August 2, 1977

$$\lambda y_n = a_n y_{n+1} + b_n y_n + a_{n-1} y_{n-1}$$

$$\frac{a_{n-1} y_{n-1}}{y_n} = \lambda - b_n - \frac{a_n^2}{\frac{a_n y_n}{y_{n+1}}}$$

when iterated leads to the l^2 solution in the $n \to \infty$ direction:

$$\frac{y_0^+}{a_{-1} y_{-1}^+} = \frac{1}{\lambda - b_0 -} \; \frac{a_0^2}{\lambda - b_1 -} \; \frac{a_1^2}{\lambda - b_2 -} \cdots$$

$$= \int \frac{d\mu^+(x)}{\lambda - x}$$

On the other hand

$$\frac{a_n y_{n+1}}{y_n} = \lambda - b_n - \frac{a_{n-1}^2}{\frac{a_{n-1} y_n}{y_{n-1}}}$$

when iterated leads to the l^2 solution in the $n \to -\infty$ direction:

$$\frac{y_{-1}^-}{a_{-1} y_0^-} = \frac{1}{\lambda - b_{-1} -} \; \frac{a_{-2}^2}{\lambda - b_{-2} -} \cdots$$

So the condition for a eigenvalue (both directions) is

$$\boxed{\lambda - b_0 - \frac{a_0^2}{\lambda - b_1 -} \; \frac{a_1^2}{\lambda - b_2 -} \cdots = \frac{a_{-1}^2}{\lambda - b_{-1} -} \; \frac{a_{-2}^2}{\lambda - b_{-2} -} \cdots}$$

Now $$W_n = \begin{vmatrix} a_{n-1} y_{n-1}^+ & a_{n-1} y_{n-1}^- \\ y_n^+ & y_n^- \end{vmatrix} = \begin{vmatrix} -a_n y_{n+1}^+ & -a_n y_{n+1}^- \\ y_n^+ & y_n^- \end{vmatrix} = W_{n+1}$$

is independent of n, and

$$W_0 = a_{-1}(y^+_{-1} y^-_0 - y^+_0 y^-_{-1}) = \left(\frac{a_{-1}y^+_{-1}}{y^+_0} - \frac{a_{-1}y^-_{-1}}{y^-_0} \right) y^+_0 y^-_0$$

$$= \left(\left(\lambda - b_0 - \frac{a_0^2}{\lambda - b_1 - } \cdots \right) - \left(\frac{a_{-1}^2}{\lambda - b_{-1} - } \cdots \right) \right) y^+_0 y^-_0$$

Hence the Wronskian of y^+, y^- is essentially the difference of the two continued fractions.

<u>August 3, 1977:</u>

Question: When is the spectrum of a S-L operator

$$\frac{d}{dx} p \frac{du}{dx} + (\lambda r - q) u = 0$$

discrete?

Make standard change: $\frac{d}{dy} = \left(\frac{p}{r}\right)^{1/2} \frac{d}{dx}$, $u = fv$

$$\frac{1}{f} \frac{1}{r} \left(\frac{r}{p}\right)^{1/2} \frac{d}{dy} \left(p \left(\frac{r}{p}\right)^{1/2} \frac{d}{dy} \right) fv + \left(\lambda - \frac{q}{r} \right) v = 0$$

$$\|$$

$$\frac{1}{f(rp)^{1/2}} \frac{d}{dy} \left(f(rp)^{1/2} \frac{1}{f} \frac{d}{dy} fv \right)$$

$$\|$$

$$\left(\frac{d}{dy} + \frac{d}{dy} \log (f(rp)^{1/2}) \right) \left(\frac{d}{dy} + \frac{d}{dy} \log f \right) v$$

$$\|$$

$$\left[\frac{d^2}{dy^2} + \left(\frac{d}{dy} \log f(pr)^{1/2} + \frac{d}{dy} \log f \right) \frac{d}{dy} + \left(\frac{d}{dy} \log (f(rp)^{1/2}) \frac{d}{dy} \log f + \frac{d^2}{dy^2} \log f \right) \right] v$$

Thus to make coeff. of $\frac{d}{dy}$ vanish we want

$$f^2(pr)^{1/2} = \text{const.} \quad \text{or}$$

$$f = (pr)^{-1/4}$$

But the formula for the potential

$$V = \left(\frac{d}{dy}\log f\right)^2 - \frac{d^2}{dy^2}(\log f) + \frac{q}{r}$$

seems too complicated. What is essential is that as $x \to \infty$ one has $y \to \infty$ and $V \to \infty$. Actually the change of independent variable cannot affect the coefficient of u. So view the change of variables as first doing u to fv which changes the coefficient $(\lambda r - q)\,u$ to $\left(\lambda r - q + \frac{d}{dx}\left(p\frac{df}{dx}\right)\right)v$, then secondly changing x to y and dividing by r. So we see the potential is

$$V = \frac{q}{r} - \frac{d}{dx}\left(p\frac{df}{dx}\right)$$

where $f = (pr)^{-1/4}$

Critical case is where $p = 1$ and $q = 0$ whence

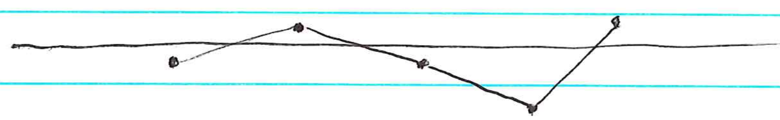
$$V = -\frac{d^2}{dx^2}\left(r^{-1/4}\right)$$

and $\frac{dy}{dx} = r^{1/2}$. I want the density to go to zero, e.g. $r(x) = x^{-a}$, then $\frac{dy}{dx} = x^{-a/2}$ so $y \to \infty$ provided $a \le 2$, and

$$V = -\frac{d^2}{dx^2}\left(x^{a/4}\right) = -(a/4)(a/4 - 1)\,x^{a/4 - 2}$$

goes to zero. So it seems that if one wants a discrete spectrum one must have $q(x) \uparrow \infty$ as $x \to \infty$.

Discrete strings: Consider a finite number of particles arranged in a line + allowed to move slightly transversally to the line. Suppose consecutive particles linked by springs



Newton's equations of motions are

$$m_i \ddot{y}_i = k_i(y_{i+1} - y_i) + k_{i-1}(y_{i-1} - y_i)$$

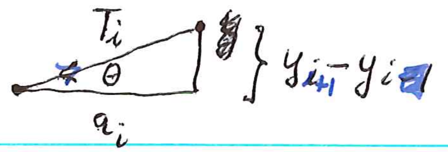
where k_i depends on the tension of the spring and separation between the $(i+1)$-th and i-th particles. If we want a sinusoidal motion $y = e^{i\lambda t} v$, then

$$-\lambda^2 m_i v_i = k_i(v_{i+1} - v_i) + k_{i-1}(v_{i-1} - v_i)$$

or

$$-\lambda^2 m_i v_i = k_i v_{i+1} - (k_i + k_{i-1}) v_i + k_{i-1} v_{i-1}.$$

I should have mentioned that since we are assuming small vibrations, the stretching of the springs is negligible, only the tension of the i-th spring matters. Thus if T_i is the tension and a_i is the separation between the $(i+1)$-th and i-th particle we have



force on ith particle due to i-th spring

$$= T_i \sin\theta \doteq T_i \tan\theta = \frac{T_i}{a_i}(y_{i+1} - y_i)$$

so

$$k_i = \frac{T_i}{a_i}$$

Krein proposes setting all $T_i = 1$. Then one can think of the particles as being tied together by weightless strings, all under tension $T = 1$.

$$(k_i + k_{i-1} - \lambda^2 m_i) v_i = k_i v_{i+1} + k_{i-1} v_{i-1}$$

$$\frac{k_{i-1} v_{i-1}}{v_i} = k_{i-1} + k_i - \lambda^2 m_i - \frac{k_i^2}{\dfrac{k_i v_i}{v_{i+1}}}$$

If the $(n+1)$-th particle is tied, i.e. $v_{n+1} = 0$, then the motion with frequency λ satisfies

$$\frac{k_0 v_0}{v_1} = k_0 + k_1 - \lambda^2 m_1 - \frac{k_1^2}{k_1 + k_2 - \lambda^2 m_2 -} \cdots \cdots \frac{k_{n-1}^2}{k_{n-1} + k_n - \lambda^2 m_n}$$

If λ^2 is not an eigenvalue, then we can arrange $k_0 v_0 = 1$ in which case if $\phi = (v_i, 1 \le i \le n)$

$$\left[(K + \lambda^2 M)\phi\right]_1 = \cancel{k_0 v_0} + (\lambda^2 m_1 - k_0 - k_1) v_1 + k_1 v_2$$

$$= -k_0 v_0 = -1$$

hence $(-\lambda^2 M - K)\phi = e_1$, so

$$\phi = (-\lambda^2 M - K)^{-1} e_1$$

Let's change $-\lambda^2$ to λ and let $u(\lambda)$ denote the solution of

$$K u(\lambda) = \lambda M u(\lambda)$$

with $\qquad u(\lambda)_0 = 0 , \quad u(\lambda)_1 = 1 .$

Then the spectral measure $d\mu(\lambda)$ is defined by

$$e_1 = \int u(x)\, d\mu(x)$$

and we know it is supported in the negative real axis. One has for some function f on the spectrum

$$\phi(\lambda) = (\lambda M - K)^{-1} e_1 = \int f(x)\, u(x)\, d\mu(x)$$

$$f(x)(\lambda M - K)\, u(x) = f(x)(\lambda M - xM)\, u(x)$$

$$e_1 = (\lambda M - K)\phi(\lambda) = M \int f(x)(\lambda - x)\, u(x)\, d\mu(x) = M(m_1^{-1} e_1)$$

so
$$f(x) = \frac{m_1^{-1}}{\lambda - x}$$

Thus
$$\phi(\lambda)_1 = \int \frac{d\mu(x)}{m_1(\lambda - x)} = v_1 = \frac{1}{k_0 + k_1 + \lambda m_1 -}\ \frac{k_1^2}{k_1 + k_2 + \lambda m_2 -} \cdots$$

and we get the formula:

$$\boxed{\int \frac{d\mu(x)}{\lambda - x} = \frac{m_1}{k_0 + k_1 + \lambda m_1 -}\ \frac{k_1^2}{k_1 + k_2 + \lambda m_2 -} \cdots\cdots \frac{k_{n-1}^2}{k_{n-1} + k_n + \lambda m_n}}$$

~~Except that the k_0 doesn't make sense. Why k_0~~

We know by physics that the eigenvalues:

$$\det(\lambda M - K) = 0$$

are all < 0. So what remains to be understood is why this is the case and why an $n \times n$ J-matrix with negative eigenvalues determines positive numbers $m_1, \dots, m_n, k_0, \dots, k_n$ in the above way.

August 4, 1977:

Discrepancy in the above: The continued fraction at the bottom of page 263 has $(2n+1)$-constants $m_1, \dots, m_n$ $k_0, \dots, k_n$ in it, but an $n \times n$ J-matrix depends on only $2n-1$ constants: $b_1, \dots, b_n, a_1, \dots, a_{n-1}$. We can without changing the spectral measure assume that $m_1 = 1$, but I don't see ~~[illegible]~~ another symmetry

What happens as a mass $m_i \to 0$.

$$k - \frac{k^2}{k+x} = \frac{kx}{k+x} = \frac{1}{\frac{1}{k} + \frac{1}{x}}$$

hence

$$k_1 - \frac{k_1}{k_1 + k_2 - \frac{k_2^2}{k_2 + y}} = \cancel{k_1 - \frac{k_1}{k_1 + \left(\frac{k_2 y}{k_2 + y}\right)}}$$

$$= \frac{1}{\frac{1}{k_1} + \frac{1}{\frac{1}{k_2} + \frac{1}{y}}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{y}} = k - \frac{k^2}{k + y}$$

where $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

The same identity allows one to write the continued fraction on p. 263 in the form

$$\cfrac{m_1}{k_0+\lambda m_1+}\;\cfrac{1}{\cfrac{1}{k_1}+\cfrac{1}{\lambda m_2+\cfrac{1}{\cfrac{1}{k_2}+\cfrac{1}{\lambda m_2+\cdots}}}}$$

$$=\frac{m_1}{k_0+\lambda m_1+}\;\frac{1}{k_1^{-1}+}\;\frac{1}{\lambda m_2+}\;\frac{1}{k_2^{-1}+}\;\frac{1}{\lambda m_3+}\;\frac{1}{k_3^{-1}+}\cdots,$$

which is the Stieltjes form. The successive convergents for this CF represent the limiting cases one obtains by letting $m_{n+1}\to\infty$ which corresponds to fixing the $(n+1)$-th particle, or letting $k_n^{-1}\to\infty$ which corresponds to letting there be no force from the $(n+1)$-th particle on the n-th.

August 5, 1977

Lets consider a system of first order equations for which the series solutions can be calculated by 2-term recursion relations:

$$\left(A_0 x\frac{d}{dx} + B_0\right) u = x\left(A_1 x\frac{d}{dx} + B_1\right) u$$

If $u = x^\mu \sum_{n\geq 0} a_n x^n$ is a series solution, then

$$\sum_{n\geq 0} \left(A_0 (n+\mu) a_n + B_0 a_n\right) x^{n+\mu} = \sum_{n\geq 0} \left(A_1 (n+\mu) a_n + B_1 a_n\right) x^{n+\mu+1}$$

hence we get the indicial equation

$$\left(\mu A_0 + B_0\right) a_0 = 0$$

and the recursion relations:

$$\left((n+\mu)A_0 + B_0\right) a_n = \left((\mu+n-1)A_1 + B_1\right) a_{n-1}$$

Review the old scheme for relating ρ to a system

$$\frac{d}{dx} u = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$$

I suppose p is even, in which case $x \mapsto -x$, $\lambda \mapsto -\lambda$, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} -u_1 \\ +u_2 \end{pmatrix}$ is a symmetry of the equation. So if $u^\pm(x,\lambda)$ is the solution decaying at $x = \pm\infty$, then

$$\begin{pmatrix} u_1^-(x,\lambda) \\ u_2^-(x,\lambda) \end{pmatrix} = \begin{pmatrix} -u_1^+(-x,-\lambda) \\ +u_2^+(-x,-\lambda) \end{pmatrix}$$

The idea is to consider

$$W = \begin{vmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{vmatrix} = u_1^+(x,\lambda)u_2^+(-x,-\lambda) + u_1^+(-x,-\lambda)u_2^+(x,\lambda)$$

which should be independent of x. One has

$$\frac{W}{u_2^+(x,\lambda)u_2^+(-x,-\lambda)} = \frac{u_1^+(x,\lambda)}{u_2^+(x,\lambda)} + \frac{u_1^+(-x,-\lambda)}{u_2^+(-x,-\lambda)}$$

Take $x=0$, put

$$f(\lambda) = \frac{u_1^+(0,\lambda)}{u_2^+(0,-\lambda)}.$$

Then $f(\lambda)$ describes the intial values of the solution u^+ decaying at ∞. We have

$$\frac{W}{u_2^+(0,\lambda)u_2^+(0,-\lambda)} = f(\lambda) + f(-\lambda).$$

Basic question - does $\xi(s)$ split as a sum $f(s)+f(1-s)$ in a natural way, better does it split into a sum

$$\xi(s) = f(x,s) + f(-x,1-s)?$$

Quite possibly some of the Θ-functions so far encountered do split up.

August 6, 1977:

Suppose p real in

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & p \\ p & -i\lambda \end{pmatrix} u$$

Then $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$, $\lambda \mapsto -\lambda$ is a symmetry of the D.E. Hence if $u^+(x,\lambda)$ is the solution decaying at $x = +\infty$ one has $u^+(x,\lambda)$ is proportional to $\begin{pmatrix} u_2^+(x,-\lambda) \\ u_1^+(x,-\lambda) \end{pmatrix}$ by a factor depending on λ. Hence

$$f(\lambda) = \frac{u_1^+(0,\lambda)}{u_2^+(0,\lambda)} = \frac{u_2^+(0,-\lambda)}{u_1^+(0,-\lambda)} = \frac{1}{f(-\lambda)}$$

So if we want the eigenvalues defined by the bdry condition $f(\lambda) = e^{i\alpha}$ to be symmetric under $\lambda \mapsto -\lambda$, we must have $e^{i\alpha} = \frac{1}{e^{i\alpha}}$ or $e^{i\alpha} = \pm 1$.

If in addition p is even, we have the symmetry $x \mapsto -x$, $\lambda \mapsto -\lambda$, $u \mapsto \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$ which changes $f(\lambda)$ to $-f(-\lambda)$, so that the global eigenvalue condition is

$$f(\lambda) + f(-\lambda) = 0$$

or

$$f(\lambda) + \frac{1}{f(\lambda)} = 0$$

or

$$f(\lambda) = \pm i .$$

If p is real & odd, we have the symmetry $x \mapsto -x$, $\lambda \mapsto -\lambda$, $u \mapsto u$, hence $f(\lambda) \mapsto f(-\lambda)$. The eigenvalue condition is

$$f(\lambda) - f(-\lambda) = 0 \qquad f(\lambda) = \frac{1}{f(\lambda)}$$

or $$f(\lambda) = \pm 1.$$

Suppose p real and even. Then the two-sided eigenvalue problem is the union of the one-sided problems for $f(\lambda) = i$ and for $f(\lambda) = -i$. If p is real and odd, then the 2-sided problem is the union of the one-sided problems for $f(\lambda) = 1$ and $f(\lambda) = -1$, each of which has symmetry under $\lambda \mapsto -\lambda$. Thus for $p = x$ which reduces to the Schroedinger for the simple harmonic oscillator, we saw the problem separated, so to speak, into even and odd Hermite polys. For p real and even however one gets eigenvalue symmetry only by taking both $f(\lambda) = i$ and $f(\lambda) = -i$.

However note that if we want for p real even

$$\mathfrak{f}(\lambda) = f(\lambda) + f(-\lambda) = f(\lambda) + \frac{1}{f(\lambda)}$$

Then one has

$$\mathfrak{f}(\lambda) \in [-2, 2] \Rightarrow |f(\lambda)| = 1 \Rightarrow \lambda \in \mathbb{R}$$

August 8, 1977

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t} \qquad \text{converges for } \mathrm{Re}(s) > 0$$

$$\Gamma(1-s) = \int_0^\infty e^{-t} t^{1-s} \frac{dt}{t} = \int_0^\infty \left(e^{-t^{-1}} t^{-1}\right) t^s \frac{dt}{t} \qquad \text{c. for } \mathrm{Re}(s) < 1$$

One has convolution formula

$$\int_0^\infty f(t)\, t^s \frac{dt}{t} \int_0^\infty g(t)\, t^s \frac{dt}{t} = \int_0^\infty \int_0^\infty f(t) t^s g(u) u^s \frac{dt}{t}\frac{du}{u} \qquad t \mapsto \frac{t}{u}$$

$$= \int_0^\infty \left(\int_0^\infty f\left(\tfrac{t}{u}\right) g(u) \frac{du}{u} \right) t^s \frac{dt}{t}$$

hence

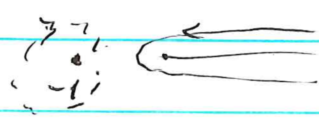
$$\Gamma(1-s)\Gamma(s) = \int_0^\infty \left(\int_0^\infty e^{-ut^{-1}} (t/u)^{-1} e^{-u} \frac{du}{u} \right) t^s \frac{dt}{t}$$

$$= \int_0^\infty \left(\int_0^\infty e^{-u(1+t^{-1})} du \right) t^{s-1} \frac{dt}{t}$$

$$= \int_0^\infty \frac{1}{1+t^{-1}} t^{s-1} \frac{dt}{t} = \int_0^\infty \frac{t^s}{1+t} \frac{dt}{t}$$

$$= \frac{1}{e^{2\pi i s} - 1} \int_C \frac{t^s}{1+t} \frac{dt}{t}$$

since $0 < \mathrm{Re}(s) < 1$ one can use contour integration



$$= \frac{1}{e^{2\pi i s} - 1} (-2\pi i) \left(e^{i\pi}\right)^{s-1}$$

$$= \frac{2\pi i}{e^{i\pi s} - e^{-i\pi s}} = \frac{\pi}{\sin(\pi s)}$$

Thus we see that the Mellin transform of $\frac{1}{1+t}$ is $\frac{\pi}{\sin(\pi s)}$

August 10, 1977

Let
$$f(t) = \frac{1}{t} + \sum_{n=1}^{\infty}\left[\frac{1}{t-n} + \frac{1}{t+n}\right]$$
$$= \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2-n^2}$$

$f(t)$ is a meromorphic function with simple poles of residue 1 at each integer. Moreover it is clear that $f(t+1) = f(t)$ and $f(-t) = -f(t)$.

Another function with the same property is
$$\pi\cot(\pi t) = \pi\frac{\cos(\pi t)}{\sin(\pi t)}$$

Now it should be that $f(t)$ is bounded as one heads vertically to ∞. (It appears at first glance that $f(t) \to 0$ as $\operatorname{Im} t \to \pm\infty$, $0 \leq \operatorname{Re}(t) \leq 1$ but this can't be so, for otherwise putting $q = e^{2\pi i t}$, one would have a meromorphic function of q in the annulus $0 < |q| < \infty$ with a simple pole at $q=1$ and tending to zero at $q \to 0$ or $q \to \infty$.) So anyway by Liouville's theorem one has to have

$$\boxed{\pi\cot(\pi t) = \frac{1}{t} + \sum_{n=1}^{\infty}\left[\frac{1}{t-n} + \frac{1}{t+n}\right]}$$

Preceding is not very convincing. For example

$$\frac{1}{e^{2\pi it}-1}$$

has simple poles at the integers with residues all equal to $\frac{1}{2\pi i}$. Hence

$$2\pi i\left(\frac{1}{e^{2\pi it}-1}\right)$$

is bounded as $\operatorname{Im} t \to +\infty$ and as $\operatorname{Im} t \to -i\infty$ so it has to coincide with $f(t)$ up to a constant. But if

$$f(t) = \quad = 2\pi i\left(\frac{1}{e^{2\pi it}-1}+c\right)$$

Then from $f(t)=-f(-t)$ we have

$$\frac{1}{e^{2\pi it}-1}+\frac{1}{e^{-2\pi it}-1}+2c=0$$

$$-1 \quad \cancel{\frac{1}{e^{2\pi it}-1}+\frac{e^{2\pi it}}{1-e^{2\pi it}}}+2c=0 \qquad c=\frac{1}{2}$$

so

$$f(t)=2\pi i\left(\frac{1}{e^{2\pi it}-1}+\frac{1}{2}\right)=\cancel{2}\pi i\left(\frac{e^{2\pi it}+1}{(e^{2\pi it}-1)\cancel{2}}\right)$$

$$=\pi\frac{\cos \pi t}{\sin \pi t}=\pi\cot(\pi t)$$

as above.

The motivation for the above comes from the following. Suppose

$$F(s) = \int_0^\infty \phi(t)\, t^{-s} \frac{dt}{t}$$

Then $$\zeta(s) F(s) = \sum_{n=1}^{\infty} \int_0^\infty \phi(t)\, n^{-s} t^{-s} \frac{dt}{t}$$

$$= \int_0^\infty \sum_{n=1}^{\infty} \phi\left(\frac{t}{n}\right) t^{-s} \frac{dt}{t}$$

at least formally. If ϕ is an even function, then

$$\sum_{n=1}^{\infty} \phi\left(\frac{t}{n}\right) = \frac{1}{2} \sum_{n \in \mathbb{Z}}' \phi\left(\frac{t}{n}\right)$$

A natural thing to look for is functions ϕ such that $\sum_{n=1}^{\infty} \phi\left(\frac{t}{n}\right)$ is meromorphic, because then one gets relations between the coefficients of the Laurent series expansion for this merom. fn. around zero, and the values of ζ at integers.

If $$\phi(t) = \sum_{k \geq 2} a_k t^k \quad, \text{ then}$$

$$\sum_{n=1}^{\infty} \phi\left(\frac{t}{n}\right) = \sum_{k \geq 2}' a_k \zeta(k) t^k$$

converges in much the same way as the series for ϕ does, because $\zeta(k) \to 1$ as $k \to \infty$

The simplest $\phi(t)$ to look at might be

$$\frac{t^2}{1+t^2} = t^2 - t^4 + t^5 - t^6 + \cdots$$

$$\int_0^\infty \frac{t^2}{1+t^2}\, t^{-s}\frac{dt}{t} = \int_0^\infty \frac{t^{-2}}{1+t^{-2}}\, t^{s}\frac{dt}{t} = \int_0^\infty \frac{t^s}{1+t^2}\,\frac{dt}{t}$$

$$= \frac{1}{2}\int_0^\infty \frac{t^{s/2}}{1+t}\,\frac{dt}{t}$$

$0 < Re(s) < 2$

$$= \frac{1}{2}\,\frac{\pi}{\sin\left(\pi \frac{s}{2}\right)}$$

$$\sum_{n=1}^\infty \frac{\left(\frac{t}{n}\right)^2}{1+\left(\frac{t}{n}\right)^2} = \sum_{n=1}^\infty \frac{t^2}{t^2+n^2} = \frac{t}{2}\sum_{n=1}^\infty\left[\frac{1}{t+in} + \frac{1}{t-in}\right]$$

But

$$\frac{1}{t} + \sum_{n=1}^\infty\left[\frac{1}{t+in} + \frac{1}{t-in}\right] = \pi\,\frac{\cosh(\pi t)}{\sinh(\pi t)}$$

hence

$$\sum_{n=1}^\infty \frac{(t/n)^2}{1+(t/n)^2} = \frac{t}{2}\left[\pi\,\frac{\cosh(\pi t)}{\sinh(\pi t)} - \frac{1}{t}\right]$$

$$= \frac{\pi t}{2}\left[\frac{2}{e^{2\pi t}-1} + 1 - \frac{1}{\pi t}\right]$$

So we seem to get the formula

$$\zeta(s)\,\frac{1}{\sin\left(\frac{\pi s}{2}\right)} = \int_0^\infty \left[\frac{2}{e^{2\pi t}-1} + 1 - \frac{1}{\pi t}\right] t^{1-s}\,\frac{dt}{t}$$

probably valid in the range $1 < Re(s) < 2$

Now recall

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{1}{e^t-1}\, t^s\,\frac{dt}{t} = \frac{1}{e^{2\pi i s}-1}\int_C \frac{1}{e^t-1}\, t^s\,\frac{dt}{t}$$

so

$$\zeta(s)\frac{1}{\sin\left(\frac{\pi s}{2}\right)} = 2\int_0^\infty \left[\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t} + \frac{1}{2}\right] t^{1-s}\frac{dt}{t}$$

$$= \frac{2}{(e^{2\pi i s}-1)}\int_C \left[\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t} + \frac{1}{2}\right] t^{1-s}\frac{dt}{t} \qquad \text{for } Re(s) > 1$$

$$= \frac{2}{e^{2\pi i s}-1}\int_C \frac{1}{e^{2\pi t}-1}\, t^{1-s}\frac{dt}{t} \qquad \text{conv. for all } s$$

$$= \frac{2}{e^{2\pi i s}-1}(2\pi)^{s-1}\int_C \frac{1}{e^{t}-1}\, t^{1-s}\frac{dt}{t}$$

$$= 2(2\pi)^{s-1}\zeta(1-s)\Gamma(1-s)$$

From this we get the functional equation for ζ using the duplication formula for the Γ-function:

$$\zeta(s)\frac{1}{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right) = 2\pi^{s-1}\frac{\Gamma(1-s)}{2^{1-s}}\zeta(1-s) = 2\pi^{s-1}\left(\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-s+1}{2}\right)\right)\zeta(1-s)$$

$$\zeta(s)\Gamma\left(\frac{s}{2}\right) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \qquad \text{or}$$

$$\pi^{-s/2}\zeta(s)\Gamma(s/2) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

It doesn't seem as if we have gained anything using $\phi(t) = \frac{t^2}{1+t^2}$.

We have now 3 proofs of the functional equation for ζ based on

1) $\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right)$ where $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$

2) $\zeta(s)\Gamma(s) = \frac{1}{e^{2\pi i s}-1}\int_C \frac{1}{e^{t}-1}\, t^{s}\frac{dt}{t}$ + contour integration

3) $\pi \cot(\pi t) = \frac{1}{t} + \sum_{n=1}^{\infty} \left[\frac{1}{t+n} + \frac{1}{t-n} \right]$

I recall that I am hoping to fit $\zeta(s)$ into the "self-adjoint on the line" operator setup, and that therefore I wanted to know if I could write

$$\pi^{-s/2}\Gamma(s/2)\,\zeta(s) = f(s) + f(1-s)$$

in a natural way. But recall one has

$$\pi^{-s/2}\Gamma(s/2)\,\zeta(s) = \int_0^\infty \tfrac{1}{2}\left[\theta(t) - 1 - t^{-1/2}\right] t^{s/2}\,\frac{dt}{t}$$

in the critical strip. Hence if we put

$$f(s) = \int_1^\infty \tfrac{1}{2}\left[\theta(t) - 1 - t^{-1/2}\right] t^{s/2}\,\frac{dt}{t}$$

$$= \int_1^\infty \tfrac{1}{2}\left[\theta(t) - 1\right] t^{s/2}\,\frac{dt}{t} - \left[\frac{1}{2}\,\frac{t^{(s-1)/2}}{(s-1)/2}\right]_1^\infty$$

$$= \underset{\text{decaying as } \mathrm{Re}(s) \to -\infty}{\text{entire fn.}} + \frac{1}{s-1}$$

Also

$$\int_0^1 \tfrac{1}{2}\left[\theta(t) - 1 - t^{-1/2}\right] t^{s/2}\,\frac{dt}{t} = \int_1^\infty \tfrac{1}{2}\left[t^{1/2}\theta(t) - 1 - t^{1/2}\right] t^{-s/2}\,\frac{dt}{t}$$

$$= f(1-s)$$

so that $\qquad \pi^{-s/2}\Gamma(s/2)\,\zeta(s) = f(s) + f(1-s)$

August 11, 1979

Consider $\zeta_C(s) = \sum_L z^{\deg(L)} \frac{q^{h^0(L)}-1}{q-1}$

where $z = q^{-s}$ and the series converges for $Re(s) > 1$. In the annulus $q^{-1} < |z| < 0$ one has the expansion

$$\zeta_C(s) = \sum_L z^{\deg(L)} \frac{q^{h^0(L)} - 1 - q^{\deg(L)+1-g}}{q-1}$$

$$= \left(\sum_{n \in \mathbb{Z}} a_n q^{n/2} z^n \right) z^{g-1}$$

and the functional equation says that

$$a_n = a_{-n}$$

and that $\qquad a_n = \frac{-h}{q-1} q^{-n/2} \qquad n \geq g$

so therefore if one puts

$$f = \frac{1}{2} a_0 z^{g-1} + \sum_{n \geq 1} a_n q^{n/2} z^{n+g-1}$$

one gets $\qquad f(z) = z^{g-1} \left(\text{poly of degree } g-1 + \frac{-h}{q-1} \frac{1}{1-z} \right)$

and

$$\zeta = f(z) + z^{2g-2} q^{g-1} f\left(\frac{1}{qz}\right)$$

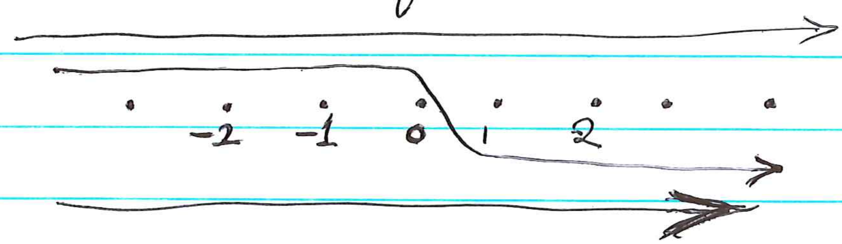
up to some minor rearrangement of the constants.

How Riemann proved the Θ-transformation formula by contour integration.

Consider the contour integral

$$f(p) = \int_L \frac{e^{-tx^2+px}}{e^{2\pi i x}-1}\,dx$$

where L is one of the contours:



Actually any path starting at ∞ in the sector $\frac{3\pi}{4}+\varepsilon < \arg(x) < \frac{5\pi}{4}-\varepsilon$ and ending at ∞ in the sector $\frac{\pi}{4}+\varepsilon < \arg(x) < \frac{\pi}{4}+\varepsilon$ will do. Then

$$(*) \qquad f(p+2\pi i) - f(p) = \int_{-\infty}^{\infty} e^{-tx^2+px}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-t(x-\frac{p}{2t})^2+\frac{p^2}{4t}}\,dx = \frac{\sqrt{\pi}}{\sqrt{t}}\,e^{\frac{p^2}{4t}}$$

~~Next suppose L is one of the horizontal contours, then if Im(p) tends to ∞ while Re(p) fixed, the Riemann-Lebesgue lemma shows that~~

Next fix L to be the contour $i\varepsilon-\infty$ to $i\varepsilon+\infty$. Then $\mathrm{Re}(px) = \mathrm{Re}(p)\,\mathrm{Re}(x) - \mathrm{Im}(p)\,\varepsilon$ so that if $\mathrm{Im}(p)\to+\infty$ and $\mathrm{Re}(p)$ stays fixed, we see $f(p)$ decays fast. Hence we can iterate $(*)$ to get

$$-f(p) = \frac{\sqrt{\pi}}{\sqrt{t}} \sum_{n=0}^{\infty} e^{(p+2\pi i n)^2/4t}$$

A simpler way of getting the same result is to use the geometric series:

$$f(p) = -\int_L e^{-tx^2+px} \sum_{n\geq 0} e^{2\pi i n x}\, dx$$

$$= -\sum_{n\geq 0} \frac{\sqrt{\pi}}{\sqrt{t}}\, e^{\frac{(p+2\pi i n)^2}{4t}}$$

Now however one uses symmetry $x \mapsto -x$

$$f(p) = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{e^{-tx^2+px-\pi i x}}{e^{\pi i x}-e^{-\pi i x}}\, dx = \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{e^{-tx^2-px+\pi i x}}{e^{-\pi i x}-e^{\pi i x}}\, dx$$

$$= -\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{e^{-tx^2-px+2\pi i x}}{e^{2\pi i x}-1}\, dx$$

or

$$-f(2\pi i - p) = \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{e^{-tx^2+px}}{e^{2\pi i x}-1}\, dx$$

so

$$-f(p) - f(2\pi i - p) = \int \frac{e^{-tx^2+px}}{e^{2\pi i x}-1}\, dx$$

$$= \cancel{2\pi i} \sum_{n\in\mathbb{Z}} \frac{e^{-tn^2+pn}}{\cancel{2\pi i}}$$

and we have the formula

$$\sum_{n\in\mathbb{Z}} e^{-n^2t+pn} = \frac{\sqrt{\pi}}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{(p+2\pi i n)^2/4t}$$

August 12, 1977

Consider again

$$\frac{du}{dx} = \begin{pmatrix} i\lambda & \bar{p} \\ p & -i\lambda \end{pmatrix} u$$

on the real line. Assume there is for each λ a unique solution u^+ decaying as $x \to +\infty$, unique up to a scalar multiple. Put

$$m(\lambda) = \frac{u_1^+(x,\lambda)}{u_2^+(x,\lambda)}$$

If the signs are correct, then $\operatorname{Im}(\lambda) > 0 \Rightarrow$ $|m(\lambda)| < 1$. not quite.

$$\cancel{\frac{d}{dx}\left(\frac{u_1}{u_2}\right) = \frac{i\lambda u_1 + \bar{p}u_2}{u_2} - \frac{u_1}{u_2^2}(pu_1 - i\lambda u_2)}$$

$$\begin{aligned}
\frac{d}{dx}(u_1\bar{u}_1 - u_2\bar{u}_2) &= (i\lambda u_1 + \bar{p}u_2)\bar{u}_1 + u_1\overline{(i\lambda u_1 + \bar{p}u_2)} \\
&\quad - (pu_1 - i\lambda u_2)\bar{u}_2 - u_2\overline{(pu_1 - i\lambda u_2)} \\
&= i(\lambda - \bar{\lambda})(u_1\bar{u}_1 + u_2\bar{u}_2) = -2\operatorname{Im}(\lambda)(|u_1|^2 + |u_2|^2) < 0
\end{aligned}$$

hence $|u_1|^2 - |u_2|^2$ decreases as x increases when $\operatorname{Im}(\lambda) > 0$. The solution u^+ is the result of taking the limit as $x_0 \to +\infty$ of a solution with $|u_1| = |u_2|$ at x_0, hence $|u_1^+|^2 > |u_2^+|^2$ and so

$$|m(\lambda)| = \frac{|u_1^+|}{|u_2^+|} > 1$$

for $\operatorname{Im}(\lambda) > 0$.

Check: Recall

$$r\frac{d}{dr}\begin{pmatrix} r^{1/2}K_{s-\frac{1}{2}} \\ -r^{1/2}K_{s+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} s & r \\ r & -s \end{pmatrix}\begin{pmatrix} r^{1/2}K_{s-\frac{1}{2}} \\ -r^{1/2}K_{s+\frac{1}{2}} \end{pmatrix}$$

so that $\quad m(\lambda) = \dfrac{-K_{i\lambda-\frac{1}{2}}(r)}{K_{i\lambda+\frac{1}{2}}(r)}$. This takes the value ∞ when the denominator vanishes, which implies $i\lambda + \frac{1}{2} \in i\mathbb{R}$ so $\lambda \in \frac{i}{2} + \mathbb{R}$ has $\operatorname{Im}(\lambda) > 0$. as it should.

The above D.E can be put in the form

$$\begin{pmatrix} \frac{d}{dx} & -\bar{p} \\ p & -\frac{d}{dx} \end{pmatrix} u = i\lambda u$$

or

$$Lu = \left(A\frac{d}{dx} + B\right)u = \lambda u$$

where $\quad A = \begin{pmatrix} \frac{1}{i} & 0 \\ 0 & -\frac{1}{i} \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -\frac{\bar{p}}{i} \\ \frac{p}{i} & 0 \end{pmatrix}$,

so that $L = L^*$.

The system $\quad \dfrac{d}{dx}u = \begin{pmatrix} -p & \lambda \\ -\lambda & p \end{pmatrix} u$

with p real can be put in the form

$$\lambda u = \begin{pmatrix} 0 & -\frac{d}{dx}+p \\ \frac{d}{dx}+p & 0 \end{pmatrix} u$$

so that in this case:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Now one should note that conjugate A matrices (conjugate by unitary transformations) lead to equivalent systems, however the meromorphic functions $m(\lambda)$ change. The two-sided eigenvalue condition doesn't change, although the meromorphic function $m^+(\lambda) - m^-(\lambda)$ does. Note that only the denominators are affected because:

$$\frac{az+b}{cz+d} - \frac{az'+b}{cz'+d} = \frac{\overbrace{(ad-bc)}^{=1}(z-z')}{(cz+d)(cz'+d)}$$

August 13, 1977

Review of the integral

$$\int_L \frac{e^{-\pi t x^2 + 2\pi i p x}}{e^{2\pi i x} - 1}\, dx$$

where L is a directed straight line avoiding the poles $n \in \mathbb{Z}$ and which goes from one of the sectors in which $e^{-\pi t x^2}$ decays to the other. For example if $\mathrm{Re}(t) > 0$ then we can take L to be the line $\mathrm{Im}(x) = \varepsilon$, $\varepsilon \neq 0$. Then on this line $|e^{2\pi i x}| < 1$, so

$$\int_{+\infty + i\varepsilon}^{-\infty + i\varepsilon} (\qquad) = \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} \frac{e^{-\pi t x^2 + 2\pi i p x}}{1 - e^{2\pi i x}}\, dx = \sum_{n \geq 0} \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} e^{-\pi t x^2 + 2\pi i (p+n) x}\, dx$$

$$= \frac{1}{\sqrt{t}} \sum_{n \geq 0} e^{-\frac{\pi}{t}(p+n)^2}$$

because $\displaystyle\int_{-\infty}^{\infty} e^{-\pi t x^2 + 2\pi i p x}\, dx = \int_{-\infty}^{\infty} e^{-\pi t \left(x - \frac{ip}{t}\right)^2 - \frac{\pi p^2}{t}}\, dx = e^{-\frac{\pi}{t} p^2} \int_{-\infty}^{\infty} e^{-\pi t x^2}\, dx$

$$= \frac{1}{\sqrt{t}} e^{-\frac{\pi}{t} p^2}.$$

Similarly

$$\int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} (\qquad) = \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{e^{-\pi t x^2 + 2\pi i p x}}{e^{2\pi i x}(1 - e^{-2\pi i x})}\, dx$$

$$= \frac{1}{\sqrt{t}} \sum_{n \geq 0} e^{-\frac{\pi}{t}(p-1-n)^2}$$

Thus using residues:

$$\frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{t}(p+n)^2} = \int \frac{e^{-\pi t x^2+2\pi i p x}}{e^{2\pi i x}-1}\,dx = \sum_{n\in\mathbb{Z}} e^{-\pi t n^2+2\pi i p n}$$

Suppose we next consider a line of small positive slope crossing the x-axis between 0 and 1.

$$\int_{0\nearrow 1}(\quad) + \int_{\infty+i\varepsilon}^{-\infty+i\varepsilon}(\quad) = \int = \sum_{n\geq 0} e^{-\pi t n^2-2\pi i p n}$$

$$-2\ -1\ \ 0$$

So

$$\int_{0\nearrow 1}(\quad) = \sum_{n\geq 0} e^{-\pi t n^2-2\pi i p n} - \frac{1}{\sqrt{t}}\sum_{n\geq 0} e^{-\frac{\pi}{t}(p^2+2pn+n^2)}$$

Now Riemann's integral is obtained by letting $t\to -i$ from the convergence region $Re(t)>0$. One gets

$$\sum_{n\geq 0} e^{-\pi i n^2-2\pi i p n} - e^{+i\frac{\pi}{4}}e^{-i\pi p^2}\sum_{n\geq 0} e^{-\pi i n^2 - 2\pi i p n}$$

Now $e^{-\pi i n^2} = (-1)^{n^2} = (-1)^n = e^{\pm\pi i n}$, so

$$\sum_{n\geq 0} e^{-\pi i n^2 - 2\pi i p n} = \sum_{n\geq 0} e^{-2\pi i(p+\frac{1}{2})n}$$

$$= \frac{1}{1-e^{-2\pi i(p+\frac{1}{2})}} = \frac{1}{1+e^{-2\pi i p}}$$

So

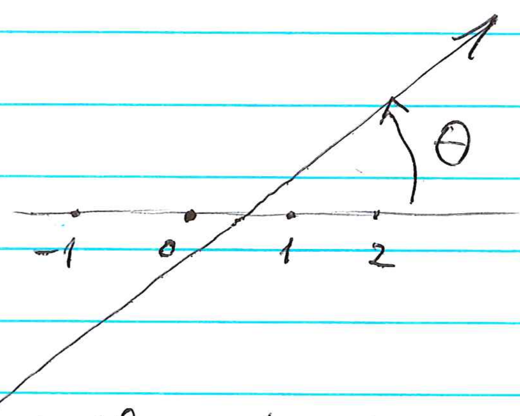
$$\int_{0\nearrow 1} \frac{e^{-\pi t x^2+2\pi i p x}}{e^{2\pi i x}-1}\,dx = \frac{1}{1+e^{-2\pi i p}} + \frac{-e^{+\frac{i\pi}{4}}e^{-i\pi p^2}}{1+e^{-2\pi i p}} = \frac{e^{2\pi i p}-e^{-\frac{i\pi}{4}+i\pi p^2}}{1+e^{2\pi i p}}$$

$$\cancel{= \frac{e^{\pi i p} - e^{-i\frac{\pi}{4} + i\pi p^2 - \pi i p}}{e^{\pi i p} + e^{-\pi i p}}}$$

Let's consider analytic continuation with respect to t. Put

$$f(t) = \int_{L_\Theta} \frac{e^{-\pi t x^2 + 2\pi i p x}}{e^{2\pi i x} - 1}\, dx$$

where L_Θ is a straight line crossing the x-axis between 0 and 1 with angle Θ.



In order that the integral converges we suppose

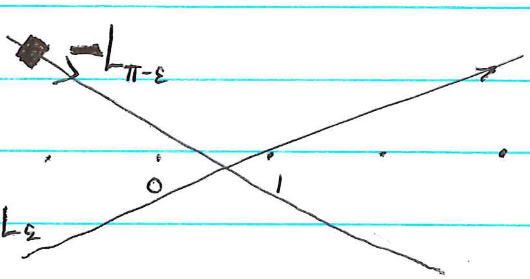
$$-\frac{\pi}{2} < 2\Theta + \arg(t) < \frac{\pi}{2} \qquad \text{and } 0 < \Theta < \pi$$

Now [redacted] start with $t > 0$, and pick [redacted] Θ slightly positive. [redacted] Better: The above integral defines f in the sector

$$-\frac{\pi}{2} - 2\Theta < \arg(t) < \frac{\pi}{2} - 2\Theta.$$

Now increase Θ to get the analytic continuation of f in different half-planes. [redacted] As Θ goes from ε to $\frac{\pi}{2} - \varepsilon$ we get the analytic continuation of f along $\arg(t)=\alpha$ where α goes from 0 to -2π, in which case the integrand is the same as when we started. So we wish to compare

$$\int_{L_\varepsilon} - \int_{L_{\pi-\varepsilon}} = \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} + \int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty}$$



$$= -\frac{1}{\sqrt{t}}\sum_{n\geq 0} e^{-\frac{\pi}{t}(p+n)^2} + \frac{1}{\sqrt{t}}\sum_{n\geq 0} e^{-\frac{\pi}{t}(p-1-n)^2}$$

$$\int_{0\nearrow 1} \frac{e^{-\pi t x^2+2\pi i p x}}{e^{2\pi i x}-1}\,dx = \int_{\hookleftarrow} - \int_{\leftarrow}$$

$$= \sum_{n\geq 0} e^{-\pi t n^2-2\pi i p n} - \frac{1}{\sqrt{t}}\sum_{n\geq 0} e^{-\frac{\pi}{t}(n^2+2pn+p^2)}$$

Now let $t\to -i$ and suppose p real

$$= \sum_{n\geq 0} e^{\pi i n^2-2\pi i p n} - e^{\frac{\pi i}{4}-\pi i p^2}\sum_{n\geq 0} e^{-\pi i n^2-2\pi i p n}$$

$$= \left(1-e^{\frac{\pi i}{4}-\pi i p^2}\right)\sum_{n\geq 0}(-1)^n e^{-2\pi i p n} = \frac{1-e^{\frac{\pi i}{4}-\pi i p^2}}{1+e^{-2\pi i p}}$$

so $$\int_{0\nearrow 1} \frac{e^{\pi i x^2+2\pi i p x}}{e^{\pi i x}-e^{-\pi i x}}\,dx = \frac{1-e^{\frac{\pi i}{4}-\pi i\left(p+\frac{1}{2}\right)^2}}{1-e^{-2\pi i p}} = \frac{1-e^{\pi i(-p^2-p)}}{1-e^{-2\pi i p}}$$

or $$\int_{0\nearrow 1} \frac{e^{+\pi i x^2+2\pi i p x}}{e^{\pi i x}-e^{-\pi i x}}\,dx = \frac{1}{1-e^{-2\pi i p}} - \frac{e^{-\pi i p^2}}{e^{\pi i p}-e^{-\pi i p}}$$

Maybe I should express things so that one sees this is an entire function of p.

$$\frac{1-e^{-2\pi i \frac{p(p+1)}{2}}}{1-e^{-2\pi i p}}$$

This is entire because the zeroes of the denominator are simple at $p\in\mathbb{Z}$ and then $\frac{p(p+1)}{2}\in\mathbb{Z}$. Change p to $-p$

$$\int_{0\nearrow 1} \frac{e^{\pi i x^2-2\pi i p x}}{e^{\pi i x}-e^{-\pi i x}}\,dx = \frac{e^{-2\pi i \frac{p(p-1)}{2}}-1}{e^{2\pi i p}-1}$$

Put $\rho = 2\pi i p$.

$$\int_{0\nearrow 1} \frac{e^{\pi i x^2-\rho x}}{e^{\pi i x}-e^{-\pi i x}}\,dx = \frac{e^{\frac{\rho}{2}\left(1-\frac{\rho}{2\pi i}\right)}-1}{e^{\rho}-1}$$

Now multiply by $\rho^{s-1}d\rho$ and integrate in a good direction from 0 to ∞.

$$\Gamma(s)\int_{0\nearrow 1} \frac{e^{\pi i x^2}\,x^{-s}\,dx}{e^{\pi i x}-e^{-\pi i x}} = \int_0^\infty \frac{e^{\frac{\rho}{2}-\frac{\rho^2}{4\pi i}}}{e^{\rho}-1}\,\rho^s\,\frac{d\rho}{\rho} - \underbrace{\int_0^\infty \frac{\rho^s}{e^{\rho}-1}\,\frac{d\rho}{\rho}}_{\zeta(s)\Gamma(s)}$$

I'd like to understand contour integrals of the form

$$\int \frac{e^{ax^2} \quad x^s}{e^{\pi i x} - e^{-\pi i x}} \frac{dx}{x}$$

and ultimately with ax^2 replaced by [redacted] ax^2+bx. (We should replace x by t) Idea is to use the decomposition

$$\frac{1}{e^{\pi i x} - e^{-\pi i x}} = \frac{e^{\pi i x}}{e^{2\pi i x}-1} = \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{x-n}$$

$$= \frac{1}{2\pi i}\left[\frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n 2x}{x^2-n^2}\right]$$

This reduces us maybe to evaluating integrals like

$$\int \frac{e^{at^2} \, t^s}{t^2 - n^2} \frac{dt}{t}$$

and if we replace t^2 by t and scale properly we should get down to evaluating

$$[redacted] \quad f(y) = \int_0^\infty \frac{e^{-yt}}{t+1} t^s \frac{dt}{t}$$

which ought to be a confluent hypergeometric function of some sort.

$$f'(y) = \int_0^\infty \frac{e^{-yt}}{t+1}(-t-1+1)\, t^s \frac{dt}{t} = +f(y) - \int_0^\infty e^{-yt} t^s \frac{dt}{t}$$

$$= f(y) - \frac{\Gamma(s)}{y^s}$$

So

$$\left(e^{-y} f(y)\right)' = -e^{-y} y^{-s} \Gamma(s)$$

$$e^{-y} f(y) = \Gamma(s) \int_y^\infty e^{-u} u^{-s}\, du$$

because $f(y) \to 0$ as $y \to +\infty$ when $Re(s) < 0$

$$f(y) = \Gamma(s)\, e^{+y} \int_y^\infty e^{-u} u^{-s}\, du$$

As a check let $y \to 0^+$. Then

$$\int_0^\infty \frac{1}{t+1} t^s \frac{dt}{t} = \Gamma(s)\Gamma(1-s)$$

which we have derived before. So we have

$$\int_0^\infty \frac{e^{-yt}}{t+1} t^s \frac{dt}{t} = \Gamma(s)\, e^{y} \int_y^\infty e^{-u} u^{1-s} \frac{du}{u}$$

$$= \Gamma(s)\, e^{y} \int_1^\infty e^{-yu} (yu)^{1-s} \frac{du}{u}$$

$$\cancel{\int_0^\infty \frac{e^{-yt}}{t+1} t^s}$$

August 14, 1978

$$f(y) = \int_0^\infty \frac{e^{-yt}}{t+a}\, t^s \,\frac{dt}{t}$$

$$\left(\frac{d}{dy} - a\right) f(y) = -\int_0^\infty e^{-yt}\, t^s \,\frac{dt}{t} = -\Gamma(s)\, y^{-s}$$

$$\frac{d}{dy}\left(e^{-ay} f(y)\right) = -\Gamma(s)\, e^{-ay}\, y^{-s}$$

$$e^{-ay} f(y) = \Gamma(s) \int_y^\infty e^{-au}\, u^{-s}\, du$$

because $f(y)$ and $e^{-ay} \to 0$ as $y \mapsto +\infty$

$$\therefore \boxed{\int_0^\infty \frac{e^{-yt}}{t+a}\, t^s \,\frac{dt}{t} = \Gamma(s)\, e^{ay} \int_y^\infty e^{-au}\, u^{-s}\, du}$$

Note

$$\frac{1}{e^{\pi t} - e^{-\pi t}} = \frac{e^{\pi t}}{e^{2\pi t} - 1} = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}}' \frac{(-1)^n}{t - in}$$

$$= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}}' \frac{(-1)^n 2t}{t^2 + n^2}$$

off by a 2

$$\int_0^\infty \frac{e^{-yt^2}}{e^{\pi t} - e^{-\pi t}}\, t^s \,\frac{dt}{t} = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}}' (-1)^n \int_0^\infty \frac{e^{-yt^2}}{t^2+n^2}\, t^{s+1}\, \frac{2dt}{t}$$

$$= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} (-1)^n \int_0^\infty \frac{e^{-yt}}{t+n^2}\, t^{\frac{s+1}{2}}\, \frac{dt}{t}$$

$$= \frac{1}{2\pi}\, \Gamma\!\left(\frac{s+1}{2}\right) \sum_{n\in\mathbb{Z}}' (-1)^n e^{n^2 y} \int_y^\infty e^{-n^2 u}\, u^{1-\frac{s+1}{2}}\, \frac{du}{u}$$

Put $y = i\pi$ $\qquad e^{n^2 i\pi} = (-1)^{n^2} = (-1)^n$

$$\boxed{\int_0^\infty \frac{e^{-i\pi t^2}}{e^{\pi t} - e^{-\pi t}} t^s \frac{dt}{t} = \frac{1}{2\pi}\Gamma\left(\frac{s+1}{2}\right)\int_{i\pi}^{\infty} \theta\left(\frac{u}{\pi}\right) u^{\frac{1-s}{2}} \frac{du}{u}}$$

(Actually it would be better to say, let $y \to i\pi$ from the good region $Re(y) > 0$).

$$\int_0^\infty \frac{e^{-i\pi t^2}}{e^{\pi t} - e^{-\pi t}} t^s \frac{dt}{t} = \frac{1}{2}\pi^{-\left(\frac{1+s}{2}\right)}\Gamma\left(\frac{s+1}{2}\right)\int_i^\infty \theta(t)\, t^{\frac{1-s}{2}}\frac{dt}{t}$$

$Re(s) > 1$

$\theta(t) \to 1$ rapidly as $Re(t) \to +\infty$ and

$$\frac{1}{2}\int_i^\infty t^{\frac{1-s}{2}}\frac{dt}{t} = \left[\frac{t^{\frac{1-s}{2}}}{2\left(\frac{1-s}{2}\right)}\right]_i^\infty = \frac{e^{\frac{i\pi}{4}(1-s)}}{s-1}.$$

$$\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{e^{-i\pi t^2}}{e^{\pi t} - e^{-\pi t}} t^s \frac{dt}{t} = \left(1 + e^{i\pi s}\right)\int_0^\infty \frac{e^{-i\pi t^2}}{e^{\pi t} - e^{-\pi t}} t^s \frac{dt}{t}$$

odd fn. $\qquad\qquad$ vanishes if $s-1 \in 2\mathbb{Z}$

$$\frac{\pi^{-\frac{1+s}{2}}\left(1+e^{i\pi s}\right)}{\Gamma\left(\frac{s+1}{2}\right)} = \frac{\pi^{-\frac{1+s}{2}} e^{i\pi\frac{s}{2}} 2\cos\frac{\pi s}{2}}{\Gamma\left(\frac{s+1}{2}\right)} = \frac{\pi^{-\frac{1+s}{2}} e^{\frac{i\pi s}{2}} 2\sin\frac{\pi}{2}(1-s)}{\Gamma\left(\frac{s+1}{2}\right)}.$$

$$= \frac{\pi^{-\frac{1+s}{2}} e^{\frac{i\pi s}{2}}\, 2\, \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{1-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}$$

$$\left(1+e^{i\pi s}\right)\pi^{-\left(\frac{1+s}{2}\right)}\Gamma\left(\frac{1+s}{2}\right) = e^{i\frac{\pi s}{2}}\, 2\cos\frac{\pi}{2}s\ \pi^{-\left(\frac{1+s}{2}\right)}\Gamma\left(\frac{1+s}{2}\right)$$

$$= e^{i\frac{\pi s}{2}}\ \frac{\sin\frac{\pi}{2}(1-s)}{\pi}\ \pi^{\frac{1-s}{2}}\,\Gamma\left(\frac{1+s}{2}\right)$$

$$= e^{i\frac{\pi s}{2}}\ \frac{1}{\Gamma\left(\frac{1-s}{2}\right)\cancel{\Gamma\left(1-\frac{1-s}{2}\right)}}\ \pi^{\frac{1-s}{2}}\,\cancel{\Gamma\left(\frac{1+s}{2}\right)} = e^{i\frac{\pi s}{2}}\Big/\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)$$

Hence

$$\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)e^{-i\frac{\pi s}{2}}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{e^{-i\pi t^2}}{e^{\pi t}-e^{-\pi t}}\,t^s\,\frac{dt}{t} = \int_i^\infty \frac{\theta(t)}{2}\,t^{\frac{1-s}{2}}\,\frac{dt}{t}$$

$$\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\int_{\varepsilon+i\infty}^{\varepsilon-i\infty}\frac{e^{i\pi t^2}}{e^{\pi i t}-e^{-\pi i t}}\,t^s\,\frac{dt}{t} = \int_i^\infty \frac{\theta(t)}{2}\,t^{\frac{1-s}{2}}\,\frac{dt}{t}$$

Or changing $s \mapsto 1-s$. Kuzmin's formula (possibly off by 2)

$$\boxed{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\int_{\varepsilon+i\infty}^{\varepsilon-i\infty}\frac{e^{i\pi t^2}}{e^{\pi i t}-e^{-\pi i t}}\,t^{-s}\,dt = \int_i^\infty \frac{\theta(t)}{2}\,t^{s/2}\,\frac{dt}{t}}$$

The integral on the left is the same as $-\int_{0\nearrow 1}$ as in Edwards book.

Point: Put

$$F_b(s) = \int_b^\infty \frac{\theta(t)}{2}\,t^{s/2}\,\frac{dt}{t} \qquad \text{conv. for } \operatorname{Re}(s)<0$$

$$= \underbrace{\int_b^\infty \frac{(\theta(t)-1)}{2}\,t^{s/2}\,\frac{dt}{t}}_{\text{entire}} - \frac{b^{s/2}}{s} \qquad \text{for all } s.$$

following Kuzmin. This is interesting because I would [redacted] have tried

$$\int_b^\infty \frac{\theta(t)-1-t^{-1/2}}{2}\, t^{s/2}\frac{dt}{t} = \int_b^\infty \frac{\theta(t)-1}{2}\, t^{s/2}\frac{dt}{t} + \frac{b^{(s-1)/2}}{s-1}$$

on the basis of my experience with $\int$ of curves.
[redacted] Notice that

$$\int_b^\infty \frac{\theta(t)-1}{2}\, t^{s/2}\frac{dt}{t}$$

is entire and it decays as $Re(s) \longrightarrow -\infty$, at least if $Re(b) \geq$ [redacted] 1.

I recall my idea was to consider

$$\hat{\xi}(s) = \int_0^\infty \frac{\theta(t)-1-t^{-1/2}}{2}\, t^{s/2}\frac{dt}{t} \qquad 0 < Re(s) < 1$$

and to break it into $f(s) + f(1-s)$ where

$$f(s) = \int_1^\infty \frac{\theta(t)-1-t^{-1/2}}{2}\, t^{s/2}\frac{dt}{t} = \int_0^1 \frac{\theta(t)-1-t^{-1/2}}{2}\, t^{\frac{1-s}{2}}\frac{dt}{t}$$

Moreover

$$f(s) = \int_1^\infty \frac{\theta(t)-1}{2}\, t^{s/2}\frac{dt}{t} + \frac{1}{s-1}$$

where the [redacted] integral is like a Laplace transform and hence it decays ~~at~~ $Re(s) \rightarrow -\infty$.