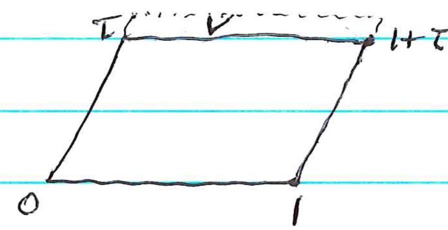
July 19, 1977:

Idea now is to make clear the relation between q-difference equations and vector bundles on the elliptic curve $X = \mathbb{C}^*/\langle q \rangle$.

We can consider X as the quotient of the annulus $\tilde{X} = \{x \mid |q| < |x| < 1+\varepsilon\}$ obtained by identifying qx and x for $x \in V = \{x \mid 1 < |x| < 1+\varepsilon\}$. It is convenient perhaps to draw pictures in the z-plane where $x = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$, so that X is the elliptic curve with the period parallelogram



and everything is periodic in the horizontal direction.

Let A be a invertible holomorphic matrix defined in the annulus V. Then A gives rise to a vector bundle $\mathcal{E}_A$ as follows. First observe that an open set $U \subset X$ can be identified with an open set $\tilde{U} \subset \tilde{X}$ such that for $x \in V$ one has $x \in \tilde{U} \Leftrightarrow qx \in \tilde{U}$. Then

$$\Gamma(U, \mathcal{E}_A) = \{ f : \tilde{U} \to \mathbb{C}^n \text{ holom.} \mid f(qx) = A(x) f(x) \quad \forall x \in V \cap \tilde{U} \}$$

When do two matrices A, A' give rise to isomorphic vector bundles? A homomorphism $\mathcal{E}_A \to \mathcal{E}_{A'}$ is given by a holomorphic matrix Θ defined on $\tilde{X}$ such that when f "is" a section of $\mathcal{E}_A$, then Θf is a section of $\mathcal{E}_{A'}$, i.e.

$$\Theta(qx) A(x) f(x) = \Theta(qx) f(qx) = A'(x) \Theta(x) f(x)$$

i.e.

$$\theta(qx)A(x) = A'(x)\theta(x) \qquad \forall x \in V$$

Suppose $\mathcal{E}$ is a vector bundle on X. Because $\tilde{X}$ is a Stein manifold, I think it follows that $\mathcal{E}$ becomes trivial when lifted to $\tilde{X}$.

$$\tilde{\mathcal{E}} \qquad \mathcal{E}$$

$$\begin{tikzcd} V \arrow[r, shift left, "1"] \arrow[r, shift right, "q"'] & \tilde{X} \arrow[r] & X \end{tikzcd}$$

Hence it's clear that if we choose a trivialization $\tilde{X} \times \mathbb{C}^n \xrightarrow{\sim} \tilde{\mathcal{E}}$, then over V we get an automorphism of the trivial bundle:

$$\begin{tikzcd} & 1^*\tilde{\mathcal{E}} \arrow[dr, phantom, "\shortparallel"] & \\ V \times \mathbb{C}^n \arrow[ur] \arrow[dr] & & h^*\mathcal{E} \\ & q^*\tilde{\mathcal{E}} \arrow[ur, phantom, "\shortparallel"] & \end{tikzcd} \qquad h : V \longrightarrow X \quad \text{canon. map.}$$

so we get an invertible holomorphic matrix A over V such that $\mathcal{E} \cong \mathcal{E}_A$.

The next point is to trivialize over a larger part of $\mathbb{C}^*$ than $\tilde{X}$. The observation is that $\mathbb{C}^*$ is a Stein manifold so given a vector bundle $\mathcal{E}$ on X we can always describe it in terms of an A which is holomorphic and invertible on $\mathbb{C}^*$. To be more precise, this time take $\tilde{X} = \mathbb{C}^*$ and identify $U \subset X$ with $\tilde{U} \subset \tilde{X}$ such that $x \in \tilde{U} \iff qx \in \tilde{U}$. Then $\Gamma(U, \mathcal{E}_A)$ consists of holomorphic $f : \tilde{U} \to \mathbb{C}^n$ such that $f(qx) = A(x)f(x)$.

Question: Can I assume $A(x)$ is a Laurent polynomial matrix, i.e. $A \in GL_n(\mathbb{C}[x, x^{-1}])$? Do for line bundles first.

<u>Question:</u> Call $A, A' \in GL_n(\mathbb{C}[x,x^{-1}])$ equivalent if there exists $\theta \in GL_n(\mathbb{C}[x,x^{-1}])$ such that

$$A'(x) = \theta(qx)A(x)\theta(x)^{-1}.$$

Can the equivalence classes be identified with iso. classes of n-dimensional vector bundles on $\mathbb{C}^*/\langle q\rangle$?

<u>$n=1$.</u> $A(x) = cx^n$, $c \in \mathbb{C}^*$, $n \in \mathbb{Z}$. If $\theta(x) = bx^m$, then

$$\theta(qx)A(x)\theta(x)^{-1} = \cancel{b}q^m \cancel{x^m} c x^n \cancel{b^{-1}} \cancel{x^{-m}} = (q^m c)x^n$$

hence the equivalence classes are

$$\mathbb{C}^*/\langle q\rangle \times \mathbb{Z}$$

so the answer is OKAY for $n=1$.

Let's consider the case where $A(x)$ is a rational function of x, which is invertible near $|x|=1$. The interesting case is $A(x) = 1+\lambda x$ where $|\lambda| \neq 1$. Now if $|\lambda|<1$ I can let $\lambda \to 0$ without the singularities of $A(x)$ crossing $|x|=1$, hence I see that the degree of the line bundle is 0.

Let $\theta(x) = x$; then

$$\theta(qx)(1+\lambda x)\theta(x)^{-1} = q(1+\lambda x)$$

doesn't change the singularities.

If $|\lambda|<1$, then $\theta(x) = 1+\lambda x$ is non-singular in the annulus $q \leq |x| \leq 1$, so $A(x) = 1+\lambda x$ defines the same line bundle as

$$\theta(qx)(1+\lambda x)\theta(x)^{-1} = (1+\lambda qx)(1+\lambda x)(1+\lambda x)^{-1} = (1+\lambda qx)$$

But even better we can solve

$$\frac{\theta(qx)}{\theta(x)} = \frac{1}{1+\lambda x}$$

by the series $\theta(x) = \prod (1+\lambda q^j x)$ which is non-singular for $|x| \leq 1$, hence $1+\lambda x$ for $|\lambda|<1$ defines the trivial bundle on X.

If $|\lambda| > 1$, then $\theta(x) = (1+\lambda q^{-1}x)^{-1}$ is non-singular for $|x| \geq q$, so $(1+\lambda x)$ is equivalent to

$$(1+\lambda x)^{-1}(1+\lambda x)(1+\lambda q^{-1}x) = (1+\lambda q^{-1}x).$$

Replace $1+\lambda x$ by $\lambda x(1+\lambda^{-1}x^{-1})$, and note that

$$\frac{\theta(qx)}{\theta(x)} = \frac{1}{1+\lambda^{-1}x^{-1}} \quad \text{or} \quad \theta(x) = \frac{\theta(q^{-1}x)}{1+\lambda^{-1}qx^{-1}}$$

has the solution

$$\theta(x) = \prod_{j\geq 1} \frac{1}{1+\lambda^{-1}q^j x^{-1}}$$

which is non-vanishing for $|x| \geq q$. Thus $1+\lambda x$ determines the same line bundle as λx for $|\lambda| > 1$.

So if we factor

$$A(x) = c\,\frac{\prod(x-\lambda_i)}{\prod(x-\lambda_i')}\,\frac{\prod(1+\mu_i x)}{\prod(1+\mu_i' x)} \qquad \begin{array}{l} |\lambda_i|, |\lambda_i'| < 1 \\ |\mu_i|, |\mu_i'| < 1 \end{array}$$

according to the singularities inside and outside the unit circle, then $A(x)$ is equivalent to cx^n where n is the net number of zeroes inside $|x| = 1$.

Suppose that A is a meromorphic invertible matrix on $\mathbb{C}^*$ with finitely many singular points and let Y be $\mathbb{C}^*$ with the points $q^n x$, $n \in \mathbb{Z}$, x a singular pt of A, removed. Let's suppose Y disjoint from $|x| = 1$, and let U be open in X, $\tilde{U}$ = its inverse image in $\tilde{X} = \mathbb{C}^*$. We have defined $\Gamma(U, \mathcal{E}_A)$ to be the set of $f : \tilde{U} \cap \{|q| \le |x| \le 1\} \to \mathbb{C}^n$ holomorphic such that $f(qx) = A(x) f(x)$ for $|x| = 1$. Using this formula we can define $f(qx)$ for $|q| \le |x| \le 1$, $x \in \tilde{U}$ and x not a singular point of A. Similarly we can define

$$f(q^2 x) = A(qx) A(x) f(x)$$

provided qx, x are not singular. Thus I can define $f(x)$ for all $x \in Y \cap \tilde{U}$. So it seems that a section of $\mathcal{E}_A$ over U can be viewed as a holomorphic map $f : Y \to \mathbb{C}^n$ such that $f(qx) = A(x) f(x)$ and such that f extends holomorphically to the annulus $|q| \le |x| \le 1$ intersected with $\tilde{U}$. In particular a global section is a meromorphic solution of $f(qx) = A(x) f(x)$ on all of $\mathbb{C}^*$ which is holomorphic in the annulus $|q| \le |x| \le 1$.

The first case to understand well is when $A(x)$ has no singular points, say for example, $A \in GL_n(\mathbb{C}[x, x^{-1}])$. Then $\Gamma(\mathcal{U}, \mathcal{E}_A)$ consists of holom solutions $f: \tilde{\mathcal{U}} \to \mathbb{C}^n$ of the difference equation. In particular, global sections of $\mathcal{E}_A$ are the same thing as Laurent series solutions of the difference equation which converge for all $x \neq 0$. For example if $A(x) = cx^m$, then a solution has to be a series

$$f(x) = \sum_{n \in \mathbb{Z}} a_n x^n$$

such that

$$f(qx) = \sum a_n q^n x^n = cx^m \sum a_n x^n$$

or

$$a_n q^n = c a_{n-m}$$

For example if $m = 0$ we get no sections unless $q^n = c$ for some $n \in \mathbb{Z}$.

If $m = -1$, then we get the recursion fmla

$$a_n = c^{-1} q^{n-1} a_{n-1}$$

which forces $a_n = c^{-n} q^{\frac{n(n-1)}{2}} a_0$, hence

$$f(x) = \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} c^{-n} x^n$$

is the only global section up to a scalar multiple.

Next if $m \geq 1$, then

$$a_n = cq^{-n} a_{n-m}$$

$$a_{dm} = cq^{-dm} a_{(d-1)m} = c^d q^{-\frac{md(d+1)}{2}} a_0$$

which leads to a divergent series. Hence there are no global solutions.

If $m \leq -1$, then $a_0, a_1, \ldots, a_{m-1}$ can be prescribed arbitrarily so there are m independent global sections. This agrees with what RR tells us – $h^0(L) = \deg L$ for $\deg(L) \geq 1$.

Example: Suppose $A(x) = \begin{pmatrix} 1 & g(x) \\ & 1 \end{pmatrix}$ $g(x) \in \mathbb{C}[x, x^{-1}]$. This has to define an extension of the trivial bundle by itself. Global sections are defined by the equation

$$f(x) = A(x) f(qx)$$

or

$$\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} 1 & g(x) \\ & 1 \end{pmatrix} \begin{pmatrix} f_1(qx) \\ f_2(qx) \end{pmatrix}$$

where f_1, f_2 are holomorphic outside $x = 0$. Clearly $f_2(x) = \text{const.}$ so one solution is $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Another solution would be $\begin{pmatrix} f_1 \\ 1 \end{pmatrix}$ where f_1 satisfies

$$f_1(x) = f_1(qx) + g(x)$$

or

$$\sum a_n (1 - q^n) x^n = g(x).$$

This can be solved iff g has zero constant term. Note that if $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is any solution then f_2 has to be constant, and if $f_2 = 0$, then $f_1(x) = f_1(qx)$ so also $f_1(x) = 0$. Hence one has two independent solutions iff g has zero constant term. From the structure of vector bundles on elliptic curves we see that the bundle $\mathcal{E}_A$ is the unique non-trivial extension of $\mathcal{O}$ by $\mathcal{O}$ when g has non-zero constant terms, and otherwise $\mathcal{E}_A = \mathcal{O}^2$.

July 20, 1977.

General facts about q-difference equations:

$$f(x) = A(x) f(qx).$$

What I am interested in are meromorphic functions on [redacted] $\mathbb{C}^*$. These form a field K with an automorphism $\sigma: f \mapsto f(qx)$ and the fixed field K^σ is the field of meromorphic functions on the elliptic curve $X = \mathbb{C}^*/\langle q \rangle$. The matrix $A(x)$ is an $(n\times n)$-invertible matrix over K and the solutions we seek are elements $f \in K^n$.

Think of this in terms of descent. Let $V = K^n$ be equipped with the [redacted] semi-linear endom.

$$(\theta f)(x) = A(x) f(qx)$$

$$(\theta\, gf)(x) = A(x)\, g(qx) f(qx) = (\sigma g)(x)(\theta f)(x)$$

for $g \in K, f \in V$. Then V^θ is the space of solutions. It is a vector space over K^σ obviously. General arguments should show that the canonical map

$$K \otimes_{K^\sigma} V^\sigma \longrightarrow V$$

is injective. In effect let $v_1, .., v_m$ be linearly independent and let's suppose we have a relation

$$\sum a_j v_j = 0$$

in V with $a_j \in K$. Take a primordial relation apply σ and you see all $a_j \in K^\sigma$. (Thus we look in the space K^m at the subspace of relations which is σ-invariant and hence is generated by its σ-invariants)

There seems to be no reason why V^σ should be n-dimensional in general even for $n=1$. (In the case of Lang's theorem the one-dimensional equation was $f = af^\sigma$ or $f^{q-1} = a^{-1}$ which is solvable when K is algebraically closed).

Consider the case $n=1$ where $A(x)$ is rational. Then by factoring f into linear factors and looking at this case one can see that solutions exist.

How does one get a vector bundle out of a q-difference equation? There is no canonical procedure unless $A(x)$ has no singularities. Thus consider $A(x) = 1+x$.

$$f(x) = (1+x)f(qx) = (1+x)(1+qx)f(q^2x)$$

Thus we get the solution

$$f(x) = \prod_{j\geq 0}(1+q^jx)$$

which is an entire function of x vanishing at the points $x = -1, -q^{-1}, -q^{-2}, \ldots\ldots$.

To get a line bundle from $A(x)$ one chooses a circle $|x| = r$ on which $A(x)$ is non-singular. The global sections of the corresponding line bundle are holomorphic functions in the annulus $|q|r \leq |x| \leq r$ such that $f(x) = A(x) f(qx)$. Thus for $A(x) = 1 + x$ we get the trivial line bundle for $r < 1$ because the function $\prod_{j \geq 0} (1 + q^j x)$ gives us an everywhere non-vanishing function on this annulus. But for $r > 1$ we get a line bundle of degree 1.

The other solutions of the difference equation $f(x) = (1 + x) f(qx)$ will be multiples

$$f(x) = h(x) \prod_{j \geq 0} (1 + q^j x)$$

where $h(x)$ is an elliptic function. You see there is no way to choose $h(x)$ so that the solution $f(x)$ will have a power series expansion at $x = \infty$, because one has at least 2 poles in a period parallelogram.

Example: $A = \dfrac{(1+x)}{(1+\lambda x)}$. Then we have the solution

$$f(x) = \prod_{j \geq 0} \frac{(1 + q^j x)}{(1 + \lambda q^j x)}$$

which has simple zeros at $-q^{-j}$ and simple poles at $-\lambda^{-1} q^{-j}$, $j \geq 0$. Assume λ not of the form q^m. ~~There is an elliptic function having simple zeroes at $-q^{-j}$ simple poles at $-\lambda^{-1}q^{-j}$ for all $j \in \mathbb{Z}$, hence we get a solution $h(x) f(x)$ which might have a power series expansion at $x = \infty$. In fact~~

It requires 2 zeroes and 2 poles at least for an elliptic function and the positions of the zeroes and poles have to add up to zero on the elliptic curve.

For example put

$$\theta(x) = \sum q^{n(n-1)/2} x^n$$

so that $\theta(x) = \sum q^{(n+1)n/2} x^{n+1} = x \sum q^{n(n-1)/2} (qx)^n = x\,\theta(qx)$.

[redacted] I know [redacted] already that $\theta(x)$ has simple zeroes at $x =$ [redacted] $-q^j$, $j \in \mathbb{Z}$.

$$\frac{\theta(\lambda_1 x)}{\theta(\lambda_2 x)} = \frac{\lambda_1}{\lambda_2} \frac{\theta(\lambda_1 (qx))}{\theta(\lambda_2 (qx))}$$

Hence if $\frac{\lambda_1}{\lambda_2} = \frac{\lambda_3}{\lambda_4}$, then $\frac{\theta(\lambda_1 x)\,\theta(\lambda_3 x)}{\theta(\lambda_2 x)\,\theta(\lambda_4 x)}$ is an elliptic function with poles at $-\lambda_2^{-1} q^j, -\lambda_4^{-1} q^j$ and zeroes at $-\lambda_1^{-1} q^j, -\lambda^{-1} q^j$.

If $A(x)$ is a rational function with the value 1 at 0, (or more generally [redacted] a meromorphic function), then a solution of

$$f(x) = A(x) f(qx)$$

is given by $A(x)A(qx)A(q^2x)\cdots\cdots$

In effect, we have $A(x) = 1 + a_1 x + a_2 x^2 + \cdots\cdots$ which converges near 0 so for n large enough $q^n x$ will be in the convergence circle. Similarly if $A(x)$ has the value 1 at $x = \infty$ we get [redacted] the solution:

$$f(x) = A^{-1}(q^{-1}x)\,A^{-1}(q^{-2}x)\cdots\cdots\cdots$$

Consider $f(x) = A(x) f(qx)$ when $A(x) = 1+\lambda x$. Then there is a unique solution

$$f(x) = \prod_{j\geq 0} (1+\lambda q^j x)$$

holomorphic at $x=0$ with $f(0)=1$. The same will be true in general if $A(x) = 1+a_1 x + \cdots$ and in n-dimensions, except $f(0)$ has to be prescribed. The singularities of $f(x)$ begin at the singularities of $A(x)$ and include points $q^{-j}x_i$ where $j \geq 0$ and x_i is a singularity of $A(x)$. If also $A(x) = 1 + a_1 x^{-1} + \cdots\cdots$, then we get a solution holomorphic at ∞. The ratio of these solutions is an elliptic function.

So we should think of $f(x) = A(x) f(qx)$ with A holomorphic near 0 and $A(0)=1$ and being the analogue of a regular point at $x=0$. We then have a nice series solution

$$f(x) = A(x) A(qx) \cdots\cdots v$$

with initial value v. Since A is meromorphic, the formula $f(x) = A(x) f(qx)$ enables one to analytically continue f to a meromorphic function in the whole plane.

Now we should view $f(x) = \lambda f(qx)$ as the simplest example of a regular singular point. When $\lambda \neq q^m$ for some m it has no solutions regular near 0. ~~The general solution is $f(x)x^{\mu}$ where ... and ... is an elliptic function.~~ However, it has the solutions

$$f(x) = \frac{\theta(ax)}{\theta(bx)} \qquad \text{where} \quad \frac{a}{b} = \lambda .$$

Now if one were to restrict x to be on the axis $0 < x < \infty$ and take q real, then it seems more natural to look at the solutions x^μ where μ ranges over solutions of $1 = \lambda q^\mu$. These are somehow the solutions analogous to the ones in the theory of D.E.'s.

Finally the equations $f(x) = \lambda x^m f(qx)$ $\quad m \neq 0$ have nice solutions in terms of the θ-function which makes me think of this equation as having an irregular singular point.

Next let us consider 2nd order difference equations

$$c_1 f(x) + c_2 f(qx) + c_3 f(q^2 x) = x\left(c_4 f(x) + c_5 f(qx) + c_6 f(q^2 x)\right)$$

If $f(x) = \sum a_n x^n$ is a Laurent series solution, then

$$\sum (c_1 + c_2 q^n + c_3 q^{2n}) a_n x^n = \sum (c_4 + c_5 q^n + c_6 q^{2n}) a_n x^{n+1}$$

so we get the recursion relations

$$(c_1 + c_2 q^n + c_3 q^{2n}) a_n = (c_4 + c_5 q^{n-1} + c_6 q^{2n-2}) a_{n-1}$$

Let's concentrate on solutions beginning with $x^\mu + a_{\mu+1} x^{\mu+1} + \cdots$. Then $a_\mu = 1$, $a_{\mu-1} = 0$, so we get the indicial equation

$$c_1 + c_2 q^\mu + c_3 q^{2\mu} = 0$$

Suppose $c_3 \neq 0$ and let the two roots of $c_1 + c_2 X + c_3 X^2 = 0$

be r_1, r_2. In the good cases r_1, r_2 will both be $\neq 0$ and $r_1/r_2 \notin \langle q \rangle$ so we get 2 series solutions.

Actually it would be better if we wrote

$$f(x) = x^{\mu} \sum_{n \geq 0}' a_n x^n \qquad a_0 = 1$$

and

$$a_n = \frac{c_4 + c_5 q^{n+\mu-1} + c_6 q^{2n+2\mu-2}}{c_1 + c_2 q^{n+\mu} + c_3 q^{2n+2\mu}} a_{n-1}$$

The point is that the coefficients a_n depend only on q^{μ} which is a root of the indicial equation

$$c_1 + c_2 q^{\mu} + c_3 q^{2\mu} = 0$$

Different choices for μ differ by $\frac{2\pi i}{\log q} n \quad n \in \mathbb{Z}$ hence the sequence of values $f(q^j)$ $j \in \mathbb{Z}$ do not depend on the choice of μ.

Very significant point: Even though $f(x) = \lambda f(qx)$ has many solutions, non of which have any distinguishing analytic properties, they differ multiplicatively by q-periodic functions, hence their values on the sequence $\{xq^j \mid j \in \mathbb{Z}\}$ do not depend (up to a multiplicative constant) on the different solutions. Consequently we are going to be able to make sense of the asymptotic behavior of solutions to difference equations as we approach $x = 0$ or $x = +\infty$.

July 21, 1977

Study the difference equation:

$$c_1 f(x) + c_2 f(qx) + c_3 f(q^2x) = x\left(c_4 f(x) + c_5 f(qx) + c_6 f(q^2x)\right)$$

Look for a Laurent series solution $\sum a_\nu x^\nu$ where ν will run over a coset $\mu + n$, $n \in \mathbb{Z}$. Recursion relation for coefficients is

$$a_\nu = \frac{c_4 + c_5 q^{\nu-1} + c_6 q^{2\nu-2}}{c_1 + c_2 q^\nu + c_3 q^{2\nu}}\, a_{\nu-1}$$

There are at most four values of q^ν such that the numerator or denominator vanish, hence for $|\nu|$ large a_ν and $a_{\nu-1}$ will both be zero or both non-zero. We can determine the annulus of convergence by the ratio test. First consider the direction $\nu \to +\infty$: Convergence occurs when

$$\lim_{\nu\to+\infty} \left|\frac{a_\nu x^\nu}{a_{\nu-1}x^{\nu-1}}\right| = \frac{|c_4|}{|c_1|}|x| < 1 \qquad \text{since } q^\nu \to 0$$

or $$|x| < \frac{|c_1|}{|c_4|}$$

(We suppose that c_1, c_4 are not both 0 other the original diff. eqn. is not of second order) In the direction $\nu \to -\infty$ convergence occurs when

$$\lim_{\nu\to-\infty}\left|\frac{a_{\nu-1}x^{\nu-1}}{a_\nu x^\nu}\right| = \lim_{\nu\to-\infty}\left|\frac{c_1 + c_2 q^\nu + c_3 q^{2\nu}}{c_4 + c_5 q^{\nu-1} + c_6 q^{2\nu-2}}\right|\frac{1}{|x|} = \frac{|c_3|}{|c_6|}|q|^2\frac{1}{|x|} < 1$$

or $$|x| > \frac{|c_3|}{|c_6|}|q|^2$$

Particularly interesting will be the case when $c_4 = 0$

and $c_3 = 0$ because then all these Laurent series, no matter what μ is, converge in the whole annulus $0 < |x| < \infty$.

If the equation is written in the form

$$(c_1 - c_4 x) f(x) + (c_2 - c_5 x) f(qx) + (c_3 - c_6 x) f(q^2 x) = 0$$

then $x = \frac{c_1}{c_4}$ is a singular point which prevents one from calculating $f(x)$ from $f(qx)$ and $f(q^2x)$. Hence it is a singular point of any solution regular in an annulus $r/|q| \leq |x| \leq r$ with $r < \left|\frac{c_1}{c_4}\right|$. This is why the series can't converge except for $|x| < \frac{|c_1|}{|c_4|}$. Similarly for the other singularity $c_3 - c_6 q^{-2} x = 0$ or $x = \frac{c_3}{c_6} q^2$.

So suppose $c_4 = c_3 = 0$. If we replace $f(x)$ by $x^a g(x)$, then we change c_6 to $c_6 q^{2a}$, so we can suppose $c_1 = c_6 = 1$.

Go back to original equation in the following form

(1) $$c_1 f(x) + c_2 f(qx) + c_3 f(q^2x) = qx\left(c_4 f(x) + c_5 f(qx) + c_6 f(q^2x)\right)$$

and replace x by $\frac{1}{q^2 x}$. You get

$$c_1 f\left(\frac{1}{q^2x}\right) + c_2 f\left(\frac{1}{qx}\right) + c_3 f\left(\frac{1}{x}\right) = \frac{1}{qx}\left(c_4 f\left(\frac{1}{q^2x}\right) + c_5 f\left(\frac{1}{qx}\right) + c_6 f\left(\frac{1}{x}\right)\right)$$

which if we put $g(x) = f\left(\frac{1}{x}\right)$ can be written

(2) $$c_6 g(x) + c_5 g(qx) + c_4 g(q^2x) = qx\left(c_3 g(x) + c_2 g(qx) + c_1 g(q^2x)\right)$$

which is the same equation as above except that c_i has been replaced by c_{7-i}. Thus if $f(x)$ satisfies (1), $f\left(\frac{1}{x}\right)$ satisfies (2).

With this new notation the recursion formula becomes $$a_\nu = \frac{qc_4 + c_5 q^\nu + c_6 q^{2\nu-1}}{c_1 + c_2 q^\nu + c_3 q^{2\nu}}\, a_{\nu-1}$$

There seems to be no simplification, so go back to the old form with $c_1 = c_6 = 1$:

$$f(x) + c_2 f(qx) = x\left(c_5 f(qx) + f(q^2 x)\right)$$

$$f(x) = (c_5 x - c_2) f(qx) + x f(q^2 x)$$

$$\begin{pmatrix} f(qx) \\ f(x) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ x & c_5 x - c_2 \end{pmatrix}}_{A(x)} \begin{pmatrix} f(q^2 x) \\ f(qx) \end{pmatrix}$$

Since $A(x) \in GL_2(\mathbb{C}[x, x^{-1}])$ it defines a 2-diml vector bundle on the elliptic curve, whose global sections are given by Laurent series solutions of the difference equation. Thus we see from the recursion relation

$$a_n = \frac{c_5 q^{n-1} + q^{2n-2}}{1 + c_2 q^n} a_{n-1}$$

(Notice that n is an integer now because I want holomorphic functions on $\mathbb{C}^*$), that where $c_5, c_2 \notin \langle q \rangle$, then there is exactly one non-trivial global section. Since $\det(A(x)) = -x$ the Λ^2 of this bundle is the line bundle of degree 1 with section $\Theta(-x)$ vanishing simply at $x = 1$, i.e. at the origin of the elliptic curve. Thus we get for vector bundle the unique non-trivial extension

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0$$

once we show this vector bundle is indecomposable. Otherwise it could be $L \oplus L^{-1}(1)$ where L is of degree 0 and non-trivial.

However from the difference equation viewpt it perhaps [redacted] isn't important that ν be integral.

Let's go back to the recursion formula for the coefficients

$$a_\nu = \frac{c_5 q^{\nu-1} + \text{[redacted]}\, q^{2\nu-2}}{1 + c_2 q^{\nu}}\, a_{\nu-1}$$

If $c_2 \neq 0$ we can μ so that $1 + c_2 q^\mu = 0$ and grind out $a_{\mu+1}, a_{\mu+2}, \cdots$ starting from $a_\mu = 1$ and $a_{\mu-n} = 0$, $n \geq 1$. This gives a solution behaving like x^μ as $x \to 0$. It will be the only solution provided the numerator doesn't vanish.

$$c_5 \,\text{[redacted]} + q^{n+\mu-1} \neq 0 \qquad \text{if } c_2 q^\mu \,\text{[redacted]} = -1$$

$$c_2 c_5 \neq q^{n-1} \qquad \text{all i.e.}$$

i.e. if $c_2 c_5 \notin \langle q \rangle$.

<u>Question</u>: Is there exactly one solution up to a scalar multiple which is regular at $x = 0$?

Recall that the solutions of the difference equation, meromorphic on $\mathbb{C}^*$, form a 2-dim'l vector space over the field of elliptic functions. Recall also that x^μ is not in this category [redacted] unless $\mu \in \mathbb{Z}$. However in so far as the difference equation

$$f(x) = \lambda f(qx) \qquad \lambda = q^\mu$$

is concerned, we can replace the [redacted] pseudo-solution x^μ by $\dfrac{\theta(\lambda x)}{\theta(x)}$. [redacted] The same principle holds for the 2nd order difference equation. Thus I should hunt for solutions of the form

$$\frac{\theta(\lambda x)}{\theta(x)} \sum_{n \geq 0} a_n x^n$$

with $a_0 = 1$.

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$$c_1 f(x) + c_2 f(qx) + c_3 f(q^2 x) = x\left(c_4 f(x) + c_5 f(qx) + c_6 f(q^2 x)\right)$$

has singularities in $\mathbb{C}^*$ at those x such that

$$c_1 - c_4 x = 0 \qquad q^2 c_3 - c_6 x = 0$$

The first case to understand well, I think, is the case where there are no singular points. The reason is that in this case I do have associated to the difference equation a definite 2-dim'l vector bundle over the elliptic curve $\mathbb{C}^*/\langle q \rangle$.
There are no singular points in $\mathbb{C}^*$ when

$$c_1 \text{ or } c_4 = 0 \qquad \text{and} \qquad c_3 \text{ or } c_6 = 0$$

Recall that not both c_1, c_4 is zero and not both c_3 and c_6 is zero, otherwise the difference equation is of order < 2.

$$(1) \qquad (c_1 - c_4 x) f(x) + (c_2 - c_5 x) f(qx) + (c_3 - c_6 x) f(q^2 x) = 0$$

If f, g are solutions, their Wronskian

$$W(x) = \begin{vmatrix} f(x) & g(x) \\ f(qx) & g(qx) \end{vmatrix}$$

satisfies

$$W(qx) = \begin{vmatrix} f(qx) & g(qx) \\ \frac{c_1 - c_4 x}{c_3 - c_6 x} f(x) & \frac{c_1 - c_4 x}{c_3 - c_6 x} g(x) \end{vmatrix} = \frac{c_1 - c_4 x}{c_3 - c_6 x} W(x)$$

or

$$\boxed{W(x) = \frac{c_3 - c_6 x}{c_1 - c_4 x} W(qx)}$$

Try a solution of the form

(2) $$\frac{\theta(x)}{\theta(\lambda x)} \sum_{n \geq 0} a_n x^n \qquad a_0 = 1$$

since $$\frac{\theta(qx)}{\theta(\lambda q x)} = \frac{\theta(x)}{x} \frac{\lambda x}{\theta(\lambda x)} = \lambda \frac{\theta(x)}{\theta(\lambda x)}$$

we get the recursion relations

$$\sum (c_1 + c_2 \lambda q^n + c_3 \lambda^2 q^{2n}) a_n x^n = \sum (c_4 + c_5 \lambda q^n + c_6 \lambda^2 q^{2n}) a_n x^{n+1}$$

or

$$a_n = \frac{c_4 + c_5 \lambda q^{n-1} + c_6 \lambda^2 q^{2n-2}}{c_1 + c_2 \lambda q^n + c_3 \lambda^2 q^{2n}} a_{n-1}$$

with the indicial equation

$$c_1 + c_2 \lambda + c_3 \lambda^2 = 0$$

Generic case: The indicial equation has two non-zero roots λ_1, λ_2 such that $\lambda_1/\lambda_2 \notin \langle q \rangle$. In this case we get two solutions of the equation (1) of the form (2) which are convergent series near zero times meromorphic functions (hence meromorphic) in $\mathbb{C}^*$. These two solutions are necessarily linearly independent over the field of elliptic functions because of their asymptotic behavior as $x \to 0$.

Note that since $(c_1, c_4) \neq (0,0)$ if a_n does not eventually become zero, the circle of convergence of the series is

$$|x| < \left|\frac{c_1}{c_4}\right|$$

hence we must have $c_1 \neq 0$. Can suppose $c_1 = 1$.

Very special case: $c_1 = 1$, $c_2 = c_3 = 0$ so there are no roots of the indicial equation!

Ultra-special case: $c_1 = 1$, $c_2 = c_3 = c_4 = c_5 = 0$, $c_6 \neq 0$:

$$f(x) = c_6 x f(q^2 x)$$

The series method gives the solutions

$$f(x) = \frac{\theta_q(x)}{\theta_q(\lambda x)} \theta_{q^2}(c_6 \lambda^2 x)$$

In effect $f(q^2 x) = \lambda^2 \dfrac{1}{c_6 \lambda^2} f(x)$. Simplest solutions are perhaps those for $\lambda = 1$, $\lambda = -1$. Suppose $c_6 = 1$. What can one say about the asymptotic behavior of the solutions as $x \to 0$?

From the equation we get

$$f(q^{2n} x) = \frac{1}{q^{2(n-1)} x} f(q^{2(n-1)} x) = \cdots = \frac{1}{q^{n(n-1)} x^n} f(x)$$

$$= q^{-n(n-1)} x^{-n} f(x)$$

which goes to infinity fairly fast as $n \to +\infty$ or $n \to -\infty$. The ratio of any two functions is q^2-periodic, which means that for any solution we get the same basic asymptotic behavior for the two sequences $\{q^{2n}x\}, \{q^{2n+1}x\}$, but the constants are independent.

The solution $\theta_{q^2}(x)$ is a basis for the solutions over the q^2-periodic function. The q^2-periodic fns. form a degree 2 extension of the q-periodic fns. with basis 1, and $\frac{\theta(x)}{\theta(-x)}$. Hence $\theta_{q^2}(x)$ and $\frac{\theta(x)}{\theta(-x)}\theta_{q^2}(x)$ form a basis of solutions over the field of q-periodic functions.

Next consider $c_1 = 1$, $c_2 = c_3 = 0$. Then as we go toward zero only the terms in a solution $\frac{\theta(x)}{\theta(\lambda x)}\sum_{n\in\mathbb{Z}} a_n x^n$ with n large and negative should matter in the asymptotic behavior of the solution. Recursion relation is

$$a_n = \left(c_4 + c_5\lambda q^{n-1} + c_6\lambda^2 q^{2n-2}\right) a_{n-1}$$

and as $n \to -\infty$ only the $c_6\lambda^2 q^{2n-2}$ matters. Hence we can expect the solutions to behave like the solutions of $f(x) = c_6 x f(q^2 x)$.

The interesting problem for me is the following. Suppose c_3 and $c_4 = 0$ so we have no singular pts. in $\mathbb{C}^*$. (Note that we can't have c_1 or $c_6 = 0$ otherwise the series we are trying to construct don't converge. In this situation we might have something like asymptotic expansions.) Suppose c_2 and $c_5 \neq 0$. We get the recursion formulas

$$a_n = \frac{c_5 + c_6 \lambda q^{n-1}}{1 + c_2 \lambda q^n} (\lambda q^{n-1}) a_{n-1}$$

If we choose λ so that $1 + c_2 \lambda = 0$, then we get a solution of the form

$$\frac{\theta(x)}{\theta(\lambda x)} \Big(\sum_{n \geq 0} a_n x^n \Big) \qquad a_0 = 1$$

which is in some sense regular at $x = 0$.
If we choose λ so that

$$c_5 + c_6 \lambda = 0$$

then we get a solution of the form

$$\frac{\theta(x)}{\theta(\lambda x)} \Big(\sum_{n \leq 0} a_n x^n \Big) \qquad a_0 = 1$$

which is regular at ∞. <u>Question:</u> What is the Wronskian of these two solutions?

By scaling f, i.e. replacing f by $\frac{\theta(x)}{\theta(\lambda x)} f$ I can suppose $c_2 = -1$, and by scaling x I can suppose $c_6 = -1$, whence the recursion relations become

$$a_n = \frac{c_5 - q^{n-1}}{1 - q^n} q^{n-1} a_{n-1}$$

for the series regular at zero. If $c_5 = q^n$ for some $n \geq 0$, then this solution is a polynomial in x and hence coincides with a multiple of the other solution. Hence the Wronskian vanishes if $c_5 = q^n$, $n \geq 0$.

Question: Is there a 2nd order differential equation which exhibits this sort of behavior?

One tries a D.E. of the form

$$\left(c_1 x^2 \frac{d^2}{dx^2} + c_2 x \frac{d}{dx} + c_3\right) y = x\left(c_4 x^2 \frac{d^2}{dx^2} + c_5 x \frac{d}{dx} + c_6\right) y$$

leading to a recursion relation of the form

$$a_n = \frac{c_4 (n-1)(n-2) + c_5 (n-1) + c_6}{c_1 n(n-1) + c_2 n + c_3} a_{n-1}$$

If this give a Laurent series extending in both directions, one calculates its annulus of convergence by the ratio test:

$$\lim_{n \to -\infty} \left|\frac{a_{n-1}}{a_n}\right| < |x| < \lim_{n \to +\infty} \left|\frac{a_{n-1}}{a_n}\right|$$

But because $\frac{a_{n-1}}{a_n}$ is a rational function of n these two limits are the same, hence one can't have Laurent series solutions of equations of the above type which aren't regular at either 0 or ∞.