix} = \lambda \begin{pmatrix} d\alpha & d\beta \\ d\beta & d\gamma \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

Notice that if $E_n = A_n - iB_n$ is of degree n and $E_n(0) = 1$, and if the associated system is

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = M_{n,n-1} \cdots M_{1,0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then changing A_n to $A_n\, c B_n$ corresponds to conjugating each $M_{i,i-1}$ by $\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix} \begin{pmatrix} -\beta & -\gamma \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & +c \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix} \begin{pmatrix} -\beta & -c\beta-\gamma \\ \alpha & c\alpha+\beta \end{pmatrix}$$

$$= \begin{pmatrix} -\beta-c\alpha & -\gamma-2c\beta-c^2\alpha \\ \alpha & \beta+c\alpha \end{pmatrix}$$

Hence without changing the de Branges spaces we can change the system to

$$d\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \longmapsto d\begin{pmatrix} \alpha & \beta+c\alpha \\ \beta+c\alpha & \gamma+2c\beta+c^2\alpha \end{pmatrix}$$

for any real c.

Homogeneous de Branges spaces. These are ones $\mathcal{H}$ such that for each a, $0<a<1$ there exists a $k(a)>0$ such that $F \mapsto k(a)F(a\lambda)$ is an [redacted] isometric [redacted] embedding of $\mathcal{H}$ into itself. It can be shown that $k(a) = a^{\mu}$ for some exponent μ, [redacted] and the classification of such $\mathcal{H}$ reduces easily to the case where $E(0) \neq 0$, whence [redacted], in virtue of the homogeneity one has $E(\lambda) \neq 0$ for all real λ.

Let $\mathcal{H}_a$ = image of $F \mapsto a^{\mu}F(a\lambda)$ with $\|a^{\mu}F(a\lambda)\| = \|F\|$. If $F \in \mathcal{H}_a$ one has

[redacted] $$\left(F, a^{\mu}J_{az}(a\lambda)\right) = \left(a^{-\mu}F(a^{-1}\lambda), [redacted] J_{az}(\lambda)\right)$$

$$= a^{-\mu}F(a^{-1}az) = a^{-\mu}F(z)$$

hence $a^{2\mu}J_{az}(a\lambda)$ is the point evaluator [redacted] ${}_aJ_z(\lambda)$ in $\mathcal{H}_a$. Hence

$$\frac{-1}{\lambda-\bar{z}}\begin{vmatrix} A_a(\lambda) & A_a(\bar{z}) \\ B_a(\lambda) & B_a(\bar{z}) \end{vmatrix} = a^{2\mu}\frac{-1}{a\lambda - a\bar{z}}\begin{vmatrix} A(a\lambda) & A(a\bar{z}) \\ B(a\lambda) & B(a\bar{z}) \end{vmatrix}$$

hence there exists $P(a) \in SL_2(\mathbb{R})$ such that

$$P(a)\begin{pmatrix} A_a(\lambda) \\ B_a(\lambda) \end{pmatrix} = a^{\mu-\frac{1}{2}} [redacted] \begin{pmatrix} A(a\lambda) \\ B(a\lambda) \end{pmatrix}$$

We [redacted] suppose E, E_a chosen so that $E(0) = E_a(0) = 1$, whence

$$P(a)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a^{\mu-\frac{1}{2}} [redacted] \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $$P(a) = \begin{pmatrix} a^{\mu-\frac{1}{2}} & h(a) \\ 0 & a^{-\mu+\frac{1}{2}} \end{pmatrix}$$

In particular we get

$$B_a(\lambda) = a^{2\mu-1} B(a\lambda).$$

From general symmetry considerations and uniqueness in the way de Branges picks E_b starting from E_a when $b<a$, one has to have

$$P(ab) = P(a)P(b)$$

hence

$$h(ab) = a^{\mu-\frac{1}{2}} h(b) + b^{-\mu+\frac{1}{2}} h(a)$$

Consider a system

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$$

where p is even, and $|p|$ grows fast enough so that the spectrum on $(-\infty, \infty)$ is discrete. Because p is even we have a symmetry of the system:

$$x \longmapsto -x, \quad \lambda \longmapsto -\lambda, \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} -u_1 \\ u_2 \end{pmatrix}$$

Suppose v is an eigenfunction for the eigenvalue λ. Is there anything I can say about the initial values of v? We can suppose that $v_2^{\#} = v_1$.

Let $v^+(x,\lambda)$ denote the solution decaying at $x = +\infty$. I can suppose $(v_2^+)^{\#} = v_1^+$ in which case v is unique up to a real multiple.

To be more precise, suppose λ real. Then I know for any choice of v that $|v_1(x,\lambda)| = |v_2(x,\lambda)|$ and I also know that

$$\begin{pmatrix} \overline{v_2(x,\lambda)} \\ \overline{v_1(x,\lambda)} \end{pmatrix} = c \begin{pmatrix} v_1(x,\lambda) \\ v_2(x,\lambda) \end{pmatrix}$$

for some constant c. So I can require $|v_1(0,\lambda)| = 1$ and also that $\overline{v_2(0,\lambda)} = v_1(0,\lambda)$ in which case $c = 1$. In this case v is unique up to ± 1.

Therefore for λ real I can require $\overline{v_2(0,\lambda)} = v_1(0,\lambda)$ and $|v_1(0,\lambda)| = 1$, in which case $v(x,\lambda)$ is unique up to ± 1. However then

$$\begin{pmatrix} -v_1(-x,-\lambda) \\ v_2(-x,-\lambda) \end{pmatrix} = v^-(x,\lambda)$$

For λ an eigenvalue, v^- and $v = v^+$ are proportional hence one has to have [redacted]

$$\begin{pmatrix} -v_1(0,-\lambda) \\ v_2(0,-\lambda) \end{pmatrix} = \pm \begin{pmatrix} v_1(0,\lambda) \\ v_2(0,\lambda) \end{pmatrix}$$

So what seems to be happening is that we have [redacted] a map

$$\begin{aligned} \mathbb{R} &\xrightarrow{\ \varphi\ } [\text{redacted}]\ \mathbb{R}/2\pi\mathbb{Z} \\ \lambda &\longmapsto \arg\left(\frac{v_1(0,\lambda)}{v_2(0,\lambda)}\right) = \arg\left(\frac{\overline{v_2(0,\lambda)}}{v_2(0,\lambda)}\right) \end{aligned}$$

and the eigenvalues are determined by

$$(*) \qquad \varphi(\lambda) = \varphi(-\lambda) + \pi \quad \mathrm{mod}(2\pi\mathbb{Z})$$

If one has a de Branges style system

$$\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \frac{du}{dx} = \lambda \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} u \qquad C = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \geq 0$$

or

$$\frac{du}{dx} = \lambda \underbrace{\begin{pmatrix} -\beta & -\gamma \\ +\alpha & +\beta \end{pmatrix}}_{J^{-1}C(x)} u$$

then in order to effect the symmetry $x \mapsto -x$, $\lambda \mapsto -\lambda$ one needs a constant matrix T such that

$$T^{-1}\left(J^{-1}C(-x)\right)T = J^{-1}C(x)$$

For example either take C to be even [redacted] and $T =$ identity or take $T = J$ [redacted]

$$T(J^{-1}C)J^{-1} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\beta & -\gamma \\ +\alpha & +\beta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} = \begin{pmatrix} [\text{redacted}]\alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ \gamma & -\beta \end{pmatrix}$$

and suppose that $\alpha(-x) = \gamma(x)$ and $\beta(x) = -\beta(-x)$

The interpretation of (*) is that $\varphi(\lambda) = T(\varphi(-\lambda))$ and the symmetry results from the fact T^2 is effectively the identity: $\varphi(+\lambda) = T\varphi(-\lambda) \Rightarrow T\varphi(\lambda) = \varphi(-\lambda)$.

Additional points: Given

$$J \frac{du}{dx} = \lambda \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} u \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

de Branges introduces the function $t(x)$ such that

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} - i\frac{dt}{dx} J = \begin{pmatrix} \alpha & \beta - it' \\ \beta + it' & \gamma \end{pmatrix}$$

is ≥ 0 but not > 0, i.e. such that

$$\alpha\gamma - (\beta - it')(\beta + it') = 0$$

$$\parallel$$

$$\alpha\gamma - \beta^2 - (t')^2$$

or

$$\frac{dt}{dx} = \sqrt{\alpha\gamma - \beta^2}$$

$$t(x) = \int_0^x \sqrt{\alpha\gamma - \beta^2}\, dx$$

$t(x)$ represents the time for a disturbance at $x=0$ to propagate to x. In the systems I look at $\alpha\gamma - \beta^2 = 1$ so $t(x) = x$. $t(x)$ is also the type of transform of functions u supported in $[0, x]$.

The following seems to be de Branges Thm. 51 in perhaps less general form: Consider the system:

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$$

on $0 \le x < \infty$ where p is supposed integrable.

Then

$$(*) \qquad \lim_{x \to \infty} u_2(x,\lambda) e^{ix\lambda} = W(\lambda)$$

is an analytic function in the UHP continuous in the closed UHP with no zeroes in the closed UHP. Moreover $\mathcal{H}(u_2(a,\lambda))$ is the space of entire functions $f(\lambda)$ such that $f(\lambda)/W(\lambda)$ and $f^{\#}(\lambda)/W(\lambda)$ are of finite type $\le a$ in the closed UHP and such that

$$\int |f(\lambda)/W(\lambda)|^2 dx < \infty$$

(this integral equals $\|f\|^2$ in $\mathcal{H}(u_2(a,\lambda))$.).

Proof of existence of the limit $(*)$. One has

$$\frac{d}{dx}\left(u_2(x,\lambda) e^{ix\lambda}\right) = p(x) u_1(x,\lambda) e^{ix\lambda}$$

$$\frac{d}{dx} \log\left(u_2(x,\lambda) e^{ix\lambda}\right) = p(x) \frac{u_1(x,\lambda)}{u_2(x,\lambda)}$$

Since $\left|\frac{u_1(x,\lambda)}{u_2(x,\lambda)}\right| \le 1$ for $\operatorname{Im}\lambda \ge 0$ one gets

$$\left| \log\left(u_2(x,\lambda)e^{ix\lambda}\right)\Big|_{x=a}^{x=b} \right| \le \int_a^b |p(x)|\,dx \to 0 \quad \text{as}$$

$a < b \to \infty$, hence $\log(u_2(x,\lambda) e^{ix\lambda})$ converges by Cauchy's criterion.