July 23, 1977

Mellin transform:

$$f(s) = \int_0^\infty g(t)\, t^s \frac{dt}{t} \qquad g(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f(s)\, t^{-s}\, ds$$

is really the Fourier transform in disguise: Put $t = e^u$, $s = i\lambda$. Then

$$f(i\lambda) = \int_{-\infty}^{\infty} g(e^u)\, e^{i\lambda u}\, du$$

so

$$g(e^u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(i\lambda)\, e^{-i\lambda u}\, d\lambda = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f(s)\, t^{-s}\, ds$$

Typically as with the Fourier integral, $f(s)$ is analytic in a vertical strip $a < \mathrm{Re}(s) < b$ and the integral giving $g(t)$ is to be taken in that strip.

Example 1: $\Gamma(s) = \int_0 e^{-t} t^s \frac{dt}{t}$ so

$$(*) \qquad e^{-t} = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \Gamma(s)\, t^{-s}\, ds \qquad \varepsilon > 0$$

Recall
$$\Gamma(s) = \int_0^1 e^{-t} t^{s-1} dt + \int_1^\infty = \sum_{n\geq 0} \frac{(-1)^n}{n!}\frac{1}{s+n} + \text{entire}$$

so $\Gamma(s)$ has a simple pole at $-n$ with residue $\frac{(-1)^n}{n!}$. Contour integration applied to $(*)$ yields the series for e^{-t}.

Example 2:
$$\int_0^\infty (1+t)^{-a} t^s \frac{dt}{t} = \int_0^\infty (1+t)^{-a+s-1}\left(\frac{t}{1+t}\right)^{s-1} dt$$

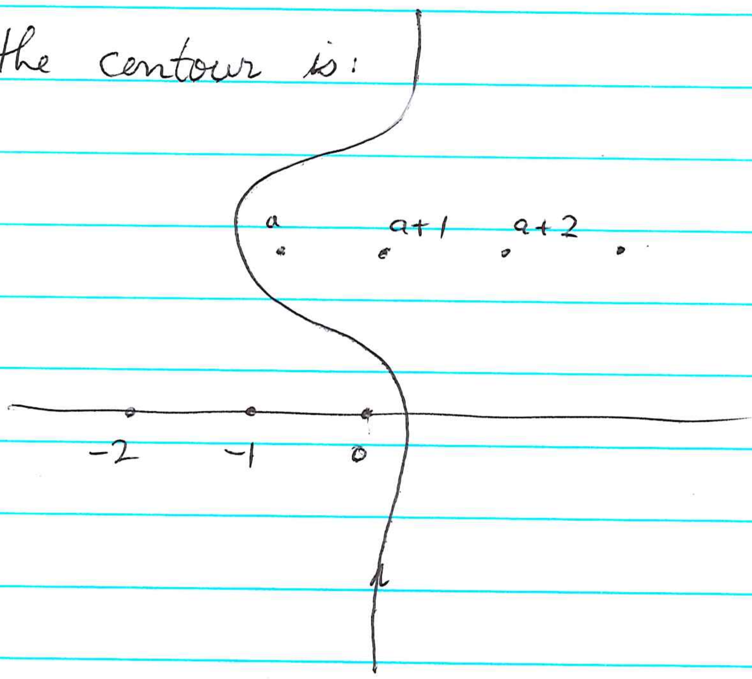
$$= \int_0^\infty \left(\frac{1}{1+t}\right)^{a-s+1}\left(\frac{t}{1+t}\right)^{s-1} dt \qquad u = \frac{t}{1+t} \qquad du = d\left(1 - \frac{1}{1+t}\right) = \frac{dt}{(1+t)^2}$$

$$= \int_0^\infty \left(\frac{1}{1+t}\right)^{a-s-1}\left(\frac{t}{1+t}\right)^{s-1}\frac{dt}{(1+t)^2} = \int_0^1 (1-u)^{a-s-1}(u)^{s-1}\,du = \frac{\Gamma(a-s)\Gamma(s)}{\Gamma(a)}.$$

for $\mathrm{Re}(s)>0$, $\mathrm{Re}(a-s)>0$, Hence in the strip $0<\mathrm{Re}(s)<\mathrm{Re}(a)$. Thus

$$\Gamma(a)(1+t)^{-a} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Gamma(a-s)\Gamma(s)\,t^{-s}\,ds \qquad 0<t<\infty$$

where the contour is:



Residues about $0,-1,-2,\cdots$ give the series

$$\sum_{n\ge 0}\frac{\Gamma(a+n)}{n!}(-1)^n t^n = \Gamma(a)\sum_{n\ge 0}\frac{(-a)(-a-1)\cdots(-a-n+1)}{n!}t^n$$

which converges for $t<1$, whereas residues about the poles $a, a+1, a+2, \cdots$ give the series

$$\sum_{n\ge 0}\Gamma(a+n)\frac{(-1)^n}{n!}t^{-a-n} = \Gamma(a)t^{-a}\sum\frac{(-a)\cdots(-a-n+1)}{n!}t^{-n}$$

convergent for $t>1$. (Note the residue of $\Gamma(a-s)$ at $s=n$ is $-\frac{(-1)^n}{n!}$, but the path circles these poles in the wrong way.)

Example: Take the Kummer equation
$$\left(x\frac{d^2}{dx^2} + c\frac{d}{dx}\right)y = \left(x\frac{d}{dx}+a\right)y$$
leading to the recursion relation
$$a_n = \frac{a+n-1}{(c+n-1)\,n}\,a_{n-1} \qquad \text{better} \quad \frac{a+\mu+n-1}{(c+\mu+n-1)(\mu+n)}$$
which gives the series [redacted] solutions $F(a,c;x)$ and $x^{1-c}F(a+1-c,2-c;x)$. Try to solve the equation with an integral
$$y(x) = \int \phi(s)\,x^s\,ds$$
$$\left(x\frac{d^2}{dx^2}+c\frac{d}{dx}\right)y = \int s(c+s-1)\phi(s)\,x^{s-1}ds$$
$$\left(x\frac{d}{dx}+a\right)y = \int (a+s)\,\phi(s)\,x^s\,ds$$
If we can substitute $s \mapsto s-1$ in the latter integral without changing the contour, then we will get a solution provided
$$\phi(s) = \frac{a+s-1}{(c+s-1)\,s}\,\phi(s-1)$$

To simplify suppose $a=c$ whence this becomes

(*) $$s\,\phi(s) = \phi(s-1)$$

which is satisfied by $e^{i\pi s}\Gamma(-s)$ [redacted] leading to the solution [redacted] $F(c,c;x) = e^x$. [redacted] Any other solution of (*) is a periodic times $e^{i\pi s}\Gamma(-s)$. None of these it seems can produce [redacted] a solution independent of e^x since c doesn't appear. [redacted] However one can grind out a formal series solution running in the negative direction:

[redacted]

$$y = x^{\mu} \sum_{n \geq 0}' a_n x^{-n}$$

$$\sum_{n \geq 0} \Big((\mu-n)(\mu-n-1) + c(\mu-n) \Big) a_n x^{\mu-n-1} = \sum_{n \geq 0}' \big((\mu-n) + a \big) a_n x^{\mu-n}$$

indicial equation: [redacted] $\mu + a = 0$

recursion formula:

$$a_n = \frac{(-a-n+1)(-a-n+1-1+c)}{(-n)} a_{n-1}$$

$$a_n = \frac{(a+n-1)(a-c+1+n-1)}{n} (-1)\, a_{n-1}$$

Thus for $a = c$ one gets the formal series

$$x^{-c} \sum' c(c+1) \cdots (c+n-1) (-1)^n x^{-n}$$

which should be the asymptotic expansion of the solution

$$e^{x} \int_{\infty}^{x} e^{-x} x^{-c}\, dx .$$

Maybe the moral of the above is that q-difference equations are fundamentally simpler than differential equations, in that the singular pts at $0, \infty$ are more accessible to Laurent series calculations.

Return to

$$(c_1 - c_4 x) f(x) + (c_2 - c_5 x) f(qx) + (c_3 - c_6 x) f(q^2 x) = 0$$

[redacted] The Wronskian of two solutions satisfies

$$W(x) = \frac{c_3 - c_6 x}{c_1 - c_4 x} W(qx)$$

Suppose $c_1 = 1$, $c_3 = c_4 = 0$, $c_6 = 1$. Then

$$W(x) = -x\, W(qx)$$

which gives the line bundle over $\mathbb{C}^*/\langle q \rangle$ having the section $\theta(-x)$ vanishing at $x = 1$. Hence the [redacted] line bundle is $\mathcal{O}(1)$. Since the equation has no singularities one gets a rank 2 vector bundle E over the curve with $\Lambda^2 E = \mathcal{O}(1)$.

I have seen that global sections of the [redacted] vector bundle E are the same as Laurent series solutions of the difference equation, i.e. $\sum a_n x^n$ such that

$$a_n = \frac{c_5 q^{n-1} + \text{[redacted]}\, q^{2n-2}}{1 + c_2 q^n} a_{n-1}$$

How many global sections are there? Notice that if the denominator $1 + c_2 q^n \neq 0$, then a_{n-1} determines a_n and if $c_5 + q^{n-1} \neq 0$, then a_n determines a_{n-1}. [redacted] Suppose the denominator never vanishes. Then we start at a spot a_n to the left of where the numerator vanishes, hence can determine $a_{n-1}, a_{n-2}, \ldots$ from a_n; also $a_{n+1}, a_{n+2}, \ldots$ are determined from a_n since the denominator doesn't vanish, hence there is exactly one solution. [redacted] This argument shows that there is one solution where the spot that the denominator vanishes (if it exists) is to the left of where the numerator vanishes. If both numerator and denominator vanish i.e.

$$1+c_2 q^n = 0$$

$$c_5 + q^{m-1} = 0$$

and if the denominator vanishes to the right or at the same spot that the numerator does, i.e.

$$n \geq m \qquad \text{or} \qquad c_2 q^n = -1 = c_5^{-1} q^{m-1}$$

$$\text{or} \qquad \boxed{c_2 c_5 = q^k \qquad k<0}$$

then there are two solutions. As a check suppose

$$c_5 + q^{n-1} = 0$$

$$1 + c_2 q^n = 0$$

so that there is no relation between a_{n-1} and a_n, i.e. they can be arbitrarily prescribed. Then one has $c_5 = -q^{n-1}$, $c_2 = -q^{-n}$ so $c_2 c_5 = q^{-1}$.

What does one know about rank 2 bundles of degree 1 on an elliptic curve? They are either decomposable or the unique non-trivial extension

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow L \longrightarrow 0$$

with L of degree 1 $(H^1(L^\vee) = k)$. Such an indecomposable bundle has a unique section, and this remains true even upon tensoring with a line bundle of degree 0.

Changing f to $\dfrac{\theta(x)}{\theta(\lambda x)} f(x)$ changes the difference eqn. to $(c_1, c_2, \ldots, c_6) \longmapsto (c_1, \lambda c_2, \lambda^2 c_3, c_4, \lambda c_5, \lambda^2 c_6)$. and corresponds to tensoring the vector bundle E with a line bundle of degree 0.

For a suitable choice of line bundle, we could make E have two sections if it were decomposable. If $E = L_1 \oplus L_2$ and L_2 had degree ≥ 2, then $E^{\otimes L_\lambda}$ would have ≥ 2 sections no matter what λ is. Hence if E is decomposable one must have $\deg(L_1) = 0$ and $\deg(L_2) = 1$. This occurs when $c_2 c_5 = g^k$ with k integral and < 0. Otherwise E is the non-trivial extension

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

July 24, 1977

Recall $\quad \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + x\right) u = 2su \qquad u = e^{-x^2/2} v$

$$\left(\frac{d}{dx} - 2x\right) \frac{d}{dx} v = 2sv$$

Solution decaying at $+\infty$ is

$$v_s(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t^2 - 2xt} t^{s-1}\, dt = \frac{\Gamma(1-s)}{2\pi i\, e^{i\pi s}} \int_C \cdots$$

$$\left(\frac{d}{dx} - 2x\right) v_{s+1} = -v_s \qquad v_s' = 2x v_s - v_{s-1}$$

$$\frac{d}{dx} v_s = -2s v_{s+1}$$

(*) $\qquad v_{s-1} = 2x v_s + 2s v_{s-1}$

Let $W_s(x)$ be the Wronskian of the solutions $v_s(x)$, $v_s(-x)$:

$$W_s(x) = \begin{vmatrix} v_s(x) & v_s(-x) \\ v_s'(x) & -v_s'(-x) \end{vmatrix} = \begin{vmatrix} v_s(x) & v_s(-x) \\ -v_{s-1}(x) & v_{s-1}(-x) \end{vmatrix}$$

But this is not far from the Wronskian of two solns of the difference equation (*). Notice that

$$e^{i\pi s} V_s(-x)$$

is another solution of (*) and

$$w(s) = \begin{vmatrix} V_s(x) & e^{i\pi s} V_s(-x) \\ V_{s-1}(x) & e^{i\pi(s-1)} V_{s-1}(-x) \end{vmatrix} = -e^{i\pi s} W_s(x)$$

Now this determinant satisfies

$$w(s) = -2s\, w(s+1)$$

or

$$+e^{i\pi s} W_s(x) = -2s\left(+e^{i\pi(s+1)} W_{s+1}(x)\right)$$

$$W_s(x) = 2s\, W_{s+1}(x)$$

Calculation shows that in fact $W_s(x) = e^{x^2}\sqrt{\pi}\,\dfrac{1}{2^{s-1}\Gamma(s)}$.

The point of the above is the following: If we propose to produce $\zeta(s)$ as a Wronskian in a fashion similar to the above, it might be forced upon us that $\zeta(s)$ satisfies a difference equation of the first order, which is inconsistent with a lot of zeroes in a vertical strip.

For example suppose $k_s(r) = \int e^{-r\left(\frac{t+t^{-1}}{2}\right)} t^s \dfrac{dt}{t}$ for some suitable path of integration. Then we have

$$\frac{s}{r} k_s(r) = \frac{1}{2}\left(k_{s+1}(r) - k_{s-1}(r)\right)$$

$$\frac{d}{dr} k_s = -\frac{1}{2}\left(k_{s+1} + k_{s-1}\right)$$

$$\left(\frac{d}{dr} + \frac{s}{r}\right) k_s = -k_{s-1}.$$

So consider the Wronskian of two k_s functions k_s^1, k_s^2 obtained from different contours.

$$W_s(r) = \begin{vmatrix} k_s^1 & k_s^2 \\ \frac{d}{dr}k_s^1 & \frac{d}{dr}k_s^2 \end{vmatrix} = \begin{vmatrix} k_s^1 & k_s^2 \\ -k_{s-1}^1 & -k_{s-1}^2 \end{vmatrix}$$

since $\quad k_{s-1} = k_{s+1} - \frac{2s}{r} k_s \quad$ one has

$$\begin{pmatrix} k_s \\ k_{s-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2s}{r} \end{pmatrix} \begin{pmatrix} k_{s+1} \\ k_s \end{pmatrix}$$

so one has $\quad W_s(r) = (-1)\, W_{s+1}(r)$

as well as

$$\frac{d}{dr} W_s(r) = -\frac{1}{r} W_s(r)$$

so that $\quad W_s(r) = \frac{f(s)}{r} \quad$ where $\quad f(s+1) = -f(s)$.

Formulas: $\quad \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{e^t-1} t^s \frac{dt}{t} = \frac{\Gamma(1-s)}{2\pi i e^{i\pi s}} \int_C \frac{1}{e^t-1} t^s \frac{dt}{t}$

$$\frac{t}{e^t-1} = \sum_{\nu \geq 0} \frac{t^\nu}{\nu!} B_\nu \qquad \frac{1}{e^t-1} = \frac{1}{t} + \sum_{n\geq 0} \frac{t^n}{n!} \frac{B_{n+1}}{n+1}$$

so

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} = \underset{\substack{\uparrow \\ n=0}}{-\frac{1}{2}},\ \frac{1}{12},\ 0,\ \frac{1}{120},\ 0$$

By the fnl. equation:

$$\zeta(2n) = \frac{(-1)^{n-1}}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n}$$

Return to

$$(c_1 - c_4 x) f(x) + (c_2 - c_5 x) f(qx) + (c_3 - c_6 x) f(q^2 x) = 0$$

with $c_1 = 1$, $c_3 = c_4 = 0$; if $f(x) = \frac{\theta(x)}{\theta(\lambda x)} \sum a_n x^n$, then

$$a_n = \frac{c_5 \lambda q^{n-1} + c_6 \lambda^2 q^{2n-2}}{1 + c_2 \lambda q^n} a_{n-1}$$

I will suppose $c_2, c_5, c_6 \neq 0$; by choosing λ suitably we can suppose $c_2 = -1$, and by scaling in x we can suppose that $c_5 = 1$; denote c_6 by $-a$. Then we have the solution

$$u_a(x) = \sum_{n \geq 0} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})}{(1-q)\cdots\cdots(1-q^n)} q^{n(n-1)/2} x^n$$

which is nice at $x = 0$. [redacted] Notation:

$$\pi(x) = \prod_{j \geq 0} (1 - x q^j)$$

Note that from the asymptotic behavior of the n-th term of the above series we should have

$$u_a(x) \sim \frac{\pi(a)}{\pi(q)} \theta(x) \qquad \text{as} \quad x \longrightarrow \infty$$

Notice that

$$\theta(q^n x) = \frac{1}{q^{n-1} x} \theta(q^{n-1} x) = \cdots = \left(q^{n(n-1)/2} x^n\right)^{-1} \theta(x),$$

hence

$$\frac{\pi(q)}{\pi(a)} u_a(x) = \sum_{n \geq 0} \frac{\pi(a q^n)}{\pi(q q^n)} \frac{\theta(x)}{\theta(q^n x)}$$

or
$$u_a(x) = \frac{\pi(a)}{\pi(q)}\,\theta(x)\left\{\sum_{n\geq 0}\frac{\pi(aq^n)}{\pi(q^{n+1})}\,\frac{1}{\theta(q^n x)}\right\}$$

where the term in brackets should approach 1 as $x\to\infty$; because for x large only the large n terms should count, and $\frac{\pi(aq^n)}{\pi(q^{n+1})}\to 1$ as $n\to\infty$, and

$$\sum_{n\in\mathbb{Z}}\frac{1}{\theta(q^n x)} = 1$$

(Multiply by $\theta(x)$ and compare both sides). Because $\theta(q^n x)$ goes to infinity fast as $|n|\to\infty$, for $x\notin\langle q\rangle$, the series in brackets converges.

Go back + get solution nice at ∞.

$$a_n = \frac{1-\lambda a q^{n-1}}{1-\lambda q^n}\,\lambda q^{n-1}\,a_{n-1}$$

Take $\lambda = a^{-1}$

$$a_{n-1} = \frac{1-a^{-1}q^n}{1-q^{n-1}}\,a q^{-n+1}\,a_n$$

$$a_n = \frac{a-q^{n+1}}{1-q^n}\,q^{-n}\,a_{n+1}$$

$$a_{-n} = \frac{a-q^{-n+1}}{1-q^{-n}}\,q^n\,a_{-n+1} = \frac{q^{n-1}a-1}{q^n-1}\,q^{n+1}\,a_{-n+1}$$

$$a_{-n} = \frac{1-aq^{n-1}}{1-q^n}\,q^{n+1}\,a_{-n+1}$$

$$\frac{\theta(x)}{\theta(a^{-1}x)}\quad\sum_{n\geq 0}\frac{(1-a)\cdots(1-aq^{n-1})}{(1-q)\cdots(1-q^n)}\,q^{n(n-1)/2}\,q^{2n}x^{-n}$$

$$= \frac{\theta(x)}{\theta(a^{-1}x)} \, u_a\left(\frac{q^2}{x}\right)$$

is the other solution. Check: Put $f(x) = \frac{\theta(x)}{\theta(a^{-1}x)} g\left(\frac{q^2}{x}\right)$ in the original diff. eqn.:

$$f(x) + (-1-x) f(qx) + ax f(q^2x) = 0$$

$$g\left(\frac{q^2}{x}\right) + (-1-x)\, a^{-1} g\left(\frac{q}{x}\right) + ax\, a^{-2} g\left(\frac{1}{x}\right) = 0 \qquad y = \frac{1}{x}$$

$$g(q^2y) + \left(-1-\frac{1}{y}\right) a^{-1} g(qy) + \frac{a^{-1}}{y} g(y) = 0$$

$$\text{or} \quad ay\, g(q^2y) + (-y-1)\, g(qy) + g(y) = 0$$

same eqn.

The problem is to compute the Wronskian of these two solutions:

$$W(x) = \begin{vmatrix} u_a(x) & \frac{\theta(x)}{\theta(a^{-1}x)} u_a\left(\frac{q^2}{x}\right) \\ u_a(qx) & a^{-1}\frac{\theta(x)}{\theta(a^{-1}x)} u_a\left(\frac{q}{x}\right) \end{vmatrix}$$

We know that $W(x) = \begin{vmatrix} -ax\, u_a(q^2x) & \cdots \\ u_a(qx) & \cdots \end{vmatrix} = ax\, W(qx)$, so

that $\frac{W(x)}{\theta(ax)}$ is ~~constant~~ q-periodic. It appears that it would be better perhaps to take the second solution to be $\frac{\theta(ax)}{\theta(x)} u_a\left(\frac{q^2}{x}\right)$. But in any case the real point is to compute

(*) $$a^{-1} u_a(x)\, u_a\!\left(\frac{q}{x}\right) - u_a(qx)\, u_a\!\left(\frac{q^2}{x}\right)$$

which hopefully ~~should~~ should be a multiple of $\theta(x)$.

Asymptotic behavior as $x \to +\infty$

$$a^{-1}\frac{\pi(a)}{\pi(q)}\theta(x) - \cancel{\frac{\pi(a)}{\pi(q)}\frac{\theta(x)}{x}}.$$

Asymptotic behavior as $x \to 0$.

$$a^{-1}\frac{\pi(a)}{\pi(q)}\theta\!\left(\frac{q}{x}\right) - \frac{\pi(a)}{\pi(q)}\theta\!\left(\frac{q^2}{x}\right)$$

$$\theta\!\left(\frac{q}{x}\right) = \sum q^{n(n-1)/2} q^n x^{-n} = \sum q^{n(n+1)/2} x^{-n}$$

$$= \sum q^{-n(-n+1)/2} x^n = \theta(x)$$

$$\theta\!\left(\frac{q^2}{x}\right) = \theta\!\left(\frac{x}{q}\right) = \frac{x}{q}\theta(x) < \theta(x) \quad \text{as } x \to 0$$

So the conjecture is that $(*) = a^{-1}\dfrac{\pi(a)}{\pi(q)}\theta(x)$.

It's clear this has to be true on general grounds because (*) is a Laurent series satisfying the same difference equation that $\theta(x)$ does. Thus we have proved:

$$\boxed{u_a(x)\, u_a\!\left(\frac{q}{x}\right) - a\, u_a(qx)\, u_a\!\left(\frac{q^2}{x}\right) = \frac{\pi(a)}{\pi(q)}\theta(x)}$$

As a check, let $a \to 0$ and use that (p. 184)

$$u_0(x) = \sum_{n\geq 0} \frac{q^{n(n-1)/2}}{(1-q)\cdots(1-q^n)} x^n = \prod_{j\geq 0}(1+q^j x)$$

You get the Jacobi identity: $\theta(x) = \left(\prod_{j\geq 1} 1-q^j\right)\left(\prod_{j\geq 0}(1+q^j x)\right)\left(\prod_{j\geq 1}(1+q^j x^{-1})\right)$

Principle involved in the proof is to look at the asymptotic expansions as $x \to \infty$. Thus if we have

$$\theta(x) = c \prod_{j\geq 0}(1+q^j x)\prod_{j\geq 1}(1+q^j x^{-1})$$

we have
$$\pi(q) = c\sum_{n\geq 0}\pi(q^{n+1})\frac{q^{n(n-1)/2}x^n}{\theta(x)}\cdot\prod_{j\geq 1}(1+q^j x^{-1})$$

$$= c\sum_{n\geq 0}\frac{\pi(q^{n+1})}{\theta(q^n x)}\cdot\prod_{j\geq 1}(1+q^j x^{-1})$$

Now let $x \to \infty$ and use that $\displaystyle\sum_{n\in\mathbb{Z}}\frac{1}{\theta(q^n x)} = 1$ and you see (heuristically at least) that $c = \pi(q)$. Actually there should be no problem in making this rigorous since the series converges absolutely and $\pi(q^{n+1}) \to 1$ etc.

July 25, 1977.

$$u_a(x) = \sum_{n\geq 0}\frac{(1-a)\cdots(1-aq^{n-1})}{(1-q)\cdots(1-q^n)}q^{n(n-1)/2}x^n$$

$$(1-aq^n)-(1-a) = a(1-q^n)$$

$$u_{aq}(x) - u_a(x) = \sum_{n\geq 1} a\frac{(1-aq)\cdots(1-aq^{n-1})}{(1-q)\cdots(1-q^{n-1})}q^{n(n-1)/2}x^n$$

$$u_a(x) - u_a(qx) = \sum_{n\geq 1}\frac{(1-a)\cdots(1-aq^{n-1})}{(1-q)\cdots(1-q^{n-1})}q^{n(n-1)/2}x^n$$

$$= \sum_{n\geq 1}\frac{(1-a)\,(1-aq)\cdots(1-aq^{n-1})}{(1-q)\cdots(1-q^{n-1})}q^{(n-1)(n-2)/2}x^{n-1}q^{n-1}x$$

$$= (1-a)x \, u_{aq}(qx)$$

$$u_{aq}(x) - u_a(x) = \sum_{n\geq 1} a \frac{(1-aq)\cdots(1-aq^{n-1})}{(1-q)\cdots(1-q^{n-1})} q^{(n-1)(n-2)/2} x^{n-1} q^{n-1} x$$

$$= ax \, u_{aq}(qx)$$

$$a \left[u_a(x) - u_a(qx)\right] = (1-a)\left[u_{aq}(x) - u_a(x)\right]$$

$$(1-a) u_{aq}(x) = u_a(x) - a u_a(qx)$$

Can write the recursion relations in the form:

$$u_a(x) = u_a(qx) + (1-a)x \, u_{qa}(qx)$$

$$u_{qa}(x) - \cancel{u_a(x)} + \cancel{u_a(x)} - u_a(qx) = x \, u_{aq}(qx)$$

$$u_{qa}(x) = u_a(qx) + x \, u_{aq}(qx)$$

or

$$\begin{pmatrix} u_a(x) \\ u_{qa}(x) \end{pmatrix} = \begin{pmatrix} 1 & (1-a)x \\ 1 & x \end{pmatrix} \begin{pmatrix} u_a(qx) \\ u_{qa}(qx) \end{pmatrix}$$

Look at self-adjoint first order operators

$$L = A\frac{d}{dx} + B$$

where A, B are square matrix functions of x.

$$L^* = -\frac{d}{dx} A^* + B^* = -A^* \frac{d}{dx} + \left(B^* - \frac{dA^*}{dx}\right)$$

so $L = L^*$ when

$$\boxed{A^* = -A \qquad \frac{dA}{dx} = B - B^*}$$

Look at this operator on $0 \le x \le 1$ and determine what are the self-adjoint boundary conditions:

$$(Lu, v) = \int_0^1 v^* Lu \, dx$$

$$v^*Lu - (Lv)^* u = v^*\left(A\frac{du}{dx} + Bu\right) - \left(A\frac{dv}{dx} + Bv\right)^* u$$

$$= v^*A\frac{du}{dx} + v^*Bu - \frac{dv^*}{dx}A^*u - v^*B^*u$$

$$= v^*A\frac{du}{dx} + \frac{dv^*}{dx}Au + v^*\frac{dA}{dx}u = \frac{d}{dx}(v^*Au)$$

Thus

$$(Lu, v) - (u, Lv) = \left[v^*Au\right]_0^1$$

and the boundary conditions have to make this vanish. I will suppose that the boundary conditions are separate at 0 and 1 and make v^*Au vanish at these points. Thus the boundary condition will be a subspace W_0 of $\mathbb{C}^n$ (L is an $n \times n$ matrix) such that

$$u, v \in W \implies v^*Au = 0$$

i.e. W_0 is isotropic for the skew-hermitian form v^*Au. So the good situation seems to be this: A is non-degenerate (so the DE can be solved), n is even and the maximal isotropic subspaces for A are of dimension $\frac{n}{2}$.

But before trying to find boundary conditions to give a self-adjoint operator, suppose u is a solution of $Lu = \lambda Cu$ i.e.

$$A\frac{du}{dx} + Bu = \lambda Cu \qquad \text{with } C = C^*$$

Then

$$\frac{d}{dx}(u^*Au) = \frac{du^*}{dx}Au + u^*(B - B^*)u + u^*A\frac{du}{dx}$$

$$= -\left(A\frac{du}{dx}+Bu\right)^* u + u^*\left(A\frac{du}{dx}+Bu\right)$$
$$= -(\lambda C u)^* u + u^* \lambda C u$$
$$= (\lambda - \bar{\lambda})\, u^* C u.$$

or $\frac{d}{dx}\left(\frac{1}{i} u^* A u\right) = 2\,\mathrm{Im}(\lambda)\, u^* C u$. Thus we see that

for $C > 0$, the real number $\frac{1}{i}(u^* A u)$ decreases if $\mathrm{Im}(\lambda) < 0$. Hence in $\mathbb{P}(\mathbb{C}^n)$ we have something like the unit disk described by $\frac{1}{i} u^* A u \leq 0$.

~~Now go back to the assumption where the maximal isotropic subspaces for A have dimension $\frac{n}{2}$, take one W_0 for A_0 and W_1 for A_1.~~

Start with u_0 such that $u_0^* A u_0 = 0$ and let $u(x,\lambda)$ be the solution of $Lu = \lambda C u$ with $u(0,\lambda) = u_0$. Then $f(\lambda) = u(1,\lambda)^* A\, u(1,\lambda)$ vanishes only for λ real. However f is not analytic in λ.

Instead suppose A has maximal isotropic subspaces of dim. $\frac{n}{2}$, let W_0 be one for A_0 and W_1 one for A_1. Let $v^1, \ldots, v^m$ be a basis for W_0 and $w^1, \ldots, w^m$ a basis for W_1. Let $v^i(x,\lambda)$ be solutions of $Lu = \lambda C u$ starting at v^i. Then consider

$$f(\lambda) = \det \left\{ (w^j)^* A_1\, v^i(1,\lambda) \right\}$$

which is holomorphic in λ. If $f(\lambda) = 0$, then there exists a non-zero $c \in \mathbb{C}^n$ such that $(w^j)^* A_1 \sum_i v^i(1,\lambda)\, c_i = 0$ for all j. But because W_1 is a maximal isotropic subspace, this means $\sum c_i v^i(1,\lambda) \in W_1$, and so we we have for the solution $u = \sum c_i v^i(x,\lambda)$,

$$u^* A u = 0 \quad \text{at} \quad x = 1$$

forcing λ to be real. This argument doesn't suppose that the isotropic subspaces be of dimension $\frac{n}{2}$.

July 26, 1977

$$L = A\frac{d}{dx} + B = \frac{d}{dx}\cdot A + \left(B - \frac{dA}{dx}\right) = L^* = -\frac{d}{dx}A^* + B^*$$

$$L = L^* \iff A = -A^* \quad \text{and} \quad \frac{dA}{dx} = B - B^*.$$

Green's fmla:

$$v^* L u - (Lv)^* u = v^*\left(A\frac{du}{dx} + Bu\right) - \left(A\frac{dv}{dx} + Bv\right)^* u$$

$$= v^* A \frac{du}{dx} + v^*(B - B^*)u - \frac{dv^*}{dx}A^* u$$

$$= v^* A \frac{du}{dx} + v^* \frac{dA}{dx} u + \frac{dv^*}{dx} A u$$

or

$$\boxed{v^* L u - (Lv)^* u = \frac{d}{dx}(v^* A u)}$$

As an application, define the "unit circle" in the projective space of u-values at the point x to be = $\{u \mid u^* A u = 0\}$. Then if $\operatorname{Im}(\lambda) > 0$ and if u is a solution of $Lu = \lambda C u$ with $C = C^* > 0$, we have

$$\frac{d}{dx}\left(\frac{1}{i} u^* A u\right) = \frac{1}{i}\left(u^* \lambda C u - (\lambda C u)^* u\right) = \frac{\lambda - \bar{\lambda}}{i} u^* C u > 0$$

hence if u starts on the unit circle it ends "outside" i.e. $\frac{1}{i} u^* A u > 0$.

Examples: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then

$$u^* A u = (\bar{u}_1, \bar{u}_2)\begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} = \bar{u}_1 u_2 - u_1 \bar{u}_2 = 0$$

means $\frac{u_1}{u_2}$ is real.

If $A = \begin{pmatrix} i & \\ & -i \end{pmatrix}$, then $u^* A u = i|u_1|^2 - i|u_2|^2 = 0$ means that $|u_1| = |u_2|$.

If $A = iI$, then $\frac{1}{i} u^* A u = u^* u$ which doesn't vanish. This shows that we maybe should think in terms of the circles $\frac{1}{i} u^* A u = r$ for different r, except that this is not a function on $\mathbb{P}(\mathbb{C}^n)$?

General theory of a first order DE

$$Lu = A\frac{du}{dx} + Bu = \lambda C u$$

on $0 \leq x \leq 1$. Here A is non-singular throughout the interval. Denote by V_0, V_1 the vector spaces of boundary values at $x = 0, x = 1$; hence $V_0 = V_1 = \mathbb{C}^n$. Then one has

$$\begin{tikzcd} & \mathrm{Ker}(L - \lambda C) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ V_0 & & V_1 \end{tikzcd}$$

giving an isomorphism $S(\lambda) : V_0 \longrightarrow V_1$. A set of boundary conditions for L is a subspace W of $V_0 \times V_1$ of dimension n such that $\exists \lambda$ with graph $S(\lambda)$ transversal
" $\mathrm{Ker}(L - \lambda C)$

to W. One gets as non-identically zero entire function whose zeroes are the eigenvalues by taking the determinant of the map

$$V_0 \xleftarrow{\sim} \mathrm{Ker}(L-\lambda C) \hookrightarrow V_0 \times V_1 \longrightarrow V_0 \times V_1 / W \simeq \mathbb{C}^n$$

For λ not eigenvalues one can construct the Green's operator $f \mapsto G_\lambda f$ = unique solution of $(L-\lambda C)G_\lambda f = f$ satisfying the boundary conditions. One first solves $(L-\lambda C)u = f$ and then adjusts u by a solution of the homogeneous equation so that it satisfies the boundary values. G_λ should be pseudo-diffl. operator of order -1, hence completely continuous on $L^2(0,1)$. The equation $Lu = \lambda u$ + u satisfies bdry conditions can be replaced by the integral equation

$$u = \lambda G u$$

where $G = G_0$, $\lambda = 0$ being assumed not to be an eigenvalue.

Example: $Lu = \frac{1}{i}\frac{d}{dx}u$ on $0 \le x \le 1$ with boundary conditions $u(1) = e^{i\alpha}u(0)$. Eigenfunctions are $e^{i\lambda x}$ where $e^{i\lambda} = e^{i\alpha}$, i.e. $\lambda = \alpha + 2\pi n$, $n \in \mathbb{Z}$. The entire function giving eigenvalues is up to a scalar factor

$$f(\lambda) = e^{i\lambda} - e^{i\alpha}$$

The Green's operator G has the eigenvalues $\frac{1}{\alpha + 2\pi n}$ $n \in \mathbb{Z}$ so it isn't a trace class operator, although the trace does exist in a conditionally convergent sense.

$$\begin{cases} u_{qa}(x) - u_a(x) = ax\, u_{qa}(qx) \\ u_a(x) - u_a(qx) = (1-a)x\, u_{qa}(qx) \end{cases}$$

$$u_{qa}(x) - u_{qa}(qx) = (1-qa)x\, u_{q^2a}(qx)$$

$$u_{qa}(x) - \frac{u_{qa}(x)-u_a(x)}{ax} = (1-qa)x\, \frac{u_{q^2a}(x)-u_{qa}(x)}{qax}$$

$$qaxu_{qa}(x) - q u_{qa}(x) + q u_a(x) = (1-qa)x\{u_{q^2a}(x) - u_{qa}(x)\}$$

$$-q u_{qa}(x) + q u_a(x) = (1-qa)x\, u_{q^2a}(x) - xu_{qa}(x)$$

$$\boxed{(1-qa)x\,u_{q^2a}(x) + (q-x)\,u_{qa}(x) - q u_a(x) = 0}$$

This shows that ~~at~~ a function of a, $u_a(x)$ satisfies a difference equation of the second order of the type studied, except there is a singularity at $a=q$. Notice that this is not the recursion formula for orthogonal polynomials.

Gauss polys: $$\prod_{j=0}^{m-1}(1+q^j x) = \sum_{n=0}^{m} \frac{(1-q^m)\cdots(1-q^{m-n+1})}{(1-q)\cdots(1-q^n)}\, q^{n(n-1)/2} x^n$$

According to Szego: Orthogonal Polys, p. 33 these are orthogonal for the weight function

$$\pi^{-1/2} k\, e^{-k^2(\log x)^2} \qquad q = e^{-\frac{1}{2k^2}}$$

on $0 < x < \infty$

but this doesn't make sense for these polys satisfy the 2 term recursion formula

$$P_{m+1}(x) = (1+q^m x)P_m(x)$$

instead of a 3-term formula. Also the zeroes don't spread out.

July 27, 1977

Eisenstein's continued fraction

$$u_a(x) = u_{qa}(x) - ax\,u_{qa}(qx) \qquad \frac{u_a(x)}{u_{qa}(x)} = 1 - \cfrac{ax}{\cfrac{u_{qa}(x)}{u_{qa}(qx)}}$$

$$u_a(x) = u_a(qx) + (1-a)x\,u_{qa}(qx) \qquad \frac{u_{qa}(x)}{u_{qa}(qx)} = 1 + \cfrac{(1-qa)x}{\cfrac{u_{qa}(qx)}{u_{q^2a}(qx)}}$$

so

$$\frac{u_a(x)}{u_{qa}(x)} = 1 - \frac{ax}{1+}\,\frac{(1-qa)x}{1-}\,\frac{qa\,qx}{1+}\,\frac{(1-q^2a)qx}{1-}\,\frac{q^2a\,q^2x}{1+}\cdots$$

Taking $a=1$ one gets Eisenstein's formula

$$\frac{1}{\sum_{n\geq 0} q^{n(n-1)/2}x^n} = 1 - \frac{x}{1+}\,\frac{(1-q)x}{1-}\,\frac{q^2x}{1+}\,\frac{(1-q^2)qx}{1-}\cdots\cdots$$

which shows for q a root of unity that the function on the left is rational.

Padé table: Start with a formal series

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots$$

with $c_0 \neq 0$. Let μ, ν be integers ≥ 0. We seek a rational function $\frac{P(x)}{Q(x)}$ with $\deg Q \leq \mu$ and $\deg P \leq \nu$ such that

(1) $$f(x) - \frac{P(x)}{Q(x)} = O(x^{\mu+\nu+1})$$

(One is working in the field of formal Laurent series $\mathbb{C}[[x]][x^{-1}]$, and $g \in O(x^{\mu+\nu+1})$ means that $g \in x^{\mu+\nu-1}\mathbb{C}[[x]]$.)

If $\frac{\bar{P}(x)}{\bar{Q}(x)}$ is another such function, then one has

$$P(x)\bar{Q}(x) - \bar{P}(x)Q(x) = O(x^{\mu+\nu+1})$$

and $P(x)\bar{Q}(x) - \bar{P}(x)Q(x)$ is a poly of degree $\leq \mu+\nu$, hence it vanishes and $\frac{P(x)}{Q(x)} = \frac{\bar{P}(x)}{\bar{Q}(x)}$. Thus the rational function is unique if it exists.

To prove existence, let $P(x) = a_0 + \cdots + a_\nu x^\nu$, $Q(x) = b_0 + \cdots + b_\mu x^\mu$. We want $Q(x)f(x) \equiv P(x) \mod x^{\mu+\nu+1}$ i.e.

$$\sum c_i x^i \sum b_i x^i = \sum a_i x^i + O(x^{\mu+\nu+1})$$

or

$$\begin{cases} a_0 = c_0 b_0 \\ \vdots \\ a_\nu = c_\nu b_0 + \cdots + c_0 b_\nu \end{cases}$$

$$\begin{cases} 0 = c_{\nu+1} b_0 + \cdots + c_{\nu-\mu+1} b_\mu \\ \\ 0 = c_{\nu+\mu} b_0 + \cdots + c_\nu b_\mu \end{cases}$$

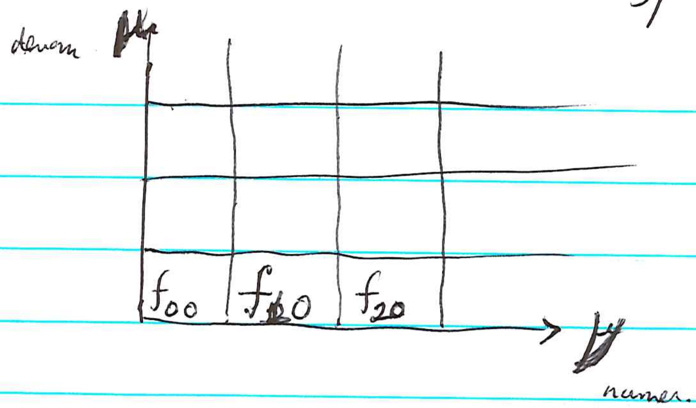
where $c_i = 0$ for $i < 0$. The second group of equations has a non-zero solution as there are ν-equations in $\nu+1$ unknowns. Once the b's are found, the a's can be found from the first set. Thus we can find $P(x), Q(x)$ not zero with $\deg Q \leq \mu$, $\deg P \leq \nu$ such that

(2) $$Q(x)f(x) - P(x) = O(x^{\mu+\nu+1})$$

This condition is to be preferred to (1), since we can always find P, Q satisfying it. Moreover if $(\bar{P}, \bar{Q})$ is another solution then

$$P\bar{Q} - Q\bar{P} \equiv fQ\bar{Q} - Qf\bar{Q} = 0 \mod x^{\mu+\nu+1}$$

so again $\frac{P}{Q} = \frac{\bar{P}}{\bar{Q}}$. Therefore condition (2) leads to a definite rational fraction $f_{\nu,\mu}(x)$ and one can form the Padé table

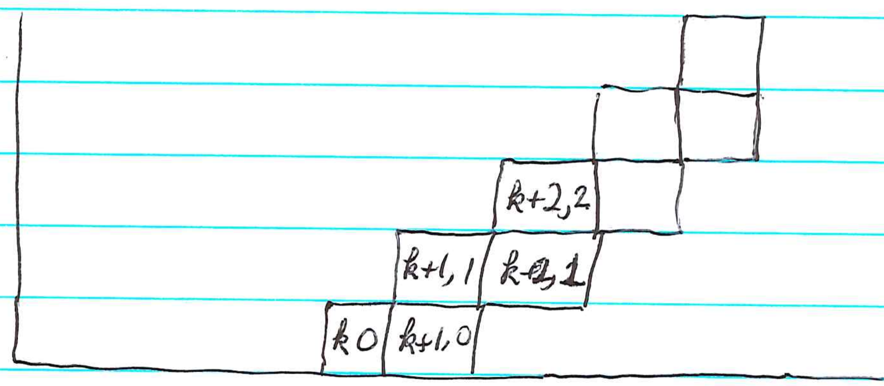


Clearly $f_{\nu,0} = c_0 + c_1 x + \cdots + c_\nu x^\nu$

The relation of the Padé table with continued fractions is roughly as follows. The sequence of approximants for the continued fraction

$$f = c_0 + c_1 x + \cdots + c_k x^k + \frac{c_{k+1} x^{k+1}}{1+} \frac{a_2 x}{1+} \frac{a_3 x}{1+} \cdots$$

is the sequence of entries in the Padé table:



IDEA: Is there any relation between the fact that $\tau = i$ is the symmetry point for the functional equation of $\theta = \sum e^{+\frac{n^2}{2}(2\pi i \tau)}$ and the fact that $\mathbb{Z} + \mathbb{Z}\tau$ is the Gaussian integers. Also do quadratic imaginary fields relate to elliptic curves over finite fields, so that ζ for latter relates to elliptic functions of formers.

July 28, 1977

Identity:
$$\frac{a_1x}{1\ +}\ \frac{a_2x}{1+y} = \frac{a_1x(1+y)}{1+y+a_2x} = \frac{a_1x(1+y+a_2x)-a_1a_2x^2}{1+y+a_2x}$$
$$= a_1x + \frac{-a_1a_2x^2}{1+a_2x+y}$$

This enables one to transform the continued fraction
$$1+\frac{a_1x}{1+}\ \frac{a_2x}{1+}\ \frac{a_3x}{1+}\ \cdots$$

into
$$(*)\qquad 1+a_1x+\frac{-a_1a_2x^2}{1+a_2x+a_3x+}\ \frac{-a_3a_4x^2}{1+(a_4+a_5)x+}\ \frac{-a_5a_6x^2}{1+(a_6+a_7)x+}$$

Hence
$$\frac{u_a(x)}{u_{qa}(x)} = 1+\frac{(-a)x}{1+}\ \frac{(1-qa)x}{1+}\ \frac{(-qa)qx}{1+}\ \frac{(1-q^2a)qx}{1+}\ \frac{(-q^2a)(q^2x)}{1+}$$
$$= 1+(-a)x+\frac{a(1-qa)x^2}{1+(1-qa-q^2a)x+}\ \frac{qa(1-q^2a)(qx)^2}{1+(1-q^2a-q^3a)qx+}$$

Suppose we put $z^{-1}=x$ into $(*)$.
$$1+a_1z^{-1}+\frac{-a_1a_2z^{-2}}{1+(a_2+a_3)z^{-1}+}\ \frac{-a_3a_4z^{-2}}{1+(a_4+a_5)z^{-1}+\cdots}\ \frac{-a_5a_6z^{-2}}{1+(a_6+a_7)z^{-1}+\cdot}$$
$$= z^{-1}\left(z+a_1+\frac{-a_1a_2}{z+(a_2+a_3)+}\ \frac{-a_3a_4}{z+(a_4+a_5)+}\ \frac{-a_5a_6}{z+(a_6+a_7)+\cdots}\right)$$

This is in Jacobi form, which should indicate that the polys $u_a(x)$ for $a=q^{-n}$, $n=0,1,2,\ldots$ might form an orthonormal system after replacing x by x^{-1}.

Review continued fractions again: Start with a prob. measure $d\mu$ on $\mathbb{R}$, and construct the associated orthonormal sequence of polynomials $\phi_n(x)$, $n \geq 0$ by applying Gram-Schmidt to x^n, $n \geq 0$. One gets recursion relations

$$x\phi_n = a_n \phi_{n+1} + b_n \phi_n + a_{n-1}\phi_{n-1}$$

with $a_n > 0$, $n \geq 0$, and $b_n \in \mathbb{R}$. The associated orthogonal system of monic polys is

$$p_n(x) = a_0 \cdots a_{n-1}\phi_n(x)$$

which satisfies the recursion relation

$$xp_n = p_{n+1} + b_n p_n + a_{n-1}^2 p_{n-1}$$

starting with $p_0 = 1$, $p_{-1} = 0$. One has the determinant formula

$$p_{n+1}(x) = \det\left(xI_{n+1} - \begin{pmatrix} b_0 & a_0^2 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1}^2 \\ & & 1 & b_n \end{pmatrix}\right)$$

More generally if we put

$$p_{n+1}^{\nu}(x) = \det\left(xI_{n-\nu+1} - \begin{pmatrix} b_\nu & a_\nu^2 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1}^2 \\ & & 1 & b_n \end{pmatrix}\right)$$

we get a poly of degree $n+1-\nu$ satisfying the relations

$$xp_n^{\nu} = p_{n+1}^{\nu} + b_n p_n^{\nu} + a_{n-1}^2 p_{n-1}^{\nu}$$

except one has the starting values

$$p_\nu^\nu(x) = 1 \qquad p_{\nu-1}^{\nu}(x) = 0.$$

Since

$$\begin{pmatrix} p_n^\nu(x) \\ p_{n+1}^\nu(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_{n-1}^2 & x-b_n \end{pmatrix} \begin{pmatrix} p_{n-1}^\nu(x) \\ p_n^\nu(x) \end{pmatrix}$$

one has on choosing some $a_{-1}^2 > 0$

$$\begin{pmatrix} p_n^1 & p_n^0 \\ p_{n+1}^1 & p_{n+1}^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_{n-1}^2 & x-b_n \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & 1 \\ -a_{-1}^2 & x-b_0 \end{pmatrix} \begin{pmatrix} p_{-1}^1 & p_{-1}^0 \\ p_0^1 & p_0^0 \end{pmatrix}$$

or transposing

$$\begin{pmatrix} p_n^1 & p_{n+1}^1 \\ p_n^0 & p_{n+1}^0 \end{pmatrix} = \underbrace{\begin{pmatrix} p_{-1}^1 & p_0^1 \\ p_{-1}^0 & p_0^0 \end{pmatrix}}_{= \begin{pmatrix} -\frac{1}{a_{-1}^2} & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} 0 & -a_{-1}^2 \\ 1 & x-b_0 \end{pmatrix} \cdots \begin{pmatrix} 0 & -a_{n-1}^2 \\ 1 & x-b_n \end{pmatrix}$$

or

$$\begin{pmatrix} p_n^1 & p_{n+1}^1 \\ p_n^0 & p_{n+1}^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & x-b_0 \end{pmatrix} \begin{pmatrix} 0 & -a_0^2 \\ 1 & x-b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -a_{n-1}^2 \\ 1 & x-b_n \end{pmatrix}$$

So

$$\frac{p_{n+1}^1}{p_{n+1}^0} = \begin{pmatrix} 0 & 1 \\ 1 & x-b_0 \end{pmatrix} \begin{pmatrix} 0 & -a_0^2 \\ 1 & x-b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -a_{n-1}^2 \\ 1 & x-b_n \end{pmatrix} (\infty)$$

$$= \frac{1}{x-b_0+}\,\frac{-a_0^2}{x-b_1+}\,\frac{-a_1^2}{x-b_2+}\cdots\cdots\frac{-a_{n-1}^2}{x-b_n}$$

By Cramer's rule

$$\lim_{n\to\infty} \frac{p_n^1(z)}{p_n^0(z)} = \left((zI-J)^{-1}e_0, e_0\right)$$

$$= \int \frac{d\mu(x)}{z-x} = \frac{1}{z}\int \frac{d\mu}{1-\frac{x}{z}}$$

$$= \sum_{n\geq 0} \frac{1}{z^{n+1}} \int x^n \, d\mu$$

Hence we have the formula

$$\boxed{\sum_{n\geq 0} \frac{1}{x^{n+1}} \int \lambda^n \, d\mu(\lambda) = \frac{1}{x-b_0+}\,\frac{-a_0^2}{x-b_1+}\,\frac{-a_1^2}{x-b_2+}\cdots\cdots}$$

relating moments of the measure $d\mu$ to the coefficients of the recursion formula for the associated orthogonal polynomials.

Consider a symmetric two-sided Jacobi matrix

$$J = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & b_{-1} & a_{-1} & & \\ & a_{-1} & b_0 & a_0 & \\ & & a_0 & b_1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \qquad a_i > 0, \; b_i \in R$$

describing a recursion relation:

(1) $$\lambda y_n = a_n y_{n+1} + b_n y_n + a_{n-1} y_{n-1}$$

Denote by ϕ^ν the solution of these recursion relations with the initial conditions

(2) $$\phi^\nu_\nu(\lambda) = 1 \qquad \Rightarrow \qquad \underbrace{\cancel{\lambda\phi^\nu_{\nu-1}}}_{0} = a_\nu \phi^\nu_\nu + \underbrace{\cancel{b_\nu \phi^\nu_{\nu-1}}}_{0} + a_{\nu-2}\phi^\nu_{\nu-2}$$

$$\phi^\nu_{\nu-1}(\lambda) = 0 \qquad\qquad \phi^\nu_{\nu-2} = -\frac{a_{\nu-1}}{a_{\nu-2}}$$

Thus $\phi^\nu, \phi^{\nu+1}$ form a basis for the solutions of (1), and $\phi^\nu_n(\lambda)$ is a polynomial of degree $n-\nu$ in λ for $n \geq \nu$.

Any solution not vanishing at $n=-1$ is proportional to

$$\phi^1 - c\,\phi^0$$

for a unique number c. This solution vanishes at $n=m$ provided

(3) $$c = \frac{\phi^1_m(\lambda)}{\phi^0_m(\lambda)}$$

Suppose λ such that the numbers (3) converge as $m \to \infty$ to a number $f(\lambda)$. Then $\phi^1(\lambda) - f(\lambda)\phi^0(\lambda)$ is "the" solution vanishing at $n = \infty$ with value $\phi^1_{-1}(\lambda) = -\frac{a_0}{a_{-1}}$.

How this relates to cont. fractions: Put recursion relation in matrix form

$$\begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} \frac{\lambda - b_n}{a_{n-1}} & -\frac{a_n}{a_{n-1}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+1} \end{pmatrix}$$

Then

$$\frac{y_{n-1}}{y_n} = \frac{\lambda - b_n}{a_{n-1}} - \frac{a_n/a_{n-1}}{\frac{y_n}{y_{n+1}}}$$

$$\boxed{\left(\frac{a_{n-1} y_{n-1}}{y_n}\right) = \lambda - b_n - \frac{a_n^2}{\left(\frac{a_n y_n}{y_{n+1}}\right)}}$$

so

$$\frac{y_{-1}}{y_0} = \frac{\lambda - b_0}{a_{-1}} - \frac{a_0/a_{-1}}{(\lambda - b_1)/a_0 -} \; \frac{a_1/a_0}{(\lambda - b_2)/a_1 -} \cdots\cdots \frac{a_n/a_{n-1}}{\frac{y_n}{y_{n+1}}}$$

or

$$\boxed{a_{-1}\frac{y_{-1}}{y_0} = \lambda - b_0 - \frac{a_0^2}{\lambda - b_1 -} \; \frac{a_1^2}{\lambda - b_2 -} \cdots\cdots \frac{a_n^2}{\frac{a_n y_n}{y_{n+1}}}}$$

Now letting $n \to \infty$ and assuming the continued fraction converges we get the initial value ratio for the solution decaying at $n = +\infty$. Since

$$\frac{\phi'_{-1} - f\phi^0_{-1}}{\phi'_0 - f\phi^0_0} = \frac{-\frac{a_0}{a_1}}{0 - f} = \frac{a_0}{a_1 f}$$

one has

$$a_{-1}\frac{a_0}{a_{-1} f} = \lambda - b_0 - \frac{a_0^2}{\lambda - b_1 -} \cdots$$

or

$$f = \frac{a_0}{\lambda - b_0 -} \; \frac{a_0^2}{\lambda - b_1 -} \cdots$$

so this is not the exact $f(\lambda)$

Another version: Start with the difference equation

$$\lambda y_n = a_n y_{n+1} + b_n y_n + a_{n-1} y_{n-1}$$

or

$$\frac{a_{n-1} y_{n-1}}{y_n} = \lambda - b_n - \frac{a_n^2}{\left(\frac{a_n y_n}{y_{n+1}}\right)} .$$

The latter difference equation has the solution

$$\frac{a_{n-1} y_{n-1}}{y_n} = \lambda - b_n - \frac{a_n^2}{\lambda - b_{n+1} -} \; \frac{a_{n+1}^2}{\lambda - b_{n+2} -} \cdots$$

provided the continued fraction converges.

Let J be the matrix

$$\begin{pmatrix} b_0 & a_0 & & \\ a_0 & b_1 & a_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

operating on the space of ℓ^2 sequences (c_n) $n \geq 0$. Let ϕ satisfy

$$(\lambda - J)\phi = e_0$$

Then ϕ satisfies the difference equation

$$a_{n-1}\phi_{n-1} + (\lambda - b_n)\phi_n + a_n \phi_{n+1} = 0$$

for $n \geq 1$. It follows by a passage to the limit from a finite J, that one can alter y by a non-zero constant $\Rightarrow$ $\phi_n = y_n$ for $n \geq 1$, and

$$a_{-1} y_{-1} + \underbrace{(\lambda - b_0)\phi_0 + a_0 \phi_1}_{-1} = 0$$

Thus

$$\frac{a_{-1} y_{-1}}{y_0} = \frac{1}{\phi_0} = \frac{1}{((\lambda - J)^{-1} e_0, e_0)}$$

We can improve the preceding by working first in the interval $[0, N]$. Start with the difference equation

$$\lambda y_n = a_n y_{n+1} + b_n y_n + a_{n-1} y_{n-1}$$

or $$\frac{a_{n-1} y_{n-1}}{y_n} = \lambda - b_n - \frac{a_n^2}{\left(\frac{a_n y_n}{y_{n+1}}\right)}$$

Consider a nonzero solution which vanishes at $N+1$. By iteration one has (for those $\lambda \ni$ no y_i vanishes $0 \le i \le N$)

$$\frac{a_{-1} y_{-1}}{y_0} = \lambda - b_0 - \frac{a_0^2}{\lambda - b_1 -} \cdots \cdots \frac{a_{N-1}^2}{\lambda - b_N}.$$

~~and we can suppose $y_0 = 1$~~. Let $J_N = \begin{pmatrix} b_0 & a_0 & & \\ a_0 & \ddots & \ddots & \\ & \ddots & \ddots & \\ & & & b_N \end{pmatrix}$ and let ϕ satisfy (λ not an eigenvalue)

$$(\lambda - J_N)\phi = e_0$$

Then we have $y_n = c\phi_n$ for $n \geq 0$, for some constant c, which we can suppose to be 1. Since

$$a_{-1} y_{-1} + \underbrace{(\lambda - b_0) y_0 + a_0 y_1}_{\displaystyle (\lambda - b_0)\phi_0 + a_0 \phi_1 = -1} = 0$$

we have $a_{-1} y_{-1} = 1$, hence

$$y_0 = \phi_0 = (\phi, e_0) = \left((\lambda - J_N)^{-1} e_0, e_0\right)$$

$$= \int \frac{d\mu_N(x)}{\lambda - x}$$

Thus we get the formula

$$\boxed{\int \frac{d\mu_N(x)}{\lambda - x} = \frac{1}{\lambda - b_0 -}\ \frac{a_0^2}{\lambda - b_1 -}\ \frac{a_1^2}{\lambda - b_2 -} \cdots \frac{a_{N-1}^2}{\lambda - b_N}}$$

Passing to the limit we get the continued fraction expansion for $m(\lambda) = \int \frac{d\mu(x)}{\lambda - x} = \sum \lambda^{-n-1} \int x^n d\mu(x)$

Final thing to get straight is the relation between the measure $d\mu(x)$ and the Plancherel formula. Let $u(\lambda)$ be the solution of the difference eqn. with $u_{-1}(\lambda) = 0$, $u_0(\lambda) = 1$. The eigenvalues of J_N are the roots of $u_{N+1}(\lambda) = 0$. If these are λ_i one has

$$e_0 = \sum_{i=0}^{N} a_i\, u(\lambda_i)$$

hence
$$1 = (e_0, u(\lambda_i)) = a_i \|u(\lambda_i)\|^2_{[0,N]}$$

so

(*) $$e_0 = \int u(\lambda)\, d\mu_N(\lambda)$$

where
$$d\mu_N(\lambda) = \sum_{i=0}^{N} \frac{1}{\|u(\lambda_i)\|^2_{[0,N]}} \delta(\lambda - \lambda_i)$$

It follows from (*) that for all polys. $f(\lambda)$

$$f(J_N) e_0 = \int f(\lambda) u(\lambda) d\mu_N(\lambda) \qquad \therefore (f(J_N)e_0, e_0) = \int f(\lambda)\, d\mu(\lambda)$$

and
$$\|f(J_N)e_0\|^2 = (|f(J_N)|^2 e_0, e_0) = \int |f(\lambda)|^2 d\mu(\lambda)$$

which is the Plancherel formula. The rest follows by letting $N \to \infty$. Note that the key formula is

$$e_0 = \int u(\lambda)\, d\mu(\lambda)$$

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$$K_s(r) = \int_0^\infty e^{-\frac{r}{2}(t+t^{-1})} t^s \frac{dt}{t} = \int_0^\infty e^{-\frac{r}{2}(t+t^{-1})} t^{-s} \frac{dt}{t}$$

Another contour leads to another solution:

$$y = \int_C e^{-\frac{r}{2}(t+t^{-1})} t^{-s} \frac{dt}{t} = \qquad \Big| \ \frac{r}{2}t = u \qquad t = \frac{2u}{r} \quad t^{-1} = \frac{r}{2u}$$

$$\left(\frac{r}{2}\right)^s \int_C e^{-u-\frac{r^2}{4u}} u^{-s} \frac{du}{u} = \left(\frac{r}{2}\right)^s \int_C e^{-u} \sum_{n\geq 0} \frac{1}{n!}\left(-\frac{r^2}{4}\right)^n u^{-s-n} \frac{du}{u}$$

Now $\displaystyle\int_C e^{-u} u^{+s} \frac{du}{u} = \left(e^{2\pi i s}-1\right)\Gamma(s) = e^{i\pi s}\, 2\pi i \frac{\sin \pi s}{\pi}\Gamma(s)$

$$= 2\pi i e^{i\pi s} \frac{1}{\Gamma(1-s)}$$

So

$$y = \left(\frac{r}{2}\right)^s \sum_{n\geq 0} \frac{1}{n!}\left(-\frac{r^2}{4}\right)^n 2\pi i\, e^{-i\pi(s+n)} \frac{1}{\Gamma(s+n+1)}$$

$$= 2\pi i e^{-i\pi s} \left(\frac{r}{2}\right)^s \sum_{n\geq 0} \frac{1}{n!\,\Gamma(s+n+1)} \left(\frac{r}{2}\right)^{2n}$$

Thus if I put

$$\boxed{I_s(r) = \sum_{n\geq 0} \frac{1}{n!\,\Gamma(s+n+1)} \left(\frac{r}{2}\right)^{s+2n}}$$

then I get a solution of the modified Bessel DE

$$\left[\left(r\frac{d}{dr}\right)^2 - r^2 - s^2\right] u = 0$$

Recursion relation

$$I_{s-1} = \sum_{n\ge 0} \frac{s+n}{n!\,\Gamma(s+n+1)}\left(\frac{r}{2}\right)^{s-1+2n} = \frac{2s}{r} I_s + \sum_{n\ge 0} \frac{\left(\frac{r}{2}\right)^{s+1+2n}}{n!\,\Gamma(s+n+2)}$$

$$\boxed{I_{s-1} = \frac{2s}{r} I_s + I_{s+1}}$$

$$\frac{d}{dr} I_s = \frac{1}{2}\sum_{n\ge 0} \frac{s+2n}{n!\,\Gamma(s+n+1)}\left(\frac{r}{2}\right)^{s-1+2n} = \frac{1}{2}\,\frac{2s}{r} I_s + \frac{1}{2}\, 2 I_{s+1}$$

$$\boxed{\left(\frac{d}{dr} - \frac{s}{r}\right) I_s = I_{s+1}}$$

$$\frac{d}{dr} I_s = \frac{s}{r} I_s + I_{s+1} = \frac{s}{r} I_s + I_{s-1} - \frac{2s}{r} I_s$$

$$\boxed{\left(\frac{d}{dr} + \frac{s}{r}\right) I_s = I_{s-1}}$$

Generating function

$$\sum_{k\in\mathbb{Z}} t^k I_k = \sum_{k\in\mathbb{Z}} \sum_{n\ge 0} \frac{t^k \left(\frac{r}{2}\right)^{k+2n}}{n!\,\Gamma(n+k+1)} \qquad m = k+n$$

$$= \sum_{m\ge 0,\, n\ge 0} \frac{t^{m-n}\left(\frac{r}{2}\right)^{m+n}}{n!\, m!} = e^{\frac{r}{2t}}\, e^{\frac{r}{2}t}$$

$$\boxed{\sum_{k\in\mathbb{Z}} t^k I_k(r) = e^{\frac{r}{2}(t+t^{-1})}}$$

I want a q-analogue of Bessel functions. Put

$$f_s(x) = \sum_{n\geq 0} \frac{\pi(q^{s+1})}{(1-q)\cdots(1-q^n)(1-q^{s+1})\cdots(1-q^{s+n})}\, x^{s+2n}$$

where $\pi(a) = \prod_{j\geq 0}(1-q^j a)$ is the analogue of $\frac{1}{\Gamma(s)}$ for $a = q^s$. In effect

$$\pi(q^{s\,\text{[redacted]}}) = (1-q^s)\,\pi(q^{s+1})$$

or

$$\frac{1}{\pi(q^{s+1})} = (1-q^s)\frac{1}{\pi(q^s)}$$

is the analogue of $\Gamma(s+1) = s\,\Gamma(s)$. Note

$$\boxed{f_s(x) = \sum_{n\geq 0} \frac{\pi(q^{s+n+1})}{(1-q)\cdots\cdots(1-q^n)}\, x^{s+2n}}$$

and if k is an integer ≥ 0, then

$$f_{-k}(x) = \sum_{n\geq k} \frac{\pi(q^{n-k+1})\,\pi(q^{n+1})}{\pi(q)}\, x^{-k+2n}$$

$$= \sum_{n\geq 0} \frac{\pi(q^{n+1})}{\pi(q)}\,\pi(q^{n+k+1})\, x^{k+2n} = f_k(x)$$

$$f_{s-1} = \sum_{n\geq 0} \frac{(1-q^{s+n})\,\pi(q^{s+n+1})}{(1-q)_n}\, x^{s-1+2n}$$

$$1-q^{s+n} = q^s(1-q^n) + 1-q^s$$

$$= \frac{1-q^s}{x} f_s + q^s \sum_{\cancel{n\geq 1}\; n\geq 0} \frac{\pi(q^{s+n+1+1})}{(1-q)_{n-1}}\, x^{s+1+2n}$$

or

$$\boxed{f_{s-1} = \frac{1-q^s}{x} f_s + q^s f_{s+1}}$$

$$q^{-s/2} f_s(q^{1/2}x) = \sum \frac{\Pi(q^{s+n+1})}{(1-q)_n} q^n x^{s+2n}$$

$$= f_s(x) - \sum_{n\geq 1} \frac{\Pi(q^{s+n+1})(1-q^n)}{(1-q)_n} x^{s+2n}$$

$$= f_s(x) - \sum_{n\geq 1} \frac{\Pi(q^{s+1+n-1+1})}{(1-q)_{n-1}} x^{s+2+2(n-1)}$$

$$\boxed{q^{-s/2} f_s(q^{1/2}x) = f_s(x) - x f_{s+1}(x)}$$

$$q^{s/2} f_s(q^{1/2}x) = \sum \frac{\Pi(q^{s+n+1})}{(1-q)_n} q^{s+n} x^{s+2n}$$

$$= f_s(x) - \sum \frac{\Pi(q^{s+n+1})(1-q^{s+n})}{(1-q)_n} x^{s+2n}$$

$$= f_s(x) - \sum_{n\geq 0} \frac{\Pi(q^{s-1+n+1})}{(1-q)_n} x^{s+2n}$$

$$\boxed{q^{s/2} f_s(q^{1/2}x) = f_s(x) - x f_{s-1}(x)}$$

$$x f_s(x) = f_{s+1}(x) - q^{\frac{s+1}{2}} f_{s+1}(q^{1/2}x)$$

$$= \left[\frac{f_s(x) - q^{-s/2} f_s(q^{1/2}x)}{x}\right] - q^{\frac{s+1}{2}} \left[\frac{f_s(q^{1/2}x) - q^{-s/2} f_s(qx)}{q^{1/2}x}\right]$$

$$x^2 f_s(x) = f_s(x) - q^{-s/2} f_s(q^{1/2}x) - q^{s/2} f_s(q^{1/2}x) + f_s(qx) = 0$$

$$\boxed{f_s(qx) - (q^{s/2} + q^{-s/2}) f_s(q^{1/2}x) + (1-x^2) f_s(x) = 0}$$

Actually, it's interesting to write this equation in the form:

$$f_s(qx) - (q^{s/2}+q^{-s/2}) f_s(q^{1/2}x) + f_s(x) = \sum_{n\geq 0} \frac{\Pi(q^{s+n+1})}{(1-q)_n} \left(q^{s+2n} - q^{s/2+\frac{s}{2}+n} - q^{-s/2+s/2+n} + 1\right) x^{s+2n}$$

$$= \sum_{n\geq 1} \frac{\Pi(q^{s+n-1})}{(1-q)_n} (1-q^n)(1-q^{s+n}) x^{s+2n} = \sum_{n\geq 1} \frac{\Pi(q^{s+n})}{(1-q)_{n-1}} x^{s+2n-2+2}$$

$$= x^2 f_s(x) \qquad \text{or}$$

$$\boxed{f_s(qx) - (q^{s/2}+q^{-s/2}) f_s(q^{1/2}x) + f_s(x) = x^2 f_s(x)}$$

Generating function. Put

$$g(t) = \sum_{k\in\mathbb{Z}} f_k(x) t^k$$

From the recursion relation on top p. 248 one get

$$t g(t) = \frac{1}{x}\left(g(t) - g(qt)\right) + \frac{g(qt)}{qt}$$

$$\left(1 - \frac{x}{qt}\right) g(qt) = g(t)(1-xt)$$

This has the solution

$$g(t) = \frac{1}{\pi(xt)\,\pi(xt^{-1})}$$

On the other hand

$$\sum_{k\in\mathbb{Z}} \sum_{n\geq 0} \frac{\pi(q^{k+n+1})}{(1-q)_n} x^{k+2n} t^k$$

have 0 unless $k+n\geq 0$
so put $m = k+n$

$$= \sum_{m\geq 0} \sum_{n\geq 0} \frac{\pi(q^{m+1})}{(1-q)_n} x^{m+n} t^{m-n}$$

$$= \pi(q) \sum_{m,n\geq 0} \frac{x^{m+n} t^{m-n}}{(1-q)_n (1-q)_m} = \pi(q) \sum_{m\geq 0} \frac{(xt)^m}{(1-q)_m} \sum_{n\geq 0} \frac{(xt^{-1})^n}{(1-q)_n}$$

$$= \pi(q) \frac{1}{\pi(xt)\,\pi(xt^{-1})}.$$

$$\boxed{\frac{\pi(q)}{\pi(xt)\,\pi(xt^{-1})} = \sum_{k\in\mathbb{Z}} t^k f_k(x)}$$

As a check put $x=0$. Then $f_k(0) = f_{-k}(0) = 0$ for $k\neq 0$, while $f_0(0) = \pi(q)$.

Curious thing here is that $\pi(x)$ is analogous to $\Gamma(x)^{-1}$ in some respects (location of zeroes + recursion formula) and analogous to e^{-x} in other respects:

$$\lim_{q\nearrow 1} \pi((1-q)x)^{-1} = \lim_{q\to 1} \sum_{n\geq 0} \frac{(1-q)^n x^n}{(1-q)_n} = e^x$$

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Simplify notation a little: Put

$$u_a(x) = \sum_{n\geq 0} \frac{\pi(aq^{n+1})}{(1-q)_n} x^n$$

Then

$$x u_a(x) = \sum_{n\geq 1} \frac{\pi(aq^{n+1})}{(1-q)_n} \frac{(1-aq^n)(1-q^n)}{} x^n$$

$$= u_a(x) - (1+a) u_a(qx) + a u_a(q^2 x)$$

or

$$\boxed{(1-x)\, u_a(x) - (1+a)\, u_a(qx) + a\, u_a(q^2x) = 0}$$

This equation has a singular point at $x=1$ which gives $u_a(x)$ a simple pole at $x = 1, q^{-1}, q^{-2}, \dots$ so put

$$V_a(x) = \pi(x) u_a(x)$$

Multiplying the above diff. eqn. by $\pi(qx)$ one gets

$$(1) \qquad \boxed{V_a(x) - (1+a) V_a(qx) + a(1-qx)\, V_a(q^2x) = 0}$$

This is the case of the diff. eqn. studied earlier with $c_1 = 1$, $c_2 = -(1+a)$, $c_3 = a$, $c_4 = c_5 = 0$, $c_6 = aq$, hence it has solutions $\frac{\theta(x)}{\theta(\lambda x)} \sum a_n x^n$ with

$$a_n = \frac{\lambda^2 a q^{2n-1}\ a_{n-1}}{1-(1+a)\lambda q^n + a\lambda^2 q^{2n}} = \frac{\lambda^2 a\, q^{2n-1}}{(1-\lambda q^n)(1-\lambda a q^n)} a_{n-1}$$

One has $\lim_{n\to+\infty} \frac{a_{n-1}}{a_n} = \infty$

$$\lim_{n\to-\infty}\frac{a_{n-1}}{a_n} = q$$

hence if we take λ to be either 1 or a^{-1} we get series converging for all x, and otherwise a series converging only for $|x| > |q|$. Note the DE for v has a singularity at $x = q$ which propagates, $q, q^2, q^3, \ldots$. A solution with $\lambda = 1$ is

$$\sum_{n\geq 0}\frac{\Pi(aq^{n+1})\,a^n q^{n^2}}{(1-q)_n}x^n$$

As this is the unique power series solution with $a_0 = \Pi(aq)$ one has

$$V_a(x) = \Pi(x)\sum_{n\geq 0}\frac{\Pi(aq^{n+1})\,x^n}{(1-q)_n} = \sum_{n\geq 0}\frac{\Pi(aq^{n+1})\,a^n q^{n^2}x^n}{(1-q)_n}$$

The solution belonging to $\lambda = a^{-1}$ is

$$\frac{\theta(x)}{\theta(a^{-1}x)}V_{a^{-1}}(x).$$

These two solutions are linearly independent provided $a \notin \langle q\rangle$ because they have different asymptotic behavior as $x \to 0$. If $a = q^k$ with $k = 0, 1, 2, \ldots$, then

$$\frac{\theta(x)}{\theta(q^{-k}x)}V_{q^{-k}}(x) = \frac{\theta(x)}{q^{-k}x\,\theta(q^{-k+1}x)}\sum_{n\geq k}\frac{\Pi(q^{-k+n+1})}{(1-q)_n}q^{n^2}(q^{-k}x)^n$$

$$= \frac{q^{k(k+1)/2}}{x^k}\sum_{m\geq 0}\frac{\Pi(q^{m+1})}{(1-q)_{m+k}}\,q^{m^2+2mk+k^2}\,q^{-k(m+k)}\,x^{m+k}$$

$$= q^{k(k+1)/2} \sum_{m\geq 0} \frac{\cancel{\frac{\pi(q^{k+m+1})}{\pi(q)}} q^{m^2} (q^k x)^m}{\cancel{(1-q)_{m+k}} (1-q)_m} = q^{k(k+1)/2} V_{q^k}(x)$$

so these two solutions are dependent.

General solution of (1) is

$$\alpha(x) V_a(x) + \beta(x) \frac{\theta(x)}{\theta(a^{-1}x)} V_{a^{-1}}(x)$$

as long as $a \notin \langle q \rangle$, where α, β are elliptic fns. Can you see the solutions regular for $|x| > q$ with poles at $x = q$ in this form?

Wronskian $W(x) = \begin{vmatrix} V_1(x) & V_2(x) \\ V_1(qx) & V_2(qx) \end{vmatrix}$ of two solutions of (1) satisfies

$$W(qx) = \begin{vmatrix} V_1(qx) & \cdot \\ -\frac{1}{a(1-qx)} V_1(x) & \cdot \end{vmatrix} = \frac{1}{a(1-qx)} W(x)$$

or

$$W(x) = a(1-qx) W(qx)$$

so

$$W(x) = (\text{periodic}) \cdot \frac{\theta(x)}{\theta(a^{-1}x)} \pi(qx)$$

Now

$$W(x) = \begin{vmatrix} V_a(x) & \frac{\theta(x)}{\theta(a^{-1}x)} V_{a^{-1}}(x) \\ V_a(qx) & \frac{\theta(x)}{\theta(a^{-1}x)} a^{-1} V_{a^{-1}}(qx) \end{vmatrix} \approx \frac{\theta(x)}{\theta(a^{-1}x)} \begin{vmatrix} \pi(qa) & \pi(qa^{-1}) \\ \pi(qa) & a^{-1}\pi(qa^{-1}) \end{vmatrix}$$

$$= \frac{\theta(x)}{\theta(a^{-1}x)} \pi(qa)(a^{-1}-1)\pi(qa^{-1}) \qquad \text{as } x \to 0$$

So we conclude that

$$\begin{vmatrix} V_a(x) & \frac{\theta(x)}{\theta(a^{-1}x)} V_{a^{-1}}(x) \\ V_a(qx) & \frac{\theta(qx)}{\theta(a^{-1}qx)} V_{a^{-1}}(qx) \end{vmatrix} = \frac{\theta(x)}{\theta(a^{-1}x)} \pi(qx) \quad \frac{\pi(a)\pi(qa^{-1})}{a}$$

This might look nicer if one used, instead of $\frac{\theta(x)}{\theta(a^{-1}x)}$, $\frac{\theta(ax)}{\theta(x)}$ since then there would be no poles at $x = q^{-1}, q^{-2}, \ldots$

Asymptotic behavior as $x \to \infty$:

$$V_a = \sum \frac{\pi(aq^{n+1})}{(1-q)_n} a^n q^{n^2} x^n$$

since

$$\frac{\pi(aq^{n+1})}{(1-q)_n} q^{n^2}(ax)^n \sim \frac{q^{n^2}(ax)^n}{\pi(q)}$$

one should have

$$V_a(x) \sim \frac{1}{\pi(q)} \sum_{n\in\mathbb{Z}} q^{n^2}(ax)^n = \frac{1}{\pi(q)} \sum q^{n^2-n}(qax)^n$$

$$= \frac{1}{\pi(q)} \theta_{q^2}(qax)$$

similarly

$$\frac{\theta(x)}{\theta(a^{-1}x)} V_{a^{-1}}(x) \sim \frac{\theta(x)}{\theta(a^{-1}x)} \frac{1}{\pi(q)} \theta_{q^2}(qa^{-1}x)$$

These asymptotic limits are solutions of

$$f(x) = aqxf(q^2x)$$

so their ratio:

$$\frac{\theta(x)}{\theta(a^{-1}x)}\frac{\theta_{q^2}(qa^{-1}x)}{\theta_{q^2}(qax)}$$

is q^2-periodic.

What I want to find is a solution

$$\alpha(x)\, V_a(x) + \beta(x)\frac{\theta(x)}{\theta(a^{-1}x)}\, V_{a^{-1}}(x)$$

with $\alpha(x), \beta(x)$ q-periodic, [redacted] which decays as $x \to \infty$. If such a thing exists, then on dividing by $V_a(x)$ and taking the asymptotic limit we find

$$\alpha(x) + \beta(x)\left(\frac{\theta(x)}{\theta(a^{-1}x)}\frac{\theta_{q^2}(qa^{-1}x)}{\theta_{q^2}(qax)}\right) = 0$$

i.e. the q^2-periodic function in parenthesis is q-periodic. I expect this can only happen when we restrict [redacted] x to lie in certain cosets $x_0\langle q\rangle$ depending on a. For example if $a=-1$, then we get $\frac{\theta(x)}{\theta(-x)}$ which is not [redacted] q-periodic, because it changes sign, unless numerator or denominator vanish, i.e. [redacted] $x \in \langle q\rangle$ or $x \in -\langle q\rangle$. Not accurate:

Take $a=-1$. Then any solution is of the form

$$\alpha(x)V_{-1}(x) + \beta(x)\frac{\theta(x)}{\theta(-x)}V_{-1}(x) = \left(\alpha(x)+\beta(x)\frac{\theta(x)}{\theta(-x)}\right)V_{-1}(x)$$

with α, β q-periodic. This solution decays at ∞ iff $V_{-1}(x)$ does. Now I believe that

$$V_{-1}(x) \sim \frac{1}{\pi(q)} \theta_{q^2}(-qx)$$

as $x \to \infty$. The right side vanishes when $x = \dots, q^{-3}, q^{-1}, q^{1}, q^{3}, \dots$ and otherwise increases rapidly in size as $x \to \infty$. So what seems to happen is this. One rigs $\alpha(x) + \beta(x)\frac{\theta(x)}{\theta(-x)}$, which is an arbitrary q^2-periodic function, to be zero on q^{2j} $j \in \mathbb{Z}$ and 1 on q^{2j+1}.

In fact if $a = -1$, the diffence equation becomes

$$f(x) = (1-qx) f(q^2 x)$$

so one has the solution

$$\prod_{j \geq 0} (1 - q^{2j+1} x)$$

which has to be a constant times $V_{-1}(x)$. As $V_{-1}(0) = \pi(-q)$ one has

$$\cancel{\prod_{j\geq 0}(1-q^{2j+1}x) =}$$

$$V_{-1}(x) = \pi(-q) \prod_{j \geq 0} (1 - q^{2j+1} x)$$

But recall that

$$\pi(x) = \prod_{j \geq 0} (1 - q^j x) = \sum_{n \geq 0} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(1-q)\cdots(1-q^n)} x^n$$

So

$$V_{-1}(x) = \sum_{n\ge 0} \frac{\pi(-q^{n+1})(1+q)_n}{(1-q)_n(1+q)_n}\, q^{n^2}(-1)^n x^n$$

$$= \sum_{n\ge 0} \frac{\pi(-q)\,(-1)^n}{(1-q^2)\cdots(1-q^{2n})}\, q^{n^2-n}(qx)^n$$

$$\boxed{V_{-1}(x) = \pi(-q)\prod_{j\ge 0}\left(1-q^{2j+1}x\right)}$$

The above shows that $a=-1$ is the analogue of $s=\frac{1}{2}$ for Bessel functions. It also shows that $V_{-1}(x)$ vanishes at $x=q^{-1}, q^{-3}, q^{-5}, \ldots\ldots$ and hence that the only possible candidate for a solution vanishing at $x=\infty$, even when restricted to the sequence $\{q^n\}$, in fact vanishes for $q^{-1}, q^{-2}, q^{-3}, \ldots$ and all q^{2j}, $j\in\mathbb{Z}$.